The algebra of Wilson-'t Hooft operators

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Abstract

We study the Operator Product Expansion of Wilson-'t Hooft operators in a twisted $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group G . The Montonen-Olive duality puts strong constraints on the OPE and in the case $G = SU(2)$ completely determines it. From the mathematical point of view, the Montonen-Olive duality predicts the L^2 Dolbeault cohomology of certain equivariant vector bundles on Schubert cells in the affine Grassmannian. We verify some of these predictions. We also make some general observations about higher categories and defects in Topological Field Theories.

1 Introduction

An important property of Yang-Mills theory is that it contains Wilson loop operators labeled by irreducible representations of the gauge group G [1]. Their product is controlled by the representation ring of G and therefore determines G uniquely. The work of Goddard, Nuyts, and Olive [2] on magnetic sources can be reinterpreted [3] as saying that Yang-Mills theory admits another class of loop operators labeled by irreducible representations of the Langlands-dual group ${}^L G$. Such operators are called 't Hooft loop operators. The Montonen-Olive duality conjecture [4] states that $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group G is isomorphic to $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group ${}^L G$, and this isomorphism exchanges Wilson and 't Hooft loop operators. This conjecture therefore predicts that the product of 't Hooft loop operators is controlled by the representation ring of ${}^L G$.

This implication of the Montonen-Olive conjecture has been verified in [5] for suitably supersymmetrized versions of 't Hooft loops. The idea is to twist $\mathcal{N} = 4$ SYM theory into a 4d Topological Field Theory (TFT), so that either Wilson or 't Hooft loop operators become topological observables. One can show then that the product of loop operators is independent of the distance between them, and in fact loop operators form a commutative ring. In the case of Wilson loop operators, it is straightforward to show that this ring is the representation ring of G . In the case of 't Hooft loop operators, it has effectively been argued in [5] that the ring is the K^0 -group of the category of equivariant perverse sheaves on the affine Grassmannian Gr_G . It has been shown by Lusztig [6] that this ring is the representation ring of ${}^L G$; a categorification of this statement, known as the geometric Satake correspondence, has been proved in [7, 8, 9]. As explained in [5], the geometric Satake correspondence can also be interpreted in physical terms, by replacing loop operators with line operators.

Yang-Mills theory also admits mixed Wilson-'t Hooft loop operators. As explained in [3], they are labeled by elements of the set

$$\widehat{\Lambda}(G)/\mathcal{W} = (\Lambda_w(G) \oplus \Lambda_w({}^L G))/\mathcal{W},$$

where $\Lambda_w(G)$ is the weight lattice of G and $\mathcal{W}$ is the Weyl group (which is the same for G and ${}^L G$). It is natural to ask what controls the product of such more general operators. The answer must somehow unify the representation theory of G and ${}^L G$. In this paper we partially answer this question. A natural framework for it is the holomorphic-topological twisted version of

the $\mathcal{N} = 4$ SYM theory described in [10], since it admits Wilson-'t Hooft loop operators labeled by arbitrary elements of $\widehat{\Lambda}/\mathcal{W}$.¹ As explained in [10], Wilson-'t Hooft loop operators in the twisted theory form a commutative ring, and this ring is abstractly isomorphic to the Weyl-invariant part of the group algebra $\widehat{\Lambda}(G)$. But this does not completely determine the operator product, since we do not yet know which element of the group algebra corresponds to a particular element of the set $\widehat{\Lambda}(G)/\mathcal{W}$ labeling Wilson-'t Hooft loop operators.

In this paper we determine the answer for $G = PSU(2)$ and $G = SU(2)$ assuming S-duality, and then verify the prediction in a special case by a direct gauge-theory computation at weak coupling. We also outline a procedure for computing the product of Wilson-'t Hooft loop operators for arbitrary G . The procedure is very similar to that for 't Hooft operators in [5]. As in [5], an important role is played by the fact that loop operators can be promoted to line operators, i.e. “open” analogs of loop operators. While loop operators form a commutative ring, line operators form a monoidal category (i.e. an additive category with a “tensor product”). We argue below that the ring of loop operators can be thought of as the K^0 -group of the category of line operators. The Montonen-Olive duality predicts that these categories for gauge groups G and ${}^L G$ are equivalent. In some sense, this can be viewed as the classical limit of the geometric Satake correspondence, but G and ${}^L G$ enter more symmetrically. As discussed in the concluding section, this conjecture, when interpreted in mathematical terms, has previously appeared in [15].

2 A brief review of the Hitchin moduli space

In this preliminary section we review some basic facts about the moduli space of Hitchin equations $\mathcal{M}_H(G, C)$ and the sigma-model with target $\mathcal{M}_H(G, C)$. The reader familiar with this material may skip this section. A more detailed discussion may be found in [5].

Given a gauge group G , let us consider a principal G -bundle E over a Riemann surface C , a connection A on E , and a 1-form ϕ with values in

¹In the topological field theory described in [5], depending on the choice of a BRST operator, either Wilson or 't Hooft loop operators may exist, but not both at the same time. In what follows we will refer to this TFT as the GL-twisted theory, where GL stands for “geometric Langlands”.

$\text{ad}(E)$. The Hitchin equations are

$$F - i\phi \wedge \phi = 0, \quad D\phi = 0, \quad D \star \phi = 0,$$

where $D = d + iA$ is the covariant differential, $F = -iD^2$ is the curvature of A , and $\star$ is the Hodge star operator. The space of solutions of this equations modulo gauge transformations is known as the Hitchin moduli space and will be denoted $\mathcal{M}_H(G, C)$ or simply $\mathcal{M}_H$ (we suppress E from the notation, because we regard $\mathcal{M}_H(G, C)$ as a disconnected sum of components corresponding to all possible topological types of E).

A crucial fact for us is that $\mathcal{M}_H$ is a hyperkähler manifold. In particular, it has three complex structures I, J, K satisfying $IJ = K$. One way to describe these complex structures explicitly is to specify holomorphic coordinates on $\mathcal{M}_H$. For a local complex coordinate z on C we write

$$A = A_z dz + A_{\bar{z}} d\bar{z}, \quad \phi = \phi_z dz + \phi_{\bar{z}} d\bar{z}.$$

For the complex structure I the holomorphic coordinates are $A_{\bar{z}}$ and ϕ_z . For the complex structure J the holomorphic coordinates are $A_{\bar{z}} + i\phi_{\bar{z}}$ and $A_z + i\phi_z$. Finally, the complex structure K is defined by the quaternion relation $K = IJ$. In the present paper we mostly work with complex structure I and use notation $\mathcal{M}_{Higgs}(G, C)$ for $\mathcal{M}_H(G, C)$ with this choice of complex structure. The reason for this notation is that $\mathcal{M}_H$ equipped with the complex structure I is naturally identified with the moduli space of Higgs bundles, i.e. pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a holomorphic $G_{\mathbb{C}}$ bundle, and φ is a holomorphic section of $K_C \otimes \text{ad}(\mathcal{E})$. This identification maps the triple (E, A, ϕ) to the holomorphic $G_{\mathbb{C}}$ -bundle defined by the $(0, 1)$ part of D and the holomorphic Higgs field $\varphi = \phi^{1,0}$. Note that the subset of $\mathcal{M}_{Higgs}(G, C)$ given by $\varphi = 0$ is the moduli space of stable holomorphic $G_{\mathbb{C}}$ bundles, which we will denote $\mathcal{M}(G, C)$.

In the complex structure J the Hitchin moduli space can be identified with the moduli space of flat $G_{\mathbb{C}}$ connections on C ; this moduli space was denoted $\mathcal{M}_{flat}(G, C)$ in [5]. But this identification will not play a role in this paper.

Consider now the supersymmetric sigma-model with target $\mathcal{M}_H$. Since $\mathcal{M}_H$ is hyperkähler, such a sigma-model has $\mathcal{N} = (4, 4)$ supersymmetry. One may twist this sigma-model into a topological field theory by picking a pair of complex structures (J_+, J_-) on $\mathcal{M}_H(G, C)$. For $J_+ = J_-$ one gets a B-model, while for $J_+ = -J_-$ one gets an A-model. In this paper we will be mostly

interested in the special case $J_+ = J_- = I$, i.e. the B-model in complex structure I .

Given a topological twist of the sigma-model, one can consider the corresponding category of topological branes. This is a category of boundary conditions for the sigma-model on a worldsheet of the form $\mathbb{R} \times I$ where I is the unit interval. The boundary conditions are required to be invariant with respect to the BRST operator of the twisted model. Equivalently, one may say that the boundary conditions are required to preserve one complex supercharge (in the untwisted theory). But since the untwisted model has $(4, 4)$ supersymmetry, there also exist branes which preserve two complex supercharges. Such branes are compatible with more than one topological twist. In this paper we will encounter (B, B, B) -branes, which are B-branes in complex structures I, J, K , as well as (B, A, A) -branes which are of B-type in complex structure I and of A-type in the other two complex structures.

3 Holomorphic-topological twist of $\mathcal{N} = 4$ SYM

Let us recall how one can twist $\mathcal{N} = 4$ gauge theory on $\Sigma \times C$ into a holomorphic-topological theory [10] which upon reduction gives the B-model on Σ with target $\mathcal{M}_{Higgs}(G, C)$. It is convenient to treat $\mathcal{N} = 4$ SYM as $\mathcal{N} = 2$ SYM with a hypermultiplet in the adjoint representation. The theory has $SU(2)_R \times U(1)_N \times U(1)_B$ symmetry. The holonomy group is $U(1)_C \times U(1)_\Sigma$. One twists $U(1)_C$ action by a suitable linear combination of $U(1)_R \subset SU(2)_R$ and $U(1)_B$, and twists $U(1)_\Sigma$ by $U(1)_N$.

The resulting field theory has the following bosonic fields: the gauge field A , the adjoint Higgs field $\varphi = \Phi_w dw \in K_\Sigma \otimes \text{ad}(E)$, the adjoint Higgs field $q = q_z d\bar{z} \in \bar{K}_C \otimes \text{ad}(E)$, and the adjoint Higgs field $\tilde{q} \in \text{ad}(E)$. Here K_Σ and K_C are the pull-backs of the canonical line bundles of Σ and C to $\Sigma \times C$. We also define $\Phi_{\bar{w}} = \Phi_w^\dagger$ and $q_z = q_{\bar{z}}^\dagger$.

The fermionic fields are the “gauginos” $\lambda_w, \bar{\lambda}_{\bar{w}}, \lambda_z, \bar{\lambda}_{\bar{z}}, \lambda_{\bar{z}w}, \bar{\lambda}_{\bar{z}\bar{w}}, \lambda_{w\bar{w}}, \bar{\lambda}_{w\bar{w}}$ and the “quarks” $\psi_{\bar{w}}, \bar{\chi}_w, \psi_{\bar{z}}, \bar{\chi}_{\bar{z}}, \chi_{z\bar{w}}, \bar{\psi}_{z\bar{w}}, \chi_{z\bar{z}}, \bar{\psi}_{z\bar{z}}$. The fermions are all in the adjoint representation.

The field content depends on complex structures of C and Σ . The dependence on the complex structure on C is inescapable, but the dependence on the complex structure on Σ is merely an artifact of our way of presentation. It is possible to combine fields with holomorphic and anti-holomorphic indices into form-valued fields on Σ so that the dependence on the complex

structure on Σ is eliminated [10].

In order to specify the theory completely, one has to pick a BRST operator. The twisted theory has two BRST operators Q_ℓ and Q_r which square to zero and anticommute, so the most general BRST operator is

$$Q = uQ_\ell + vQ_r,$$

where u, v are homogeneous coordinates on $\mathbb{P}^1$. It is often convenient to work with an affine coordinate $t = v/u$ taking values in $\mathbb{C} \cup \{\infty\}$. To get a theory which is topological on Σ and holomorphic on C , one needs to assume that u and v are both nonzero, i.e. $t \neq 0, \infty$ [10].² The precise choice of t then does not matter [10]; we let $t = i$ from now on.

The action of the twisted theory can be written as a sum of a BRST-exact piece and a piece which is independent of the gauge coupling e^2 and the θ -parameter (after a rescaling of fermions). Therefore semiclassical computations in the twisted theory are exact [10]. We will use this important fact throughout the rest of the paper.

The path-integral of the twisted theory localizes on Q -invariant field configurations. The conditions of Q -invariance imply, among other things, that the complex connection $\mathcal{A} = A + i\varphi + i\varphi^\dagger$ has a curvature $\mathcal{F}$ whose only nonzero components are along Σ . In the limit when the volume of C goes to zero, the equations simplify and imply the Hitchin equations for A_z and q_z

$$F_{z\bar{z}} - i[q_z, q_{\bar{z}}] = 0, \quad D_{\bar{z}}q_z = 0$$

as well as

$$D_{\bar{z}}\tilde{q}^\dagger = 0,$$

which implies that $\tilde{q}$ is generically zero. Thus in this limit the field theory reduces to a sigma-model with target $\mathcal{M}_{Higgs}(G, C)$. There are further equations which say that this sigma-model is a B-model in the natural complex structure (the one which we denote I).

The Montonen-Olive duality, as usually defined, maps G to ${}^L G$ and maps [5, 10] the BRST operator at $t = i$ to another BRST operator with

$${}^L t = \frac{|\tau|}{\tau} t.$$

²If $t = 0$ or $t = \infty$, the twisted theory is holomorphic on both C and Σ . Such a theory does not admit line operators which we are interested in.

But since the phase of t can be changed by an automorphism of the theory (an R-symmetry transformation), one can redefine the Montonen-Olive duality so that it leaves t invariant. We adopt this definition of Montonen-Olive duality from now on.

In this paper we mostly focus on the case when Σ has a flat metric. Then the twist along Σ is a trivial operation, and the theory can be regarded as twisted only along C . In the limit $\text{vol}(C) \rightarrow 0$ it becomes equivalent to an untwisted supersymmetric sigma-model with target $\mathcal{M}_H(G, C)$. Since $\mathcal{M}_H$ is hyperkähler, this sigma-model has $\mathcal{N} = (4, 4)$ supersymmetry, i.e. it has two left-moving and two right-moving complex supercharges, as well as their complex conjugates. The BRST operator defined above is a particular linear combination of these supercharges. The BRST operator of the GL twisted theory considered in [5] is another such linear combination (depending on a single complex parameter t). Both kinds of BRST operators can be included into a more general three-parameter family of BRST operators [5].

4 Wilson-'t Hooft operators in the twisted theory

4.1 Definition

In any gauge theory one can define various loop operators: Wilson, 't Hooft, and Wilson-'t Hooft. The Wilson loop operator in representation R is usually defined as

$$W_R(\gamma) = \text{Tr}_R P \exp i \int_{\gamma} A$$

where γ is a closed curve. Instead of labeling the operator by an irreducible representation, one can label it by the orbit of its highest weight under the Weyl group. The 't Hooft loop operator is a disorder operator defined by the requirement that near a curve γ the gauge field has a singularity of a Dirac-monopole kind. Such singularities are labeled by conjugacy classes of homomorphisms from $U(1)$ to G , which is equivalent to saying that they are labeled by orbits of the Weyl group in the coweight lattice Λ_{cw} of G . More generally, Wilson-'t Hooft operators are labeled by Weyl orbits in the product $\Lambda_w(G) \times \Lambda_{cw}(G)$ [3].

In the $\mathcal{N} = 4$ SYM theory there are more possibilities for loop operators, since one can construct them not only from gauge fields, but also from other

fields. By imposing natural symmetry requirements (namely, the geometric symmetries and supersymmetry), one can cut down on the number of possibilities.

In the twisted $\mathcal{N} = 4$ theory we have to require that loop operators be BRST-invariant. For $t = i$, we see that none of the components of A are BRST-invariant. But we also see that $\mathcal{A}_w = A_w + i\Phi_w$ and $\mathcal{A}_{\bar{w}} = A_{\bar{w}} + i\Phi_{\bar{w}}$ are BRST-invariant. Hence if γ is a closed curve on Σ and p is a point on C , the Wilson operator

$$W_R(\gamma, p) = \text{Tr}_R P \exp i \int_{\gamma \times p} \mathcal{A}$$

is BRST-invariant.

By MO duality, there should also be BRST-invariant 't Hooft operators at $t = i$.³ Indeed, if γ is given by the equation $x^1 = \text{Re } w = 0$ and we require the gauge field to have a Dirac-like singularity in the x^1, x^2, x^3 plane:

$$F \sim \star_3 d \left(\frac{\mu}{2r} \right) \quad (1)$$

for some $\mu \in \mathfrak{g}$, then the condition of Q -invariance requires Φ_w to be singular as well:

$$\Phi_w \sim \frac{\mu}{2r}. \quad (2)$$

It is a plausible guess that such a disorder operator is mapped to the Wilson operator by the MO duality.

