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Cusps of Hilbert modular varieties

BY D. B. MCREYNOLDS†
Department of Mathematics, California Institute of Technology, Pasadena, CA 91125*, U.S.A.*
e-mail: dmcreyn@caltech.edu

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Abstract

Motivated by a question of Hirzebruch on the possible topological types of cusp cross-sections of Hilbert modular varieties, we give a necessary and sufficient condition for a manifold M to be diffeomorphic to a cusp cross-section of a Hilbert modular variety. Specialized to Hilbert modular surfaces, this proves that every Sol 3–manifold is diffeo morphic to a cusp cross-section of a (generalized) Hilbert modular surface. We also deduce an obstruction to geometric bounding in this setting. Consequently, there exist Sol 3–manifolds that cannot arise as a cusp cross-section of a 1–cusped nonsingular Hilbert modular surface.

1. *Introduction*

Main results

It is a classicalproblem in topology to decide whether or not a closed n–manifold M bounds. Hamrick and Royster [**5**] resolved this in the affirmative for flat n–manifolds and Rohlin [**12**] for closed 3–manifolds. However, beyond these two classes there are few other settings where the story is nearly this complete. The introduction of geometry to a topological problem provides additional structure which can lead to new insight. This philosophy serves as motivation for the primary concern of this paper which is a geometric notion of bounding and its specialization to infrasolv manifolds.

Let k be a totally real number field with $[k:\mathbf{Q}]=n$, $\mathcal{O}_k$ the ring of integers of k, and $\sigma_1, \ldots, \sigma_n$ denote the n real embeddings of k. The group $\mathrm{PSL}(2;\mathcal{O}_k)$ is an arithmetic subgroup of the n–fold product $(\mathrm{PSL}(2;\mathbf{R}))^n$ (see [**2**]) via the embedding $\xi \mapsto (\sigma_1(\xi), \ldots, \sigma_n(\xi))$ for $\xi \in \mathrm{PSL}(2;\mathcal{O}_k)$. Through this embedding, $\mathrm{PSL}(2;\mathcal{O}_k)$ acts with finite volume on the n–fold product of real hyperbolic planes $(\mathbf{H}^2)^n$. The group $\mathrm{PSL}(2;\mathcal{O}_k)$ is called *the Hilbert modular group*. More generally, we call any subgroup Λ of $\mathrm{PSL}(2;k)$ which is commensurable with $\mathrm{PSL}(2;\mathcal{O}_k)$ a *Hilbert modular group* and the quotients $(\mathbf{H}^2)^n/\Lambda$, *Hilbert modular varieties*. In the case that k is a real quadratic number field, these quotients are called *Hilbert modular surfaces*. For more on Hilbert modular surfaces, see [**6**] or [**16**].

The primary focus of this paper is cusp cross-sections of Hilbert modular varieties. These infrasolv manifolds are virtual n–torus bundles over $(n-1)$–tori where $[k:\mathbf{Q}] = n$ and

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rank $\mathcal{O}_k^\times = n-1$. For brevity, we simply call these *virtual* $(n, n-1)$*–torus bundles*. Recall that an n–torus bundle over an m–torus is the total space of a fiber bundle with base manifold T^m and fiber T^n. We call such manifolds simply (n, m)*–torus bundles*. We say that N is a *virtual* (n, m)*–torus bundle* if N is finitely covered by an (n, m)–torus bundle.

In [**9**], cusp cross-sections of real, complex, and quaternionic arithmetic hyperbolic n–orbifolds were classified. We continue this theme by classifying cusp cross-sections of Hilbert modular varieties. By taking the quotient of the associated neutered space for the Hilbert modular group Λ, we obtain a compact Riemannian $2n$–orbifold whose totally geodesic boundaries are the cusp cross-sections equipped with metrics (defined up to scaling) coming from the associated solvable Lie group.

Before stating our first classification result, we introduce an additional piece of terminology.

For a totally real number field k, we say $\beta \in k$ is *totally positive* if $\sigma_j(\beta) > 0$ for $j = 1, \ldots, n$. We denote the set of totally positive elements and totally positive integers by k_+ and $\mathcal{O}_{k,+}$, and define the sets $k_+^\times = k_+ \cap k^\times$, $\mathcal{O}_{k,+}^\times = \mathcal{O}_k^\times \cap \mathcal{O}_{k,+}$. We say that a virtual torus bundle N is *k–defined* if there exists a faithful representation $\rho\colon \pi_1(N) \to k \rtimes k_+^\times$. If in addition $\rho(\pi_1(N))$ is commensurable with $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$, we say that N is *k–arithmetic*.

Our first result is:

THEOREM 1·1. *A virtual* $(n, n-1)$*–torus bundle N is diffeomorphic to a cusp cross-section of a Hilbert modular variety over k if and only if* $\pi_1(N)$ *is k–arithmetic.*

Theorem 1·1 answers a question of Hirzebruch [**6**, page 203] who asked (in our terminology) which k–arithmetic torus bundles arise as cusp cross-sections of Hilbert modular varieties. See Subsection 3·3 for more on this.

