TABLE IV
COMPARISON OF NUMERICAL SENSITIVITY FROM BOTH APPROACHES

Point	S.D. of $\hat{g}(\theta)$ (conventional approach)	S.D. of $\hat{g}(\theta)$ (proposed approach)
(20,20)	1.7010×10^5	0.5194
(0,20)	2.5966×10^4	0.1627
(20,0)	2.7249×10^4	0.1642
(10,10)	1.0437×10^3	0.0371
(0.6,0.6)	4.7266×10^{-5}	1.3687×10^{-5}

as (19), and compute the standard deviation of the values of the corresponding perturbed polynomial at the point (θ_1^*, θ_2^*). We perform the test at $(\theta_1^*, \theta_2^*) = (20,20)$, $(0,20)$, $(20,0)$, $(10,10)$ and $(0.6,0.6)$ and compare the results of both approaches in Table IV.

There are several reasons showing that our approach is promising. First, we can see from Tables I and III that the number of variables in the conventional approach grows rapidly when the degrees of the monomial bases increase, while the number of variables in our approach grows linearly with respect to the number of subregions. Moreover, our approach attains the exact optimal value with less computational cost than the conventional SOS approach. Secondly, Table IV shows that the standard deviations of the perturbed polynomial from the proposed approach are less than that from the conventional approach, for all selected points. This implies that the optimal value of the proposed approach is less numerically sensitive than that of the conventional approach. Strictly speaking, the asymptotic exactness of our scheme is not guaranteed in the case of lowest-degree monomial bases. However, it is achieved apparently in this example. We expect that the asymptotic exactness can also be proved with the monomial bases of lowest degrees, and this is the direction of our further research.

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Robust Stability Analysis of Nonlinear Hybrid Systems

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***Abstract*—We present a methodology for robust stability analysis of nonlinear hybrid systems, through the algorithmic construction of polynomial and piecewise polynomial Lyapunov-like functions using convex optimization and in particular the sum of squares decomposition of multivariate polynomials. Several improvements compared to previous approaches are discussed, such as treating in a unified way polynomial switching surfaces and robust stability analysis for nonlinear hybrid systems.**

***Index Terms*—Hybrid systems, linear matrix inequality, sum of squares, switched systems.**

I. INTRODUCTION

Hybrid systems have dynamics that are described by a set of continuous (or discrete) time differential equations in conjunction with a dis-

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crete-event (decision) process. Examples include motion control systems and robotics [1], air traffic management [2], [3] etc.

For the case of systems with continuous dynamics, stability properties are traditionally addressed using Lyapunov functions [4]. Extensions of these ideas to hybrid systems have appeared in, e.g., [5]–[8]. See also [9] for a survey of the field, as well as the recent books [10]–[12]. Piecewise quadratic Lyapunov functions have been introduced, which are constructed by concatenating several quadratic Lyapunov-like functions algorithmically by solving a set of Linear Matrix Inequalities (LMIs) [13]. However, in some cases such LMI conditions can be conservative or the number of quadratic Lyapunov-like functions needed is large, resulting in an increased computational load.

This technical note, which extends the work in [14] to treat robustness analysis in the presence of dynamic uncertainties, provides a methodology for stability analysis of switched and hybrid systems. For proving stability, polynomial and piecewise polynomial Lyapunov functions are constructed using positive polynomials and the sum of squares decomposition [15]–[18], which can be efficiently computed using semidefinite programming, e.g., using the software [19]. An advantage of this method is that it provides a less conservative test for proving stability when switching between subsystems is arbitrary, provided that a finite number of switches occurs on every bounded time interval. Moreover, we demonstrate that stability can be proven with a smaller number of Lyapunov-like functions, eliminating the need of refining the state space partition in order to find quadratic such multiple Lyapunov functions. The method can be readily applied to systems with nonlinear subsystems and nonlinear switching surfaces, therefore allowing much richer system descriptions. Finally, parametric and dynamic robustness analysis can be performed in a unified manner.

Appropriate modeling for such systems, as well as the existence and uniqueness of solutions are important research topics [20], as many times solutions may not exist, may not be unique (non-determinism) or a Zeno behaviour is observed (infinite number of discrete transitions in finite time). Here we will assume that the system models are such that these phenomena, including sliding modes and equivalent dynamics are avoided [21]. Moreover, we will say "arbitrary switching" to mean arbitrary switching in which only a finite number of switches is allowed in finite time. The tools developed in this technical note can be extended to cover the case of systems with sliding modes if the additional mode that captures the sliding mode dynamics is added to the system description and a condition that the Lyapunov function decreases along sliding-mode trajectories is imposed.

The technical note is organized as follows. We first present some preliminaries on hybrid and switched systems, as well as tools from positive polynomials that we will be using to analyze them. In Section III we will formulate various algorithms for testing stability for hybrid and switched systems, giving examples for the various cases. We then discuss robust stability analysis, before concluding the technical note.

II. Preliminaries

A. Hybrid Systems

We consider systems of the following form:

$$\dot{x} = f_l(x), \quad l \in L = \{1, \ldots, N_L\} \tag{1}$$

where $x \in \mathbb{R}^n$ is the continuous state, l is the discrete state (location), $f_l(x)$ is the vector field describing the dynamics of the l-th mode/subsystem (assumed to be sufficiently smooth), and L is the finite index set. *Executions* (trajectories) of the system are concatenations of a sequence of continuous flows and discrete transitions. During a continuous flow, the discrete location l is maintained and the continuous state evolves according to (1). The evolution of the discrete state l can be either time-dependent or state-dependent.