Finally, we may consider more general Wilson-'t Hooft loop operators which source both electric and magnetic fields. Roughly speaking, they are products of Wilson and 't Hooft operators. To define a WH loop operator more precisely, let it be localized at $x^{1,2,3} = 0$. Then we require the components of the curvature in the 123 plane to have a singularity as in (1), the real part of Φ_w to have a singularity as in (2), and insert into the path-integral a factor

$$\text{Tr}_R P \exp i \int_{\gamma \times p} \mathcal{A}$$

where R is an irreducible representation of the stabilizer subgroup $G_\mu \subset G$ of μ . This definition makes sense because in the infinitesimal neighborhood

³This is unlike the GL twisted theory, where for $t = i$ only Wilson operators are BRST-invariant.

of γ the component of $\mathcal{A}$ tangent to γ must lie in the centralizer subalgebra $\mathfrak{g}_\mu \subset \mathfrak{g}$ of μ [3]. One may describe R by specifying its highest weight ν , which is defined up to an action of the subgroup of the Weyl group which preserves μ . The net result is that the WH operator is labeled by a pair $(\mu, \nu) \in \Lambda_{cw}(G) \times \Lambda_w(G)$ defined up to the action of the Weyl group $\mathcal{W}$. We will denote the abelian group $\Lambda_{cw}(G) \times \Lambda_w(G)$ by $\widehat{\Lambda}(G) = \widehat{\Lambda}^L(G)$. The WH operator labeled by the Weyl-equivalence class of (μ, ν) will be denoted $WT_{\mu, \nu}(\gamma, p)$.

There is a natural action of the S-duality group on $\widehat{\Lambda}(G)$. It is a natural conjecture that this is how the S-duality group acts on the corresponding WH operators. One of the goals of this paper is to test this conjecture.

Note that all our loop operators are localized at points on C . If we take the volume of Σ to be small compared to that of C , then the twisted theory reduces to an effective 2d field theory on C , and in this effective 2d field theory our loop operators behave in all ways like local operators. There are no BRST-invariant operators which are localized on loops in C .

4.2 Basic properties

As explained in [10], in the twisted theory all correlators depend holomorphically on coordinates on C and are invariant under arbitrary diffeomorphisms of Σ . This puts strong constraints on the correlators of WH loop operators. We will be mostly interested in the Operator Product Expansion of WH loop operators. That is, we will assume that Σ is flat, pick a pair of points $p, p' \in C$ and a pair of straight lines γ and γ' on Σ and consider a pair of WH operators localized on $\gamma \times p$ and $\gamma' \times p'$. So far, we have assumed that the curve on which the WH operator is localized is closed; if we want to maintain this, we may assume that Σ locally looks like a cylinder with a flat metric; since the theory is diffeomorphism-invariant along Σ , the only thing that matters is that both γ and γ' are closed and isotopic to each other. One may also consider WH operators localized on lines rather than closed curves; we will return to this possibility later.

Consider now a correlator involving these WH loop operators. If γ and γ' do not have common points, then there is no singularity as one takes the limit where p coincides with p' . If z is a local complex coordinate on C centered at p , then the correlator is a holomorphic function of $z(p')$ in the neighborhood of zero. By continuity, this implies that even when γ and γ' coincide, the

correlator is a holomorphic function of z . Therefore the Operator Product of any two WH operators is nonsingular. More generally, this conclusion holds for any two BRST-invariant loop operators in the twisted theory which are localized on C .

Given this result, we can define a commutative algebra of loop operators, simply by taking the coincidence limit. For Wilson and 't Hooft loop operators this result can be more easily obtained using the GL twisted theory of [5], but here we see that it holds for general loop operators in the holomorphic-topological twisted theory.

At this stage it is natural to ask whether the subspace spanned by WH operators is closed with respect to the operator product. More optimistically, one could hope that WH operators form a basis in the space of loop operators in the twisted theory, and therefore the vector space spanned by them is automatically closed with respect to the operator product. We will argue below that both statements are true, *if only closed loops are considered*.

4.3 Line versus loop operators

As emphasized in [5], one may also consider analogs of Wilson and 't Hooft operators localized on open curves instead of loops. The endpoints of a curve must lie on the boundaries of the four-manifold. Such “operators” are called line operators in [5]. We put the word “operators” in quotes because they do not act on the Hilbert space of the theory; rather, they alter the definition of the Hilbert space of the theory.

To be concrete, suppose $\Sigma = \mathbb{R} \times X_1$, where X_1 is either S^1 or an interval I . We regard $\mathbb{R}$ as the time direction. Consider a Wilson line operator $W_R(\gamma, p)$, where $\gamma \subset \Sigma$ has the form $\mathbb{R} \times q$ for some $q \in X_1$. Insertion of such a Wilson line operator means that the Hilbert space of the gauge theory has to be modified: instead of gauge-invariant wave-functions on the space of fields on $X_1 \times C$, one has to consider gauge-invariant elements of the tensor product of the space of all wave-functions and the representation space of R . Similarly, when we insert an open 't Hooft operator, we have to change the class of fields on which the wave-functions are defined.

While loop operators form a commutative algebra, line operators form a category. A morphism between line operators **A** and **B** is a local BRST-invariant operator inserted at a junction of **A** and **B**. Composition of morphisms is defined in an obvious way. There is also an obvious structure of a complex vector space on the space of morphisms and an obvious way to

define a sum of line operators. Thus line operators form an additive $\mathbb{C}$ -linear category.

The distinction between line and loop operators has played some role in [5] and it is even more important in the context of the holomorphic-topological theory, as we will see below.

It is often convenient to relax the condition that local operators inserted at the junction of two line operators be BRST-invariant, and define the space of morphisms to be the space of all local operators. This space is graded by the ghost number and is acted upon by the BRST-differential. Thus the set of morphisms between any two line operators has the structure of a complex of vector spaces, and composition of morphisms is compatible with the differentials. That is, line operators form a differential graded category (DG-category). This viewpoint is convenient for keeping track of the dependence of various correlators on parameters, such as the insertion point on C (see below).

There is one more important operation for line operators in the twisted theory: an associative tensor product. In other words, the category of line operators is a monoidal category. The product is defined by taking two line operators “side-by-side” on Σ and “fusing” them together. The product of line operators need not be commutative, in general. But for Wilson-’t Hooft line operators it is commutative because of a discrete symmetry: parity reversal. Indeed, consider the twisted gauge theory on $\mathbb{R} \times \mathbb{R} \times C$, where we regard the first copy of $\mathbb{R}$ as time and the second one as space. It is easy to check that spatial reflection $x \rightarrow -x$ is a symmetry of the theory.⁴ Furthermore, Wilson-’t Hooft line operators are invariant under this symmetry. Therefore, we can change the order of WH line operators on the spatial line by a symmetry transformation.

4.4 Remarks on TFT in arbitrary dimension

A similar discussion applies to the GL twisted theory considered in [5], and in fact to any topological field theory in any number of dimensions. That is, in any TFT line operators form a monoidal $\mathbb{C}$ -linear additive category.

In the case of a TFT in dimension $d > 3$ the fusion product is necessarily symmetric, because there is no diffeomorphism-invariant way to order line operators. In dimension $d = 3$ there may be nontrivial braiding, so in

⁴This is particularly obvious from a 2d viewpoint, as any B-model is parity-invariant.

general the category of line operators is braided rather than symmetric. A well-known example is the Chern-Simons theory [11], where the category of Wilson line operators is equivalent to the category of representations of a quantum group. In dimension $d = 2$ the monoidal structure need not be either symmetric or braided, in general.

In this paper we are dealing with a holomorphic-topological field theory rather than a TFT, and the “topological” part of the manifold is two-dimensional. From the abstract viewpoint the situation is very much like in a 2d TFT, because every line operator in the twisted gauge theory on $\Sigma \times C$ can be regarded as a line operator in the B-model on Σ with target $\mathcal{M}_{Higgs}(G, C)$. But the converse is not necessarily true, because line operators in gauge theory are local on C , while line operators in the B-model on Σ are not subject to this constraint. (Below we will construct a large class of examples of line operators in the B-model which do not lift to ordinary line operators in the gauge theory.) To enforce locality, one has to keep track of the dependence of all correlators on the insertion point $p \in C$ of the line operator. To put it differently, if we denote by $V(q, p)$ the Hilbert space of the twisted theory on $\mathbb{R} \times X_1 \times C$ with an insertion of a line operator at $q \times p \in X_1 \times C$, then for fixed q this family of vector spaces can be thought of as a holomorphic vector bundle V_q over C . Similarly, spaces of morphisms between different line operators can be thought of as holomorphic vector bundles over C .

To make precise the idea of a “holomorphically varying space of morphisms”, it is very convenient to take the viewpoint that the space of morphisms is a differential graded vector space, i.e. a complex. Let $W(p)$ be the vector space of all (not necessarily BRST-invariant) local operators inserted at the junction of two line operators **A** and **B**, both located at a point $p \in C$. The space $W(p)$ is graded by the ghost number and carries the BRST-differential Q . The complexes $W(p)$ fit into a complex of smooth vector bundles W on C . Let us tensor this complex of vector bundles with the Dolbeault complex of C . The resulting space of sections is acted upon by both Q and $\bar{\partial}$ and carries all the information about the dependence of morphisms on p . “Holomorphic dependence” means simply that $\bar{\partial}$ is Q -exact, and therefore acts trivially on the cohomology of Q .

We can put our discussion of line operators in a more general perspective by noting that n -dimensional TFTs form a n -category. 1-Morphisms in this n -category are codimension-1 walls separating a pair of TFTs. We will call codimension-1 walls 1-walls, for short. 1-walls themselves form an $n - 1$ category: 2-morphisms are codimension-2 walls which separate different 1-

walls between the same pair of TFTs. And so on.

If we consider all 1-walls between a pair of identical TFTs, they can be "fused" together. This gives a kind of monoidal structure on an $n - 1$ category of 1-walls. In this $n - 1$ -category there is a unit object: the "trivial 1-wall" which is equivalent to no wall at all. 2-walls living on the trivial 1-wall form a monoidal $n - 2$ category with a unit, and so on. Thus line operators considered above belong to a rather special variety: they live on a trivial $n - 2$ wall which lives on a trivial $n - 3$ -wall, etc. For example, in the GL twisted theory at $t = i$ Wilson line operators form a category which is equivalent to the category of finite-dimensional representations of G . Gukov and Witten also considered nontrivial 2-walls in this theory and line operators living on such 2-walls [12].

Boundary conditions for an n -dimensional TFT also fit into this general scheme: they are 1-morphisms between a given TFT and an "empty" TFT. For this reason they form an $n - 1$ category (which is not monoidal, in general). A special case of this is the well-known fact that D-branes in a 2d TFT form a category.

In connection with possible 2-dimensional generalizations of the Geometric Langlands Duality, it would be interesting to understand the 3-category of boundary conditions for the GL twisted $\mathcal{N} = 4$ SYM, as well as the monoidal 3-category of 1-walls in the same theory. The latter acts on the former. These 3-categories appear to be suitable 2d generalizations of the derived category of $\mathcal{M}_{flat}(G, C)$ and the representation category of G , respectively.

4.5 Deformations of line operators

In the case of the GL twisted theory at $t = i$ the product of two parallel Wilson loop operators W_{R_1} and W_{R_2} is a Wilson loop operator $W_{R_1 \otimes R_2}$. This means that Wilson loop operators form a closed algebra, which happens to be commutative and associative. Wilson loop operators corresponding to irreducible representations of G form a basis in this algebra. A similar statement holds for Wilson line operators: the subcategory of Wilson line operators is closed with respect to the monoidal structure, i.e. it is a symmetric monoidal category, and any Wilson line operator is isomorphic to a direct sum of Wilson line operators corresponding to irreducible representations of G . By S-duality, similar statements hold for 't Hooft operators in the GL twisted theory (for $t = 1$).

At $t = i$ any line operator in the GL-twisted theory is isomorphic to a

Wilson line operator for some R (which can be reducible). One way to see it is to first classify line operators with the right bosonic symmetries in the untwisted theory (this has been done in [3]) and then impose the condition of BRST-invariance. A similar statement holds for 't Hooft operators at $t = 1$.

One consequence of this is that there are no infinitesimal deformations of Wilson line operators in the GL-twisted theory. This can also be checked directly. From the mathematical viewpoint, infinitesimal deformations of a line operator A are classified by degree-1 cohomology of the complex $\text{Hom}(A, A)$. One can check that this cohomology is trivial by considering BRST-invariant local operators which can be inserted at a point of the Wilson line A .

For line operators in the holomorphic-topological twisted theory the situation is more complicated. The difficulty is that twisting breaks $SO(3)$ rotational symmetry used in [3] down to $U(1)$. A generic Wilson-'t Hooft operators (i.e. not purely electric or purely magnetic) also preserves only rotation symmetry in the z -plane (which is present when $C \simeq \mathbb{C}$).

The simplest question one can ask in this regard is whether there are infinitesimal deformations of a Wilson-'t Hooft line operator. One obvious deformation arises from varying the insertion point on C . For a Wilson line $W_R(p)$, is easy to exhibit the degree-1 endomorphism corresponding to such a deformation. It is a fermionic field

$$\Gamma_z = \lambda_z + \bar{\lambda}_z.$$

It is BRST-invariant and can be inserted into a Wilson line in any representation R . The corresponding infinitesimal deformation of $W_R(p)$ is obtained as follows. First, we apply the descent procedure to Γ_z , i.e. look for a boson Δ_z such that

$$\mathcal{D}_\Sigma \Gamma_z = \delta \Delta_z.$$

Note the covariant differential on the left-hand side. Usually, descent is applied to gauge-invariant operators, in which case one uses ordinary de Rham differential. In our case, the operator becomes gauge-invariant only after insertion into a Wilson line, and this requires replacing ordinary differential with the covariant one. The descent equation is solved by

$$\Delta_z = \mathcal{F}_{zw} dw + \mathcal{F}_{z\bar{w}} d\bar{w}.$$

The deformed Wilson operator is

$$\text{Tr}_R P \exp \left(i \int \mathcal{A} + \Delta_z \epsilon^z \right)$$

where ϵ^z is an infinitesimal parameter. It is easy to see that this is the same as a Wilson operator evaluated at a nearby point, shifted from p by a vector $\epsilon^z \partial_z$.

Similarly, given any two line operators and a degree-1 morphism between them, one can construct their “bound state”, which is a deformation of the direct sum of the two line operators. In homological algebra, this is known as the mapping cone construction. In section 6.1 we will see some examples of the mapping cone construction with less obvious deformations of Wilson-’t Hooft line operators which do not correspond to changing the insertion point on C .

4.6 Line operators and K-theory

The existence of nontrivial deformations suggests that the category of Wilson-’t Hooft line operators may not be closed with respect to the tensor product. But we will argue below that the space of Wilson-’t Hooft *loop* operators is closed with respect to the product. Therefore it is important to understand the relationship between loop and line operators. We would like to argue here that loop operators should be thought of as elements of the K^0 -group of the category of line operators. The closure of the space of Wilson-’t Hooft loop operators under operator product suggests that these operators form a basis for the K^0 -group of the category of line operators, but we will not try to prove this here.

First, let us recall the definition of the K^0 -group of a DG-algebra $\mathcal{A}$. A finitely-generated projective DG-module over $\mathcal{A}$ is any DG-module which can be obtained from free DG-modules of finite rank using the following three operations: shift of grading, cone, and taking a direct summand. Consider a free abelian group generated by the isomorphism classes of finitely-generated projective DG-modules and quotient it by the relations

$$M \sim (-1)^n M[n]$$

for any integer n , and

$$M_1 \oplus M_2 \sim M$$

for any exact sequence of DG-modules

$$0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0.$$

This quotient group is $K^0(\mathcal{A})$.

The definition of the K^0 -group of a small DG-category $\mathfrak{A}$ is similar.⁵ The idea is to think about a category as an “algebra with several objects”. A DG-module $\mathfrak{M}$ over a small DG-category $\mathfrak{A}$ is a DG-functor from $\mathfrak{A}$ to the DG-category of complexes of vector spaces. In more detail, it is a collection of DG-modules $\mathfrak{M}(A)$ over the DG-algebras $\text{Hom}_{\mathfrak{A}}(A, A)$ for all $A \in \text{Ob}(\mathfrak{A})$ and DG-morphisms from the complex $\text{Hom}_{\mathfrak{A}}(A, B)$ to the complex $\text{Hom}(\mathfrak{M}(A), \mathfrak{M}(B))$ for any $A, B \in \text{Ob}(\mathfrak{A})$. These data should satisfy some fairly obvious compatibility conditions.

The analog of a free rank-1 DG-module is a presentable DG-module $\mathfrak{M}_B$ corresponding to an object B of $\mathfrak{A}$. Given any $B \in \text{Ob}(\mathfrak{A})$, we let $\mathfrak{M}_B(A) = \text{Hom}_{\mathfrak{A}}(B, A)$. It has an obvious DG-module structure over the DG-algebra $\text{Hom}_{\mathfrak{A}}(A, A)$. A finitely-generated projective DG-module over $\mathfrak{A}$ is a DG-module which is obtained from presentable modules by the operations of shift, cone, and taking a direct summand. To get the K^0 -group of $\mathfrak{A}$, we consider the free abelian group generated by isomorphism classes of finitely-generated projective DG-modules and quotient it by the relations coming from shift of grading and short exact sequences of DG-modules.

