

Sampling Lissajous and Fourier Knots

Adam Boocheer*

University of Notre Dame

Jay Daigle*

Pomona College

Jim Hoste*

Pitzer College

Wenjing Zheng*

University of California, Berkeley

July 25, 2018

Abstract

A *Lissajous knot* is one that can be parameterized as

$$K(t) = (\cos(n_x t + \phi_x), \cos(n_y t + \phi_y), \cos(n_z t + \phi_z))$$

where the *frequencies* n_x, n_y , and n_z are relatively prime integers and the *phase shifts* ϕ_x, ϕ_y and ϕ_z are real numbers. Lissajous knots are highly symmetric, and for this reason, not all knots are Lissajous. We prove several theorems which allow us to place bounds on the number of Lissajous knot types with given frequencies and to efficiently sample all possible Lissajous knots with a given set of frequencies. In particular, we systematically tabulate all Lissajous knots with small frequencies and as a result substantially enlarge the tables of known Lissajous knots.

A *Fourier- (i, j, k) knot* is similar to a Lissajous knot except that the x, y and z coordinates are now each described by a sum of i, j and k cosine functions respectively. According to Lamm, every knot is a Fourier- $(1, 1, k)$ knot for some k . By randomly searching the set of Fourier- $(1, 1, 2)$ knots we find that all 2-bridge knots up to 14 crossings are either Lissajous or Fourier- $(1, 1, 2)$ knots. We show that all twist knots are Fourier- $(1, 1, 2)$ knots and give evidence suggesting that all torus knots are Fourier- $(1, 1, 2)$ knots.

As a result of our computer search, several knots with relatively small crossing numbers are identified as potential counterexamples to interesting conjectures.

*Supported by NSF REU grant DMS-0453284.

1 Introduction

A *Lissajous* knot K in \mathbb{R}^3 is a knot that has a parameterization $K(t) = (x(t), y(t), z(t))$ given by

$$\begin{aligned} x(t) &= \cos(n_x t + \phi_x) \\ y(t) &= \cos(n_y t + \phi_y) \\ z(t) &= \cos(n_z t + \phi_z) \end{aligned}$$

where $0 \leq t \leq 2\pi$, n_x, n_y , and n_z are integers, and $\phi_x, \phi_y, \phi_z \in \mathbb{R}$.

Lissajous knots were first studied in [1] where some of their elementary properties were established. Most notably, Lissajous knots enjoy a high degree of symmetry. In particular, if the three *frequencies* n_x, n_y and n_z (which must be pairwise relatively prime—see [1]) are all odd, then the knot is strongly plus amphicheiral. If one of the frequencies is even, then the knot is 2-periodic, with the additional property that it links its axis of rotation once. These symmetry properties imply (strictly) weaker properties such as the fact that the Alexander polynomial of a Lissajous knot must be a square mod 2, which in turn implies that its Arf invariant must be zero. See [1], [7] and [11] for details. Thus for example, the trefoil and figure eight knots are not Lissajous since their Arf invariants are one. In fact, “most” knots are not Lissajous.

To date it is unknown if every knot which is strongly plus amphicheiral or 2-periodic (and links its axis of rotation once) is Lissajous. Several knots with relatively few crossings exist which meet these symmetry requirements and yet are still unknown to be Lissajous or not. For example, according to [6] there are only three prime knots with 12 or less crossings which are strongly plus amphicheiral: 10a103 (10₉₉), 10a121 (10₁₂₃), and 12a427. Here we have given knot names in both the Dowker-Thistlethwaite ordering of the Hoste-Thistlethwaite-Weeks table [6] and, in parenthesis, the Rolfsen [14] ordering (for knots with 10 or less crossings). Symmetries of the knots in the Hoste-Thistlethwaite-Weeks table were computed using *SnapPea* as described in [6]. Of these three knots, only 10a103 (10₉₉) was previously reported as Lissajous. (See [10] and [11].) However we find 12a427 to be Lissajous. (See Section 5 of this paper.) This leaves open the case of 10a121. As a further example, there are exactly four 8-crossings knots which are 2-bridge, 2-periodic, and link their axis of rotation once. Despite our extensive searching (see Section 5) only one of these knots turned up as Lissajous (and it had already been reported as such in [10]). Whether the other three are Lissajous remains unknown.