For time-dependent switching we consider switching signals which are piecewise constant and continuous from the right and which have a finite number of discontinuities on every bounded time interval. For state-dependent switching, we assume that $\mathbb{R}^n$ is partitioned into operating regions $\mathcal{X}_l$, $l = 1, \ldots, N_L$ by *guard sets* (also called switching surfaces). These operating regions may or may not intersect, and their union is $\mathbb{R}^n$. When the continuous state is x in location l and a guard set $G(l, l')$ is met, a discrete transition to l' will occur and the continuous state will take the value x', which is prescribed by the single-valued reset map $R(l, l')(x)$. Systems with infinitely fast switching, such as those that have sliding modes, are excluded from our discussion, even though analysis in the case of sliding modes can still be performed.

We describe mathematically the regions $\mathcal{X}_l$ by

$$\mathcal{X}_l = \{x \in \mathbb{R}^n : g_{lk}(x) \geq 0, \text{ for } k = 1, \ldots, m_{\mathcal{X}_l}\} \tag{2}$$

for some $g_{lk} : \mathbb{R}^n \to \mathbb{R}$. A guard set between the l and l' modes is given by

$$G(l, l') = \{x \in \mathbb{R}^n : h_{ll'0}(x) = 0, h_{ll'k}(x) \geq 0, \text{ for } k = 1, \ldots, m_{G(l,l')}\} \tag{3}$$

for some $h_{ijk} : \mathbb{R}^n \to \mathbb{R}$. Lastly, the reset map is given by

$$R(l, l')(x) = \phi_{ll'}(x). \tag{4}$$

We assume that the origin is a common equilibrium of the locations the stability of which we will investigate; this implies that $f_l(0) = 0$ for all $l \in L$.[1]

We will also consider systems of the form

$$\dot{x} = f_l(x, p), \quad l \in L = \{1, \ldots, N_L\} \tag{5}$$

where $p \in \mathcal{P} \subset \mathbb{R}^m$ denotes the uncertainty in the continuous flow which may be time-varying, in which case $\dot{p} \in \mathcal{Q} \subset \mathbb{R}^m$ is bounded.

Finally, we assume that the functions f_l, g_{lk}, $h_{ll'k}$ and $\phi_{ll'}$ are polynomials. For the case in which any of these functions is nonpolynomial, see the comment at the end of Section III-D.

B. Sum of Squares Decomposition

Our analysis is based on positive polynomials [15]–[17] and the sum of squares decomposition of multivariate polynomials. A multivariate polynomial $p(x)$ is a sum of squares if there exist polynomials $p_1(x), \ldots, p_m(x)$ such that $p(x) = \sum_{i=1}^{m} p_i^2(x)$. This in turn is equivalent to the existence of a positive semidefinite matrix Q, and a properly chosen vector of monomials $Z(x)$ such that $p(x) = Z^T(x) Q Z(x)$ [15].

What makes the sum of squares decomposition attractive is the fact that it can be computed using semidefinite programming, since the computation of Q is nothing but a search for a positive semidefinite matrix subject to some affine constraints. Coupled with the property that $p(x)$ being a sum of squares implies[2] $p(x) \geq 0$, the sum of squares decomposition provides a computational relaxation for testing polynomial positivity, which belongs to the class of NP-hard problems. Three

[1]Here we assume that $0 \in \mathcal{X}_l$ for all $l \in L$. A relaxed assumption would be that $f_l(0) = 0$ for all $l \in L_0 = \{l \in L \mid 0 \in \mathcal{X}_l\}$ and also that a transition can occur from location l at state 0 only if $l, l' \in L_0$ and $\phi_{ll'}(0) = 0$.

[2]Note that the converse implication is true only in special cases. One such instance is when the polynomial is quadratic.

TABLE I
THREE KINDS OF POLYNOMIAL POSITIVITY (ON THE LEFT) AND THE CORRESPONDING SUM OF SQUARES CONDITIONS (ON THE RIGHT). CONDITIONS ON THE RIGHT ARE SUFFICIENT FOR THOSE ON THE LEFT. THE POLYNOMIAL DEGREE N IS ASSUMED TO BE EVEN, OTHERWISE THE POLYNOMIAL WILL BE NEGATIVE FOR SOME x. HERE $\epsilon > 0$

Positive semidefinite: $p(x) \geq 0 \quad \forall x \in \mathbb{R}^n$.	$p(x)$ is a sum of squares.
Positive definite: $p(x) > 0 \quad \forall x \neq 0$; $p(0) = 0$.	
• If $p(x)$ is homogeneous of degree N.	$(p(x) - \epsilon \sum_{i=1}^{n} x_i^N)$ is a sum of squares.
• If $p(x)$ is of degree N, but nonhomogeneous.	$\begin{cases} (p(x) - \sum_{i=1}^{n} \sum_{j=1}^{N/2} \epsilon_{ij} x_i^{2j}) \text{ is a sum of squares;} \\ \sum_{j=1}^{N/2} \epsilon_{ij} \geq \epsilon \quad \forall i; \quad \epsilon_{ij} \geq 0 \quad \forall i, j. \end{cases}$ Note that $p(x)$ is also radially unbounded.
Strict positivity: $p(x) > 0 \quad \forall x \in \mathbb{R}^n$.	$(p(x) - \epsilon)$ is a sum of squares.

kinds of polynomial positivity and their corresponding sum of squares computational relaxations are shown in Table I.