Now let $\mathfrak{A}$ be the DG-category of line operators. A presentable DG-module corresponding to a line operator B is a module $\mathfrak{M}_B$ such that $\mathfrak{M}_B(A)$ is the space of local operators which can be inserted at the joining point of line operators B and A . There is a special line operator: the Wilson line corresponding to the trivial representation of G . It is a unit object with respect to the monoidal structure on $\mathfrak{A}$. The space of local operators which can be inserted at such a trivial line operator is the same as the space of “bulk” local operators.

A loop operator is a line operator with no insertions of local operators and with the endpoints identified. A convenient geometry to study a loop operator A is to take $\Sigma = S^1_\tau \times S^1_\sigma$, where S^1_τ is regarded as the compactified Euclidean time and S^1_σ is the compactified spatial direction. We consider an arbitrary number of insertions of line operators, one of which is our A . All line operators are taken to “run” along the τ direction and are located at fixed σ . We also allow arbitrary local insertions at all line operators except A . This includes bulk local operator insertions, which may be regarded as local operators sitting on the trivial line operator. If all such correlators are unchanged when one replaces A with another loop operator A' , it is natural

⁵A small category is a category whose objects are members of a set rather than a class. We sincerely hope that line operators in a twisted gauge theory form a set.

to identify A and A' . We claim that this happens if $\mathfrak{A}$ -modules $\mathfrak{M}_A$ and $\mathfrak{M}_{A'}$ are in the same K^0 class.

To see this, let us reformulate the set-up slightly. First of all, we can lump all line operators except A and all bulk local operators into a single line operator B with a single insertion. It is easy to see that the Hilbert space of the twisted gauge theory on $\mathbb{R} \times S^1_\sigma \times C$ is the homology of the complex $\text{Hom}_{\mathfrak{A}}(A, B)$. Equivalently, we can say that it is the homology of $\mathfrak{M}_A(B)$. The local operator inserted into B can be thought of as an endomorphism T of the complex $\mathfrak{M}_A(B)$, and the correlator is the supertrace of T . It is obvious that shifting the grading of $\mathfrak{M}_A$ by n changes the supertrace by a factor $(-1)^n$. The other equivalence relation has to do with short exact sequences of $\mathfrak{A}$ -modules. If $\mathfrak{M}_A$ is the middle term of a short exact sequence

$$0 \rightarrow \mathfrak{M}_1 \rightarrow \mathfrak{M}_A \rightarrow \mathfrak{M}_2 \rightarrow 0,$$

then we have a short exact sequence of complexes

$$0 \rightarrow \mathfrak{M}_1(B) \rightarrow \mathfrak{M}_A(B) \rightarrow \mathfrak{M}_2(B) \rightarrow 0$$

and the corresponding long exact sequence in homology. The endomorphism T of $\mathfrak{M}$ induces an endomorphism $\mathcal{T}$ of this long exact sequence, regarded as a complex of vector spaces. We may assume that both T and $\mathcal{T}$ are of degree zero, since otherwise all supertraces vanish for trivial reasons. Now the statement that the supertrace of T depends only on the K^0 -class of A is equivalent to the statement that the supertrace of $\mathcal{T}$ vanishes. But this is an immediate consequence of exactness: if d denotes the differential in the long exact sequence, and $\mathcal{R}$ denotes the sum of all terms in the long exact sequence regarded as a graded vector space, then by exactness one can write

$$\mathcal{T} = d\mathcal{P} + \mathcal{P}d,$$

for some linear map $\mathcal{P} : \mathcal{R} \rightarrow \mathcal{R}$ of degree -1 . The supertrace of the anti-commutator of two odd endomorphisms of a graded vector space obviously vanishes.

4.7 Line operators as functors on branes

We have seen that in the twisted theory line operators form a monoidal $\mathbb{C}$ -linear category, or, better, a monoidal DG-category. As in [5], it is useful to

think of objects of this category as functors on the category of B-branes on $\mathcal{M}_{Higgs}(G, C)$. This makes the monoidal structure more obvious: it is simply given by the composition of functors.⁶

It is particularly simple to describe the functor corresponding to a Wilson line operator $W_R(p)$. It tensors every B-brane on $\mathcal{M}_{Higgs}(G, C)$ by a holomorphic vector bundle $R(\mathcal{E}(p))$, where $\mathcal{E}(p)$ is a restriction to $p \in C$ of the universal G -bundle $\mathcal{E}$ on $\mathcal{M}_{Higgs}(G, C) \times C$ [5].

The functor corresponding to an 't Hooft operator is a Hecke transformation, as explained in [5]. Let us remind what a Hecke transformation is in the case $G = U(N)$. Instead of a principal $U(N)$ -bundle, it is convenient to work with a holomorphic vector bundle E associated via the tautological N -dimensional representation of $U(N)$. A Hecke transformation of $E_- = E$ at a point $p \in C$ is another holomorphic vector bundle E_+ of the same rank which is isomorphic to E_- on $C \setminus p$. One can always choose a basis of holomorphic sections $f_1, \dots, f_N$ of E_- near p so that E_+ is locally generated by

$$s_1 = z^{-\mu_1} f_1, \dots, s_N = z^{-\mu_N} f_N,$$

where $\mu_1, \dots, \mu_N$ are integers. The integers $\mu_1, \dots, \mu_N$ are well-defined modulo permutation and can be thought of as a coweight of $U(N)$ modulo the action of the Weyl group. For fixed E_- and μ , the space of allowed E_+ is a finite-dimensional submanifold $\mathcal{C}_\mu$ of the infinite-dimensional affine Grassmannian $GL(N, \mathbb{C}((z)))/GL(N, \mathbb{C}[[z]])$, where $\mathbb{C}((z))$ is the field of formal Laurent series and $\mathbb{C}[[z]]$ is the ring of formal Taylor series. Specifically, $\mathcal{C}_\mu$ is the orbit of the matrix

$$Z_\mu(z) = \text{diag}(z^{-\mu_1}, \dots, z^{-\mu_N})$$

under the left action of $GL(N, \mathbb{C}[[z]])$. This describes how 't Hooft transformations act on structure sheaves of points on $\mathcal{M}(G, C) \subset \mathcal{M}_{Higgs}(G, C)$. One can similarly define the transformation of a more general point with a nontrivial Higgs field, see [5] for details. One can also define how 't Hooft/Hecke operators act on more general objects of the category of B-branes, but we will not need this here.

⁶Alternatively, one can regard a B-brane as a 1-morphism between an empty theory and the B-model on $\mathcal{M}_{Higgs}(G, C)$, regarded as objects of the 2-category of 2d TFTs, and one can regard a line operator as a 1-morphism from the B-model to itself. Then the action of the line operator on the brane is given by the composition of 1-morphisms.

For a general gauge group G , the situation is similar. One defines the affine Grassmannian Gr_G as the quotient $G((z))/G[[z]]$, where $G((z))$ is the group of $G_{\mathbb{C}}$ -valued Laurent series and $G[[z]]$ is the group of $G_{\mathbb{C}}$ -valued Taylor series. Gr_G is a union of Schubert cells $\mathcal{C}_{\mu}$ labeled by the elements of $\Lambda_{cw}(G)/\mathcal{W}$. For a fixed coweight μ the space of Hecke transformations of a holomorphic G -bundle E_- is the corresponding Schubert cell $\mathcal{C}_{\mu}$.

The functor corresponding to a general Wilson-'t Hooft operator is a combination of a Hecke transformation and tensoring with a certain holomorphic vector bundle on $\mathcal{C}_{\mu}$. For simplicity, let us only consider the case when the initial B-brane is a point E_- of $\mathcal{M}(G, C) \subset \mathcal{M}_{Higgs}(G, C)$. Recall that the ‘‘electric’’ part of the Wilson-'t Hooft operator can be described by a representation R of the group H which is the stabilizer subgroup of the coweight μ (under the adjoint representation). Clearly, the electric degree of freedom will live in some vector bundle over $\mathcal{C}_{\mu}$. This bundle is associated via R to a certain principal H -bundle over $\mathcal{C}_{\mu}$.

To determine this bundle, note that over $\mathcal{C}_{\mu}$ there is a principal G -bundle whose fiber can be identified with the fiber of E_+ over $z = 0$. A formal definition, in the case $G = U(N)$, is as follows. $\mathcal{C}_{\mu}$ can be thought of as the set of equivalence classes of matrix functions of the form

$$F(z)Z_{\mu}(z)G(z), \quad F(z), G(z) \in GL(N, \mathbb{C}[[z]]), \quad Z_{\mu}(z) = \text{diag}(z^{-\mu_1}, \dots, z^{-\mu_N})$$

under the right action of $GL(N, \mathbb{C}[[z]])$. Let us now replace $GL(N, \mathbb{C}[[z]])$ with its subgroup $GL_0(N, \mathbb{C}[[z]])$ consisting of matrix functions which are identity at $z = 0$. The set of equivalence classes of matrices $F(z)Z_{\mu}(z)G(z)$ under the right $GL_0(N, \mathbb{C}[[z]])$ action is clearly a principal G -bundle $\mathcal{P}_{\mu}$ over $\mathcal{C}_{\mu}$. This G -bundle has a reduction to a principal P -bundle $\mathcal{Q}_{\mu}$, where P is the parabolic subgroup whose quotient by the maximal unipotent subgroup is $H_{\mathbb{C}}$. (This reflects the fact that the gauge group is broken down to H near $z = 0$). Explicitly, $\mathcal{Q}_{\mu}$ consists of $GL_0(N, \mathbb{C}[[z]])$ -equivalence classes of matrix functions $F(z)Z_{\mu}(z)G(z)$ such that $G(0) \in P$. The group H acts by right multiplication. The $H_{\mathbb{C}}$ -bundle we are after is the quotient of $\mathcal{Q}_{\mu}$.

In the next section, we will discuss in detail Wilson-'t Hooft line operators for $G = PSU(2)$; as a preparation, let us describe the relevant vector bundles over $\mathcal{C}_{\mu}$ in the case when μ is the smallest nontrivial coweight. The Schubert cell $\mathcal{C}_{\mu}$ in this case is simply $\mathbb{P}^1 = PSU(2)/U(1) = PSL(2, \mathbb{C})/B$, where B is the Borel subgroup of $G_{\mathbb{C}} = PSL(2, \mathbb{C})$. The B -bundle in question is simply the tautological bundle $G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}/B$, and the $H = U(1)$ bundle is the Hopf

bundle. The coweight (resp. weight) lattice of $PSL(2, \mathbb{C})$ is isomorphic to the lattice of integers (resp. even integers). The electric degree of freedom of a Wilson-'t Hooft line operator with $\mu = 1$ and $\nu \in 2\mathbb{Z}$ by definition takes values in the fiber of the line bundle L associated with the Hopf bundle via a $U(1)$ representation of charge ν . Since the Hopf bundle is the circle bundle of the line bundle $\mathcal{O}(-1)$ over $\mathbb{P}^1$, we conclude that $L = \mathcal{O}(-\nu)$.

As a rule, a functor from the derived category of X to itself is “representable” by an object of the derived category of $X \times X$. It is not known whether this is the case for all reasonable functors, but it is certainly true for functors corresponding to line operators. To show this, let $\Sigma \simeq \mathbb{R}^2$, and suppose for simplicity that the line operator has the shape of a straight line. Using the “folding trick” we can regard the field theory on $\mathbb{R}^2$ with an insertion of a straight line operator as a product of two copies of the same field theory on a half-plane, with a particular boundary condition. The product of two copies of a B-model with target X is a B-model with target $X \times X$, and the boundary condition corresponds to a B-brane on $X \times X$. This B-brane represents the functor corresponding to the line operator. For example, in the case of Wilson line operator $W_R(p)$, the corresponding object is the diagonal of $\mathcal{M}_{Higgs}(G, C) \times \mathcal{M}_{Higgs}(G, C)$ equipped with the holomorphic vector bundle $R(\mathcal{E}_p)$.

The “folding trick” reduces the study of line operators to the study of boundary conditions. There is a converse trick which reduces the study of boundary conditions to the study of line operators. Consider a B-model on a strip $I \times \mathbb{R}$ with some boundary conditions α and β . We can identify the α and β boundaries and replace I with S^1 , with an insertion of a line operator. If we think of β as a 1-wall between our B-model and the empty theory, and about α as the 1-wall between the empty theory and the B-model, then the line operator is obtained by fusing together these 1-walls to get a 1-wall between the B-model and itself. We will call this the “gluing trick”.

One application of the “gluing trick” is to produce new examples of line operators from known boundary conditions. Given any two B-branes on $\mathcal{M}_{Higgs}(G, C)$ we may produce a line operator in the B-model with target $\mathcal{M}_{Higgs}(G, C)$. However, this construction is not local on C and does not produce new line operators in the twisted gauge theory. For example, if we start with boundary conditions for the B-model which can be lifted to the gauge theory, then “gluing” them produces a 3-wall in the gauge theory rather than a 1-wall. Only upon further compactification on C does one get a line operator in the 2d TFT.

We have argued above that any correlator involving a loop operator A and any other loop, line, or local operator depends only on the K^0 -class of A . It was assumed that $\Sigma = S^1 \times S^1$. One may ask if the statement remains true if $\Sigma = \mathbb{R} \times I$ with suitable boundary conditions. Since the “gluing trick” replaces any pair of boundary conditions with a line operator, the answer appears to be “yes”. But we have to keep in mind that line operators produced by the “gluing trick” are not local on C . Therefore, to apply the above reasoning we need to work with a different K^0 -group: the K^0 -group of the category of all line operators in the B-model with target $\mathcal{M}_{Higgs}(G, C)$.

4.8 The algebra of loop operators and S-duality

We have argued above that loop operators form a commutative algebra. To identify this algebra, one can use the fact that the gauge theory becomes abelian in the infrared, if the Higgs field has a generic expectation value (with all eigenvalues distinct). More precisely, the gauge group is broken down to a semi-direct product of the maximal torus T of G and the Weyl group $\mathcal{W}$. Loop operators in such a theory are labeled by Weyl-invariant combinations of loop operators in the abelian gauge theory with gauge group T . The latter are labeled by electric and magnetic charges, i.e. by elements of $\text{Hom}(T, U(1)) = \Lambda_w(G)$ and $\text{Hom}(U(1), T) = \Lambda_{cw}(G)$. The algebra structure is also obvious: under Operator Product electric and magnetic charges simply add up, so the algebra of loop operators is isomorphic to the Weyl-invariant part of the group algebra of $\Lambda_w(G) \oplus \Lambda_{cw}(G) = \widehat{\Lambda}(G)$.

This reasoning may seem suspect, because a vacuum with a particular expectation value of a Higgs field is not BRST-invariant, and if we try to integrate over all expectation values, we have to include vacua where non-abelian gauge symmetry is restored. One can give a more careful argument as follows. Let us consider again the case where $M = \Sigma \times C$, and Σ has a nonempty boundary. From the viewpoint of the effective field theory on Σ , the theory “abelianizes” in the limit where the Higgs field q_z is large and all of its eigenvalues are distinct. The problem is that one has to integrate over all values of q_z , including those where some of the eigenvalues coincide. To argue that we can perform the computation locally in the target space $\mathcal{M}_{Higgs}(G, C)$, recall that in the B-model the path-integral localizes on constant maps. Therefore if we impose a boundary condition which keeps q_z away from the dangerous region, we can be sure that the dangerous regions of the target space will not contribute. For example, one can take a boundary

condition corresponding to a B-brane which is a generic fiber of the Hitchin fibration. If $\partial\Sigma$ has several components, it is sufficient to impose such a boundary condition only on one component of the boundary.

It is known how the S-duality group acts on the algebra of loop operators [3]. The generator T , which shifts $\tau \rightarrow \tau + 1$, does not change the magnetic charge $\mu \in \Lambda_{cw}(G)$ and acts on the electric charge $\nu \in \Lambda_w(G)$ by

$$\nu \rightarrow \nu + \mu.$$

Here we regard μ as an element of $\mathfrak{t}^*$ using the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$, defined by the canonical metric on $\mathfrak{t}$ (the Killing metric is normalized so that short coroots have length $\sqrt{2}$). The shift of the electric charge is due to the Witten effect [13]. The generator S which exchanges G and ${}^L G$ conjecturally acts by

$$(\mu, \nu) \rightarrow (\mathfrak{R} \cdot \mu, \mathfrak{R} \cdot \nu) \begin{pmatrix} 0 & -1/\sqrt{n_{\mathfrak{g}}} \\ \sqrt{n_{\mathfrak{g}}} & 0 \end{pmatrix}$$

Here $\mathfrak{R}$ is a certain orthogonal transformation which squares to an element of the Weyl group [2, 14]. For simply-laced groups one can define Montonen-Olive duality so that $\mathfrak{R} = 1$.

