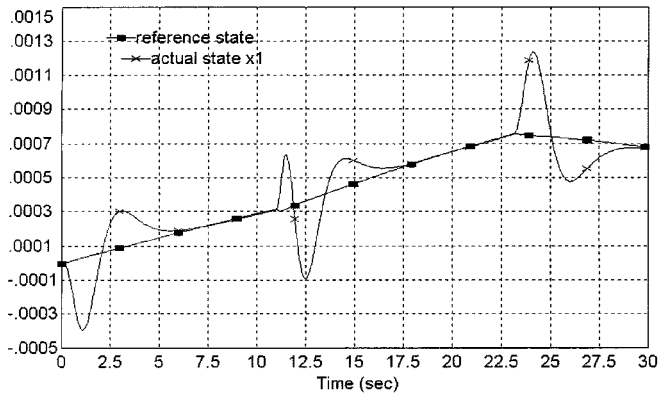

Fig. 3. State reference profile $x_{c1}(t)$ and the state x_1 versus time.

asymptotically accommodates ramp-type unmatched disturbance $f_1(t)$ and cancels out effect of matched disturbance $f_2(t)$ and uncertain nonlinear terms. Stable behavior of the internal dynamics states is demonstrated in Fig. 3.

V. Conclusion

A complete constructive algorithm to address the nonminimum-phase output-tracking problem in a class of causal nonlinearly disturbed linear MIMO systems is obtained. A sliding mode controller has been designed to provide robust tracking in the system with matched nonlinear terms and disturbances, as well as unmatched disturbances, using the method of system center and the second-order SMC-based observer. Such a controller is shown to be insensitive to matched disturbances and nonlinearities, and can accomodate unmatched terms as well. The proposed control scheme allows canceling out the error from a real-time reference input with zero high derivatives (piecewise-polynomial spline model).

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Kuhn–Tucker-Based Stability Conditions for Systems With Saturation

James A. Primbs and Monica Giannelli

***Abstract*—This note presents a new approach to deriving stability conditions for continuous-time linear systems interconnected with a saturation. The method presented here can be extended to handle a deadzone, or in general, nonlinearities in the form of piecewise linear functions. By representing the saturation as a constrained optimization problem, the necessary (Kuhn–Tucker) conditions for optimality are used to derive linear and quadratic constraints which characterize the saturation. After selecting a candidate Lyapunov function, we pose the question of whether the Lyapunov function is decreasing along trajectories of the system as an implication between the necessary conditions derived from the saturation optimization, and the time derivative of the Lyapunov function. This leads to stability conditions in terms of linear matrix inequalities (LMIs), which are obtained by an application of the S-procedure to the implication. An example is provided where the proposed technique is compared and contrasted with previous analysis methods.**

***Index Terms*—Kuhn–Tucker conditions, linear system, Lyapunov function, S-procedure, saturation.**

I. Introduction

A standard problem in stability analysis is that of a linear system interconnected with a saturation nonlinearity. The stability properties of such a system have been studied for decades, beginning from the classical absolute stability theory, with the so-called circle criterion (see, for instance, [9]) and the Popov criterion [14]. A further advancement came when Zames and Falb [19] introduced a powerful condition for stability [19], exploiting the fact that a saturation is an odd and monotonic nonlinearity. Recently, even more refined stability analysis methods have been provided in [4] and [12] for systems with saturation-like operators, using integral quadratic constraints (IQCs) [11]. Additionally, a number of linear matrix inequality (LMI) based approaches to characterizing stability have been developed (see, for instance, [6], [13] and references therein).

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From another point of view, a linear system interconnected with a saturation can be viewed as a special case of a piecewise linear system. Stability analysis of these systems has generally been attacked by partitioning the state space and constructing piecewise Lyapunov functions. (For an account of such an approach, see the thesis by Johansson [8].) A computable approach in which stability can be verified by linear matrix inequalities involves the construction of piecewise quadratic Lyapunov functions over a partitioned state space (again, see [8]). Recently, a new approach based upon quadratic Lyapunov functions on switching surfaces has been developed (see the thesis by Goncalves [7]).