The sum of squares decomposition has been exploited to algorithmically construct Lyapunov functions for nonlinear systems [15], [18], [22], [23]. For this purpose, real coefficients $c_1, \ldots, c_m$ are used to parameterize a set of Lyapunov functions in the following way:

$$\mathcal{V} = \left\{ p(x) : p(x) = p_0(x) + \sum_{i=1}^{m} c_i p_i(x) \right\} \tag{6}$$

where $p_i(x)$ are some polynomials; for example they could be monomials of degree up to some number. The search for a Lyapunov function $V(x) \in \mathcal{V}$, or equivalently some c_i, such that $V(x)$ is positive definite and dV/dt is negative definite can still be formulated as a sum of squares problem and solved using semidefinite programming. The results in the subsequent sections will be formulated in terms of inequalities such as $V(x) \geq 0$ or $V(x) > 0$. However, if the computation of Lyapunov functions is to be performed using semidefinite programming, then these inequalities have to be relaxed to sum of squares constraints, in the way summarized in Table I.

III. STABILITY ANALYSIS

A. Stability of Hybrid Systems

Stability of equilibria of hybrid systems has been addressed in [5], [6], [9], [24], [25]. We will use the following two Lyapunov theorems in the sequel. The first theorem concerns the case of what is known as a "common" Lyapunov function:

Theorem 1: Consider a hybrid system (1) with 0 an equilibrium point and let $R(l, l')(x) = x$. Suppose that there exists an open set $\mathcal{U} \subset \mathbb{R}^n$ such that $0 \in \mathcal{U}$. Let $V : \mathcal{U} \to \mathbb{R}$ be a continuously differentiable function such that:

1) $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathcal{U} \setminus \{0\}$,
2) $(\partial V(x)/\partial x) f_l(x) \leq 0$ for all $x \in \mathcal{U}, l \in L$.

Then $x = 0$ is a stable equilibrium of the hybrid system. If furthermore $(\partial V(x)/\partial x) f_l(x) < 0$ for all $x \in \mathcal{U} \setminus \{0\}, l \in L$ then $x = 0$ is an asymptotically stable equilibrium.

The proof of this theorem can be found in, e.g., [10]. Global asymptotic stability can be obtained if $\mathcal{U} = \mathbb{R}^n$ and V is radially unbounded. The above theorem has the drawback that such a V may be difficult to construct even if switching occurs between linear subsystems. But since the switching signal is not state-dependent, it is useful when investigating stability under arbitrary switching.

A more general theorem that will lead our discussion on multiple Lyapunov functions for determining stability is stated below, which includes 'impulsive jumps', even if 0 is an equilibrium point. Recall that $R(l, l')(x) = x'$ is the reset map when $G(l, l')$ is met. Let us denote the switching times by τ_i so that $R(l(\tau_i), l(\tau_i^+))(x(\tau_i)) = x(\tau_i^+)$, where τ_i^+ are the times just after the switching times.

Theorem 2: Consider a hybrid system with 0 as an equilibrium point. For each $l \in L$, suppose that there exists a continuously differentiable function $V_l : \mathcal{X}_l \to \mathbb{R}$ such that:

1) $V_l(0) = 0$ and $V_l(x) > 0$ for all $x \in \mathcal{X}_l \setminus 0$,
2) $(\partial V_l(x)/\partial x) f_l(x) \leq 0$ for all $x \in \mathcal{X}_l$.

If moreover, for all executions and for all switching times τ_i, we have $V_{l(\tau_i^+)}(x(\tau_i^+)) \leq V_{l(\tau_i)}(x(\tau_i))$ then $x = 0$ is stable.

The above theorem (a proof of which can be found in [6]) considers many Lyapunov functions, each defined for each subsystem, satisfying the familiar Lyapunov conditions. The last condition ensures that during switches, the value of the Lyapunov function is non-increasing, even if the continuous state is reset. It can be relaxed to the statement that $V_{l(\tau_i^+)}(x(\tau_i^+)) \leq V_{l(\tau_i^+)}(x(\tau_j))$ where $\tau_j < \tau_i^+$ is the time that location $l(\tau_i^+)$ was last active. This condition is difficult to impose algorithmically, so we will use the condition stated in the theorem instead, i.e., that the Lyapunov functions are non-increasing when switches occur.

Corollary 3: Consider a hybrid system with 0 as an equilibrium point. For each $l \in L$, suppose that there exists a continuously differentiable function $V_l : \mathcal{X}_l \to \mathbb{R}$ such that:

1) $V_l(0) = 0$ and $V_l(x) > 0$ for all $x \in \mathcal{X}_l \setminus 0$,
2) $(\partial V_l(x)/\partial x) f_l(x) \leq 0$ for all $x \in \mathcal{X}_l$,
3) $V_{l'}(x') - V_l(x) \leq 0$ for all $x \in G(l, l'), x' = R(l, l')(x)$.

Then $x = 0$ is a stable equilibrium.

B. Stability Under Time-Dependent Switching

Here switching is time-dependent, and the switching signal is piecewise constant and continuous from the right. A finite number of switches is allowed on every bounded time interval in order to exclude arbitrarily fast switching. We also assume $\mathcal{X}_l = \mathbb{R}^n$ and consider (1) with $G(l, l') = \mathbb{R}^n$, and $R(l, l')(x) = x$. A sufficient condition for the stability of the origin in this case is the existence of a global common Lyapunov function, as summarized in the following theorem.