These results, however, do not yet allow us to compute the OPE of any two given Wilson-'t Hooft operator. To do that, one needs to know which element of the group algebra of $\widehat{\Lambda}(G)$ corresponds to any particular Wilson-'t Hooft operator. Recall that the space of WH operators has a natural basis labeled by elements of

$$\widehat{\Lambda}(G)/\mathcal{W}$$

So what we are looking for is a basis for the Weyl-invariant part of the group algebra of $\widehat{\Lambda}(G)$ labeled by this set.

The most obvious such basis is obtained simply by taking an element of $\widehat{\Lambda}(G)$ in a particular Weyl-equivalence class and averaging it over the Weyl group. Such basis elements correspond to loop operators in the abelian gauge theory with particular electric and magnetic charges.⁷ But this is not the basis we are looking for. For example, consider a Wilson operator for an irreducible representation R_ν with highest weight ν . From the viewpoint of the effective abelian gauge theory, it is a sum of Wilson operators with electric charges given by decomposing R_ν with respect to the maximal torus of G .

⁷Averaging over the Weyl group reflects the fact that the gauge group is really a semidirect product of the Weyl group and the maximal torus of G .

All weights of R_ν appear in this decomposition, not just the weights which are in the Weyl-orbit of the highest weight. Similarly, in the phase with the broken nonabelian gauge symmetry an 't Hooft operator corresponding to a coweight μ of G decomposes as a sum over weights of the representation ${}^L R_\mu$ of the dual group. The explanation of this phenomenon is more subtle than for Wilson operators and involves “monopole bubbling” [5].

In the case $G = PSU(2)$ (or $G = SU(2)$), the desired basis is uniquely determined by imposing S-duality. To simplify notation, let us identify the group algebra of $\Lambda_{cw}(PSU(2)) \simeq \mathbb{Z}$ with the space of polynomials of x, x^{-1} , and the group algebra of $\Lambda_w(PSU(2)) \simeq 2 \cdot \mathbb{Z}$ with the space of polynomials of y^2, y^{-2} . The Weyl group acts by $x \rightarrow x^{-1}, y \rightarrow y^{-1}$. Then the algebra of WH loop operators can be identified with the space of Weyl-invariant polynomials of x, x^{-1}, y^2, y^{-2} (for $G = PSU(2)$) or of x^2, x^{-2}, y, y^{-1} (for $G = SU(2)$). We know already that the Wilson loop in the representation with highest weight $n \in \mathbb{Z}$ corresponds to the polynomial

$$WT_{0,n} = y^n + y^{n-2} + \dots + y^{-n}.$$

Here n is an arbitrary integer if $G = SU(2)$ and an even integer if $G = PSU(2)$. Similarly, the 't Hooft loop labeled by the coweight $m \in \mathbb{Z}$ corresponds to the polynomial

$$WT_{m,0} = x^m + x^{m-2} + \dots + x^{-m},$$

where $m \in \mathbb{Z}$ if $G = PSU(2)$ and $m \in 2 \cdot \mathbb{Z}$ if $G = SU(2)$. This is, of course, compatible with the Montonen-Olive duality, which acts by

$$(m, n) \mapsto (n, -m).$$

Moreover, any pair $(m, n) \in \widehat{\Lambda}(G)$ can be brought to the form $(m', 0)$ by an S-duality transformation. This determines the polynomial corresponding to an arbitrary Wilson-'t Hooft operator for $G = PSU(2)$ or $G = SU(2)$:

$$WT_{m,n} = x^m y^n + x^{m-2a} y^{n-2b} + x^{m-4a} y^{n-4b} + \dots + x^{-m} y^{-n}.$$

Here the integers a, b are defined by the condition that $m/n = a/b$, a and b have the same signs as m and n , respectively, and the fraction a/b is reduced.

For higher-rank groups, S-duality is not sufficient to fix the basis. This is because electric and magnetic charges need not be linearly dependent for higher-rank gauge groups.

In the next section, we will test some predictions of S-duality for the gauge group $PSU(2)$ by a direct computation of the OPE of WH loop operators at weak coupling. The same method could be used to determine the OPE of WH loop operators for higher-rank groups, but the computations become very complicated.

5 OPE at weak coupling

5.1 Semiclassical quantization of Wilson-'t Hooft operators

To compute the OPE of a pair of Wilson-'t Hooft line operators we will follow the same method as in [5]. We will quantize the twisted gauge theory on a manifold with boundaries $C \times I \times \mathbb{R}$, with suitable boundary conditions and with two insertions of Wilson-'t Hooft operators. From the 2d viewpoint, the boundary conditions correspond to B-branes on $\mathcal{M}_{Higgs}(G, C)$. The problem reduces to the supersymmetric quantum mechanics on the space of zero modes of the gauge theory. In principle, one has to study the limit where the two operators approach each other, but in the twisted theory this last step is not necessary, if the line operators are sitting at the same point on C .

As in [5], it is convenient to choose the branes so that in the absence of Wilson-'t Hooft line operators the Hilbert space of the twisted gauge theory is one-dimensional. One possible choice is to take the brane α at $y = 0$ to be the 0-brane at a point r of $\mathcal{M}_{Higgs}(G, C)$ with vanishing Higgs field. The brane β at $y = 1$ will be the trivial line bundle on $\mathcal{M}_{Higgs}(G, C)$. Both of these branes are of type (B, B, B) . The classical space of vacua in this case consists of a single point r , with no zero modes, so the Hilbert space is one-dimensional. Alternatively, as in [5], one could take two branes of type (B, A, A) intersecting at a single point. The former choice is somewhat easier, so we will stick to it, but in practice there is not much difference between the two.

Having chosen the boundary conditions, we can assign to any collection $A, B, \dots$, of WH line operators the graded vector space $\mathfrak{H}_{\alpha\beta}(A, B, \dots)$, or better yet the corresponding BRST complex. Note that this assignment need not be invariant with respect to S-duality. This is because the choice of branes necessarily breaks the S-duality group. Neither is this assignment compatible with the monoidal structure on the category of line operators.

That is, it is not true, in general, that $\mathfrak{H}_{\alpha\beta}(\mathbf{A}, \mathbf{B})$ is isomorphic to

$$\mathfrak{H}_{\alpha\beta}(\mathbf{A}) \otimes \mathfrak{H}_{\alpha\beta}(\mathbf{B}).$$

This is in contrast with the situation in the GL-twisted theory [5].

The ultimate reason for this difference is that the twisted gauge theory we are dealing with is not topological, but only holomorphic-topological. Suppose we fix the location of the line operator $\mathbf{B}$, but vary the location of $\mathbf{A}$ on $I \times C$. The BRST-complex $\mathfrak{H}_{\alpha\beta}(\mathbf{A}, \mathbf{B})$ is a differential graded vector bundle over $I \times C$ with a connection along I and a $\bar{\partial}$ operator along C . If one fixes $p \in C$ and varies $y \in I$ (without colliding with $\mathbf{B}$), then the BRST complexes are all naturally isomorphic. But there is no isomorphism between complexes corresponding to different p .

In the GL-twisted theory, one can choose all line operators to be at the same point on X_1 and different points on C . Because line operators are local along C , the supersymmetric quantum mechanics describing this situation decomposes as a product of supersymmetric quantum-mechanical systems corresponding to each line operator. This implies that the quantum Hilbert space also factorizes.

In the holomorphic-topological field theory, if we want to study the OPE, we have to work with all line operators inserted at the same point on C (but different points on X_1), and the arguments like in the previous paragraph do not apply.

For simplicity, let us begin with the case where all line operators are either Wilson or 't Hooft, with no “mixed” ones. When quantizing the theory at weak coupling, the roles of Wilson and 't Hooft operators are very different. 't Hooft operators directly affect the equations for the BRST-invariant configurations whose solutions determine the space of bosonic zero modes. A Wilson operator corresponds to inserting an extra degree of freedom, which couples weakly to the gauge fields, and can be treated perturbatively.

The first step is to ignore the Wilson operators completely. As explained in [5], 't Hooft operators are line operators of type (B, A, A) , i.e. they can be viewed either as line operators in the B-model on $\mathcal{M}_{Higgs}(G, C)$, or in the A-model on $\mathcal{M}_{flat}(G, \mathbb{C})$. When Σ is flat and has no boundary, we can regard the twisted gauge theory on $\Sigma \times C$ as a supersymmetric sigma-model with $(4, 4)$ supersymmetry. The introduction of boundaries (either of A or B types) breaks 3/4 of supercharges and effectively eliminates one of the spatial directions, so we end up with a supersymmetric quantum mechanics

with $N = 2$ supersymmetry. The corresponding supersymmetry algebra has a single complex supercharge Q satisfying

$$Q^2 = 0, \quad \{Q, Q^\dagger\} = 2H,$$

where H is the Hamiltonian. Q -cohomology can be identified with the space of supersymmetric ground states, i.e. states satisfying

$$Qa = Q^\dagger a = 0.$$

Strictly speaking, this is guaranteed only when the target space of the supersymmetric quantum mechanics is compact. In the case of interest to us, the target space is the Schubert cell $\mathcal{C}_\mu$ (if there is a single 't Hooft operator), or a product of several Schubert cells, which are noncompact unless all coweights are minuscule [5]. From the physical viewpoint, the correct version of Q -cohomology is the L^2 -cohomology, and we will assume some version of Hodge theory works for the L^2 -cohomology.

There are two well-known kinds of $N = 2$ supersymmetric quantum mechanics (SQM). $N = 2$ SQMs of the first kind are classified by a choice of a Riemannian target and a flat vector bundle V over it; its space of states is the space of differential forms with values in V , and the corresponding operator Q is the twisted de Rham differential. This is the kind of effective SQM which appears when considering 't Hooft operators as line operators in the A-model [5]. It is clear that this SQM is not suitable for the B-model, because once we include the Wilson operators, the bundle over $\mathcal{C}_\mu$ will not be flat. Also, in the B-model the BRST operator Q is likely to be a Dolbeault-type operator.

$N = 2$ SQMs of the second kind look more promising: they are classified by a choice of a Kähler target space and a holomorphic vector bundle over it. The space of states is the space of differential forms of type $(0, p)$ with values in a holomorphic vector bundle W , and Q acts as the Dolbeault operator.

In the next section we will perform the reduction to a SQM in some detail and show that in the absence of Wilson operators W is the bundle of forms of type $(p, 0)$ (for any p). But we can deduce this result in a simpler way by making use of both A and B-models. Indeed, if we take as our boundary conditions branes of type (B, A, A) , we can interpret the space of ground states of the SQM in terms of either model. For the Dolbeault cohomology of W to be isomorphic to the de Rham cohomology of $\mathcal{C}_\mu$, W has to be the bundle

$$\oplus_p \Omega^{p,0}(\mathcal{C}_\mu).$$

In the presence of a Wilson line, this also has to be tensored with the holomorphic vector bundle corresponding to the Wilson line.

5.2 Bosonic zero modes

Our next task is to analyze bosonic zero modes in the presence of 't Hooft operators. The BPS equations are simply the Bogomolny equations, if the boundary conditions are suitably chosen [5]. In fact, it has been shown in [5] that if in the absence of an 't Hooft operator the solution is unique, then in the presence of 't Hooft operators the moduli space of solutions is $\mathcal{C}_\mu$ (for a single 't Hooft operator), or a tower of several Schubert cells $\mathcal{C}_{\mu_i}$ fibered over each other (for several 't Hooft operators). So the bosonic zero modes span the tangent space to $\mathcal{C}_\mu$ or its generalization. However, it is useful to have an explicit description of the tangent space in terms of solutions of linearized Bogomolny equations in order to identify the fermionic zero modes.

Recall that $w = y + ix_0$ with $y \in [0, 1]$, $x_0 \in \mathbb{R}$, is a complex coordinate on Σ , while z is a complex coordinate on a closed Riemann surface C . For $t = i$ the BRST-invariant “holomorphic connection” on Σ is

$$\mathcal{A} = (A_w + i\Phi_w)dw + (A_{\bar{w}} + i\Phi_{\bar{w}})d\bar{w} \quad (3)$$

We further define the “anti-holomorphic connection”

$$\check{\mathcal{A}} = (A_w - i\Phi_w)dw + (A_{\bar{w}} - i\Phi_{\bar{w}})d\bar{w} \quad (4)$$

and introduce corresponding covariant differentials in the adjoint representation:

$$\mathcal{D} = \partial + i[\mathcal{A}, \cdot] = D - [\Phi, \cdot], \quad \check{\mathcal{D}} = \partial + i[\check{\mathcal{A}}, \cdot] = D + [\Phi, \cdot].$$

Note that holomorphic and anti-holomorphic connections are related by Hermitian conjugation:

$$\mathcal{A}^\dagger = \check{\mathcal{A}}.$$

We set background q_z and $\tilde{q}$ to zero. Then it can be shown analogously to [5] that variations of these fields are also zero. Therefore, the complete set of BPS equations is obtained by setting to zero the BRST variations of gauginos. These are written down⁸ in [10].

⁸In comparing with [10] exchange z and w .

Let us first consider one of the “real” BPS equations:

$$-i\left(D_w\Phi_{\bar{w}}+D_{\bar{w}}\Phi_w\right)=g_{w\bar{w}}g^{z\bar{z}}\left(F_{z\bar{z}}-i[q_z,q_{\bar{z}}]+2g_{z\bar{z}}[\tilde{q},\tilde{q}^\dagger]\right) \quad (5)$$

where $w = y + ix^0$ and $z = x^1 + ix^2$. Variation of (5) gives

$$-iD_w(\delta\Phi_{\bar{w}})-iD_{\bar{w}}(\delta\Phi_w)+[\delta A_w,\Phi_{\bar{w}}]+[\delta A_{\bar{w}},\Phi_w]=g_{w\bar{w}}g^{z\bar{z}}\left(D_z\delta A_{\bar{z}}-D_{\bar{z}}\delta A_z\right) \quad (6)$$

where $2D_z = D_1 - iD_2$. We further assume that all fields are independent of time x^0 and that background fields $A_0 = \Phi_0 = 0$, so that $D_w = D_{\bar{w}} = \frac{1}{2}D_y$ and $\Phi_w = \Phi_{\bar{w}} = \frac{1}{2}\Phi_y$. Then, (6) becomes

$$-D_y(\delta\Phi_w+\delta\Phi_{\bar{w}})+i[\Phi_y,\delta A_w+\delta A_{\bar{w}}]+2ig_{w\bar{w}}g^{z\bar{z}}D_z(\delta A_{\bar{z}})-2ig_{w\bar{w}}g^{z\bar{z}}D_{\bar{z}}(\delta A_z)=0 \quad (7)$$

Now we impose a gauge-fixing condition:

$$D_y(\delta\mathcal{A}_w+\delta\mathcal{A}_{\bar{w}})+[\Phi_y,\delta\mathcal{A}_w+\delta\mathcal{A}_{\bar{w}}]+4g_{w\bar{w}}g^{z\bar{z}}D_z(\delta A_{\bar{z}})=0 \quad (8)$$

From (7) and (8) follows

$$\frac{g^{w\bar{w}}}{2}\mathcal{D}_y(\delta\check{A}_w+\delta\check{A}_{\bar{w}})+2g^{z\bar{z}}D_{\bar{z}}(\delta A_z)=0 \quad (9)$$

Taking hermitean conjugate of (9) gives

$$\frac{g^{w\bar{w}}}{2}\check{\mathcal{D}}_y(\delta\mathcal{A}_w+\delta\mathcal{A}_{\bar{w}})+2g^{z\bar{z}}D_z(\delta A_{\bar{z}})=0 \quad (10)$$

Next we consider the complex BPS equations:

$$\mathcal{F}_{\bar{z}w}=0, \quad \mathcal{F}_{\bar{z}\bar{w}}=0 \quad (11)$$

Variation of these two equations gives

$$D_{\bar{z}}(\delta\mathcal{A}_w)-\mathcal{D}_w(\delta A_{\bar{z}})=0 \quad (12)$$

and

$$D_{\bar{z}}(\delta\mathcal{A}_{\bar{w}})-\mathcal{D}_{\bar{w}}(\delta A_{\bar{z}})=0 \quad (13)$$

The sum of (12) and (13) gives

$$D_{\bar{z}}(\delta\mathcal{A}_w+\delta\mathcal{A}_{\bar{w}})-\mathcal{D}_y\delta A_{\bar{z}}=0 \quad (14)$$

Taking hermitean conjugate of (14) we obtain

$$D_z (\delta \check{A}_w + \delta \check{A}_{\bar{w}}) - \check{D}_y (\delta A_z) = 0 \quad (15)$$

We conclude that $T\mathcal{M}$ splits into two parts. Holomorphic bosonic modes from the first part satisfy Dirac-like equation:

$$O_1 := \begin{pmatrix} \mathcal{D}_y & 2D_{\bar{z}} \\ 2g^{z\bar{z}}D_z & -g^{w\bar{w}}\check{D}_y \end{pmatrix} \begin{pmatrix} -\delta A_{\bar{z}} \\ \frac{1}{2}(\delta \mathcal{A}_w + \delta \mathcal{A}_{\bar{w}}) \end{pmatrix} = 0 \quad (16)$$

We impose boundary conditions

$$\delta A_{\bar{z}}(0) = 0, \quad \delta \mathcal{A}_w(1) + \delta \mathcal{A}_{\bar{w}}(1) = 0 \quad (17)$$

The difference of (12) and (13) as well as variation of the second “real” BPS condition $\mathcal{F}_{w\bar{w}} = 0$ give equations for the remaining holomorphic bosonic variation $\delta \mathcal{A}_w - \delta \mathcal{A}_{\bar{w}}$:

$$D_{\bar{z}}(\delta \mathcal{A}_w - \delta \mathcal{A}_{\bar{w}}) = 0, \quad \mathcal{D}_y(\delta \mathcal{A}_w - \delta \mathcal{A}_{\bar{w}}) = 0. \quad (18)$$

Analogously, $\bar{T}\mathcal{M}$ splits into two parts. Some of the anti-holomorphic bosonic zero modes satisfy Dirac-like equation:

$$O_2 := \begin{pmatrix} \check{D}_y & 2D_z \\ 2g^{z\bar{z}}D_{\bar{z}} & -g^{w\bar{w}}\mathcal{D}_y \end{pmatrix} \begin{pmatrix} -\delta A_z \\ \frac{1}{2}(\delta \check{A}_w + \delta \check{A}_{\bar{w}}) \end{pmatrix} = 0 \quad (19)$$

We impose boundary conditions

$$\delta A_z(0) = 0, \quad \delta \check{A}_w(1) + \delta \check{A}_{\bar{w}}(1) = 0 \quad (20)$$

The remaining anti-holomorphic bosonic variation $\delta \check{A}_w - \delta \check{A}_{\bar{w}}$ satisfy

$$D_z(\delta \check{A}_w - \delta \check{A}_{\bar{w}}) = 0, \quad \check{D}_y(\delta \check{A}_w - \delta \check{A}_{\bar{w}}) = 0. \quad (21)$$

There are no non-trivial solutions of (18) and (21) and we conclude that $T\mathcal{M}$ (resp. $\bar{T}\mathcal{M}$) is defined as the kernel of the operator O_1 (resp. O_2).