Every (2, 1)–torus bundle admits either a Euclidean, Nil or Sol geometry. Long and Reid [**8**] proved that the (2, 1)–torus bundles which admit a Euclidean structure are diffeomorphic to cusp cross-sections of arithmetic real hyperbolic 4–orbifolds. In [**9**], we proved that those that admit Nil structures are diffeomorphic to cusp cross-sections of arithmetic complex hyperbolic 2–orbifolds. In this paper, we prove (see Section 5 for the definitions):

THEOREM 1·2. *Every* Sol 3*–manifold is diffeomorphic to a cusp cross-section of a generalized Hilbert modular surface.*

We note that this shows closed 3–manifolds modelled on this three geometries bound; of course, this is not new as Rohlin proved this for any 3–manifold.

Using the Atiyah–Patodi–Singer signature formula, Long and Reid [**7**] showed that a flat 3–manifold which arises as a cusp cross-section of a 1–cusped real hyperbolic 4–manifold must have integral η–invariant. Together with Ouyang's work, this proves that certain flat 3–manifolds cannot be the cusp cross-section of a 1–cusped real hyperbolic 4–manifold. We conclude this article with a similar result. Specifically, using the work of Hirzebruch [**6**], Atiyah–Donnely–Singer [**1**] and Cheeger–Gromov [**3**], we prove:

THEOREM 1·3. *There exists a* Sol 3*–manifold which cannot be diffeomorphic to a cusp cross-section of any* 1*–cusped Hilbert modular surface with torsion free fundamental group.*

2. *Preliminary material*

2·1. *Stabilizer groups*

For $v \in \partial\mathbf{H}^n$, the group $\mathrm{Stab}(v) = \{\gamma \in \mathrm{Isom}(\mathbf{H}^n) : \gamma v = v\}$ is isomorphic to $\mathbf{R}^{n-1} \rtimes (\mathbf{R}^+ \times \mathrm{O}(n-1))$. For $v \in \partial\mathbf{H}^n$ and $H < \mathrm{Isom}(\mathbf{H}^n)$, we define the *stabilizer group of H at v* to be $\triangle_v(H) = \mathrm{Stab}(v) \bigcap H$. When $\triangle_v(H)$ contains a parabolic isometry, we call $\triangle_v(H)$ the *maximal peripheral subgroup of H at v* and say that H has a *cusp at v*. Often, we simply write $\triangle(H)$.

Cusps, horospheres, and cusp cross-sections are defined as in the hyperbolic setting via Iwasawa decompositions of $(\mathrm{PSL}(2; \mathbf{R}))^r$. For the Hilbert modular group $\mathrm{PSL}(2; \mathcal{O}_k)$ over a totally real number field k, the stabilizer of the boundary point corresponding to the Iwasawa decomposition given by the r–fold product of the groups $\mathbf{A}$, $\mathbf{N}$, $\mathbf{K}$ is the peripheral subgroup

$$\triangle = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} \ : \ \alpha \in \mathcal{O}_k,\ \beta \in \mathcal{O}_{k,+}^{\times} \right\}.$$

Every peripheral subgroup of $\mathrm{PSL}(2; \mathcal{O}_k)$ is conjugate in $\mathrm{PSL}(2; k)$ to a group commensurable with $\triangle$.

2·2. *Infrasolv manifolds and smooth rigidity*

For a simply connected, connected solvable Lie group S, the affine group of S is $\mathrm{Aff}(S) = S \rtimes \mathrm{Aut}(S)$. We say that a discrete subgroup $\Gamma < \mathrm{Aff}(S)$ is an *infrasolv group modelled on S* if $\Gamma \bigcap S$ is finite index in Γ and S/Γ is compact. An infrasolv group which is a subgroup of S will be called a *solv group modelled on S*. Any smooth manifold which is diffeomorphic to S/Γ for some infrasolv group will be called an *infrasolv manifold modelled on S*. When Γ is a solv group, we call the manifold S/Γ a *solv manifold modelled on S*.

We require the following rigidity result of Mostow [**10**].

THEOREM 2·1 (Mostow [**10**]). *Let M_1 and M_2 be infrasolv manifolds. If $\pi_1(M_1)$ and $\pi_1(M_2)$ are isomorphic, then M_1 is diffeomorphic to M_2.*

3. *Cusps of Hilbert modular varieties*

In this section, we prove Theorem 1·1. The philosophy for the proof is simple. Using the arithmeticity assumption on the torus bundle N, we construct an injective homomorphism $\rho\colon \pi_1(N) \to \triangle(\mathrm{PSL}(2; \mathcal{O}_k))$. To find a Hilbert modular group Λ for which $\triangle(\Lambda) = \rho(\pi_1(N))$, we are reduced to making a subgroup separability argument. The proof is completed by applying Theorem 2·1. The remainder of this section is devoted to the details.

3·1. *Subgroup separability*

Recall that if G is a group, $H < G$ and $g \in G \setminus H$, we say H and g are *separated* if there exists a subgroup K of finite index in G which contains H but not g. We say that $H < G$ is *separable* in G if every $g \in G \setminus H$ and H can be separated.