Theorem 4: Suppose that for system (1) there exists a polynomial $V(x)$ such that $V(0) = 0$ and

$$V(x) > 0 \quad \forall x \neq 0, V(x) \text{ radially unbounded} \tag{7}$$

$$\frac{\partial V(x)}{\partial x} f_l(x) < 0 \quad \forall x \neq 0, \ l \in L \tag{8}$$

then the origin is globally asymptotically stable for arbitrary switching.

Notice in particular that if the vector fields are linear, i.e., $f_l(x) = A_l x$, and if $V(x)$ is chosen to be quadratic, say $V(x) = x^T P x$, then the conditions in Theorem 4 correspond to the well-known LMIs $P > 0$, $A_l^T P + P A_l < 0$ for all l, which prove quadratic stability of the system but may be conservative. Several researchers have considered the use of non-quadratic Lyapunov functions for such systems, e.g., polyhedral [26], [27], piecewise-quadratic [28] and polynomial [29]. See also [30] where a sufficient condition for the existence of a homogeneous polynomial Lyapunov function is given, which is also necessary in some cases. For the case of systems with time-varying

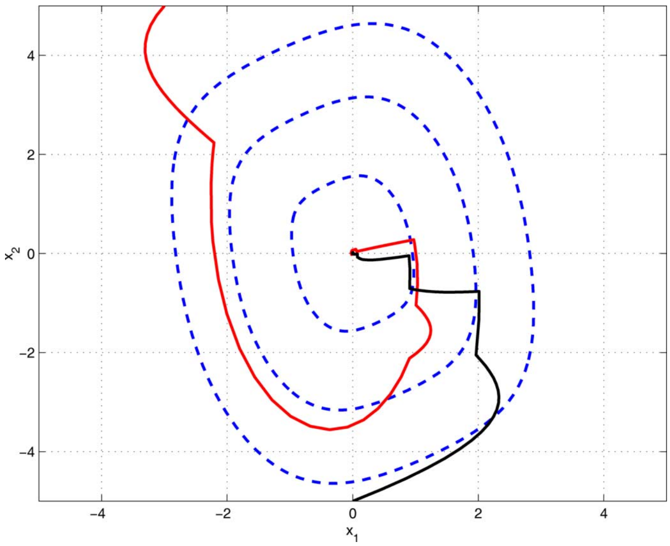


Fig. 1. Trajectories of the system in Example 5 under arbitrary, time dependent switching. The switching signal is piecewise constant and continuous from the right and switches arbitrarily between the two locations with a bounded number of discontinuities on every bounded time interval. Dashed curves are level curves of the common Lyapunov function.

uncertainties with a bounded variation rate [31], parameter-dependent homogeneous Lyapunov functions can be employed. This is related to the work of P. -A. Bliman [32], [33], which provides sufficient conditions for robust stability of linear systems, using quadratic Lyapunov functions with polynomial dependence on the parameters.

1) Example 5: Consider the system $\dot{x} = f_i(x)$, $x = [x_1, x_2]^T$ under arbitrary switching, with

$$f_1(x) = \begin{bmatrix} -5x_1 - 4x_2 \\ -x_1 - 2x_2 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} -2x_1 - 4x_2 \\ 20x_1 - 2x_2 \end{bmatrix}.$$

It can be proven using a dual semidefinite program that no global quadratic Lyapunov function exists for this system [7]. Nevertheless, a global sextic Lyapunov function

$$V(x) = 19.861x_1^6 + 11.709x_1^5x_2 + 14.17x_1^4x_2^2 + 4.2277x_1^3x_2^3 + 8.3495x_1^2x_2^4 - 1.2117x_1x_2^5 + 1.0421x_2^6$$

exists, and therefore the system is asymptotically stable under arbitrary switching (cf. Fig. 1).

For higher degree polynomial vector fields and Lyapunov functions, the search for $V(x)$ can also be performed using semidefinite programming by formulating the conditions as sum of squares conditions.

C. Piecewise Polynomial Lyapunov Functions

For state-dependent switching, the analysis method presented in Section III-B will be too conservative. Stability can be proven in a more effective way using piecewise polynomial Lyapunov functions. Such functions are "patched" from several polynomial functions $V_l(x)$ (also termed Lyapunov-like functions), typically corresponding to the regions $\mathcal{X}_l$. Theorem 2 requires that the Lyapunov-like function $V_l(x)$ and its time derivative along the trajectory of the l-th location need only be positive and negative respectively within $\mathcal{X}_l$.

The conditions in the previous paragraph can be accommodated using a method similar to the S-procedure [13] as follows. To incorporate the fact that $V_l(x)$ only needs to be positive on $\mathcal{X}_l$, where $\mathcal{X}_l$ is described by (2), we impose the relaxed condition

$$V_l(x) - \sum_{k=1}^{m_{\mathcal{X}_l}} a_{lk}(x) g_{lk}(x) > 0 \tag{9}$$

for some $a_{lk}(x) \geq 0$. Since $g_{lk}(x)$ is nonnegative on $\mathcal{X}_l$, the above condition implies that $V_l(x)$ is positive on $\mathcal{X}_l$. An analogous condition can be imposed on dV_l/dt. Note that there is no requirement in this method that the multipliers $a_{lk}(x)$ be constants (as in the S-procedure); they can also be polynomials of higher degree [15]. Thus, this condition is generally less conservative than the S-procedure.