Lissajous knots are a subset of the more general class of *Fourier* knots. A Fourier- (i, j, k) knot is one that can be parameterized as

$$\begin{aligned} x(t) &= A_{x,1} \cos(n_{x,1}t + \phi_{x,1}) + \dots + A_{x,i} \cos(n_{x,i}t + \phi_{x,i}) \\ y(t) &= A_{y,1} \cos(n_{y,1}t + \phi_{y,1}) + \dots + A_{y,j} \cos(n_{y,j}t + \phi_{y,j}) \\ z(t) &= A_{z,1} \cos(n_{z,1}t + \phi_{z,1}) + \dots + A_{z,k} \cos(n_{z,k}t + \phi_{z,k}). \end{aligned}$$

Because any function can be closely approximated by a sum of cosines, every knot is a Fourier knot for some (i, j, k) . But a remarkable theorem of Lamm [10] states that in fact every knot is a Fourier- $(1, 1, k)$ knot for some k . While k cannot equal one for all knots (these are the Lissajous knots, and not all knots are Lissajous) could k possibly be less than some universal bound M for all knots? This seems unlikely, with the more reasonable outcome being that k depends on the specific knot K . Yet no one has found a knot for which k must be bigger than two!

If K is a Fourier- $(1, 1, k)$ knot then its bridge number is less than or equal to the minimum of n_x and n_y . (The bridge number of a knot K can be defined as the smallest number of extrema on K with respect to a given direction in \mathbb{R}^3 , taken over all representations of K and with respect to all directions. See [2] or [14] for more details.) Moreover, Lamm’s proof is constructive and explicitly shows that if K has bridge number b , then K is a Fourier- $(1, 1, k)$ knot for some k and with $n_x = b$. This raises several interesting questions. For any knot K , when expressed as a Fourier- $(1, 1, k)$ knot, can the minimum values of n_x and k

be simultaneously realized? In particular, can a knot which is Lissajous and with bridge index b be realized as a Lissajous knot with $n_x = b$?

Let $\mathcal{L}(n_x, n_y, n_z)$ be the set of all Lissajous knots with frequencies n_x, n_y, n_z . (Throughout this paper we consider a knot and its mirror image to be equivalent.) One of the main goals of this paper is to investigate the set $\mathcal{L}(n_x, n_y, n_z)$. By a simple change of variables, $t \rightarrow t + c$, we may alter the phase shifts. Therefore we will assume that $\phi_x = 0$ in all that follows. This leaves the pair of parameters (ϕ_y, ϕ_z) which vary within the *phase torus* $[0, 2\pi] \times [0, 2\pi]$. In Section 2 we examine the phase torus and identify a finite number of regions in which the phase shifts must lie, with each region corresponding to a single knot type. We further show that a periodic pattern of knot types are produced as one traverses the phase torus. This allows us to prove

Theorem 1. *Let $|\mathcal{L}(n_x, n_y, n_z)|$ be the number of distinct Lissajous knots with frequencies (n_x, n_y, n_z) . Then*

$$|\mathcal{L}(n_x, n_y, n_z)| \leq 2n_x n_y.$$

If furthermore $n_x = 2$, then

$$|\mathcal{L}(2, n_y, n_z)| \leq 2n_y + 1.$$

There is also a periodicity that exists across frequencies and in Section 2 we also prove

Theorem 2. $\mathcal{L}(n_x, n_y, n_z) \subseteq \mathcal{L}(n_x, n_y, n_z + 2n_x n_y)$, with equality if $n_z \geq 2n_x n_y - n_y$.

Our analysis of the phase torus, together with these theorems allow us to efficiently sample (with the aid of a computer) all possible Lissajous knots having two of the three frequencies bounded. Even with relatively small frequencies, the three natural projections of a Lissajous knot into the three coordinate planes can have a large number of crossings. (The projection into the xy -plane has $2n_x n_y - n_x - n_y$ crossings.) With frequencies of 10 or more, diagrams with hundreds of crossings result and many, if not most, knot invariants are computationally out of reach. Thus it becomes extremely difficult to compare different Lissajous knots with large frequencies, or to try to locate them in existing knot tables. However, if one frequency is two, the knot is 2-bridge and even with hundreds of crossings it is relatively simple to compute the identifying fraction p/q by which 2-bridge knots are classified.