In this note, a new computationally tractable approach to analyzing the stability of a single-input–single-output (SISO) linear system interconnected with piecewise linear functions is presented. We specialize to the case of a saturation, but the results can easily be extended to general piecewise linear functions. The method is based on a description of the saturation as an optimization problem and can be briefly described as follows. The saturation is described through the necessary conditions that characterize the optimal solution to an optimization problem which is equivalent to the saturation. These are simply a set of linear and quadratic equalities and inequalities in the variables involved in the optimization. After selection of a candidate Lyapunov function, we pose the question of whether the Lyapunov function is decreasing along trajectories of the system as an implication between the necessary conditions derived from the saturation optimization, and the time derivative of the Lyapunov function. This allows us to apply the S-procedure [18] which results in a linear matrix inequality [1] condition, whose feasibility can be efficiently tested.

We show that by using a Lyapunov function that contains variables from the saturation optimization, we implicitly construct a piecewise quadratic Lyapunov function, without explicitly partitioning the state space. In this sense, our results relate to those of the analysis of piecewise linear systems [8], yet our approach is much more indicative of classical absolute stability theory or IQCs. The proposed method actually provides a clear advantage over the Zames–Falb criterion, precisely because the use of a piecewise quadratic Lyapunov function captures the difference between a saturation and a deadzone, a distinction which the Zames–Falb method fails to achieve. Furthermore, the proposed methodology can easily be extended to analyze the stability of a linear system interconnected with a piecewise linear function.

II. Problem Formulation

Let $\mathbb{R}^n$ denote the space of n-dimensional real vectors and $\mathbb{R}^{n\times m}$ denote real matrices of size $n \times m$. Standard inequality signs $(\geq, >, <, \leq)$ will denote element by element inequalities when applied to vectors and matrices, and $(\succeq, \succ, \prec, \preceq)$ will signify matrix inequalities. Given a matrix M, let $M_\perp$ represent a basis for the null space of M, i.e., $MM_\perp = 0$.

The system under consideration will be the feedback connection of a SISO linear time-invariant plant and a saturation, as shown in Fig. 1, where A, B, C are the matrices defining the dynamics of the continuous-time linear system with A Hurwitz. The linear dynamics are given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$. The relationship between u and y is given by

$$u(t) = \mathrm{sat}(y(t)) \tag{3}$$

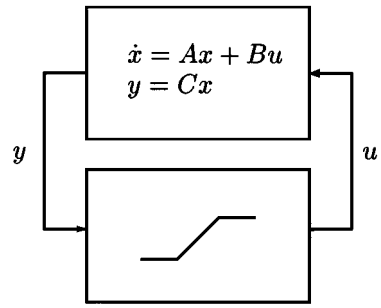

Fig. 1. Feedback connection of a SISO linear time-invariant plant and a saturation

where $\mathrm{sat}(\cdot)$ is the saturation operator defined as

$$\mathrm{sat}(y) = \frac{y}{\max\{1, |y|\}}. \tag{4}$$

The aim of this note is to provide new convex conditions to analyze the global stability of the interconnection given above.

III. The Analysis Method

In this section, we derive the main result, which is a linear matrix inequality condition for stability of the interconnected system. We begin by representing the saturation as the solution to an optimization, and then characterize that solution through the necessary conditions for optimality. Next, we consider candidate Lyapunov functions which include the state, and even the output of the saturation, u. Finally, after briefly introducing the S-procedure, Lyapunov stability is formulated as an implication involving constraints implied by the necessary conditions for the saturation optimization and the time derivative of the Lyapunov function. The S-procedure is used to convert the implication to the final result, which is a linear matrix inequality for stability.