1) Switched Systems: In this case, the guards between two locations $G(l, l')$ and $G(l', l)$ coincide. Such systems are categorized as switched systems. The transition between locations is unknown *a priori*, but will depend on the direction of the vector fields. Without characterizing the direction of switching, it is essential that the piecewise Lyapunov function used to prove stability be continuous on $G(l, l')$. Imposing

$$V_l(x) + c_{ll'0}(x) h_{ll'0}(x) - V_{l'}(x) = 0 \tag{10}$$

where $c_{ll'0}(x)$ is an arbitrary polynomial, will guarantee the continuity of $V(x)$ on $G(l, l')$ (equivalently $G(l', l)$). This results in the following Theorem for switched systems.

Theorem 6: Consider a switched system with $G(l, l') = G(l', l)$ for all $l, l' \in L$. Assume that there exist polynomials $V_l(x)$, $c_{ll'}(x)$, with $V_l(0) = 0$ and $a_{lk}(x) \geq 0$, $b_{lk}(x) \geq 0$, such that

$$V_l(x) - \sum_{k=1}^{m_{\mathcal{X}_l}} a_{lk}(x) g_{lk}(x) > 0 \quad \forall x \neq 0, l \in L, \tag{11}$$

$$\frac{\partial V_l}{\partial x} f_l(x) + \sum_{k=1}^{m_{\mathcal{X}_l}} b_{lk}(x) g_{lk}(x) < 0 \quad \forall x \neq 0, l \in L, \tag{12}$$

$$V_l(x) + c_{ll'0}(x) h_{ll'0}(x) - V_{l'}(x) = 0 \quad \forall l, l'. \tag{13}$$

Then the origin of the state space is asymptotically stable. A Lyapunov function that proves this is the piecewise polynomial function $V(x)$ defined by

$$V(x) = V_l(x), \quad \text{if } x \in \mathcal{X}_l. \tag{14}$$

Moreover, if each $V_l(x)$ is radially unbounded in the invariant $\mathcal{X}_l$ and $\cup_l \mathcal{X}_l = \mathbb{R}^n$ then the result holds globally.

Even though the switched system is stable, low degree (e.g., quadratic) $V_l(x)$ that satisfy the above conditions may not exist, as those conditions are only sufficient for stability. In this case, an improved test can be performed by dividing the continuous state space into a more refined partition than the original $\mathcal{X}_l$, and then constructing a piecewise Lyapunov function (of the same degree as before) based on this new partition. For systems with more than two state variables, this refinement is obviously not an easy matter. A simpler way for obtaining an improved test is to use a higher degree Lyapunov function based on the original partition, as illustrated by the following example.

2) Example 7: Consider the switched system $\dot{x} = f_l(x)$ with four state variables and two modes

$$f_1(x) = \begin{bmatrix} -x_1 - 23x_2 + 12x_3 - 2x_4 \\ -0.5x_1 + 8.5x_2 - 6x_3 + 0.5x_4 \\ 0.5x_1 + 26x_2 - 9.5x_3 + 5x_4 \\ -3x_1 - 35x_2 + 12x_3 - 6x_4 \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} -1.4x_1 - 18.6x_2 + 8x_3 - 1.6x_4 \\ -0.3x_1 + 8.3x_2 - 4x_3 + 1.3x_4 \\ 1.7x_1 + 20.6x_2 - 5.7x_3 + 3.6x_4 \\ -3.4x_1 - 28.6x_2 + 8x_3 - 4.6x_4 \end{bmatrix},$$

$$\mathcal{X}_1 = \{x \in \mathbb{R}^4 : g(x) \geq 0\}, \quad \mathcal{X}_2 = \{x \in \mathbb{R}^4 : g(x) \leq 0\}$$

where $g(x) = (x_1 + 0.5x_2 + 1.5x_3 + 0.5x_4)(x_1 - 0.5x_2 + 0.5x_3 - 0.5x_4)$. No piecewise quadratic Lyapunov function (using the original state space partition) exists for this system. Refining the partition for this system is not easy, thus we resort to higher order Lyapunov function instead. A homogeneous piecewise quartic Lyapunov function can be found by solving the optimization problem corresponding to the conditions in Theorem 6. This proves that the origin of the state space is globally asymptotically stable.

3) Hybrid Systems: Directions of transitions in most hybrid systems are characterized *a priori*. Because of this, a piecewise Lyapunov function for a hybrid system need not be continuous, and it is enough to have $V_{l'}(x) \leq V_l(x)$ on $G(l, l')$. This is taken into account in condition (15) of the theorem below.

Theorem 8: Consider a hybrid system and assume that there exist polynomials $V_l(x)$, $c_{ll'0}(x, x')$, $d_{ll'}(x, x')$, and $a_{lk}(x) \geq 0$, $b_{lk}(x) \geq 0$, $c_{ll'k}(x, x') \geq 0$ such that $V_l(0) = 0$ and

$$V_l(x) - \sum_{k=1}^{m_{\mathcal{X}_l}} a_{lk}(x) g_{lk}(x) > 0 \quad \forall x \neq 0, l \in L,$$

$$\frac{\partial V_l}{\partial x} f_l(x) + \sum_{k=1}^{m_{\mathcal{X}_l}} b_{lk}(x) g_{lk}(x) < 0 \quad \forall x \neq 0, l \in L,$$

$$V_{l'}(x') + c_{ll'0}(x, x') h_{ll'0}(x) + \sum_{k=1}^{m_{G(l,l')}} c_{ll'k}(x, x') h_{ll'k}(x) + d_{ll'}(x, x')(x' - \phi_{ll'}(x)) - V_l(x) \leq 0 \quad \forall l, l'. \tag{15}$$

Then the origin is asymptotically stable. Moreover, if each $V_l(x)$ is radially unbounded in the invariant $\mathcal{X}_l$ and $\cup_l \mathcal{X}_l = \mathbb{R}^n$ then the result holds globally.