In Section 2 we recall basic facts about Lissajous knots and prove several theorems, including the two already mentioned, that will allow us to efficiently sample all Lissajous knots with two given frequencies. In Section 4 we then recall some basic facts about 2-bridge knots. Using these results we then report in Section 5 on our computer experiments. Theorems similar to those given in Section 2 but for Fourier- $(1, 1, k)$ knots would necessarily be much more complicated and we only begin the analysis of the phase torus for Fourier- $(1, 1, 2)$ knots in Section 3. Without the analogous results, we have not been able to rigorously sample Fourier knots. Instead, we have proceeded by two methods, either random sampling, or a sampling based on first forming a bitmap image of the phase torus and its singular curves. However, even without exhaustive sampling, our data show that all 2-bridge knots up to 14 crossings are Fourier- $(1, 1, k)$ knots with $k \leq 2$ and with $n_x = 2$.

This research was carried out at the Claremont College's REU program in the summer of 2006. The authors thank the National Science Foundation and the Claremont Colleges for their generous support.

2 The Phase Torus—Lissajous Knots

Suppose $K(t)$ is a Lissajous knot and consider its diagram in the xy -plane. Each crossing in this diagram corresponds to a double point in the xy -projection given by a pair of times (t_1, t_2) , where $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$. The following lemma is given in [7].

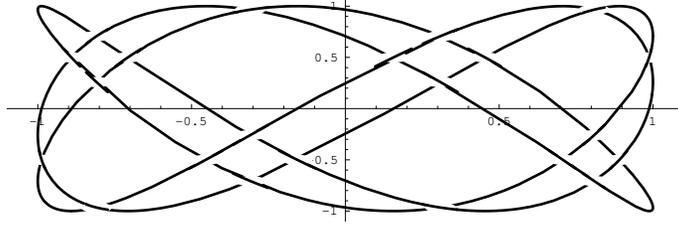


Figure 1: A Lissajous knot with frequencies $(3, 5, 7)$ and corresponding phase shifts $(0, \pi/4, \pi/12)$. The Type I crossings appear in two rows with five crossings each and the Type II crossings appear in four columns with three crossings each.

Lemma 3. *Let $K(t)$ be a Lissajous knot. There are two types of time pairs (t_1, t_2) that give double points in the xy -projection:*

$$\begin{aligned}
 \text{Type I:} \quad (t_1, t_2) &= \left(\left(-\frac{k}{n_x} + \frac{j}{n_y} \right) \pi - \frac{\phi_y}{n_y}, \left(\frac{k}{n_x} + \frac{j}{n_y} \right) \pi - \frac{\phi_y}{n_y} \right) \\
 1 \leq k \leq n_x - 1, \quad 1 + \lfloor \frac{n_y k + \frac{\phi_y}{\pi}}{n_x} \rfloor &\leq j \leq \lfloor 2n_y - \frac{n_y k + \frac{\phi_y}{\pi}}{n_x} \rfloor \\
 \text{Type II:} \quad (t_1, t_2) &= \left(\left(-\frac{k}{n_y} + \frac{j}{n_x} \right) \pi - \frac{\phi_x}{n_x}, \left(\frac{k}{n_y} + \frac{j}{n_x} \right) \pi - \frac{\phi_x}{n_x} \right) \\
 1 \leq k \leq n_y - 1, \quad 1 + \lfloor \frac{n_x k + \frac{\phi_x}{\pi}}{n_y} \rfloor &\leq j \leq \lfloor 2n_x - \frac{n_x k + \frac{\phi_x}{\pi}}{n_y} \rfloor
 \end{aligned}$$

There are $n_x n_y - n_y$ double points of Type I, and $n_x n_y - n_x$ double points of Type II.

Figure 1 shows a Lissajous knot with frequencies $(3, 5, 7)$ and corresponding phase shifts $(0, \pi/4, \pi/12)$. Since all the frequencies are odd, this knot is symmetric through the origin. It is not hard to show that in general, the Type I crossings line up in sets of size n_y on $n_x - 1$ horizontal lines, while the Type II crossings line up in sets of size n_x on $n_y - 1$ vertical lines. If $n_x = 2$ there is a single row of Type I crossings, all of which lie on the x -axis and $n_y - 1$ columns of Type II crossings with each column consisting of two crossings.