A. Necessary (Kuhn–Tucker) Conditions for a Saturation

Consider the following optimization problem:

$$\min_u \frac{1}{2}(u-y)^2 \tag{5}$$

$$\text{s.t. } |u| \leq 1. \tag{6}$$

One may easily verify that the input–output characteristics from y to u are exactly those of the saturation operator (4). The optimization problem (5) and (6) can be used to derive an alternate description of a saturation in terms of quadratic and linear constraints by exploiting the necessary (Kuhn–Tucker [10]) conditions for constrained optimality. For the above quadratic optimization problem, the Kuhn–Tucker conditions are

$$u - y + \lambda_1 - \lambda_2 = 0 \tag{7}$$

$$\lambda_1(u-1) + \lambda_2(-u-1) = 0 \tag{8}$$

$$1 - u \geq 0 \tag{9}$$

$$1 + u \geq 0 \tag{10}$$

$$\lambda_1 \geq 0 \tag{11}$$

$$\lambda_2 \geq 0 \tag{12}$$

where λ_1 is the Lagrange multiplier corresponding to the constraint $u - 1 \leq 0$ and λ_2 corresponds to $-u - 1 \leq 0$.

It is this description of a saturation that we will exploit in our analysis results. Furthermore, we will derive other constraints implied by the Kuhn–Tucker conditions which, when coupled with a candidate Lya-

punov function as described in the following section, will allow us to derive stability LMIs.

B. Lyapunov Function Candidates

In this subsection, several choices of Lyapunov function will be presented. Since a quadratic Lyapunov function in the state may lead to a restrictive stability criterion for a system under saturation, more general Lyapunov functions are expected to provide less conservative stability estimates. We will consider the following two Lyapunov functions:

1) a quadratic Lyapunov function in the state

$$\mathcal{P}(x) = x^{\mathrm{T}} P x \tag{13}$$

where P is a symmetric matrix satisfying $P \succ 0$;

2) a quadratic Lyapunov function in the variables $[x, u]$:

$$\mathcal{P}(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} x \\ u \end{bmatrix} \tag{14}$$

where P is a symmetric matrix satisfying $P \succ 0$.

Note that since u is the following piecewise linear function of x:

$$u = \begin{cases} 1 & Cx \geq 1 \\ Cx & -1 \leq Cx \leq 1 \\ -1 & Cx \leq -1. \end{cases} \tag{15}$$

$\mathcal{P}(x, u)$ is actually piecewise quadratic in x. Hence, including u in the Lyapunov function implicitly partitions the state space and constructs a piecewise quadratic function.

Remark 3.1: It is possible to consider a Lyapunov function which includes the Lagrange multipliers along with the state x and u. Through numerous examples, we have found that adding the Lagrange multipliers does not provide stronger results than a Lyapunov function in x and u, and, hence, we do not include them.

C. S-Procedure

Both the Kuhn–Tucker conditions for the saturation, and our candidate Lyapunov function will be merged into a single LMI through a technique known as the S-procedure. More specifically, the S-procedure is a method of replacing an implication of the form

$$\begin{aligned} \Lambda_1 \varphi \geq & 0, \quad \Lambda_2 \varphi = 0, \\ \varphi^{\mathrm{T}} \Omega_1 \varphi \geq & 0, \quad \varphi^{\mathrm{T}} \Omega_2 \varphi = 0 \Rightarrow \varphi^{\mathrm{T}} \Pi \varphi \geq 0 \end{aligned}$$

by the sufficient LMI condition

$$\begin{aligned} \Lambda_{2\perp}^{\mathrm{T}} \left\{ \Lambda_2^{\mathrm{T}} R \Lambda_2 + r\Omega_1 + t\Omega_2 - \Pi \right\} \Lambda_{2\perp} \preceq & 0 \\ R \geq 0, \ \ r \geq & 0, \ \ t \in \mathbb{R}. \end{aligned}$$

We refer the reader to [18] or [1] for a justification of the S-procedure.

To apply the S-procedure, we need to formulate our problem as an implication in the form given above. The following section converts the question of stability to an implication by using constraints implied by the Kuhn–Tucker conditions and a candidate Lyapunov function. Finally, the S-procedure is used to reduce the problem to an LMI condition.

D. Stability as an Implication

Given our description of a saturation through the Kuhn–Tucker conditions (7)–(12), and a candidate Lyapunov function as in (13) or (14), our interconnected system is stable if the following implication is true:

$$\text{Kuhn Tucker conditions satisfied} \ \Rightarrow \ \text{Lyapunov function decreasing.} \tag{16}$$

This implication simply asks whether the system is Lyapunov stable when y and u are related through the saturation operator, which we can equivalently describe through the Kuhn–Tucker conditions.