As in [**9**], the main technical result we make use of is:

THEOREM 3·1. *Let Λ be a Hilbert modular group and $\triangle(\Lambda)$, a maximal peripheral subgroup. Then every subgroup of $\triangle(\Lambda)$ is separable in Λ.*

3·2. *Proof of Theorem* 1·1

The following establishes a correspondence between k–arithmetic torus bundle groups and maximal peripheral subgroups of Hilbert modular groups.

THEOREM 3·2 (Correspondence theorem). *Let N be a k–arithmetic torus bundle. Then there exists a faithful representation $\psi: \pi_1(N) \to \triangle(\mathrm{PSL}(2; \mathcal{O}_k))$ such that $\psi(\pi_1(N))$ is a finite index subgroup of $\triangle(\mathrm{PSL}(2; \mathcal{O}_k))$. Moreover, there exists a finite index subgroup Λ of $\mathrm{PSL}(2; \mathcal{O}_k)$ such that $\triangle(\Lambda) = \psi(\pi_1(N))$.*

We defer the proof of Theorem 3·2 for the moment in order to prove Theorem 1·1.

Proof of Theorem 1·1. For the direct implication, since N is diffeomorphic to a cusp cross-section of a Hilbert modular variety, there exists a Hilbert modular group Λ and an isomorphism $\psi: \pi_1(N) \to \triangle(\Lambda)$. To obtain an injective homomorphism $\rho: \pi_1(N) \to k \rtimes k_+^\times$ such that $\rho(\pi_1(N))$ is commensurable with $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$, we argue as follows. By conjugating by an element γ of $\mathrm{PSL}(2; k)$, we can assume that

$$\gamma^{-1}\psi(\pi_1(N))\gamma \subset B_k = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} \ : \ \alpha \in k, \ \beta \in k_+^\times \right\}.$$

As $\gamma \in \mathrm{PSL}(2; k)$, $\gamma^{-1}\Lambda\gamma$ remains a Hilbert modular group, and moreover, $\gamma^{-1}\psi(\pi_1(N))\gamma$ is commensurable with

$$\triangle(\mathrm{PSL}(2; \mathcal{O}_k)) = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} \ : \ \alpha \in \mathcal{O}_k, \ \beta \in \mathcal{O}_{k,+}^\times \right\}.$$

To obtain the faithful representation ρ, we simply compose $\mu_\gamma \circ \psi$ with the isomorphism $\iota: B_k \to k \rtimes k_+^\times$ given by $\iota\left(\begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix}\right) = (\alpha, \beta)$.

For the reverse implication, we apply Theorem 3·2 and Theorem 2·1. Specifically, let Λ by the Hilbert modular group guaranteed by Theorem 3·2 and let N' denote an embedded cusp cross-section associated with $\triangle(\Lambda)$. As a smooth manifold, N' is of the form $\mathbf{R}^{2n-1}/\triangle(\Lambda)$. By Theorem 3·2, we have an isomorphism $\psi: \pi_1(N) \to \pi_1(N')$. Applying Theorem 2·1, we obtain the desired diffeomorphism between N and N'.

In the proof of Theorem 3·2 the following lemma is required.

LEMMA 3·3. *Let N be a k–arithmetic torus bundle. Then there exists an injective homomorphism $\rho: \pi_1(N) \to \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. Moreover, $\rho(\pi_1(N))$ is a finite index subgroups of $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$.*

Proof. Since N is k–arithmetic, we have a faithful representation $\theta: \pi_1(N) \to k \rtimes k_+^\times$ such that $\theta(\pi_1(N))$ is commensurable with $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. Hence, given $(\alpha, \beta) \in \theta(\pi_1(N))$, we have for some $m \in \mathbf{N}$,

$$(\alpha + \beta\alpha + \beta^2\alpha + \cdots + \beta^{m-1}\alpha, \beta^m) \in \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times.$$

Consequently, $\beta^m \in \mathcal{O}_{k,+}^\times$ and thus $\beta \in \mathcal{O}_{k,+}^\times$. Even so, it may be the case that (α, β) is not contained in $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. This is rectified as follows. Select a generating set for $\pi_1(N)$, say $g_1, \ldots, g_u$. For each generator, we have $\theta(g_j) = (\alpha_j, \beta_j)$ with $\alpha_j \in k$ and $\beta_j \in \mathcal{O}_{k,+}^\times$. Since k is the field of fractions of $\mathcal{O}_k$, we can select $\lambda_j \in \mathcal{O}_k$ such that $(0, \lambda_j)\theta(g_j)(0, \lambda_j)^{-1} \in \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. Note that

$$(0, \lambda_j)\theta(g_j)(0, \lambda_j)^{-1} = (\lambda_j\alpha_j, \beta_j),$$

and so the second coordinate β_j is unchanged. Finally, for $\lambda = \lambda_1 \cdots \lambda_u$, define $\rho = \mu_{(0,\lambda)} \circ \theta$, where $\mu_{(0,\lambda)}$ denotes the inner automorphism determined by $(0, \lambda)$. By construction, ρ is a faithful representation of $\pi_1(N)$ onto a finite index subgroup of $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^{\times}$.