We remark that the last condition implies that when $h_{ll'k}(x) \geq 0$, $h_{ij0}(x) = 0$ and $x' = \phi_{ll'}(x)$ we have $V_{l'}(x') \leq V_l(x)$, which is the last condition of Corollary 3.

D. Nonlinear Vector Fields and Switching Surfaces/Transition Sets

The same methodology can be applied to systems with nonlinear vector fields and nonlinear switching surfaces or transition sets. To illustrate this, consider the following example.

1) Example 9: Let the hybrid system $\dot{x} = f_i(x)$ be composed of two subsystems

$$f_1(x) = \begin{bmatrix} -2x_1 - x_1^3 - 5x_2 - x_2^3 \\ 6x_1 + x_1^3 - 3x_2 - x_2^3 \end{bmatrix}, \; f_2(x) = \begin{bmatrix} x_2 + x_1^2 - x_1^3 \\ 4x_1 + 2x_2 \end{bmatrix} \tag{16}$$

with a guard set

$$G(1,2) = \{x \in \mathbb{R}^2 | x_1 \geq 0 \text{ and } x_2 = 0\}, \tag{17}$$

$$G(2,1) = \{x \in \mathbb{R}^2 | x_2^2 = x_1^3\}. \tag{18}$$

Fig. 2 depicts some trajectories of the system, when the system is initialized with subsystem 1. Using Theorem 8 , the origin can be proven globally asymptotically stable with a sextic piecewise polynomial Lyapunov function given by $V(x(t)) = V_l(x(t))$ if l is active, for some $V_l(x)$'s that are omitted for brevity.

This way we have demonstrated how more complicated switching rules can be taken into account when analyzing a hybrid system. We note here that systems with rational or nonpolynomial vector fields can still be analyzed using the sum of squares decomposition. This has been presented in [22] and will not be discussed in this technical note. The same technique can also be applied to nonpolynomial guard sets.

IV. Robust Stability Analysis

Uncertainty in a switched or hybrid system can be present in the vector fields describing the flow of the system and/or in the switching scheme/transition law. The uncertainty can be of parametric nature, or caused by time-varying perturbations of the vector field, switching delays, etc.

A method for robustness analysis has been proposed in [5]. The approach is based on bounding the guard sets by an uncertain switching

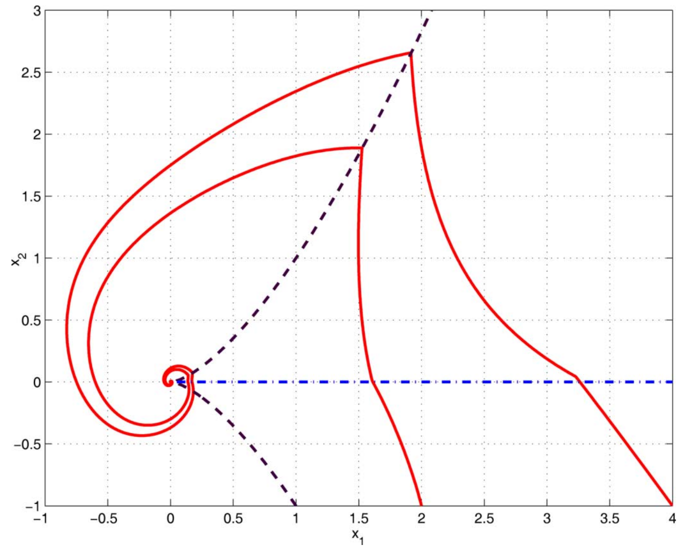


Fig. 2. Trajectories of the system in Example 9. Dash-dotted line and dashed curves show $G(1, 2)$ and $G(2, 1)$ respectively. The switching rule is given by (17)–(18).

set, and the subsystem invariants by a bigger set where the corresponding Lyapunov-like function is decreasing. Since this analysis is carried out using conditions similar to those given in Section III, it can be immediately generalized to make use of polynomial functions. The method is well-suited for robustness analysis with respect to nonparametric uncertainty, but unfortunately, although in principle parametric uncertainty can be handled in a similar fashion, it is not treated in a direct and efficient way.

Instead, in this section we present an analysis technique for handling parametric and dynamic uncertainty in a direct way, based on parameter dependent Lyapunov-like functions and multipliers. Computation of parameter dependent quadratic Lyapunov-like functions using LMIs had been previously difficult, since such functions are nonquadratic polynomials in the state and parameter variables. Using the sum of squares decomposition, computation of even higher degree functions becomes straightforward.