The above implication is actually already in a form where the S-procedure could be applied to it, but there are a couple of problems with the above formulation which make that a bad approach. The first is that if we wish to use $\mathcal{P}(x, u)$ as a Lyapunov function, then we need its time derivative, which involves the time derivative of u. At this point, we have no convenient characterization of $\dot{u}$, and the LMIs obtained from the S-procedure would fail miserably. Second, the S-procedure is only a sufficient condition, and the exact form of the equalities and inequalities that are used in it can effect the final result enormously. This means that, even though the Kuhn–Tucker conditions describe the saturation exactly, they may not be written in the best form for the S-procedure. Fortunately, both of these problems can be remedied.

To obtain a characterization for $\dot{u}$ we simply differentiate the Kuhn–Tucker (7)–(12) conditions implicitly. (Note that differentiation of u and λ_1, λ_2 is questionable since there are points $(Cx = 1, -1)$ where their derivatives do not exist. Nevertheless, it can be shown that the resulting equations are satisfied at all points where they are differentiable, and is satisfied by the limits from the left and right at points where they are not differentiable (see [15]).

To deal with the conservativeness of the S-procedure, in principle we should simply derive as many constraints as possible from the Kuhn–Tucker conditions. Stated this way, the procedure is clearly not feasible. An approach which works well in practice is to derive constraints up to quadratics in the variable $[x, u, \lambda_1, \lambda_2, \dot{u}, \dot{\lambda}_1, \dot{\lambda}_2]$. Yet only a few of those constraints are effective when used in the S-procedure. As in any analysis methodology based upon the S-procedure, *a priori* it is not clear how effective a constraint will be (this is related to the problem of duality gaps in nonlinear programming). In the case of the saturation, among the linear and quadratic constraints, we found that the following are the key constraints in the S-procedure:

$$\begin{cases} u - y + \lambda_1 - \lambda_2 = 0 \\ \dot{u} - \dot{y} + \dot{\lambda}_1 - \dot{\lambda}_2 = 0 \end{cases} \quad \begin{cases} \lambda_1 u \geq 0 \\ -\lambda_2 u \geq 0 \end{cases} \quad \begin{cases} \dot{u}\dot{\lambda}_1 = 0 \\ \dot{u}\dot{\lambda}_2 = 0 \\ \lambda_1 \lambda_2 = 0 \\ \lambda_1 \dot{u} = 0 \\ \lambda_2 \dot{u} = 0 \\ \dot{\lambda}_1 \dot{\lambda}_2 = 0. \end{cases} \tag{17}$$

(See [15] for a complete derivation of the above constraints). Hence, we will actually verify the truthfulness of the implication

$$\begin{aligned} &\text{Constraints implied by KT conditions in (17)} \\ &\qquad \Rightarrow \text{Lyapunov function decreasing.} \end{aligned} \tag{18}$$

E. Main Result

All that is left is to formulate the implication in (18) more precisely, and apply the S-procedure. The left side of the implication will be the equations in (17), while the right will be a condition on one of our candidate Lyapunov functions in (13)–(14).

To simplify notation, let $\varphi = [x, u, \lambda_1, \lambda_2, \dot{u}, \dot{\lambda}_1, \dot{\lambda}_2]^{\mathrm{T}}$ be a collection of all variables. After straightforward manipulations, the constraints, (17), can be expressed in terms of φ as:

$$\Lambda \varphi = 0 \tag{19}$$

$$\varphi^{\mathrm{T}} \Omega_1^i \varphi \geq 0, \quad i = 1 \ldots q_1 \tag{20}$$

$$\varphi^{\mathrm{T}} \Omega_2^j \varphi = 0, \quad j = 1 \ldots q_2 \tag{21}$$

where

Λ contains all linear equality constraints in (17);

Ω_1 contains all quadratic inequality constraints in (17);

Ω_2 contains all quadratic equality constraints in (17).

Therefore, (19)–(21) are the mathematical expression of the left-hand side of the implication (18).