Proof of Theorem 3·2. By Lemma 3·3, we have an injective homomorphism

$$\rho \colon \pi_1(N) \longrightarrow \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^{\times}$$

such that $\rho(\pi_1(N))$ is a finite index subgroup. To obtain the injective homomorphism ψ, we compose ρ with the isomorphism

$$\iota^{-1} \colon \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^{\times} \longrightarrow \triangle(\mathrm{PSL}(2; \mathcal{O}_k))$$

where $\iota^{-1}(\alpha, \beta) = \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix}$. That ψ is faithful and $\psi(\pi_1(N))$ is a finite index subgroup of $\triangle(\mathrm{PSL}(2; \mathcal{O}_k))$ follow immediately from the properties of ρ and ι.

To find the desired subgroup Λ, we apply Theorem 3·1. Specifically, select a complete set of coset representatives $\gamma_1, \ldots, \gamma_s$ for $\triangle(\mathrm{PSL}(2; \mathcal{O}_k))/\psi(\pi_1(N))$. By Theorem 3·1, $\psi(\pi_1(N))$ is separable. Therefore for each j we can find finite index subgroups Λ_j such that $\gamma_j \notin \Lambda_j$ and $\psi(\pi_1(N)) < \Lambda_j$. To get the desired Λ, take $\Lambda = \bigcap_{j=1}^{s} \Lambda_j$.

3·3. *A question of Hirzebruch*

Let k be a totally real number field, $M < k$ an additive group of rank n (the degree of k over $\mathbf{Q}$), and $V < \mathcal{O}_{k,+}^{\times}$ a finite index subgroup such that for all $\lambda \in V$, $\lambda M \subset M$. For each pair (M, V), we define the peripheral group

$$\triangle(M, V) = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} \; : \; \alpha \in M, \; \beta \in V \right\} < \mathrm{PSL}(2; k).$$

For any Hilbert modular variety, the peripheral groups $\triangle(\Lambda)$ are conjugate (in $\mathrm{PSL}(2; k)$) to groups of the form $\triangle(M, V)$. In [**6**, p. 203], Hirzebruch mentions that it is apparently unknown whether or not every $\triangle(M, V)$ can occur as a maximal peripheral subgroup of a Hilbert modular group. The following corollary gives an affirmative answer.

COROLLARY 3·4. *For every pair* (M, V)*, there exists a Hilbert modular group* Λ *such that* $\triangle(\Lambda) = \triangle(M, V)$.

Proof. As in the proof of Lemma 3·3, we can conjugate $\triangle(M, V)$ by an element of the form $\gamma = \left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{smallmatrix}\right)$, with $\lambda \in \mathcal{O}_k$, such that $\gamma^{-1}\triangle(M, V)\gamma$ is contained in $\mathrm{PSL}(2; \mathcal{O}_k)$. Since M and V are finite index subgroups of $\mathcal{O}_k$ and $\mathcal{O}_{k,+}^{\times}$, respectively, $\gamma^{-1}\triangle(M, V)\gamma$ is a finite index subgroup of $\triangle(\mathrm{PSL}(2; \mathcal{O}_k))$. Thus there exists a finite index subgroup $\Lambda_1 < \mathrm{PSL}(2; \mathcal{O}_k)$ such that $\triangle(\Lambda_1) = \gamma^{-1}\triangle(M, V)\gamma$. Hence, for $\Lambda = \gamma\Lambda_1\gamma^{-1}$, we have $\triangle(\Lambda) = \triangle(M, V)$. As $\gamma \in \mathrm{PSL}(2; k)$, Λ is a Hilbert modular group, as required.

4. *A simple criterion for arithmeticity*

In this section, we give a simple criterion for the arithmeticity of (n, m)–torus bundles. The need for such a result is practical, as it allows one to establish the arithmeticity of a torus bundle computationally. We encourage the reader to compare the results of this section with [**9**, Corollary 5·5].

4·1. *Linear equations and presentations of torus bundle groups*

For an (orientable) $(n, n-1)$–torus bundle M, since both the base and fiber are aspherical, we have the short exact sequence induced by the long exact sequence of the fiber bundle

$$1 \longrightarrow \mathbf{Z}^n \longrightarrow \pi_1(M) \longrightarrow \mathbf{Z}^{n-1} \longrightarrow 1.$$

The action of $\mathbf{Z}^{n-1}$ on $\mathbf{Z}^n$ induces a homomorphism $\varphi\colon \mathbf{Z}^{n-1} \to \mathrm{SL}(n;\mathbf{Z})$ called the *holonomy representation*. Since peripheral subgroups in Hilbert modular groups have faithful holonomy representation, we assume throughout that φ is faithful. In particular, we obtain a faithful representation of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \mathrm{SL}(n;\mathbf{Z})$.