Not all phase shift pairs will generate curves that are knots. Assuming $\phi_x = 0$, the knot $K(t)$ will intersect itself, and thus fail to be a knot, exactly when the phase shifts satisfy

$$\phi_z = \frac{n_z}{n_y} \phi_y + l \frac{\pi}{n_y}, \text{ or} \quad (1)$$

$$\phi_z = l \frac{\pi}{n_x}, \text{ or} \quad (2)$$

$$\phi_y = l \frac{\pi}{n_x} \quad (3)$$

for some integer l . Crossings of Type I become singular precisely when Equation 1 holds; crossings of Type II when Equation 2 holds. When Equation 3 holds, the entire xy -projection degenerates to an arc. While this alone does not imply the knot has points of self-intersection, this is indeed the case. See [8, 1, 7] for more details. In Proposition 5, we specifically identify which crossings become singular as the phase shifts move across these lines.

The slanted, horizontal and vertical lines given in Equations 1–3 obviously divide the phase torus into regions with each region defining one knot type. Thus there are only a finite number of knot types possible for a given set of frequencies. There is, however, a great deal of repetition in knot types as one traverses the phase torus due to the periodicity of the cosine function. The following theorem describes a nice choice of “fundamental domain” on the phase torus to which we may restrict our attention.

Theorem 4. Any knot in $\mathcal{L}(n_x, n_y, n_z)$ can be represented with $\phi_x = 0$ and using some phase shift pair (ϕ_y, ϕ_z) in $[0, \frac{\pi}{n_x}] \times [0, \pi]$.

Proof. Define an equivalence relation \sim on the phase torus for $\mathcal{L}(n_x, n_y, n_z)$ by $(\phi_y, \phi_z) \sim (\phi'_y, \phi'_z)$ if the Lissajous knot with phase shifts $(0, \phi_y, \phi_z)$ is the same as the knot with phase shifts $(0, \phi'_y, \phi'_z)$, or its mirror image. Clearly

$$(\phi_y, \phi_z) \sim (\phi_y \pm \pi, \phi_z) \sim (\phi_y, \phi_z \pm \pi). \quad (4)$$

If $K \in \mathcal{L}(n_x, n_y, n_z)$, a change of variable $t \rightarrow t + \frac{\pi}{n_x}$ shows that K is also parameterized as

$$\begin{aligned} x &= -\cos(n_x t) \\ y &= \cos(n_y t + \phi_y + \frac{n_y \pi}{n_x}) \\ z &= \cos(n_z t + \phi_z + \frac{n_z \pi}{n_x}). \end{aligned}$$

Therefore we also have

$$(\phi_y, \phi_z) \sim (\phi_y + \frac{n_y \pi}{n_x}, \phi_z + \frac{n_z \pi}{n_x}). \quad (5)$$

Now since n_x and n_y are relatively prime, there are integers k and l with $0 \leq \phi_y + \frac{k n_y \pi}{n_x} - l \pi < \frac{\pi}{n_x}$. Repeatedly using (4) and (5) we obtain

$$(\phi_y, \phi_z) \sim (\phi_y + \frac{k n_y \pi}{n_x} - l \pi, \phi_z + \frac{k n_z \pi}{n_x})$$

The first coordinate is already in $[0, \frac{\pi}{n_x}]$; we can shift the second coordinate by multiples of π until it is in $[0, \pi]$. Thus an arbitrary point (ϕ_y, ϕ_z) is equivalent to some point in $[0, \frac{\pi}{n_x}] \times [0, \pi]$, as desired. \square

Figure 2 shows the fundamental domain on the phase torus for $\mathcal{L}(2, 3, 5)$. The singular lines divide the domain into regions with each region determining a single knot type. Since these knots are all 2-bridge, we identify each with its classifying fraction p/q .