It remains now to characterize the right-hand side of (18). Using the Lyapunov function $\mathcal{P}(x,u)$ in (14) (the Lyapunov function $\mathcal{P}(x)$ follows similarly), the time derivative of $\mathcal{P}(x,u)$ being negative can be written as

$$2\begin{bmatrix} x \\ u \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} + \gamma(\|x\|^2 + \|u\|^2) \leq 0 \tag{22}$$

where $\gamma > 0$ is used to ensure that the Lyapunov function is strictly decreasing and can be chosen as a small number. After replacing $\dot{x}$ with $Ax + Bu$, in terms of φ we will denote the quadratic form in (22) as $\varphi^{\mathrm{T}}\Pi(P)\varphi$. Therefore, we aim to verify the following implication:

$$(19)\text{–}(21) \Longrightarrow (22). \tag{23}$$

An application of the S-procedure results in the following stability theorem.

Theorem 3.1: If there exists a matrix $P \succ 0$, and scalars $r_i^1 \geq 0$, $i = 1 \ldots q_1$, r_j^2, $j = 1 \ldots q_2$, such that the following LMI is satisfied:

$$\Lambda_\perp^{\mathrm{T}} \left\{ \sum_{i=1}^{q_1} r_i^1 \Omega_i^1 + \sum_{j=1}^{q_2} r_j^2 \Omega_j^2 + \Pi(P) \right\} \Lambda_\perp \preceq 0 \tag{24}$$

where Λ, Ω_1 and Ω_2 are given in (19)–(21), and $\Pi(P)$ in (22), then $x(t) \equiv 0$ is a globally exponentially stable fixed point of the system (1)–(3).

A proof of this theorem is contained in [15], and involves Lyapunov theory for nonsmooth systems [3]. In the following section we compare these results with previous results for determining the stability of a linear system interconnected with a saturation.

IV. Example

In this section, we test the analysis method on the following feedback system with control saturation (see [11] for a full analysis of this problem from an IQC point of view).

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= -KCx(t) \\ u(t) &= \mathrm{sat}(y(t)) \end{aligned} \tag{25}$$

where $K > 0$ and

$$P(s) = C(sI - A)^{-1}B = \frac{s^2}{s^3 + 2s^2 + 2s + 1}$$

is the transfer function of the controlled plant. A state-space representation of this plant is given by

$$\dot{x} = \begin{bmatrix} -2 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{26}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x. \tag{27}$$

Let us begin our stability analysis of this system by applying the circle criterion, which guarantees stability for $K < K_{\mathrm{circ}} \approx 8.12$.

TABLE I
Stability Results

Technique	Max Value of K
Circle	8.12
Popov	8.9
Zames-Falb	∞
Thm. 3.1 (x)	8.12
Thm. 3.1 (x, u)	∞

We may improve on the results obtained from the circle criterion by resorting to the Popov criterion [11], [14] which results in $K < K_{\mathrm{Popov}} \approx 8.90$. Next, since a saturation is a monotone and odd nonlinearity, the Zames–Falb result may be applied [19]. It is much stronger than the circle or Popov criterion and guarantees stability for all positive K.

Finally, we consider an application of Theorem 3.1 for two Lyapunov functions. First, when the Lyapunov function is only quadratic in the state, we find that the system is stable for $K < K_x \approx 8.12$. In essence, we recover the circle criterion. In fact, the conic sector bound used in the circle criterion can be derived from the Kuhn–Tucker conditions.

Next, we allowed the Lyapunov function to be a quadratic function of x and u as in $\mathcal{P}(x,u)$ in (14). In this case we find that the system is stable for all positive K. A summary of the results is presented in Table I.

Remark 4.1: In the above example we found that both the Zames–Falb IQC and our method produced stability for all K. It is important to note that since our methodology constructs a piecewise quadratic Lyapunov function, it can capture differences between nonlinearities that the Zames–Falb IQC cannot. This can be shown even in the one-dimensional case. To see this, consider the following simple example:

$$\begin{cases} \dot{x} = u \\ u = -\phi(x) \end{cases} \tag{28}$$

where $\phi(x)$ will be either $\mathrm{sat}(x)$ or $dzn(x)$ where $dzn(x)$ indicates the deadzone operator defined as $dzn(x) = x - \mathrm{sat}(x)$. The open-loop system (28) is not asymptotically stable, since it is a pure integrator. When $\phi(x) = \mathrm{sat}(x)$, our method can be easily modified to prove exponential stability for x in any arbitrarily large compact set. This is a well known result, and additionally it is known that the system is globally stable (see [2], [5], [16], [17], and references therein).