Of primary importance for us here is that the holonomy representation together with any finite presentation yields a homogenous linear system of equations with coefficients in $\mathbf{Z}$. This system arises as follows. For ease, select a presentation of the form

$$\langle x_1, \ldots, x_n, \overline{y_1}, \ldots, \overline{y_{n-1}} \ : \ R\rangle,$$

where $x_1, \ldots, x_n$ generate $\mathbf{Z}^m$, $\overline{y_1}, \ldots, \overline{y_{n-1}}$ are lifts of a generating set $y_1, \ldots, y_{n-1}$ for $\mathbf{Z}^{n-1}$, and R is a finite set of relations of the form

$$x_j \overline{y_k} = \overline{y_k} w_{j,k}, \quad w_{j,k} \in \langle x_1, \ldots, x_n\rangle .$$

Using the holonomy representation, we can write

$$x_j = (a_j, I), \quad \overline{y_j} = (b_j, \varphi(y_j)) \in \mathbf{R}^n \rtimes \mathrm{SL}(n;\mathbf{R}).$$

Each relation in the presentation yields a linear homogenous equation in the vector variables a_j and b_j (see below for an explicit example of how these equations arise). Namely, we insert the above forms for x_j and $\overline{y_k}$ into the relation and consider only the first coordinate. The equations we obtain are of the form

$$a_j + b_k - \varphi(y_k) - v_{j,k} = 0$$

where $w_{j,k} = (v_{j,k}, I)$. That this system has integral solutions which yield faithful representations follows from the fact that φ is faithful and induces a faithful representation of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \mathrm{SL}(n;\mathbf{Z})$.

4·2. *A simple criterion for arithmeticity*

The main result of this section is a simple criterion for arithmeticity based on the structure of the holonomy representation. In the statement and proof, let $\mathrm{Res}_{k/\mathbf{Q}}$ denotes restriction of scalars from k to $\mathbf{Q}$ and assume that $[k : \mathbf{Q}] = n$ and rank $\mathcal{O}_k^\times = n-1$. In particular, k is totally real.

THEOREM 4·1. *Let M be an orientable $(n, n-1)$–torus bundle. Then M is diffeomorphic to a cusp cross-section of a Hilbert modular variety defined over k if and only if $\varphi = Res_{k/\mathbf{Q}}(\chi)$, for some faithful character $\chi : \mathbf{Z}^{n-1} \to \mathcal{O}_{k,+}^\times$, where φ is some holonomy representation.*

Proof. For the direct implication, since M is diffeomorphic to a cusp cross-section of a Hilbert modular variety, by Theorem 1·1, we have a faithful representation

$$\rho\colon \pi_1(M) \longrightarrow \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times.$$

By restricting scalars from k to $\mathbf{Q}$, we obtain a faithful representation

$$\mathrm{Res}_{k/\mathbf{Q}}(\rho)\colon \pi_1(M) \longrightarrow \mathbf{Z}^n \rtimes \mathrm{SL}(n;\mathbf{Z}).$$

The proof is completed by noting that the holonomy map induced by this representation is simply $\mathrm{Res}_{k/\mathbf{Q}}(\chi)$, where $\chi : \mathbf{Z}^{n-1} \to \mathcal{O}_{k,+}^{\times}$ is the holonomy representation induced by the representation ρ.

For the converse, we seek a faithful representation $\rho : \pi_1(M) \to \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^{\times}$. Note that since $[k : \mathbf{Q}] = n$ and $\operatorname{rank} \mathcal{O}_k^{\times} = n - 1$, the image of $\pi_1(M)$ would necessarily be a finite index subgroup. By assumption, we have a faithful character $\chi : \mathbf{Z}^{n-1} \to \mathcal{O}_{k,+}^{\times}$. We extend this to a faithful representation of $\pi_1(M)$ into $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^{\times}$ as follows. Select a presentation as above for $\pi_1(M)$ with generators $x_1, \ldots, x_n, \overline{y_1}, \ldots, \overline{y_{n-1}}$. Write

$$x_i = (\alpha_i, 1),\ \overline{y_i} = (\gamma_i, \chi(y_i)) \in k \rtimes \mathcal{O}_{k,+}^{\times}, \tag{4·1}$$

where α_i and γ_i are to be determined. Using our presentation for $\pi_1(M)$, we obtain a system of linear homogenous equations $\mathcal{L}$ with coefficients in $\mathcal{O}_k$. Note, as above, solutions to $\mathcal{L}$ yield representations of $\pi_1(M)$ into $k \rtimes \mathcal{O}_{k,+}^{\times}$. We assert that there is a solution which yields a faithful representation. To see this, by restricting scalars from k to $\mathbf{Q}$, we obtain a linear system $\mathrm{Res}_{k/\mathbf{Q}}(\mathcal{L})$ with coefficients in $\mathbf{Z}$. Solutions to the system $\mathrm{Res}_{k/\mathbf{Q}}(\mathcal{L})$ yield representations of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \mathrm{SL}(n; \mathbf{Z})$. Moreover, a solution to $\mathrm{Res}_{k/\mathbf{Q}}(\mathcal{L})$ which yields a faithful representation is equivalent to a solution of $\mathcal{L}$ which yields a faithful representation into $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^{\times}$. That such a solution exists with integral coefficients for $\mathrm{Res}_{k/\mathbf{Q}}(\mathcal{L})$ follows from the faithfulness of $\mathrm{Res}_{k/\mathbf{Q}}(\chi)$ and our discussion in the previous subsection. This yields a solution for $\mathcal{L}$ with coefficients in $\mathcal{O}_k$ which yields a faithful representation. Therefore, M is k–arithmetic, since there exists a faithful representation $\psi : \pi_1(M) \to \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^{\times}$ such that $\psi(\pi_1(M))$ is a finite index subgroup of $\mathcal{O}_k \rtimes \mathcal{O}_k^{\times}$.