Recall the description of the continuous flow field introduced in Section II-A. Let the set of admissible parameters be given by

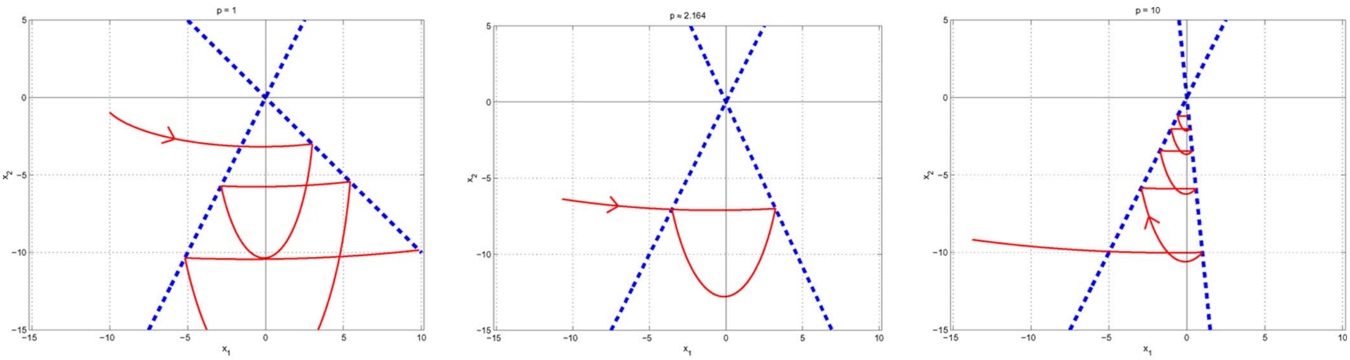
$$\mathcal{P} = \{p \in \mathbb{R}^m : r_{1k_1}(p) \geq 0, k_1 = 1, \ldots, K_1, \; r_{2k_2}(p) = 0, k_2 = 1, \ldots, K_2\} \tag{19}$$

and in the case they are time-varying, let the set $\mathcal{Q}$ be given by

$$\mathcal{Q} = \{q \in \mathbb{R}^m : s_{1k_3}(q) \geq 0, k_3 = 1, \ldots, K_3, \; s_{2k_4}(q) = 0, k_4 = 1, \ldots, K_4\} \tag{20}$$

for some polynomials $r_{1k_1}(p)$, $r_{2k_2}(p)$, $s_{1k_3}(q)$ and $s_{2k_4}(q)$. Furthermore, assume that the vector fields f_i and the polynomials describing the invariants (2), guards (3) and reset maps (4) are dependent on p. Theorems 4, 6, and 8 can be modified to accommodate parameter dependent Lyapunov functions and multipliers. For brevity, we only present the parameter dependent version of Theorem 8, for the case the parameters are time-varying.

Theorem 10: Consider a hybrid system in which $f_l(x, p)$ has unknown parameters $p \in \mathcal{P}$, where $\mathcal{P}$ is as in (19) so that their time-variation is in $\mathcal{Q}$, described by (20). Assume that there exist polynomials $V_l(x, p)$, $a_{lk}(x, p) \geq 0$, $\hat{a}_{lk}(x, p)$, $\tilde{a}_{lk}(x, p) \geq 0$, $b_{lk}(x, p, q) \geq 0$ $\hat{b}_{lk}(x, p, q)$, $\tilde{b}_{lk}(x, p, q) \geq 0$, $\check{b}_{lk}(x, p, q)$, $\breve{b}_{lk}(x, p, q) \geq 0$, $c_{ll'0}(x, x', p)$, $c_{ll'k}(x, x', p) \geq 0$, $\hat{c}_{ll'k}(x, x', p)$,



(a) $p = 1$ (b) $p \simeq 2.164$ (c) $p = 10$

Fig. 3. Trajectories of the system in Example 11 for different values of p. Dashed lines represent the guard sets.

$\tilde{c}_{ll'k}(x, x', p) \geq 0$, and $d_{ll'}(x, x')$ such that $V_i(0, p) = 0$ and conditions (21)–(23) are satisfied. Then the origin of the state space is robustly asymptotically stable with respect to the unknown parameters $p \in \mathcal{P}$ when they vary inside $\dot{p} \in \mathcal{Q}$.

$$V_l(x,p) - \sum_{k=1}^{m_{X_l}} a_{lk}(x,p) g_{lk}(x,p) - \sum_{k=1}^{K_1} \tilde{a}_{lk}(x,p) r_{1k}(p) - \sum_{k=1}^{K_2} \hat{a}_{lk}(x,p) r_{2k}(p) > 0 \quad \forall x \neq 0, i \in I, \tag{21}$$

$$\frac{\partial V_l(x,p)}{\partial x} f_l(x,p) + \frac{\partial V_l(x,p)}{\partial p} q + \sum_{k=1}^{m_{X_l}} b_{lk}(x,p,q) g_{lk}(x,p) + \sum_{k=1}^{K_1} \tilde{b}_{lk}(x,p,q) r_{1k}(p) + \sum_{k=1}^{K_2} \hat{b}_{lk}(x,p,q) r_{2k}(p) + \sum_{k=1}^{K_3} \breve{b}_{lk}(x,p,q) s_{1k}(q) + \sum_{k=1}^{K_4} \breve{b}_{lk}(x,p,q) s_{2k}(q) < 0 \quad \forall x \neq 0, i \in I, \tag{22}$$

$$V_{l'}(x',p) + c_{ll'0}(x,x',p) h_{ll'0}(x,p) + \sum_{k=1}^{m_{G(l,l')}} c_{ll'k}(x,x',p) h_{ll'k}(x,p) + d_{ll'}(x,x')(x' - \phi_{ll'}(x)) - V_l(x,p) + \sum_{k=1}^{K_1} \tilde{c}_{ll'k}(x,x',p) r_{1k}(p) + \sum_{k=1}^{K_2} \hat{c}_{ll'k}(x,x',p) r_{2k}(p) \leq 0 \quad \forall l, l'. \tag{23}$$