Suppose now that $\phi(x) = dzn(x)$. It is straightforward to derive stability constraints similar to those in (17) for the deadzone (see [15]). The system (28) is now unstable since in a neighborhood of the origin $\dot{x} = 0$. Our method captures this behavior because a piecewise quadratic Lyapunov function is used to prove stability. On the other hand, the Zames–Falb criterion fails to achieve such a characterization, since both $\phi(x) = \mathrm{sat}(x)$ and $\phi(x) = dzn(x)$ have the same derivative range $(d\phi(x)/dx) \in [1]$ (see [12]).

V. Conclusion

In this note, we presented new stability conditions for linear systems interconnected with a saturation. These results were obtained by representing the saturation as an optimization problem, and characterizing it through its necessary (Kuhn–Tucker) conditions, and constraints implied by them. The question of stability could then be posed as an implication between linear and quadratic equalities and inequalities, which were converted to a linear matrix inequality through the S-procedure.

These new stability conditions are related to the use of piecewise quadratic Lyapunov functions, but do not require an explicit partitioning of the state space. Instead, the appropriate partition falls out of the necessary conditions from the saturation operator formulated as an optimization. Furthermore, it should be clear that this approach can easily be extended to the analysis of general piecewise linear systems. When tested versus previous results, the conditions in this paper were found to match even the Zames–Falb stability conditions for an example where both the circle and Popov criteria fail to produce strong results. An important feature of the proposed method is its ability to capture the distinction between a deadzone and a saturation. This is achieved because a piecewise quadratic Lyapunov function is used in the stability analysis. Such a difference cannot be represented by the Zames–Falb criterion within the multiplier analysis context.

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Optimal Tracking Performance: Preview Control and Exponential Signals

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***Abstract*—In this note, we study tracking performance limitation problems. Two issues are addressed, concerning how earlier results developed elsewhere may be extended to more general classes of reference signals, and how tracking performance may be further improved beyond that offered by feedback control. Toward these issues we consider exponentially increasing reference inputs and examine the use of preview control for tracking. We take an optimal interpolation approach, and our purpose is to develop analytical expressions and conceptual insight which will aid in the understanding of these issues. To this effect, we derive explicit expressions for the optimal tracking error, either as exact solutions or bounds. It is found that for the exponential signals the earlier results can be directly extended, and similar conclusive statements can be drawn. It is also shown that in general preview can be used to advantage for improving tracking performance, especially in countering the effect resulted from plant nonminimum phase zeros.**

***Index Terms*—Exponential signals, nonminimum phase zeros, preview control, tracking performance, unstable poles.**

I. INTRODUCTION

The ability of tracking command input signals is a primary criterion for assessing the performance of feedback control systems and indeed it constitutes a primary objective in control system design. As such, optimal tracking problems have over the years received a considerable amount of research interest. While in many such problems a main objective is to design an optimal compensator to minimize tracking error, which from a numerical computation viewpoint can be tackled using standard techniques and routines, and thus is considered a resolved issue, more recent attention has been focused on the understanding of the inherent limitation on the best tracking performance achievable via feedback. This has led to several important discoveries. Among the notable issues and results are cheap LQR control [11], servomechanism problems [17], and optimal tracking control [5], [14], [16], [18]. By now it is generally known that in the full generality of *causal* feedback compensation, i.e., when a two-parameter causal feedback control scheme is employed, the best achievable tracking performance is limited, and in fact is only limited, by the nonminimum phase characteristics of plant [5]; here by the latter we mean both the nonminimum phase zeros as well as time delays in the plant. Consequently, such characteristics impose an intrinsic barrier which in no way may be surpassed by causal feedback alone, in that the tracking accuracy cannot be further improved by use of any causal feedback controller.

One of the main issues to be investigated in this note dwells on the use of *noncausal* actions for tracking. More specifically, can

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