Remark. If the character χ only maps into $\mathcal{O}_k^{\times}$, the above proof yields a faithful representation $\rho : \pi_1(M) \to \mathcal{O}_k \rtimes \mathcal{O}_k^{\times}$.

5. Sol 3–*manifolds*

We give a brief review of Sol 3–manifolds (see [**14**]). Let $\mathrm{Sol} = \mathbf{R}^2 \times \mathbf{R}^+$ with group operation defined by

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) \stackrel{\text{def}}{=} (x_1 + e^{t_1} x_2, y_1 + e^{-t_1} y_2, t_1 + t_2).$$

By a Sol 3–*orbifold*, we mean a manifold M which is diffeomorphic to Sol/Γ, where Γ is a discrete subgroup of Aff(Sol) such that Sol/Γ is compact and $[\Gamma : \Gamma \cap \mathrm{Sol}] < \infty$. These manifolds, in the terminology from Section 2, are infrasolv manifolds modelled on Sol. However, the terminology used in this section for these manifolds is more prevalent.

In [**14**], Scott proved that every (2, 1)–torus bundles admits either a Euclidean, Nil, or Sol structure. The following result is easily derived from [**14**]. We include a proof here for completeness.

PROPOSITION 5·1. *Let M be an orientable* (2, 1)*–torus bundle which admits a* Sol *structure. Then there exists a faithful representation $\rho : \pi_1(M) \to \mathcal{O}_k \rtimes \mathcal{O}_k^{\times}$ for some real quadratic number field k.*

Proof. For any (2, 1)–torus bundle M, let the $\mathbf{Z}$–action be given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If the order of A is finite, then $\pi_1(M)$ is a Bieberbach group and M admits a Euclidean structure. Therefore we may assume that the order of A is infinite. If A is not diagonalizable, then some power of A is conjugate to $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ with $\alpha \neq 0$. In this case, M admits a Nil structure. Thus, we may assume that A is diagonalizable. In this case we have $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ for a conjugate of A. It follows, since $A \in \mathrm{SL}(2; \mathbf{Z})$, that β and β^{-1} are algebraic integers in the real quadratic field $\mathbf{Q}(\beta)$. Thus the representation $\varphi\colon \mathbf{Z} \to \mathrm{GL}(2; \mathbf{Z})$ is conjugate to $\mathrm{Res}_{k/\mathbf{Q}}(\chi)$, where $\chi : \mathbf{Z} \to \mathcal{O}_k^\times$ is given by $\chi(1) = \beta$. Therefore by the remark following Theorem 4·1, we have a faithful representation $\rho\colon \pi_1(M) \to \mathcal{O}_k \rtimes \mathcal{O}_k^\times$, as asserted.

Via Proposition 5·1, note every Sol 3–manifold group does faithfully represent into $\mathrm{Isom}((\mathbf{H}^2)^2)$. Those that arise as cusp cross-sections of Hilbert modular surfaces are precisely the ones whose fundamental group faithfully represents into the identity component of $\mathrm{Isom}((\mathbf{H}^2)^2)$. However, the quotients of those groups which fail to map into the identity component do produce finite volume quotients which possess 2–fold covers which are Hilbert modular surfaces. For this reason, we call such quotients *generalized Hilbert modular varieties*. Given this, Theorem 1·2 follows from this discussion in combination with Theorem 1·1.

6. *Geometric bounding*

Let W be a 1–cusped Hilbert modular surface W with torsion free fundamental group—we call W a *Hilbert modular manifold* in this case. Similar to the thick-thin decomposition of a real hyperbolic n–manifold, W has a decomposition comprised of a compact manifold $\widetilde{W}$ with boundary S and cusp end $S \times \mathbf{R}^+$. Following Schwartz [**13**] (see also [**4**]), we call the universal cover of $\widetilde{W}$ the associated *neutered manifold*, and note $\widetilde{W}$ is a compact 4–manifold with Sol 3–manifold boundary. Moreover, the locally symmetric metric $\widetilde{g}$ on W restricted to S endows S with a complete Sol metric g such that $\widetilde{g}$ is a complete, finite volume metric in the interior of $\widetilde{W}$ and (S, g) is a totally geodesic boundary.