1) Example 11: Let us consider the hybrid system $\dot{x} = f_i(x)$, with vector fields

$$f_1(x) = \begin{bmatrix} -x_1 - 100(1+\delta_1(t))x_2 \\ 10x_1 - x_2 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} x_1 + 10(1+\delta_2(t))x_2 \\ -100x_1 + x_2 \end{bmatrix}$$

and guard sets

$$G(1,2) = \{x \in \mathbb{R}^2 | -px_1(t) - x_2(t) = 0\}, \quad G(2,1) = \{x \in \mathbb{R}^2 | 2x_1(t) - x_2(t) = 0\}.$$

TABLE II

RELATION BETWEEN THE DEGREE OF $V_{i,k}(x)$, $k = 1, 2$, AND THE VALUE OF C FOR WHICH ROBUST STABILITY CAN BE PROVEN. RECALL THAT THE SYSTEM IS STABLE FOR $p > 2.165$

Degree of $V_{i,k}(x)$	C
2	5.86
4	2.50
6	2.23
8	2.18
⋮	⋮

Notice the dependence of the first guard set on the unknown parameter $p \in \mathbb{R}$, and the time-varying uncertainties in the vector field, $\delta_1(t)$ and $\delta_2(t)$.

Let us first concentrate on the case $\delta_1(t) = \delta_2(t) = 0$. Obviously, stability of the system depends on the value of p. In this example, we have deliberately chosen a system with linear subsystems, so that robust stability of the system can also be analyzed in a purely analytical way for comparison purposes. By computing the flows of the subsystems when the system is initialized in subsystem 1, it can be proven that the system is stable for $p > 2.165$ and unstable for $p < 2.163$. At $p \simeq 2.164$ it exhibits a limit cycle (see Fig. 3). With parameter dependent Lyapunov-like functions of the form

$$V_i(x,p) = V_{i,1}(x) + pV_{i,2}(x) \tag{24}$$

robust stability of the system with respect to $p \in P = \{p : p - C \geq 0\}$, where C is a constant, can be proven. Using quadratic $V_{i,1}(x)$ and $V_{i,2}(x)$, we can prove robust stability for $C = 5.86$. Tighter robustness bounds can be obtained by increasing the degree of the Lyapunov-like functions, as depicted in Table II.

Now, let us fix $p = 8$, and consider the robust stability of the system under the presence of dynamic uncertainties $\delta_1(t)$ and $\delta_2(t)$. We assume that

$$|\delta_1(t)| \leq \rho_1, |\delta_2(t)| \leq \rho_2 \tag{25}$$

for some numbers $\rho_1, \rho_2 > 0$ (these inequalities define the set $\mathcal{P}$), and we consider different variation levels

$$|\dot{\delta}_1(t)| \leq v, |\dot{\delta}_2(t)| \leq v. \tag{26}$$

for some number $v \geq 0$. Fig. 4 shows robust stability regions that are obtained using the same quadratic Lyapunov function structure, for different variation rates v.

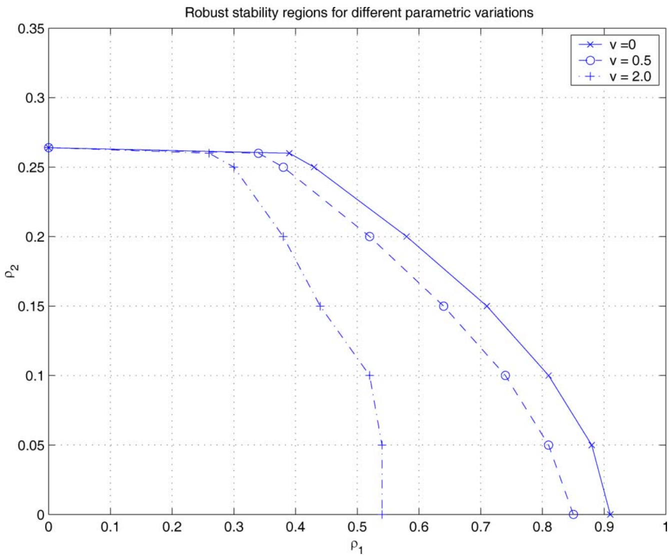


Fig. 4. Robust stability regions verified by a quadratic multiple Lyapunov function for Example (11) with $p = 8$. Here ρ_1, ρ_2 and v are given by (25)–(26).

V. Conclusion

A method for stability analysis of switched and hybrid systems has been presented. The method is based on polynomial and piecewise polynomial Lyapunov functions, whose computation can be efficiently performed using the sum of squares decomposition and semidefinite programming. Using this approach, higher degree Lyapunov functions can be constructed, thus reducing the conservatism of searching for only quadratic candidates. In the same way parametric uncertainty can be incorporated in the search. Several examples have been provided to illustrate the benefits of this approach.

At the present time, the largest system that we could analyze was a polynomial hybrid system with 10 continuous states and 6 locations, for which the safety properties were assessed [34]. It is important to note that the computation of Lyapunov functions is polynomial-time for a fixed order system, however limitations are imposed because of the size of the associated semidefinite programmes. Distributing the calculation may be possible in some cases. For example, in the case of multiple Lyapunov functions the computation can be done locally with appropriate communication between the locations where transition is possible. In the case of common Lyapunov function construction, we would need to impose some sort of synchronization condition so that at the end of the algorithm the same Lyapunov function is computed by all computers.

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