5.3 Fermionic zero modes

The gaugino equations of motion are:

$$D_z(\bar{\lambda}_{\bar{w}}) + D_{\bar{w}}(\bar{\lambda}_z) + [\Phi_{\bar{w}}, \lambda_z] = 0 \quad (22)$$

$$D_z(\lambda_w) + D_w(\lambda_z) + [\Phi_w, \bar{\lambda}_z] = 0 \quad (23)$$

$$D_w \bar{\lambda}_w + [\lambda_w, \Phi_w] - g_{w\bar{w}} g^{z\bar{z}} D_{\bar{z}} \bar{\lambda}_z = 0 \quad (24)$$

$$D_{\bar{w}} \lambda_w + [\bar{\lambda}_w, \Phi_w] - g_{w\bar{w}} g^{z\bar{z}} D_{\bar{z}} \lambda_z = 0 \quad (25)$$

The sum of (22) and (23) gives (recall $\Phi_w = \Phi_{\bar{w}} = \frac{1}{2}\Phi_y$, $D_w = D_{\bar{w}} = \frac{1}{2}D_y$)

$$2D_z(\lambda_w + \bar{\lambda}_w) + D_y(\lambda_z + \bar{\lambda}_z) + [\Phi_y, (\lambda_z + \bar{\lambda}_z)] = 0 \quad (26)$$

Meanwhile, the sum of (24) and (25) gives

$$2g^{z\bar{z}} D_{\bar{z}}(\lambda_z + \bar{\lambda}_z) - g^{w\bar{w}} D_y(\lambda_w + \bar{\lambda}_w) + g^{w\bar{w}} [\Phi_y, (\lambda_w + \bar{\lambda}_w)] = 0 \quad (27)$$

The two equations (26) and (27) can be recast as a Dirac-like equation:

$$\begin{pmatrix} \check{D}_y & 2D_z \\ 2g^{z\bar{z}} D_{\bar{z}} & -g^{w\bar{w}} \mathcal{D}_y \end{pmatrix} \begin{pmatrix} \lambda_z + \bar{\lambda}_z \\ \lambda_w + \bar{\lambda}_w \end{pmatrix} = 0 \quad (28)$$

Similarly, the difference of equations (22) and (23) combines with the difference of equations (24) and (25) into another Dirac-like equation:

$$\begin{pmatrix} \mathcal{D}_y & 2D_z \\ 2g^{z\bar{z}} D_{\bar{z}} & -g^{w\bar{w}} \check{D}_y \end{pmatrix} \begin{pmatrix} \bar{\lambda}_z - \lambda_z \\ \bar{\lambda}_w - \lambda_w \end{pmatrix} = 0 \quad (29)$$

We impose boundary conditions at $y = 0$ or $y = 1$:

$$\lambda_z(0) + \bar{\lambda}_z(0) = 0, \quad \lambda_w(1) + \bar{\lambda}_w(1) = 0 \quad (30)$$

$$\lambda_z(1) - \bar{\lambda}_z(1) = 0, \quad \lambda_w(0) - \bar{\lambda}_w(0) = 0 \quad (31)$$

Note that (30) are BRST invariant boundary condition, moreover they are BRST variations of the bosonic boundary conditions (20). Meanwhile the BRST variation of (31) gives

$$q_z T \tilde{q}^\dagger|_{y=1} = 0$$

which is zero in the background we consider, i.e. with $q_z = 0$ and $\tilde{q} = 0$. Comparing (28) with equations of motion for the anti-holomorphic bosonic

zero modes (19), we conclude that solutions of (28) are in one-to-one correspondence with elements of $\overline{T}\mathcal{M}$.

Eq. (29) has no nontrivial solutions for the following reason. Let us denote by O the operator in (29). In addition to (31) we impose boundary conditions on ghost number -1 fermions

$$\lambda_{\bar{z}w}(0) - \bar{\lambda}_{\bar{z}\bar{w}}(0) = 0, \quad \lambda_{w\bar{w}}(1) - \bar{\lambda}_{w\bar{w}}(1) = 0 \quad (32)$$

The boundary conditions (31) and (32) are chosen so that in computing hermitean conjugate $O^\dagger$ we can drop boundary terms obtained from integration by parts. Then we find

$$O^\dagger O = -\left(2\Delta_C + \frac{1}{2}\Delta_\Sigma\right)I_{2\times 2} \quad (33)$$

where $\Delta_C = g^{z\bar{z}}(D_{\bar{z}}D_z + D_zD_{\bar{z}})$ and $\Delta_\Sigma = g^{w\bar{w}}(\check{D}_y\mathcal{D}_y + \mathcal{D}_y\check{D}_y)$. In obtaining (33) we used BPS equations for the background fields. Since both $-\Delta_C$ and $-\Delta_\Sigma$ are nonnegative operators, the kernel of the operator O must be annihilated by both Laplacians. This implies, in particular, that $\bar{\lambda}_z - \lambda_z$ is constant on the interval $y \in [0, 1]$. However, such a mode is necessarily zero due to boundary conditions (32).

Equations of motion for matter fermions (using the $\mathcal{N} = 2$ language) are

$$\begin{pmatrix} \check{D}_y & 2D_{\bar{z}} \\ 2g^{z\bar{z}}D_z & -g^{w\bar{w}}\mathcal{D}_y \end{pmatrix} \begin{pmatrix} \psi_{\bar{z}} + \bar{\chi}_{\bar{z}} \\ -(\psi_{\bar{w}} + \bar{\chi}_w) \end{pmatrix} = 0 \quad (34)$$

and

$$\begin{pmatrix} \mathcal{D}_y & 2D_{\bar{z}} \\ 2g^{z\bar{z}}D_z & -g^{w\bar{w}}\check{D}_y \end{pmatrix} \begin{pmatrix} \bar{\chi}_{\bar{z}} - \psi_{\bar{z}} \\ \psi_{\bar{w}} - \bar{\chi}_w \end{pmatrix} = 0 \quad (35)$$

We impose the following boundary conditions at $y = 0$ or $y = 1$:

$$\psi_{\bar{w}}(1) - \bar{\chi}_w(1) = 0, \quad \bar{\chi}_{\bar{z}}(0) - \psi_{\bar{z}}(0) = 0 \quad (36)$$

$$\psi_{\bar{w}}(0) + \bar{\chi}_w(0) = 0, \quad \bar{\chi}_{\bar{z}}(1) + \psi_{\bar{z}}(1) = 0 \quad (37)$$

Note that (37) are BRST invariant, meanwhile the BRST variation of (36) gives

$$D_{\bar{z}}\tilde{q}^\dagger|_{y=0} = 0, \quad \mathcal{D}_y\tilde{q}^\dagger|_{y=1} = 0$$

Matter fermions (35) belong to $T\mathcal{M}$, as can be seen by comparing with (16). Eq. (34) has no nontrivial solutions. The proof is similar to that for

operator O above. Let us denote by O' the operator in (34). In addition to (37) we impose boundary conditions on ghost number -1 fermions

$$\chi_{z\bar{w}}(0) + \bar{\psi}_{zw}(0) = 0, \quad \chi_{z\bar{z}}(1) + \bar{\psi}_{z\bar{z}}(1) = 0 \quad (38)$$

The boundary conditions (37) and (38) are chosen so that when computing hermitean conjugate $O'^{\dagger}$ we can drop boundary terms obtained from integration by parts. Then we use BPS equations for the background to show

$$O'^{\dagger}O' = -\left(2\Delta_C + \frac{1}{2}\Delta_{\Sigma}\right)I_{2\times 2} \quad (39)$$

where $\Delta_C = g^{z\bar{z}}(D_{\bar{z}}D_z + D_zD_{\bar{z}})$ and $\Delta_{\Sigma} = g^{w\bar{w}}(\check{\mathcal{D}}_y\mathcal{D}_y + \mathcal{D}_y\check{\mathcal{D}}_y)$. Since both $-\Delta_C$ and $-\Delta_{\Sigma}$ are non-negative operators, the kernel of the operator O' must be annihilated by both Laplacians. This implies, in particular, that $\bar{\chi}_{\bar{z}} + \psi_{\bar{z}}$ is constant on the interval $y \in [0, 1]$. However, such a mode is necessarily zero due to boundary conditions (37).

The result of this analysis is that fermionic zero modes span $T\mathcal{M} \oplus \bar{T}\mathcal{M}$. Therefore the Hilbert space of the effective SQM is the space of L^2 sections of the vector bundle

$$\oplus_p \Lambda^p \left(T^*\mathcal{M} \oplus \bar{T}^*\mathcal{M} \right) = \oplus_{p,q} \Omega^{p,q}(\mathcal{M}).$$

From the formulas for BRST transformation we see that BRST variation of bosonic zero modes are precisely the fermionic zero modes spanning $\bar{T}\mathcal{M}$, while BRST variations of fermionic zero modes vanish. This means that the BRST operator acts as the Dolbeault operator.

6 OPE of Wilson-'t Hooft operators for $G = PSU(2)$

In this section we study in detail the OPE of WH loop operators in the special case $G = PSU(2)$. The main goal is to test the predictions of S-duality explained in 4.8.

6.1 OPE of a Wilson and an 't Hooft operator

Let us begin by considering the OPE of a Wilson and an 't Hooft operator. The most naive approach is to regard an 't Hooft operator as creating a

classical field configuration, and analyze the electric degree of freedom corresponding to the Wilson operator in this classical background. As explained above, the field singularity at the insertion point of an 't Hooft operator T_μ breaks the gauge group $G = PSU(2)$ down its subgroup $H = U(1)$, so it seems that all we have to do is to decompose the representation R associated to the Wilson operator into irreducibles with respect to H . If we label representations of $PSU(2)$ by an even integer n which is twice the isospin, and denote the magnetic charge of the 't Hooft operator by $m \in \mathbb{N}$, then the OPE at weak coupling appears to be

$$T_m \cdot W_n = WT_{m,n} + WT_{m,n-2} + \dots + WT_{m,-n}.$$

But this contradicts S-duality, which requires that there be a symmetry under $n \rightarrow -m, m \rightarrow n$. In fact, S-duality predicts that the OPE also contains contributions from WH operators with smaller magnetic charge. As explained in [5], this is due the “monopole bubbling”: the magnetic charge of an 't Hooft operator can decrease by 2 when it absorbs a BPS monopole. Such process is possible because the moduli space of solutions of the Bogomolny equations is noncompact for $m > 1$; configurations with smaller magnetic charge can be associated with points at infinity. The naive argument ignored monopole bubbling and therefore missed all such contributions.

This explanation also suggests that for $m = 1$, where the moduli space is simply $\mathbb{P}^1$ and therefore is compact, the naive argument is valid. To compare this with the S-duality predictions, we follow the procedure outlined in section 4.8. To the loop operators T_1 and W_n one associates Laurent polynomials

$$WT_{1,0}(x) = x + x^{-1}, \quad WT_{0,n}(y) = y^n + y^{n-2} + \dots + y^{-n}.$$

To the WH loop operator $WT_{1,k}$ one associates the Laurent polynomial

$$WT_{1,k}(x, y) = xy^k + x^{-1}y^{-k}.$$

We see that

$$WT_{1,0}(x)WT_{0,n}(y) = \sum_{j=0}^n WT_{1,n-2j}(x, y), \quad (40)$$

in agreement with the naive formula.

This example also provides a nice illustration of the difference between line and loop operators. Recall that the Hilbert space $\mathfrak{H}(A)$ associated to the

line operator $WT_{1,k}$ is the space of sections of the differential graded vector bundle

$$\mathcal{O}(-k) \otimes \oplus_{p,q} \Omega^{p,q}$$

over the Schubert cell $\mathcal{C}_\mu \simeq \mathbb{P}^1$, with the differential being the Dolbeault differential. Here the first factor comes from the electric degree of freedom, and the rest comes from fermionic zero modes. Instead of this differential graded vector bundle, we can think of the corresponding coherent sheaf

$$\mathcal{O}(-k) \otimes \Omega^*(\mathbb{P}^1) = \mathcal{O}(-k) + \mathcal{O}(-k-2).$$

The SQM Hilbert space is the Dolbeault resolution of this coherent sheaf, so instead of thinking about the BRST cohomology, we can think about the cohomology of this sheaf. Thus the sum of the WH line operators on the right-hand side of eq. (40) corresponds to the coherent sheaf

$$(\mathcal{O}(-n) + \mathcal{O}(-n+2) + \dots + \mathcal{O}(n)) \otimes \Omega^*(\mathbb{P}^1).$$

On the other hand, the product of a Wilson operator W_n and an 't Hooft operator T_1 gives a trivial vector bundle of rank $n+1$ over $\mathbb{P}^1$, tensored with $\Omega^*(\mathbb{P}^1)$. Clearly, the equality between left-hand side and right-hand side of eq. (40) does not hold on the level of line operators, because

$$\mathcal{O} \otimes \mathbb{C}^{n+1} \neq \mathcal{O}(-n) + \mathcal{O}(-n+2) + \dots + \mathcal{O}(n). \quad (41)$$

But the equality does hold on the level of K-theory.⁹ To see this, we will exhibit a filtration of $\mathcal{O} \otimes \mathbb{C}^{n+1}$ whose cohomology is precisely the right-hand-side of eq. (41). Recall that $\mathbb{P}^1 = G_{\mathbb{C}}/B$, where $G_{\mathbb{C}} = SL(2, \mathbb{C})$ and B is the group of upper-triangular matrices with unit determinant. The fiber of the trivial vector bundle V of rank $n+1$ carries the representation of $G_{\mathbb{C}}$ of isospin n ; for example, we can realize it by thinking of the fiber of V as the space of homogeneous degree- n polynomials in variables u and v , which we denote $\mathcal{D}_n(u, v)$. $SL(2, \mathbb{C})$ acts on it by linear substitutions. To define a filtration on V , we can specify a B -invariant filtration on $\mathcal{D}_n(u, v)$. The obvious filtration is to take F_k to be the subspace of $\mathcal{D}_n(u, v)$ consisting of polynomials of degree k or lower in u , with k ranging from 0 to n . It is easy to check that F_k is B -invariant for any k . Obviously, F_k/F_{k-1} is one-dimensional and the maximal torus of $SL(2, \mathbb{C})$ acts on it with weight $2k-n$. Hence V acquires a filtration of length $n+1$ whose k -th cohomology is $\mathcal{O}(2k-n)$.

⁹We are grateful to Roman Bezrukavnikov for providing the following argument.