The goal of this section is the establishment of a nontrivial obstruction for this geometric situation. The obstruction is obtained by mimicking the argument of Long–Reid [**7**] for flat 3–manifolds. This in combination with a calculation of Hirzebruch bears Theorem 1·3 from the introduction.

In [**6**], Hirzebruch extended his signature formula to Hilbert modular surfaces. The formula relates the signature of the neutered manifold $\widetilde{W}$ to a Hirzebruch L–polynomial evaluated on the Pontrjagin classes of $\widetilde{W}$ but with a correction term associated to $\partial \widetilde{W}$. When $\pi_1(W)$ contains torsion, the elliptic singularities also contribute nontrivially to this correction term, and so for simplicity, we assume throughout that $\pi_1(W)$ is torsion free. In this case, Hirzebruch's formula becomes

$$\sigma(\widetilde{W}) = \delta(E_1) + \cdots + \delta(E_r),$$

where $E_1, \ldots, E_r$ is a complete set of cusp ends of W given from the thick-thin decomposition and $\sigma(\widetilde{W})$ denotes the signature of $\widetilde{W}$. The definition of the terms $\delta(E_j)$ are given as follows. Associated to each cusp end is the $\pi_1(W)$–conjugacy class of a maximal peripheral subgroup Γ_j. The group Γ_j is conjugate in $\mathrm{PSL}(2; k)$ to a subgroup of the familiar form $\triangle(M_j, V_j)$. In turn, for the pair (M_j, V_j), we have an associated Shimuzu L–function $L(M_j, V_j, s)$—see [**15**]—defined by

$$L(M, V, s) = \sum_{\beta \in (M_j \setminus \{0\})/V_j} \frac{\mathrm{sign}(N_{k/\mathbf{Q}}(\beta))}{(N_{k/\mathbf{Q}}(\beta))^s},$$

where $N_{k/\mathbf{Q}}$ is the norm map. With this, the invariant $\delta(E_j)$ is defined to be

$$\delta(E_j) = \frac{-\operatorname{vol}(M_j)}{\pi^2} L(M_j, V_j, 1),$$

where $\operatorname{vol}(M_j)$ is the volume of $\mathbf{R}^2/M$ with respect to the pairing $\operatorname{Tr}_{k/\mathbf{Q}}$. Equivalently,

$$\operatorname{vol}(M_j) = \left|\det(\beta_i^{(j)})\right|,$$

where β_1, β_2 is a $\mathbf{Z}$–module basis for M_j and $\beta_i^{(1)}$ and $\beta_i^{(2)}$ denote the image of β_i under the two real embeddings of k into $\mathbf{R}$.

THEOREM 6·1 (Hirzebruch [**6**]). *If W is a Hilbert modular manifold with exactly one cusp, then*

$$\sigma(\widetilde{W}) = \frac{-\operatorname{vol}(M)}{\pi^2} L(M, V, 1)$$

for the unique $\pi_1(W)$–conjugacy class $\triangle(M, V)$.

As we seek an integrality condition, it is convenient to change the pair M, V. Associated to the $\mathbf{Z}$–module M is the *dual lattice* M^* defined to be the image of M under the duality pairing provided by $\operatorname{Tr}_{k/\mathbf{Q}}$.

PROPOSITION 6·2. *For a horosphere $\mathcal{H}$ stabilized by $\triangle(M, V)$ and $\triangle(M^*, V)$, $\mathcal{H}/\triangle(M, V)$ and $\mathcal{H}/\triangle(M^*, V)$ are diffeomorphic* Sol *3–manifolds.*

Proof. Let $\varphi_M, \varphi_{M^*} \colon V \to \mathrm{SL}(2; \mathbf{Z})$ be the holonomy representations for $\triangle(M, V)$ and $\triangle(M^*, V)$. The pairing $\operatorname{Tr}_{k/\mathbf{Q}}$ can be viewed as an element of $\lambda \in \mathrm{SL}(2; \mathbf{Z})$ such that $\lambda M = M^*$. By construction $\varphi_{M^*} = \lambda(\varphi_M)\lambda^{-1}$, and so we have an isomorphism $\rho \colon \triangle(M, V) \to \triangle(M^*, V)$ given by

$$\rho(\beta, \varphi_M(\alpha)) = (\lambda\beta, \lambda\varphi_M(\alpha)\lambda^{-1}).$$

The proof is completed by appealing to the smooth rigidity theorem of Mostow Theorem 2·1.