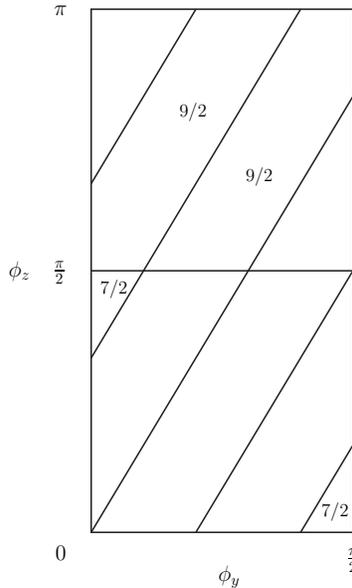


Figure 2: The fundamental domain of the phase torus for $\mathcal{L}(2, 3, 5)$. Each region defines a single 2-bridge knot which is identified by its classifying fraction p/q . Unlabeled regions define unknots.

Our next result specifically describes what happens as we cross a singular line of the type given in Equation 1 or 2.

Proposition 5. *Let K and K' be two Lissajous knots with frequencies (n_x, n_y, n_z) and phase shifts (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) respectively.*

1. *Suppose (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) lie in two adjacent regions of the phase torus separated by a diagonal line L given by $\phi_z = \frac{n_z}{n_y}\phi_y + l\frac{\pi}{n_y}$. Then K and K' differ by changing all Type I crossings with parameters (k, j) such that with $jn_z + l \equiv 0 \pmod{n_y}$. The number of such crossings is $n_x - 1$.*
2. *Suppose (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) lie in two adjacent regions of the phase torus separated by a horizontal line L given by $\phi_z = l\frac{\pi}{n_x}$. Then K and K' differ by changing all Type II crossings with parameters (k, j) such that $jn_z + l \equiv 0 \pmod{n_x}$. The number of such crossings is $n_y - 1$.*

Proof. If (t_1, t_2) is a Type I crossing with parameters (k, j) , then $z(t_1) = z(t_2)$ if and only if

$$\cos(n_z t_1 + \phi_z) = \cos(n_z t_2 + \phi_z)$$

which will occur if and only if

$$n_z(t_1 - t_2) = 2m\pi \quad \text{or} \quad n_z(t_1 + t_2) + 2\phi_z = 2m'\pi \tag{6}$$

for some integers m, m' . For Type I crossings,

$$t_1 - t_2 = -\frac{2k}{n_x}\pi \quad \text{and} \quad t_1 + t_2 = \frac{2j}{n_y}\pi - \frac{2\phi_y}{n_y}.$$

If (6) is to hold, then in the first case, we have

$$-\frac{2kn_z}{n_x}\pi = 2m\pi$$

which is equivalent to $-kn_z = mn_x$. This is impossible since n_x and n_z are relatively prime and $1 \leq k \leq n_x - 1$.

In the second case, we have

$$n_z \left(2\frac{j}{n_y}\pi - 2\frac{\phi_y}{n_y} \right) + 2\phi_z = 2m'\pi$$

which is equivalent to

$$\phi_z = \frac{n_z}{n_y}\phi_y + (m'n_y - jn_z)\frac{\pi}{n_y}.$$

Thus Type I crossings only become singular on lines of the form given in Equation 1 with $l = mn_y - jn_z$.

If $\phi_z = \frac{n_z}{n_x}\phi_y + l\frac{\pi}{n_y} + \varepsilon$ and $jn_z + l = mn_y$ for some integer m then it is straightforward to check that

$$z(t_1) - z(t_2) = (-1)^m 2 \sin \varepsilon \sin \frac{kn_z\pi}{n_x}.$$

Hence, as we move across the line L by letting ε go from a small positive value to a small negative value, the difference $z(t_1) - z(t_2)$ changes sign. Thus the Type I crossings with parameters (k, j) actually change from over to under or vice versa, rather than simply becoming singular and then “rebounding” to their original positions.

Finally, note that once l is fixed this does not necessarily uniquely determine j and thus the corresponding Type I crossing. If both $jn_z + l \equiv 0 \pmod{n_y}$ and $j'n_z + l \equiv 0 \pmod{n_y}$ then $j \equiv j' \pmod{n_y}$ since n_y and n_z are