6.2 OPE of WH operators with minuscule coweights

In this subsection we compute the product of WH with the smallest nontrivial coweights (for $G = PSU(2)$). The weights may be arbitrary. This case is very special, because when the WH operators are not coincident, the moduli space of Bogomolny equations is compact. This happens because the smallest nontrivial coweight of $G = PSU(2)$ is minuscule.¹⁰ Therefore the monopole bubbling is absent, as discussed in [5]. The main difficulty is to determine the behavior of the zeromode wavefunctions in the limit when the two WH operators coincide.

Let us recall what the moduli space of Bogomolny equations looks like for two noncoincident WH operators with $\mu = 1$ located at the same point on C [5]. It is a Hirzebruch surface F_2 which is a fibration of $\mathbb{P}^1$ over $\mathbb{P}^1$. One can think of it as a blow-up of the weighted projective plane $\mathbb{WP}_{112}$ at the $\mathbb{Z}_2$ -orbifold point. This blow-up is associated with moving the WH operators apart in the y directions. Thus $\mathbb{WP}_{112}$ is the coincidence limit of the moduli space. The orbifold point corresponds to the trivial solution of the Bogomolny equations (without the monopole singularity), while the complement of the orbifold point is isomorphic to $T\mathbb{P}^1$ and corresponds to solutions of the Bogomolny equations with one singularity of coweight $\mu = \pm 2$. This implies [5] that the product of two WH operators with coweight $\mu = \pm 1$ may contain WH operators with coweight $\mu = \pm 2$ and WH operators with coweight $\mu = 0$. To understand which WH operators appear in the product, one has to understand the zeromode wavefunctions in the coincidence limit. Those which remain spread-out on the complement of the orbifold point correspond to WH operators with $\mu = \pm 2$, while those which concentrate in the neighborhood of the exceptional divisor correspond to WH operators with $\mu = 0$.

As explained above, the wavefunctions of the effective SQM in the presence of WH operators are square-integrable forms on the moduli space with values in a certain holomorphic line bundle which satisfy the equations

$$\bar{D}\rho = 0, \quad \bar{D}^\dagger \rho = 0. \quad (42)$$

where $\bar{D}$ is the covariant Dolbeault differential¹¹. In the coincidence limit, the Kähler metric on the moduli space degenerates so that in the neighbor-

¹⁰The corresponding representation of ${}^L G = SU(2)$ has the property that all its weights lie in a single Weyl orbit.

¹¹ See sections 6.3 and 6.4 for appropriate $\bar{D}$.

hood of the orbifold point it becomes a flat metric on $\mathbb{C}^2/\mathbb{Z}_2$. More generally, when the WH operators are close to each other, the metric in the neighborhood of the exceptional divisor is well-approximated by a hyperkähler metric on the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ [5]. This is because this region in the moduli space corresponds to solutions of the Bogomolny equations which are trivial everywhere except in a small neighborhood of a point on $C \times I$ (the point at which one of the WH operators is inserted). Such solutions are arbitrarily well approximated by patching together solutions on $\mathbb{R}^3$ with the trivial solution on $C \times I$. Therefore the metric will be arbitrarily well approximated by the metric on the moduli space of Bogomolny equations on $\mathbb{R}^3$, which is hyperkähler.

The blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ is isomorphic to $T^*\mathbb{P}^1$ and has a unique asymptotically flat hyperkähler metric: the Eguchi-Hanson metric. Therefore, one can produce approximate solutions of equations (42) on F_2 by first solving them on the Eguchi-Hanson space and on $T\mathbb{P}^1$, assuming square-integrability in both cases, and patching them with the zero solution on the remainder of F_2 . The solutions coming from the Eguchi-Hanson space will represent contributions to the zero-mode Hilbert space from WH operators with $\mu = 0$, while the solutions coming from $T\mathbb{P}^1$ will represent contributions from $\mu = \pm 2$.

The contribution to the product of WH operators coming from $T\mathbb{P}^1$ will be called the “bulk” contribution, while the one coming from $T^*\mathbb{P}^1$ will be called the “bubbled” contribution, because it is due to monopole bubbling. The “bulk” contribution is rather trivial and in fact can be determined without any computations: the magnetic charges of the singularities simply add up, the same applies to the electric charges, and therefore the bulk contribution must be simply

$$WT_{2,2m+2k}.$$

The “bubbled” contributions are much more subtle and will be determined below by solving the equations (42) on $T^*\mathbb{P}^1$. We will also solve the same equations on $T\mathbb{P}^1$, not because it is required to determine the operator product, but because this computation will provide a consistency check on our approach, see section 6.5.

As a preliminary step, let us exhibit the predictions of $SL(2, \mathbb{Z})$ duality for the product of WH operators with coweight $\mu = \pm 1$:

$$WT_{1,2m} \cdot WT_{1,2k} = WT_{2,2m+2k} + WT_{0,2m-2k} - WT_{0,0} - WT_{0,2m-2k-2}$$

Here m and k are integers, and we assume $m \neq k$. We can simplify our problem a bit by noting that by applying the T -transformation several times,

we can reduce to the case $k = 0$, in which case the duality predicts that for $m \neq 0$ we have

$$WT_{1,2m} \cdot WT_{1,0} = WT_{2,2m} + WT_{0,2m} - WT_{0,0} - WT_{0,2m-2} \quad (43)$$

The “bulk” contribution is as expected, while the “bubbled” contributions are far from obvious. Note that some of the coefficients are negative, unlike for ’t Hooft operators in [5]. This is because we are working in the K-theory of the category of line operators, where negative signs occur naturally.

Similar manipulations in the case $m = 0$ lead to

$$T_1 \cdot T_1 = T_2 + T_0. \quad (44)$$

This is S-dual to the fact that the tensor square of the defining representation of $SU(2)$ is a sum of the adjoint representation (corresponding to the ’t Hooft operator T_2) and the trivial representation (corresponding to T_0). This prediction was checked in [5] for the GL-twisted theory. Briefly speaking, in the GL-twisted theory we are looking for harmonic square-integrable forms on F_2 and study their behavior in the limit when F_2 degenerates to $\mathbb{WP}_{112}$. Since topologically F_2 is the same as $\mathbb{P}^1 \times \mathbb{P}^1$, and harmonic forms can be interpreted in topological terms (as cohomology classes), we know a priori that the dimension of the space of harmonic forms is the same as the dimension of $H^*(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{C})$, which is four. It is also well-known that there is a unique square-integrable harmonic form on the Eguchi-Hanson space (in degree 2). Therefore the Eguchi-Hanson space contributes one state, and $T\mathbb{P}^1$ contributes three states. The latter states arise precisely from the quantization of the moduli space of the Bogomolny equations with a single singularity of coweight $\mu = \pm 2$. This leads to the formula (44), as predicted by S-duality.

The case $m \neq 0$ is different in two respects. First of all, we have to consider forms with values in a holomorphic line bundle $\mathcal{L}$. Second, the equations we have to solve (42) involve the Dolbeault operator rather than the de Rham operator.

To fix $\mathcal{L}$, let us use the same boundary conditions as before, i.e. assume that the boundary condition on which $WT_{1,2m}$ acts corresponds to a particular $PSU(2)$ bundle on C . Then the line bundle on F_2 is the pull-back of $\mathcal{O}(-2m)$ from the base $\mathbb{P}^1$. (This is because the electric degree of freedom is associated, via weight $2m$, with the $U(1)$ bundle coming from the first Hecke transformation and does not care about the second Hecke transformation. The base $\mathbb{P}^1$ is the parameter space for the first Hecke transformation, while

the fiber $\mathbb{P}^1$ is the parameter space for the second Hecke transformation.) Therefore, in the “bulk” part of the computation, $\mathcal{L}$ is simply the pull-back of $\mathcal{O}(-2m)$ from the base of $T\mathbb{P}^1$ to the total space.

Similarly, in the “bubbled” part of the computation the line bundle is a pull-back of $\mathcal{O}(-2m)$ from the base of $T^*\mathbb{P}^1$ to the total space. To see this, we can make use of an explicit description of F_2 as a Kähler quotient of $\mathbb{C}^4$ by $U(1)^2$ [5]. Let the coordinates on $\mathbb{C}^4$ be u, v, b , and b' . The first $U(1)$ action has weights $1, 1, 2, 0$, and the second $U(1)$ action has weights $0, 0, 1, 1$. The moment map equations are

$$|u|^2 + |v|^2 + 2|b|^2 = 1, \quad |b|^2 + |b'|^2 = d.$$

where d is assumed to be positive and smaller than $1/2$. These equations imply that u and v cannot vanish simultaneously and can be regarded as homogeneous coordinates on $\mathbb{P}^1$. Therefore the map $(u, v, b, b') \mapsto (u, v)$ defines a fibration over $\mathbb{P}^1$. Its fiber is also a $\mathbb{P}^1$ with homogeneous coordinates b and b' . To degenerate F_2 into $\mathbb{WP}_{112}$ one need to take the limit $d \rightarrow 1/2$. The exceptional divisor is given by $b' = 0$. The neighborhood of the exceptional divisor is the subset given by $b \neq 0$. We can see that it is a copy of $T^*\mathbb{P}^1$ by letting $a = b'/b$. Since u, v, a have zero weights with respect to the second $U(1)$ and since every orbit of the second $U(1)$ action contains a unique representative with $\arg(b) = 0$, we conclude that the subset $b \neq 0$ can be identified with the Kähler quotient of $\mathbb{C}^3$ parameterized by u, v, a by the first $U(1)$. Since the weights of these variables are $1, 1, -2$, and u and v cannot vanish simultaneously, this quotient is the total space of the line bundle $\mathcal{O}(-2)$ over $\mathbb{P}^1$, which is the same as $T^*\mathbb{P}^1$.

Now, the line bundle $\mathcal{L}$ on F_2 can be defined as the quotient of the space of quintuples u, v, b, b', ρ by the $(\mathbb{C}^*)^2$ action with weights

$$(1, 0), (1, 0), (2, 1), (0, 1), (-2m, 0).$$

The variable ρ parameterizes the fiber of $\mathcal{L}$. When we restrict to the subset $b \neq 0$, we may forget about the second $\mathbb{C}^*$, and replace b and b' with $a = b'/b$. Thus the restriction of $\mathcal{L}$ to this subset is the quotient of the space of quadruples u, v, a, ρ by the $\mathbb{C}^*$ action with weights $1, 1, -2, -2m$. This is clearly the total space of the line bundle over $T^*\mathbb{P}^1$ which is a pull-back of $\mathcal{O}(-2m)$ on the $\mathbb{P}^1$ base.

6.3 Wavefunctions on $T\mathbb{P}^1$.

Let u, v, b be homogeneous coordinates on $T\mathbb{P}^1$, with $\mathbb{C}^*$ weights $1, 1, 2$. Let us work in the patch $u \neq 0$ and define inhomogeneous “coordinates”

$$z = \frac{v}{u}, \quad w = \frac{\sqrt{b}}{u} \quad (45)$$

We put the word “coordinates” in quotation marks, because w is defined up to a sign and is not really a good coordinate. The good coordinate is w^2 .

Our goal is to solve equations (42) on $T\mathbb{P}^1$, i.e. to find harmonic representatives of the L^2 Dolbeault cohomology groups

$$H^p(\Omega^q(-2m), T\mathbb{P}^1), \quad p, q = 0, 1, 2.$$

Here

$$\Omega^q(-2m) = \Omega^q \otimes \mathcal{O}(-2m).$$

The sum of these cohomology groups is nothing but the vector space $\mathfrak{H}(\mathcal{A})$, where $\mathcal{A}$ is the WH operator $WT_{2,2m}$. In section 6.5 we will use the knowledge of $\mathfrak{H}(\mathcal{A})$ for this and other WH operators on the r.h.s. of eq. (43) to make a consistency check on our computations.

6.3.1 The metrics

While we do not know the precise form of the Kähler metric on $T\mathbb{P}^1$ coming from the Bogomolny equations, it is tightly constrained by symmetry considerations. Indeed, $PSU(2)$ gauge transformations act on the moduli space by isometries which preserve the complex structure, and the orbits have real codimension 1, therefore the most general ansatz will depend on functions of a single variable. The $PSU(2)$ action in question acts on u, v as a two-dimensional projective representation, and acts trivially on b . Using this, it is easy to show that the most general $PSU(2)$ -invariant $(1, 1)$ -form on TP^1 is

$$J = f_1(\lambda)e_1 \wedge \bar{e}_1 + f_2(\lambda)e_2 \wedge \bar{e}_2$$

where

$$e_1 = \frac{dw}{w} - \bar{z}e_2, \quad e_2 = \frac{dz}{1 + |z|^2} \quad (46)$$

and f_1, f_2 are functions of the $PSU(2)$ invariant

$$\lambda = \frac{|w|^2}{1 + |z|^2} \quad (47)$$

The Kähler condition $dJ = 0$ implies $f_1 = -\lambda f_2'$ so that geometry is specified in terms of a single function $f_2(\lambda)$ on $[0, \infty)$. Its behavior at zero is constrained by the requirement that the metric be smooth at $w = 0$. Its behavior at infinity is constrained by the requirement that after one-point compactification of $T\mathbb{P}^1$ the neighborhood of infinity looks like $\mathbb{C}^2/\mathbb{Z}_2$ with a flat metric. These two conditions are equivalent to

$$f_2 \rightarrow \frac{1}{2\lambda} \quad \text{for } \lambda \rightarrow \infty, \quad f_2 \rightarrow \text{const} \quad \text{for } \lambda \rightarrow 0. \quad (48)$$

The standard Fubini-Study metric on $\mathbb{WP}_{112}$ corresponds to specific f_1, f_2 with these asymptotics:

$$f_1 = \frac{(\sqrt{1+8\lambda^2}-1)^2}{4\lambda^2\sqrt{1+8\lambda^2}}, \quad f_2 = \frac{\sqrt{1+8\lambda^2}-1}{4\lambda^2}$$

Let us consider the line bundle $\mathcal{O}(n)$ over $T\mathbb{P}^1$. The $PSU(2)$ action on $T\mathbb{P}^1$ lifts to a $PSU(2)$ action on $\mathcal{O}(n)$ if n is even, or to an $SU(2)$ action if n is odd. We are mainly interested in even n . In a unitary trivialization, the most general $SU(2)$ -invariant connection on $\mathcal{O}(n)$ is:

$$A^{(n)} = \frac{\lambda f_{(n)}'}{f_{(n)}} e_1 - \frac{n}{2} \bar{z} e_2, \quad \bar{A}^{(n)} = -\frac{\lambda f_{(n)}'}{f_{(n)}} \bar{e}_1 + \frac{n}{2} z \bar{e}_2 \quad (49)$$

and covariant differentials are defined as

$$D = \partial + A^{(n)}, \quad \bar{D} = \bar{\partial} + \bar{A}^{(n)}$$

For $n = -2m$, $m \in \mathbb{Z}$ the function $f_{(-2m)}$ has the following asymptotics:

$$f_{(-2m)} \rightarrow \lambda^m \quad \text{for } \lambda \rightarrow \infty, \quad f_{(-2m)} \rightarrow 1 \quad \text{for } \lambda \rightarrow 0. \quad (50)$$

The asymptotic at $\lambda \rightarrow \infty$ is chosen in such a way that the norm of the holomorphic section w^{-2m} approaches a constant, i.e. we go to the unitary trivialization

$$s_{unit} = s_{hol}(1 + |z|^2)^m f_{(-2m)}(\lambda) \quad (51)$$

and require the pointwise norm $|s_{unit}|^2$ to approach a constant. The reason is that in the neighborhood of the orbifold point $u = v = 0$ w^{-2m} represents a section which transforms trivially between the two charts $u \neq 0$ and $v \neq 0$, and provides a local holomorphic trivialization of $\mathcal{O}(-2m)$. We would like its norm neither to diverge nor to become zero at the orbifold point. The postulated behavior at $\lambda \rightarrow 0$ ensures that the connection is smooth at the zero section of $T\mathbb{P}^1$.

6.3.2 A heuristic argument

Since solving partial differential equations is hard, it is useful to have some idea about the kind of solutions one expects to find. There is a heuristic argument, explained to us by Roman Bezrukavnikov, which gives the dimensions of the cohomology groups we are after. Let us start with the case $m = 0$ where we already know the structure of solutions [5]: all of cohomology is of type (p, p) , and there is a single solution for $p = 0, 1, 2$:

$$H^p(T\mathbb{P}^1, \Omega^q) = \delta_{pq} V_1$$

where V_{2j+1} stands for $(2j + 1)$ -dimensional irreducible representation of $SL(2, \mathbb{C})$.