Hecke (see [**1**]) related the L–functions $L(M, V, s)$ and $L(M^*, V, s)$ by the functional equation $H(M, V, s) = (-1)^s H(M^*, V, 1 - s)$, where

$$H(M, V, s) = \left[\Gamma\left(\frac{s+1}{2}\right)\right]^2 \pi^{-(s+1)} \left[\operatorname{vol}(M)\right]^s L(M, V, s).$$

The specialization of this functional equation at $s = 1$ produces

$$(\Gamma(1))^2 \pi^{-2} \operatorname{vol}(M) L(M, V, 1) = -\left(\Gamma\left(\frac{1}{2}\right)\right)^2 \pi^{-1} L(M^*, V, 0)$$
$$L(M^*, V, 0) = -\frac{\operatorname{vol}(M)}{\pi^2} L(M, V, 1),$$

and thus from this and Theorem 6·1, we obtain

$$\sigma(\widetilde{W}) = L(M^*, V, 0). \tag{6·1}$$

Let us take stock what has been done. For a 1–cusped Hilbert modular manifold W with cusp cross-section S, we have associated to S the invariant $\delta(S \times \mathbf{R}^+)$. As both M and V depend on the associated Sol metric on S afforded by its embedding as a cusp cross-section, the invariant $\delta(S \times \mathbf{R}^+)$ depends on the associated Sol metric on S. Our goal is to use the integrality of $\sigma(\widetilde{W})$ and (6·1) to produce an obstruction for S to topologically occur in this geometric setting. For this, it remains to show the invariant $\delta(S \times \mathbf{R}^+)$ is independent of the Sol structure on S.

Given a peripheral group $\triangle(M, V)$ and stabilized horosphere $\mathcal{H}$, the metric on $\mathbf{H}^2_{\mathbf{R}} \times \mathbf{H}^2_{\mathbf{R}}$ endows $\mathcal{H}$ with a $\triangle(M, V)$–invariant metric $g_{\mathcal{H},M,V}$. Consequently the metric $g_{\mathcal{H},M,V}$ descends to quotient $\mathcal{H}/\triangle(M, V)$ and endows $\mathcal{H}/\triangle(M, V)$ with a complete Sol structure that depends on the horosphere $\mathcal{H}$ only up to similarity.

The formula (6·1) was also established in [**1**] where $L(M^*, V, 0)$ was reinterpreted as the η–invariant of an adiabatic limit.

THEOREM 6·3 (Atiyah–Donnely–Singer [**1**]).

$$L(M^*, V, 0) = \lim_{\varepsilon \longrightarrow 0} \eta(\mathcal{H}/\triangle(M^*, V), g_{\mathcal{H},M^*,V}/\varepsilon).$$

More generally, given any Sol structure g on S, we can define

$$\delta(S, g) = \lim_{\varepsilon \longrightarrow 0} \eta(S, g/\varepsilon).$$

The last ingredient for proof of Theorem 1·3 is the independence of $\delta(S, g)$ from g, a result established by Cheeger and Gromov [**3**].

THEOREM 6·4 (Cheeger–Gromov [**3**]). *$\delta(S, g)$ is a topological invariant of the* Sol 3–*manifold S.*

We are now in position to state and prove the principal observation needed in the proof of Theorem 1·3 (compare with [**7**]).

THEOREM 6·5. *If S is diffeomorphic to a cusp cross-section of a* 1*–cusped Hilbert modular manifold, then $\delta(S) \in \mathbf{Z}$.*

Proof. If (S, g) arises as a cusp cross-section of a 1–cusped Hilbert modular manifold W, then there is an isometric embedding $f\colon (S, g) \to W$ onto a cusp cross-section of W. Let $f_*(\pi_1(S)) = \triangle(M, V)$ with associated horosphere $\mathcal{H}$ selected such that $\mathcal{H}/\triangle(M, V)$ is embedded in W. By Proposition 6·2, $\mathcal{H}/\triangle(M^*, V)$ is diffeomorphic to S, though equipped with the metric $g_{\mathcal{H},M^*,V}$. From the computation above in combination with Theorem 6·3, $\sigma(\widetilde{W}) = \delta(S, g_{\mathcal{H},M^*,V})$ and by Theorem 6·4, the right hand side depends only on the topological type of S. Since $\sigma(\widetilde{W})$ is in $\mathbf{Z}$, $\delta(S)$ is in $\mathbf{Z}$ as asserted.

Proof of Theorem 1·3. To prove Theorem 1·3, by Theorem 6·5, it suffices to find a Sol 3–manifold S for which $\delta(S) \notin \mathbf{Z}$. For $k = \mathbf{Q}(\sqrt{3})$, the standard Hilbert modular surface W over k has precisely one cusp, since the number of cusps of a standard Hilbert modular surface over k is the ideal class number of k. Setting S to be an embedding cusp cross-section of W, the proof is completed by appealing to [**6**]. Specifically, Hirzebruch showed $\delta(S) = -1/3$.

Remark. It is unknown to the author whether or not there exist 1*–cusped Hilbert modular manifolds*. In addition, the number fields $\mathbf{Q}(\sqrt{6})$, $\mathbf{Q}(\sqrt{21})$ and $\mathbf{Q}(\sqrt{33})$ also have standard Hilbert modular surfaces with precisely one cusp for which the associated invariant $\delta(S) \notin \mathbf{Z}$. In each of these cases, $\delta(S) = -2/3$ (see [**6**, p. 236]).

Using the generalized Riemann hypothesis, K. Petersen [**11**] constructed infinite many 1–cusped Hilbert modular surfaces. However, the nature of the construction likely produces Hilbert modular surface groups with 2–torsion.

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