relatively prime. If $n_x = 2$, then $k = 1$ and j lies in an interval of length n_y . Thus with $n_x = 2$ we have that j is uniquely determined by l and a single crossing changes as we move across L . But if $n_x > 2$ and $k = 1$ then j lies in an interval of length greater than n_y . Hence two admissible values, j and $j + n_y$, are possible. Using j , we must have $1 \leq k \leq \left\lfloor \frac{n_x}{n_y} \left(j - \frac{\phi_y}{\pi} \right) \right\rfloor$ while for $j + n_y$ we must have $1 \leq k \leq \left\lfloor \frac{n_x}{n_y} \left(-j + n_y + \frac{\phi_y}{\pi} \right) \right\rfloor$. Thus the total number of possible points (k, j) is $\left\lfloor \frac{n_x}{n_y} \left(j - \frac{\phi_y}{\pi} \right) \right\rfloor + \left\lfloor \frac{n_x}{n_y} \left(-j + n_y + \frac{\phi_y}{\pi} \right) \right\rfloor = n_x - 1$.

A similar discussion handles the Type II crossings. \square

Corollary 6. *Suppose K and K' are Lissajous knots with frequencies (n_x, n_y, n_z) and phase shifts which belong to regions separated by $2n_y$ singular lines of the type given in Equation 1. Then all Type I crossings are the same for both knots.*

Proof. From Proposition 5 we know that crossing the line $\phi_z = \frac{n_z}{n_y} \phi_y + l \frac{\pi}{n_y}$ changes exactly those Type I crossings with parameters (k, j) for which $j n_z + l \equiv 0 \pmod{n_y}$. Thus if we cross the singular line corresponding to l and then later cross the line corresponding to $l + n_y$ the same set of Type I crossings will first be changed and then changed back again. Hence, after crossing over $2n_y$ such lines all Type I crossings will be restored to their original position. \square

If $n_x = 2$ there is even more repetition due to additional symmetry as is shown in the following result.

Proposition 7. *Let K and K' be Lissajous knots with frequencies $(2, n_y, n_z)$ and phase shifts (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) respectively. If (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) are symmetric with respect to the point $(\pi/4, \pi/4)$ or the point $(\pi/4, 3\pi/4)$ then K and K' are equivalent.*

Proof. Suppose (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) are symmetric with respect to the point $(\pi/4, \pi/4)$. Then $\phi'_y = \pi/2 - \phi_y$ and $\phi'_z = \pi/2 - \phi_z$. Thus

$$\begin{aligned} K'(-t + \pi/2) &= (\cos(-2t + \pi), \cos(-n_y t + n_y \pi/2 + \pi/2 - \phi_y), \cos(-n_z t + n_z \pi/2 + \pi/2 - \phi_z)) \\ &= (-\cos(2t), (-1)^{(n_y+1)/2} \cos(n_y t + \phi_y), (-1)^{(n_z+1)/2} \cos(n_z t + \phi_z)) \end{aligned}$$

which is either $K(t)$ or its mirror image $\overline{K}(t)$.

The second case follows similarly. \square

We may now prove Theorem 1.

Proof of Theorem 1: The fundamental domain is divided into n_x “boxes” of the form $[0, \frac{\pi}{n_x}] \times [k \frac{\pi}{n_x}, (k+1) \frac{\pi}{n_x}]$ for $0 \leq k \leq n_x - 1$. Within each box all the knots have the same Type II crossings and hence by Corollary 6 there are at most $2n_y$ different knot types in that box. Since there are n_x boxes we obtain at most $2n_x n_y$ different knots.

If $n_x = 2$ there is the additional rotational symmetry in each box given by Proposition 7. The center of each box either lies on a slanted singular line, or midway between two such lines. Moreover, one of the two boxes will be one way and the other box will be the other way. There are at most n_y knot types in the box where the center of the box lies on a singular line, and there are at most $n_y + 1$ knot types in the box otherwise. Thus there are at most $2n_y + 1$ knot types in total. \square

Our results thus far allow us to efficiently sample all Lissajous knots with a given set of frequencies (n_x, n_y, n_z) . We can easily pick one set of phase shifts from each region on the phase torus and Corollary 6, and Proposition 7 in the case when $n_x = 2$, allows us to further restrict the regions that we must sample. However, once n_x, n_y , and ϕ_y are given, the xy -projection has been fixed and it is natural to ask if all possible choices for n_z are necessary. Theorem 2, which is stated in the introduction, shows that in fact, only a finite number of values for n_z are needed to produce all possible knots.