Next we note that the line bundle $\mathcal{O}(2)$ corresponds to the divisor D , where D is the zero section of $T\mathbb{P}^1$. Hence we have a short exact sequence of coherent sheaves on $T\mathbb{P}^1$:

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(D) \rightarrow N_D \rightarrow 0,$$

where N_D is the normal bundle of D . This gives a long exact sequence for sheaf cohomology groups. We are of course interested not in sheaf cohomology groups, but in L^2 Dolbeault cohomology of the corresponding line bundles. But let us cheat and ignore this distinction. Then the long exact sequence implies

$$H^0(T\mathbb{P}^1, \mathcal{O}(2)) = V_1 + V_3, \quad H^i(T\mathbb{P}^1, \mathcal{O}(2)) = 0, \quad i = 1, 2.$$

Similarly, if we tensor the short exact sequence with the sheaf Ω^i , $i = 1, 2$, and then write down the corresponding long exact sequences, we infer:

$$H^0(T\mathbb{P}^1, \Omega^1(2)) = V_1, \quad H^0(T\mathbb{P}^1, \Omega^2(2)) = 0, \quad H^i(T\mathbb{P}^1, \Omega^j(2)) = 0, \quad i, j = 1, 2.$$

Having determined all relevant cohomology groups for $m = -1$, we can move on to $m = -2$ and write down a short exact sequence involving $\mathcal{O}(4) \simeq \mathcal{O}(2D)$:

$$0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(2D) \rightarrow N_D(D),$$

which implies

$$H^0(T\mathbb{P}^1, \mathcal{O}(4)) = V_1 + V_3 + V_5, \quad H^0(T\mathbb{P}^1, \Omega^1(4)) = V_3 + V_1 + V_3, \quad (52)$$

$$H^0(T\mathbb{P}^1, \Omega^2(4)) = V_1,$$

with all higher cohomologies vanishing. Continuing in this fashion, we can determine cohomology groups for all negative m . We find that only degree-0 cohomology is nonvanishing. If we let $k = -m > 0$, then degree-0 cohomology groups are

$$H^0(T\mathbb{P}^1, \mathcal{O}(2k)) = \sum_{j=0}^k V_{2j+1}, \quad (53)$$

$$H^0(T\mathbb{P}^1, \Omega^1(2k)) = V_{2k-1} + \sum_{j=1}^{k-1} V_{2j-1} + \sum_{j=1}^{k-1} V_{2j+1}, \quad (54)$$

$$H^0(T\mathbb{P}^1, \Omega^2(2k)) = \sum_{j=0}^{k-2} V_{2j+1}. \quad (55)$$

If $m > 0$, we can find cohomology groups by applying Kodaira-Serre duality to the results for $m < 0$:

$$H^p(T\mathbb{P}^1, \Omega^q(2m)) = H^{2-p}(T\mathbb{P}^1, \Omega^{2-q}(-2m)).$$

Thus for positive m only degree-2 cohomology is nontrivial.

Below we write down an explicit basis for degree-0 cohomology groups and check that all elements of the basis are square-integrable. By Kodaira-Serre duality, this also verifies the predictions for degree-2 cohomology. We have not been able to prove that degree-1 L^2 cohomology groups vanish for all m . We only checked that degree-1 L^2 cohomology, if it exists, does not contain irreducible $PSL(2, \mathbb{C})$ representations of dimensions 1 and 3. (For larger $PSL(2, \mathbb{C})$ representations, the analysis becomes very complicated, and we were not able to push it through.)

6.3.3 $H^0(T\mathbb{P}^1, \mathcal{O}(2k))$. $k > 0$.

First we find holomorphic sections of the line bundle $\mathcal{O}(2k)$ on $T\mathbb{P}^1$. In a holomorphic trivialization these sections are $b^{k-j} P_{2j}(u, v)$ for $j = 0, \dots, k$ where $P_{2j}(u, v)$ is a homogeneous polynomial of degree $2j$ in variables u, v . For each j these sections transform in a representation V_{2j+1} .

To see that all these sections are in L^2 we go to the unitary trivialization (51) and compute the norm. For $s_{hol} = w^{2(k-j)} z^p$ with $p \leq 2j \leq 2k$ we find

$$\int_{T\mathbb{P}^1} |s_{unit}|^2 f_1(\lambda) f_2(\lambda) e_1 \wedge \bar{e}_1 \wedge e_2 \wedge \bar{e}_2 = \frac{\pi}{2} \int \frac{d\lambda}{\lambda} \lambda^{2(k-j)} f_1 f_2 f_{(2k)}^2 \int \frac{|z|^{2p} dz d\bar{z}}{(1 + |z|^2)^{2+2j}} \quad (56)$$

where we used (46) and (51). The z -integral is convergent for $p \leq 2j$, while the integral over λ is finite since the integrand behaves¹² at infinity as $\frac{1}{\lambda^{2j+3}}$ and at zero as $\lambda^{1+2(k-j)}$ for $j = 0, \dots, k$.

6.3.4 $H^0(T\mathbb{P}^1, \Omega^1(2k)), \quad k > 0$.

Next we find holomorphic sections of the vector bundle $\Omega^1(2k)$ on $T\mathbb{P}^1$. Again it is easy to do it in a holomorphic trivialization. $2k-1$ sections pulled back from the base transform in a representation V_{2k-1} :

$$\rho_{hol} = uvP_{2k-2}(u, v)\left(\frac{du}{u} - \frac{dv}{v}\right).$$

All these sections are square-integrable. Indeed, in the chart $u \neq 0$ they are of the form $\rho_{hol} = z^p dz$ for $p = 0, \dots, 2k-2$ and their norm is finite:

$$\rho_{unit} \wedge \bar{*} \rho_{unit} = \frac{\pi}{2} \int \frac{d\lambda}{\lambda} f_1 f_{(2k)}^2 \int \frac{|z|^{2p} dz d\bar{z}}{(1 + |z|^2)^{2k}} < \infty. \quad (57)$$

Also, there are holomorphic sections of the form

$$\rho_{hol} = \left(db - b\frac{du}{u} - b\frac{dv}{v}\right)F_{2k-2}(u, v, b) + b\tilde{F}_{2k-2}(u, v, b)\left(\frac{du}{u} - \frac{dv}{v}\right),$$

where F_{2k-2} and $\tilde{F}_{2k-2}(u, v, b)$ must satisfy (to ensure non-singular behavior)

$$\tilde{F}_{2k-2} - F_{2k-2} = ug_{2k-3}(u, v, b), \quad \tilde{F}_{2k-2} + F_{2k-2} = v\tilde{g}_{2k-3}(u, v, b).$$

We further write

$$g_{2k-3}(u, v, b) = \sum_{j=1}^{k-1} b^{k-1-j} P_{2j-1}(u, v), \quad \tilde{g}_{2k-3}(u, v, b) = \sum_{j=1}^{k-1} b^{k-1-j} \tilde{P}_{2j-1}(u, v)$$

So the total number of mixed-type sections is $4 \sum_{j=1}^{k-1} j$. To see that these sections decompose as $\sum_{j=1}^{k-1} (V_{2j+1} + V_{2j-1})$ we write them in a unitary trivialization (in the chart $u \neq 0$)

$$\rho_{unit} = w^2 f_{(2k)} (1 + |z|^2)^{-k} \left((z\tilde{g}_{2k-3} - g_{2k-3})e_1 - (\bar{z}g_{2k-3} + \tilde{g}_{2k-3})e_2 \right).$$

¹²We use the asymptotics (48) and (50) of $f_1, f_2, f_{(2k)}$.

where

$$\tilde{g}_{2k-3} = \sum_{j=1}^{k-1} w^{2k-2-2j} \sum_{n=0}^{2j-1} a_n^{(j)} z^{2j-1-n}, \quad g_{2k-3} = \sum_{j=1}^{k-1} w^{2k-2-2j} \sum_{n=0}^{2j-1} c_n^{(j)} z^{2j-1-n}$$

Then, ρ_{unit} is brought to the form

$$\begin{aligned} \rho_{unit} = & w^{2k} f_{(2k)}(\lambda) (1 + |z|^2)^{-k} \left(e_1 \sum_{j=1}^{k-1} w^{-2j} \left(a_0^{(j)} z^{2j} - c_{2j-1}^{(j)} + \sum_{n=0}^{2j} \beta_n^{(j)} z^{2j-n} \right) + \right. \\ & \left. (-) \frac{\bar{w}}{w} e_2 \sum_{j=1}^{k-1} \frac{1}{w^{2j-2} |w|^2} \left(a_0^{(j)} z^{2j-1} + c_{2j-1}^{(j)} \bar{z} + \sum_{n=1}^{2j-1} (a_n^{(j)} + c_{n-1}^{(j)} |z|^2) z^{2j-1-n} \right) \right) \end{aligned}$$

where

$$\beta_0^{(j)} = a_0^{(j)}, \quad \beta_{2j}^{(j)} = -c_{2j-1}^{(j)}, \quad \beta_n^{(j)} = a_n^{(j)} - c_{n-1}^{(j)}, \quad n = 1, \dots, 2j-1$$

Now recall that e_1 and $\frac{\bar{w}}{w} e_2$ are $SL(2, \mathbb{C})$ invariant (1,0) forms, and $\lambda = \frac{|w|^2}{1+|z|^2}$ is $SL(2, \mathbb{C})$ invariant. We see that the e_1 piece in ρ_{unit} transforms as $\sum_{j=1}^{k-1} V_{2j+1}$, i.e. for each $j = 0, \dots, k-1$

$$w^{-2j} \sum_{n=0}^{2j} \beta_n^{(j)} z^{2j-n}$$

transforms as V_{2j+1} .

The e_2 piece in ρ_{unit} transforms as $\sum_{j=1}^{k-1} V_{2j-1}$ if we impose $2j+1$ constraints for each $j = 0, \dots, k-1$

$$a_0^{(j)} = 0, \quad c_{2j-1}^{(j)} = 0, \quad a_n^{(j)} = c_{n-1}^{(j)} \quad n = 1, \dots, 2j-1.$$

All these sections are in L^2 . Indeed, the norm of each section in V_{2j+1} is not greater than

$$\frac{\pi}{2} \int \frac{d\lambda}{\lambda} f_2 f_{(2k)}^2 \int \frac{|z^{2j} w^{2(k-j)}|^2 dz d\bar{z}}{(1 + |z|^2)^{2k+2}},$$

where $j = 1, \dots, k-1$.

Using (46) the integral is brought to the form

$$\frac{\pi}{2} \int \frac{d\lambda}{\lambda} \lambda^{2(k-j)} f_2 f_{(2k)}^2 \int \frac{|z|^{4j} dz d\bar{z}}{(1 + |z|^2)^{2+2j}},$$

which is finite in the relevant range, i.e. for $j = 1, \dots, k-1$.

Analogously, the norm of each section in V_{2j-1} is not greater than

$$\frac{\pi}{2} \int \frac{d\lambda}{\lambda} f_1 f_{(2k)}^2 \lambda^{2(k-j)} \int \frac{|z|^{2(2j-2)} dz d\bar{z}}{(1 + |z|^2)^{2j}},$$

which is finite in the relevant range, i.e. for $j = 1, \dots, k-1$.

6.3.5 $H^0(T\mathbb{P}^1, \Omega^2(2k))$, $k > 0$.

Finally we find holomorphic sections of the line bundle $\Omega^2(2k)$ on $T\mathbb{P}^1$. In a holomorphic trivialization they are

$$\rho_{hol} = F_{2k-4}(u, v, b) (vdu - u dv) \wedge (db - b \frac{du}{u} - b \frac{dv}{v}),$$

where

$$F_{2k-4} = \sum_{j=0}^{k-2} b^{k-2-j} P_{2j}(u, v).$$

In a unitary trivialization they have the form

$$\rho_{unit} = w^{2k} \frac{f_{(2k)}(\lambda)}{\lambda} (1 + |z|^2)^{-k} \sum_{j=0}^{k-2} w^{-2j} \sum_{p=0}^{2j} a_p^{(j)} z^{2j-p} e_1 \wedge \left(e_2 \frac{\bar{w}}{w} \right),$$

so we conclude that they transform in a representation $\sum_{j=0}^{k-2} V_{2j+1}$.

All these sections have finite L^2 norm, since for $j = 0, \dots, k-2$ and $p = 0, \dots, 2j$ we find

$$\frac{\pi}{2} \int \frac{d\lambda}{\lambda} \lambda^{2(k-j-1)} f_{(2k)}^2 \int \frac{|z|^{2p} dz d\bar{z}}{(1 + |z|^2)^{2+2j}} < \infty.$$

6.4 Wavefunctions on $T^*\mathbb{P}^1$.

We regard $T^*\mathbb{P}^1$ as the total space of the line bundle $\mathcal{O}(-2)$ over $\mathbb{P}^1$ and use homogeneous coordinates u, v, b' with $\mathbb{C}^*$ weights $1, 1, -2$. In the patch $u \neq 0$ we define inhomogeneous coordinates

$$z = \frac{v}{u}, \quad w' = \sqrt{b'}u \quad (58)$$

Our goal is to compute the L^2 Dolbeault cohomology of the bundles $\Omega^i(-2m)$, $i = 0, 1, 2$.

6.4.1 Metrics

The most general $SU(2)$ -invariant Kähler form on $T^*\mathbb{P}^1$ is

$$J = f_1(x)e'_1 \wedge \bar{e}'_1 + f_2(x)e_2 \wedge \bar{e}_2,$$

where

$$e'_1 = \frac{dw'}{w'} + \bar{z}e_2, \quad e_2 = \frac{dz}{1+|z|^2} \quad (59)$$

and f_1, f_2 are functions of THE $SU(2)$ invariant

$$x := |w'|^2(1+|z|^2). \quad (60)$$

From $dJ = 0$ we find $f_1 = xf'_2$ so that geometry is specified in terms of a single function $f_2(x)$, which we take to be a positive function with the following asymptotics:

$$f_2 \rightarrow x \quad \text{for } x \rightarrow \infty, \quad f_2 \rightarrow \text{const} \quad \text{for } x \rightarrow 0. \quad (61)$$

The first condition ensures that at $x \rightarrow \infty$ the metric becomes flat. The second condition is required so that for $x = 0$ the metric is nonsingular.

Next consider the line bundle $\mathcal{O}(2k)$ over $T^*\mathbb{P}^1$. In a unitary trivialization the connection on this bundle is

$$A^{(2k)} = \frac{xf'_{(2k)}}{f_{(2k)}}e'_1 - k\bar{z}e_2, \quad \bar{A}^{(2k)} = -\frac{xf'_{(2k)}}{f_{(2k)}}\bar{e}'_1 + kz\bar{e}_2 \quad (62)$$

and covariant differentials are defined as

$$D = \partial + A^{(2k)}, \quad \bar{D} = \bar{\partial} + \bar{A}^{(2k)}$$

For $k = -m$, $m \in \mathbb{Z}$, the function $f_{(-2m)}$ has the asymptotics

$$f_{(-2m)} \rightarrow x^{-m} \quad \text{for } x \rightarrow \infty, \quad f_{(-2m)} \rightarrow 1 \quad \text{for } x \rightarrow 0. \quad (63)$$

The behavior for $x \rightarrow \infty$ is chosen in such a way that asymptotically the holomorphic section w'^{2m} of $\mathcal{O}(-2m)$ has constant pointwise norm. The reason for this choice is that w'^{2m} continues in the limit $x \rightarrow \infty$ to a section which transforms trivially between the two charts $u \neq 0$ and $v \neq 0$. The behavior for $x \rightarrow 0$ ensures that we have a nonsingular metric when restricting to the zero section of $T^*\mathbb{P}^1$.

6.4.2 A heuristic argument

Again we begin with a heuristic argument. The sheaf $\mathcal{O}(2)$ on $T^*\mathbb{P}^1$ can be identified with $\mathcal{O}(-D)$, where D is the zero section $b' = 0$. A short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_D \rightarrow 0 \quad (64)$$

implies a long exact sequence for sheaf cohomology. Let us assume that it holds also for L^2 Dolbeault cohomology. We also recall [5] that for $m = 0$ the only square-integrable solution of equations (42) on $T^*\mathbb{P}^1$ is of type $(1, 1)$, so

$$H^1(T^*\mathbb{P}^1, \Omega^1) = V_1,$$

and all other L^2 Hodge numbers on $T^*\mathbb{P}^1$ vanish. Then the long exact sequence coming from (64) and its relatives obtained by tensoring (64) with Ω^i imply

$$H^1(T^*\mathbb{P}^1, \mathcal{O}(2)) = V_1, \quad H^1(T^*\mathbb{P}^1, \Omega^2(2)) = V_1, \quad H^1(T^*\mathbb{P}^1, \Omega^1(2)) = V_3,$$

and all other cohomologies for $m = -1$ vanish. Now that we know cohomology for $m = -1$, we can tensor (64) with $\Omega^i(2)$ and determine cohomology for $m = -2$, etc. In this way we obtain the following predictions for dimensions of L^2 cohomology groups for $k = -m > 0$:

$$H^1(T^*\mathbb{P}^1, \Omega^1(2k)) = V_1 + V_{2k-1} + V_{2k+1} + 2 \sum_{j=1}^{k-2} V_{2j+1}, \quad k \geq 3,$$

$$H^1(T^*\mathbb{P}^1, \Omega^1(2)) = V_3, \quad H^1(T^*\mathbb{P}^1, \Omega^1(4)) = V_1 + V_3 + V_5,$$

$$H^1(T^*\mathbb{P}^1, \Omega^2(2k)) = H^1(T^*\mathbb{P}^1, \mathcal{O}(2k)) = \sum_{j=0}^{k-1} V_{2j+1}.$$

The results for $k < 0$ are obtained by Kodaira-Serre duality; in fact, from the above formulas it is easy to see that cohomology groups depend only on $|k|$.

