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Mathematics

An inequality among determinants

(exterior algebra/Fredholm theory)

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ABSTRACT **We prove that for any trace class operators, A,B, $\det(1+|A+B|) \leq \det(1+|A|)\det(1+|B|)$ where $|C| = (C^*C)^{1/2}$.**

It is often difficult to control non-normal operators because the absolute value of an operator C, defined by $|C| = (C^*C)^{1/2}$ fails to obey the triangle inequality:

$$|A+B| \leq |A| + |B|. \qquad [1]$$

Not only is [1] false, but various weaker statements also fail. For example (ref. 1), the inequality:

$$\| |A| - |B| \| \leq c\|A-B\| \qquad [2]$$

with $\|\cdot\|$ = operator norm is false for any constant c. Moreover, we have found that

$$\det(|A+B|) \leq \det(|A|+|B|) \qquad [3]$$

is false as can be seen by considering:

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In our work on the Y_2 constructive quantum field theory, we have discovered an inequality, which appears to be new, that we feel may be of sufficient interest to warrant separate publication (motivated by our work, E. Lieb (*Advances in Mathematics*, in press) has proven a general inequality that provides another proof of [4]):

THEOREM 1. Let $\mathcal{H}$ be a separable Hilbert space and let A,B be trace class operators on $\mathcal{H}$. Then

$$\det(1+|A+B|) \leq \det(1+|A|)\det(1+|B|). \qquad [4]$$

By a simple approximation argument, we can suppose that $\mathcal{H}$ is finite dimensional. The difficulty in proving something like [4] is that $|A+B|$ is a nonlinear function of A and B. In the finite dimensional case, we have $|A+B| = U|A| + V|B|$ for suitable unitaries U and V. U and V are nonlinear functions of A and B, of course, but we can ignore the nonlinearity by proving the stronger:

THEOREM 1′. For any unitaries U, V and trace class operators, A and B:

$$|\det(1+U|A|+V|B|)| \leq \det(1+|A|)\det(1+|B|). \qquad [5]$$

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Since [5] only depends on $|A|$ and $|B|$ and not otherwise on A or B, we can suppose that $A = U|A|$, $B = V|B|$ so that [5] is clearly implied by:

THEOREM 1″. For any trace class operators A and B:

$$|\det(1+A+B)| \leq \det(1+|A|)\det(1+|B|). \qquad [6]$$

Remark: Since $|\det(1+C)| \leq \det(1+|C|)$ (see, e.g., below) *Theorems 1* and *1″* are actually equivalent.

We will prove [6] by expanding $\det(1+\mu A+\lambda B)$ and $\det(1+\mu|A|)\det(1+\lambda|B|)$ as polynomials in μ,λ and prove the coefficients of one polynomial dominate those of the others. Thus, we have other inequalities implicit in our proof. For example, let $d_m(A)$ be the sum of all products of m eigenvalues of A (i.e., if $\alpha_1 \leq \ldots \leq \alpha_n$ are the eigenvalues of A counting multiplicity, then

$$d_m(A) = \sum_{i_1 \cdots i_m} \alpha_{i_1} \ldots \alpha_{i_m} \qquad [7]$$

where the sum is over all $\binom{n}{m}$ choices of $i_1 < i_2 < \ldots < i_m$). Let $d_0(A) \equiv 1$. Then our proof implies that:

THEOREM 2. For A,B trace class

$$d_m(|A+B|) \leq \sum_{k=0}^{m} d_k(|A|)\,d_{m-k}(|B|). \qquad [8]$$

THEOREM 2′. For A,B trace class

$$|d_m(A+B)| \leq \sum_{k=0}^{m} d_k(|A|)\,d_{m-k}(|B|). \qquad [9]$$

Remark: We have

$$\det(1+C) = \sum_{k=0}^{n} d_k(C) \qquad [10]$$

if C is an operator on a finite dimensional space $\mathcal{H}$ of dimension n. Thus, the right-hand side of [6] is

$$\sum_{j,k=0}^{n} d_k(|A|)\,d_j(|B|)$$

while by [9], the left-hand side of [6] is bounded by

$$\sum_{\substack{j,k=0 \\ j+k\leq n}}^{n} d_k(|A|)\,d_j(|B|)$$

so [6] can be strengthened when n is finite.

Our proofs will be exercises in multilinear algebra (refs. 2 and 3). Suppose that $\mathcal{H}$ has $n < \infty$ and let $\Lambda_m(\mathcal{H})$ be the m-fold alternating product of $\mathcal{H}$. Given A: $\mathcal{H} \to \mathcal{H}$, let