Below we will find exactly the right number of square-integrable solutions of (42) in cohomological degree 1, with the correct transformation properties under $PSU(2)$. We also checked that in degree zero (and by Kodaira-Serre duality, in degree 2) all L^2 cohomology vanishes, just as the long exact sequence predicts. We have not been able to verify that we have found all square-integrable solutions of (42) in degree 1. We only checked that if other solutions in degree 1 exist, they cannot transform in $PSU(2)$ representations of dimensions 1 and 3.

6.4.3 $H^1(\Omega^1(2k))$, $k > 0$.

The most general ansatz (in the unitary trivialization and in the chart $u \neq 0$) for the component of the $(n+1)$ -plet in $H^1(\Omega^1(2k))$ with the $PSU(2)$ isospin projection $J_3 = -(n/2)$ is:

$$\omega = \frac{f_{(2k)}}{(1+|z|^2)^k} w'^{-2k} \left(a e'_1 \wedge \bar{e}'_1 + b e_2 \wedge \bar{e}_2 + c \frac{\bar{w}'}{w'} e'_1 \wedge \bar{e}_2 + d \frac{w'}{\bar{w}'} e_2 \wedge \bar{e}'_1 \right), \quad (65)$$

where

$$a = \sum_{p=0}^n a_n(x) w'^{n-p} (\bar{w}' \bar{z})^p$$

and the functions b, c, d have a similar form. We have used that e'_1 and $\frac{w'}{\bar{w}'} e_2$ are $SU(2)$ -invariant $(1,0)$ forms. Various terms in a correspond to different ways of building up the component of a $(n+1)$ -plet with $J_3 = -(n/2)$, i.e. $u^n, u^{n-1}\bar{v}, \dots, \bar{v}^n$.

Imposing $D(*\omega) = 0$ and $\bar{D}(\omega) = 0$ we found that non-trivial cohomology groups come from using two simple special cases of the general ansatz (65).

I. The first simplified ansatz has the form:

$$\omega = \frac{w'^{-2k} f_{(2k)}}{(1+|z|^2)^k} \Omega_n, \quad (66)$$

where

$$\Omega_n = w'^n \left(a(x) e_1' \wedge \bar{e}_1' + b(x) e_2 \wedge \bar{e}_2 + d(x) \bar{z} e_2 \wedge \bar{e}_1' \right).$$

From $\bar{D}\omega = 0$ we find

$$a - xb' + d = 0. \quad (67)$$

Meanwhile, $D(*\omega) = 0$ gives

$$2k \frac{f_1}{f_2} b - xd' - 2xd \frac{f'_{(2k)}}{f_{(2k)}} + (n - 2k) \left(\frac{f_1 b}{f_2} - d \right) = 0, \quad (68)$$

$$x \left(\frac{f_2 a}{f_1} \right)' - \frac{f_1}{f_2} b + \left(n - 2k + 2x \frac{f'_{(2k)}}{f_{(2k)}} \right) \frac{f_2}{f_1} a = 0. \quad (69)$$

• Let us first assume $n = 2k$, then a linear combination of (68) and (69) gives

$$\left(2k \frac{f_2}{f_1} a - d \right)' + 2 \frac{f'_{(2k)}}{f_{(2k)}} \left(2k \frac{f_2}{f_1} a - d \right) = 0,$$

which can be integrated to express d in terms of a as

$$d = 2k \frac{f_2}{f_1} a - \frac{C_0}{f_{(2k)}^2} \quad (70)$$

where C_0 is an integration constant. From (69) b can also be expressed in terms of a and its derivative, so that the system (67-69) reduces to a second-order inhomogeneous differential equation:

$$-x^2 \frac{f_2}{f_1} \phi'' - \left(x \left(\frac{x f_2}{f_1} \right)' + 2x^2 \frac{f_2}{f_1} \frac{f'_{(2k)}}{f_{(2k)}} \right) \phi' + \left(2k + \frac{f_1}{f_2} - 2x \left(\frac{x f_2}{f_1} \frac{f'_{(2k)}}{f_{(2k)}} \right)' \right) \phi = \frac{C_0}{f_{(2k)}^2}, \quad (71)$$

where $\phi = \frac{f_2 a}{f_1}$. Near $x \rightarrow \infty$ (71) becomes

$$-x^2 \phi'' - (1 + 2k)x \phi' + (1 + 2k)\phi = \frac{C_0}{x^{2k}},$$

and its general solution behaves at infinity as

$$\phi = \frac{C_0}{(1 + 2k)x^{2k}} + C_1 x + \frac{C_2}{x^{1+2k}} \quad (72)$$

where C_1 and C_2 parameterize the general solution of the homogeneous equation.

Near $x \rightarrow 0$ (71) becomes

$$-x\phi'' + \phi' + 2kx\phi = C_0x,$$

and its general solution behaves at zero as

$$\phi = \frac{C_0}{2k} + C_3x^2 + C_4x^2 \text{Log}x, \quad (73)$$

where C_3 and C_4 parameterize the general solution of the homogeneous equation.

To ensure that ω is well-behaved near the origin we must choose $C_4 = 0$. This is always possible. Starting from any decaying solution of the inhomogeneous equation at infinity

$$\phi_{inhom} = \frac{C_0}{(1+2k)x^{2k}} + \frac{\tilde{C}_2}{x^{2k+1}}$$

we may always add a decaying solution of the homogeneous equation so that

$$\phi = \phi_{inhom} + \frac{C_2}{x^{2k+1}}$$

continues to small x in a desired way, i.e. $C_4 = 0$.

Finally we note that ω has a finite L^2 norm:

$$\frac{\pi}{2} \int \frac{dx}{x} f_{(2k)}^2 \int \frac{dz d\bar{z}}{(1+|z|^2)^{2+2k}} \left(a^2 \frac{f_2}{f_1} + b^2 \frac{f_1}{f_2} + d^2 |z|^2 \right)$$

Indeed, using the asymptotics at $x \rightarrow \infty$

$$a \sim \frac{1}{x^{2k}}, \quad b \sim \frac{1}{x^{2k}}, \quad d \sim \frac{1}{x^{2k}}$$

we find that integral converges for $2k \geq 2$. We conclude that using ansatz (66) for $n = 2k$ we found a well-behaved $(2k+1)$ -plet with finite L^2 norm.

- $n \leq 2k - 2, n > 0$

Let us consider (67-69) with $n \neq 2k$. Then a linear combination of (68) and (69) can be integrated to express d in terms of a as

$$d = n \frac{f_2}{f_1} a - \frac{C_0 x^{2k-n}}{f_{(2k)}^2}, \quad (74)$$

where C_0 is an integration constant. From equation (69) b can also be expressed in terms of a and its derivative, so that the system (67-69) reduces to a second-order inhomogeneous differential equation:

$$-x^2 \frac{f_2}{f_1} \phi'' - \left(x \left(\frac{x f_2}{f_1} \right)' + 2x^2 \frac{f_2}{f_1} \frac{f'_{(2k)}}{f_{(2k)}} + \frac{x f_2 (n-2k)}{f_1} \right) \phi' + \left(n + \frac{f_1}{f_2} - 2x \left(\frac{x f_2}{f_1} \frac{f'_{(2k)}}{f_{(2k)}} \right)' + (2k-n)x \left(\frac{f_2}{f_1} \right)' \right) \phi = \frac{C_0 x^{2k-n}}{f_{(2k)}^2}, \quad (75)$$

where $\phi = \frac{f_2 a}{f_1}$. Near $x \rightarrow \infty$ (75) becomes

$$-x^2 \phi'' - (1+n)x \phi' + (1+n)\phi = \frac{C_0}{x^n},$$

and its general solution at infinity is

$$\phi = \frac{C_0}{(1+n)x^n} + C_1 x + \frac{C_2}{x^{1+n}} \quad (76)$$

where C_1 and C_2 parameterize the general solution of the homogeneous equation.

We must set $C_1 = 0$ to obtain ω with a finite L^2 norm for $n > 0$:

$$\int_{T^*\mathbb{P}^1} \omega \wedge * \bar{\omega} = \int \frac{dx}{x^{1+n}} \int \frac{dz d\bar{z}}{(1+|z|^2)^{2+n}}.$$

Near $x \rightarrow 0$ (75) becomes

$$-x^2 \phi'' + (2k-n+1)x \phi' + 2(n-2k)\phi = C_0 x^{2k-n+2}$$

For $n < 2k-2$ general solution near zero is

$$\phi = \frac{C_0 x^{2k-n+2}}{2(n-2k)} + C_3 x^2 + C_4 x^{2k-n}, \quad (77)$$

and for $n = 2k-2$

$$\phi = -\frac{C_0 x^4}{4} + C_3 x^2 + C_4 x^2 \text{Log}(x), \quad (78)$$

where C_3 and C_4 parameterize the general solution of the homogeneous equation.

For $n = 2k - 2$ there is a good solution if $C_4 = 0$. For even¹³ n such that $n < 2k - 2$ there is a good solution if $C_3 = 0$. Such solutions always exist. Starting from any decaying solution of the inhomogeneous equation at infinity

$$\phi_{inhom} = \frac{C_0}{(1+n)x^n} + \frac{\tilde{C}_2}{x^{n+1}}$$

we may always add a decaying solution of the homogeneous equation so that

$$\phi = \phi_{inhom} + \frac{C_2}{x^{n+1}}$$

continues to small x in the desired way, i.e. $C_4 = 0$ or $C_3 = 0$. We conclude that using the ansatz (66) we found a well-behaved $(n+1)$ -plet with a finite norm for even n such that $n > 0$ and $n \leq 2k - 2$.

II. The second simplified ansatz has the form:

$$\omega = \frac{w'^{n-2k} f_{(2k)}}{(1+|z|^2)^k} d(x) \frac{w'}{\bar{w}'} e_2 \wedge \bar{e}_1'. \quad (79)$$

Imposing $D(*\omega) = 0$ and $\overline{D}(\omega) = 0$ gives

$$d(x) = \frac{x^{2k-n-1}}{f_{(2k)}^2}. \quad (80)$$

For $x \rightarrow 0$ ω is well-behaved if n is even and satisfies the inequality $n \leq 2k - 4$. Also, this solution has finite L^2 norm for $n > 0$:

$$\int \omega \wedge * \bar{\omega} \sim \int \frac{dx}{x^{n+3}} \int \frac{dz d\bar{z}}{(1+|z|^2)^{2+n}}.$$

6.4.4 $H^1(\mathcal{O}(2k))$ and $H^1(\Omega^2(2k))$, $k > 0$

For $k > 0$ we start from an ansatz (in the unitary trivialization and in the chart $u \neq 0$) for the component of the $(n+1)$ -plet in $H^1(\mathcal{O}(2k))$ with $J_3 = -(n/2)$:

$$\omega = \frac{f_{(2k)}}{(1+|z|^2)^k} w'^{n-2k} \left(\beta(x) \bar{e}_1' + \alpha(x) \frac{\bar{w}'}{w'} \bar{e}_2 \right). \quad (81)$$

¹³Recall that x^2 is a good coordinate, but x is not, so odd powers of x are ill-behaved.

Imposing $\overline{D}(\omega) = 0$ and $D(*\omega) = 0$ gives the following result. For even n such that $0 \leq n \leq 2k - 2$

$$\omega = w'^n \left(\frac{\overline{w}'}{w'} \right)^k \frac{x^{k-n}}{f_{(2k)} f_2} \overline{e}'_1$$

belongs to $H^1(\mathcal{O}(2k))$, has finite L^2 norm and is well-behaved for $x \rightarrow 0$.

The component of the $(n+1)$ -plet in $H^1(\Omega^2(2k))$ with $J_3 = -(n/2)$ can be found analogously. For even n such that $0 \leq n \leq 2k - 2$

$$\omega = w'^n \left(\frac{\overline{w}'}{w'} \right)^{k-1} \frac{x^{k-n-1} f_1}{f_{(2k)}} e'_1 \wedge e_2 \wedge \overline{e}'_1$$

belongs to $H^1(\Omega^2(2k))$, has finite L^2 norm and is well-behaved for $x \rightarrow 0$.

6.5 Testing S-duality

We are now ready to perform a test of the S-duality prediction (43). Summing up all cohomology groups $H^p(T^*\mathbb{P}^1, \Omega^q(-2m))$ with the sign $(-1)^{p+q}$, we find the “bubbled” contribution to the zeromode Hilbert space:

$$V_{2m+1} - V_1 - V_{2m-1},$$

where V_{2j+1} is the $2j+1$ -dimensional representation of $PSL(2, \mathbb{C})$. This corresponds to the sum of Wilson loops

$$W_{2m} - W_0 - W_{2m-2},$$

in precise agreement with the S-duality prediction (43).

As a consistency check on our computation, let us consider the Euler characteristics of the graded vector spaces $\mathfrak{H}(\mathbf{A}, \mathbf{B}, \dots)$ associated to the left-hand side and right-hand side of eq. (42). According to our computations, the “bulk” contribution to the Euler characteristic of the right-hand side is

$$1 + (2m+1) - (2m-1) = 3.$$

The “bubbled” contribution is

$$(2m+1) - 1 - (2m-1) = 1.$$

Therefore the Euler characteristic of the right-hand side is 4. We can compute the Euler characteristic of the left-hand side by moving the WH operators so that they are inserted at the same point on the interval I but at different points on C .¹⁴ If the WH operators are inserted at different points on C , the space of zero modes factorizes, and so does the Euler characteristic. The Hilbert space $\mathfrak{H}(WT_{1,0})$ is purely even and two-dimensional. The Hilbert space $\mathfrak{H}(WT_{1,2m})$ is

$$\bigoplus_{p,q=0}^1 H^p(\mathbb{P}^1, \Omega^q(-2m)),$$

and its Euler characteristic is 2 for any m . Therefore the Euler characteristic of the left-hand side is also 4.

7 Concluding remarks

As mentioned in the introduction, 't Hooft line operators can be interpreted mathematically as objects of the category of equivariant perverse sheaves on the affine Grassmannian Gr_G . Then the algebra of loop operators can be identified with the K-theory of this category, and the S-duality prediction is equivalent to the geometric Satake correspondence. 't Hooft loop operators labeled by coweights of G define a distinguished basis in the K^0 -group.

It was suggested by R. Bezrukavnikov that the algebra of Wilson-'t Hooft loop operators can be similarly interpreted as the K^0 -group of the equivariant derived category of coherent sheaves on a certain subset Λ_G of the cotangent bundle of Gr_G . Λ_G is defined as the union of the conormal bundles to the Schubert cells in Gr_G and is invariant under the left $G[[z]]$ action on Gr_G . Just like Gr_G parameterizes Hecke transformations of holomorphic G -bundles, Λ_G parameterizes Hecke transformations of Higgs bundles. Thus any object of the $G[[z]]$ -equivariant derived category of Λ_G can be used to define a functor from the derived category of $\mathcal{M}_{Higgs}(G, C)$ to itself and can be thought of as a line operator. It was proved in [15] that the K^0 -group of $D_{eq}^b(\Lambda_G)$ is the Weyl-invariant part of the group algebra of $\widehat{\Lambda}(G)$, in agreement with the physical arguments. Further, it was conjectured in [15] that the obvious invariance of $\widehat{\Lambda}(G)$ under the exchange of G and ${}^L G$ comes from

¹⁴Unlike in [5], there is no natural flat connection on the sheaf of the zero mode Hilbert spaces $\mathfrak{H}(A, B, \dots)$, and in principle the stalk of this sheaf might depend on the locations of the insertion points. Nevertheless, while the dimensions of the individual graded components might jump, the Euler characteristic must be constant.

an equivalence between categories $D_{eq}^b(\Lambda_G)$ and $D_{eq}^b(\Lambda_{LG})$. From the physical viewpoint, this conjecture means that the categories of line operators for G and ${}^L G$ are equivalent and thus follows from the S-duality conjecture.

Note also that the physical definition of the Wilson-'t Hooft loop operator suggests that there is a distinguished basis in the K-theory of $D_{eq}^b(\Lambda_G)$ labeled by elements of $\widehat{\Lambda}(G)/\mathcal{W}$, and that the S-duality group acts on this basis in a natural way. The mathematical significance of this basis remains unclear. Moreover, Wilson-'t Hooft *line* operators should correspond to some distinguished objects in $D_{eq}^b(\Lambda_G)$. It was conjectured by R. Bezrukavnikov that these distinguished objects are certain perverse coherent sheaves on Λ_G .

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