$\Lambda_m(A) : \Lambda_m(\mathcal{H}) \to \Lambda_m(\mathcal{H})$ be $A\Lambda \ldots \Lambda A$. Finally, let $\Lambda(\mathcal{H}) = \oplus_{m=0}^{n} \Lambda_m(\mathcal{H})$ and $\Lambda(A) = \oplus_{m=0}^{n} \Lambda_m(A)$. It is then easy to see that

$$d_m(A) = Tr(\Lambda_m(A)) \qquad [11]$$

and

$$\det(1 + A) = Tr(\Lambda(A)). \qquad [12]$$

Remarks 1: For a unitary, U, and positive operator B $|Tr(UB)| \leq Tr(B)$ so [11], [12], and the functorial nature of Λ_m immediately imply that $|d_m(A)| \leq d_m(|A|)$ and $|\det(1+A)| \leq \det(1+|A|)$ 2. It is easy to see that $|Tr(\Lambda_m(A))| \leq Tr(\Lambda_m(|A|)) \leq (m!)^{-1}(Tr(|A|))^m$. In the infinite dimensional case one can thus recover the well-known results that $\Lambda(A)$ is trace class if A is and that the Fredholm determinant $\det(1+A)$ is well-defined and obeys $|\det(1+A)| \leq \exp(tr(|A|))$.

Now let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathcal{H}$. Identifying the $\mu^m \nu^k$ term in $\det(1+\mu A+\nu B)$, we see that *Theorems 1″,2′* follow if we prove that:

$$\left| \sum_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_k}} \langle e_{i_1}\Lambda \ldots \Lambda e_{i_m}\Lambda e_{j_1}\Lambda \ldots e_{j_k}, Ae_{i_1}\Lambda \ldots \Lambda Ae_{i_m}\Lambda Be_{j_1}\Lambda \cdots \Lambda Be_{j_k}\rangle \right| \leq Tr(\Lambda_m(|A|))\, Tr(\Lambda_k(|B|)). \qquad [13]$$

In [13], the i's and j's are distinct and $i_1 < \ldots < i_m$, $j_1 < \ldots < j_k$. For later reference, we note that the left side of [13] is independent of basis, since it has the basis independent form

$$\left| \frac{\partial^m}{\partial \mu^m} \frac{\partial^k}{\partial \nu^k} \det(1 + \mu A + \nu B)\bigg|_{\mu=\nu=0} \right|.$$

To prove [13], we use the notion of interior product. Let $w \in \mathcal{H}$. Then Λw: $\Lambda_k(\mathcal{H}) \to \Lambda_{k+1}(\mathcal{H})$ by $(\Lambda w)(u) = w\Lambda u$ is an operator of norm $\|w\|$ (the easiest way of seeing this is to extend $w/\|w\|$ to a basis $e_1 = w/\|w\|, \ldots, e_n$ whence $\Lambda w/\|w\|$ takes an orthonormal set into an orthogonal set of vectors each of norm 0 or 1). The adjoint $\rfloor w = (\Lambda w)^*$: $\Lambda_{k+1}(\mathcal{H}) \to \Lambda_k(\mathcal{H})$ is called interior product and clearly has norm $\|w\|$.

In [13], let $e_1, \ldots, e_n$ be a basis of eigenvectors for $|A|$ and let $A = U|A|$, $B = V|B|$ be the polar decompositions for A and B. Write $|A|\, e_i = \alpha_i e_i$. [13] follows from

$$\left| \sum_{j_1,\ldots,j_k} \langle e_{i_1}\Lambda \ldots \Lambda e_{i_m}\Lambda e_{j_1}\Lambda \ldots e_{j_k}, Ae_{i_1}\Lambda \ldots \Lambda Be_{j_k}\rangle \right| \leq \alpha_{i_1} \ldots \alpha_{i_m} Tr(\Lambda_k(|B|)) \qquad [14]$$

Let $K = (\rfloor e_{i_m})(\rfloor e_{i_{m-1}}) \ldots (\rfloor e_{i_1}) \ldots (\Lambda U e_{i_1}) \ldots (\Lambda U e_{i_m})$ as a map from $\Lambda_k(\mathcal{H})$ to itself. K is a map of norm less than one, as a product of contractions. Moreover, in [14] we can take the sum of all j_l with $j_1 < \ldots < j_k$ without worrying about the condition $j_m \neq$ any i_l (since the terms with $j_m =$ some i_l are automatically zero). Thus, the left-hand side of [14] is equal to $\alpha_{i_1} \ldots \alpha_{i_m} \left| Tr_{\Lambda_k(\mathcal{H})} (K\Lambda_k(V)\, \Lambda_k(|B|)) \right|$. Now taking a basis of $\Lambda_k(\mathcal{H})$, $v_1, \ldots, v_{\binom{n}{k}}$ of eigenvectors for $\Lambda_k(|B|)$ we see that $\left| Tr(K\Lambda_k(V)\Lambda_k(|B|)) \right|$

$$\begin{aligned} &= \left|\Sigma \langle v_i, K\Lambda_k(V)v_i\rangle \langle v_i, \Lambda_k(|B|)v_i\rangle\right| \\ &\leqq \Sigma \left|\langle v_i, K\Lambda_k(V)v_i\rangle\right| \left|\langle v_i, \Lambda_k(|B|)v_i\rangle\right| \\ &\leq \Sigma \langle v_i, \Lambda_k(|B|)v_i\rangle = Tr(\Lambda_k|B|)) \end{aligned}$$

because $K\Lambda_k(V)$ is a contraction. It follows that [14] holds and, hence, the theorems are proven.

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