Proof of Theorem 2: Suppose that $K \in \mathcal{L}(n_x, n_y, n_z)$, $K' \in \mathcal{L}(n_x, n_y, n_z + 2n_x n_y)$ and that both knots have the same phase shifts. We will show first that each knot has its Type II crossings arranged the same way.

Let (t_1, t_2) be a Type II crossing with parameters (k, j) and let

$$\begin{aligned} \Delta_{II}(n_x, n_y, n_z, \phi_z, k, j) &= \cos(n_z t_1 + \phi_z) - \cos(n_z t_2 + \phi_z) \\ &= 2 \sin \left(n_z \left(\frac{t_1 + t_2}{2} \right) + \phi_z \right) \sin \left(n_z \left(\frac{t_1 - t_2}{2} \right) \right) \\ &= -2 \sin \left(n_z \frac{j\pi}{n_x} + \phi_z \right) \sin \left(n_z \frac{k\pi}{n_y} \right) \end{aligned}$$

be the height difference between the two points on the knot directly above the crossing.

It is easy to verify that

$$\Delta_{II}(n_x, n_y, n_z, \phi_z, k, j) = \Delta_{II}(n_x + 2n_x n_y, n_y, n_z, \phi_z, k, j) \quad \text{for all } k, j.$$

Thus if n_z is increased by $2n_x n_y$, not only do all Type II crossings remain unchanged, they each maintain the same height difference between upper and lower strand.

We now shift our focus to Type I crossings. Let K have phase shifts $(\phi_y, \frac{n_z}{n_y} \phi_y - \varepsilon)$ and choose ε small enough so that K corresponds to the region just below the singular line $\phi_z = \frac{n_z}{n_y} \phi_y$. Let K' correspond to the “same” region, that is, let K' have phase shifts $(\phi_y, \frac{n_z + 2n_x n_y}{n_y} \phi_y - \varepsilon)$. As before, let (t_1, t_2) be a Type I crossing with parameters (k, j) and let

$$\begin{aligned} \Delta_I(n_x, n_y, n_z, \phi_y, \phi_z, k, j) &= \cos(n_z t_1 + \phi_z) - \cos(n_z t_2 + \phi_z) \\ &= 2 \sin \left(n_z \left(\frac{t_1 + t_2}{2} \right) + \phi_z \right) \sin \left(n_z \left(\frac{t_1 - t_2}{2} \right) \right) \\ &= -2 \sin \left(n_z \frac{j\pi}{n_y} - \frac{n_z \phi_y}{n_y} + \phi_z \right) \sin \left(n_z \frac{k\pi}{n_x} \right) \end{aligned}$$

be the height difference between the two points on the knot directly above the crossing. It is easy to check that

$$\Delta(n_x, n_y, n_z, \frac{n_z}{n_y} \phi_y - \varepsilon, k, j) = \Delta(n_x, n_y, n_z + 2n_x n_y, \frac{n_z + 2n_x n_y}{n_y} \phi_y - \varepsilon, k, j).$$

Thus K and K' are the same knot since both the Type I and Type II crossings are arranged the same way in each. If the phase shifts for K are now changed by moving into an adjacent region, and if the phase shifts for K' are changed in the same way, then the same set of crossings is changed for both K and K' and hence K and K' remain the same knot. Therefore

$$\mathcal{L}(n_x, n_y, n_z) \subseteq \mathcal{L}(n_x, n_y, n_z + 2n_x n_y). \quad (7)$$

According to Corollary 6, the pattern of knot types in each square $[0, \pi/n_x] \times [k\pi/n_x, (k+1)\pi/n_x]$, as we move from the upper left corner to the lower right corner, is periodic with period $2n_y$. Thus if each box contains

at least $2n_y$ regions the inclusion in (7) is equality. Now the distance between successive singular lines of the type given in Equation 1 is $\frac{\pi}{\sqrt{n_y^2+n_z^2}}$ and the distance between lines of slope $\frac{n_z}{n_y}$ containing opposite corners of the square is $\frac{(n_2+n_3)\pi}{n_1\sqrt{n_y^2+n_z^2}}$. Thus there are at least $\left\lfloor \frac{n_y+n_z}{n_x} \right\rfloor$ regions in each square. Hence the inclusion in (7) is equality if $2n_y \leq \left\lfloor \frac{n_y+n_z}{n_x} \right\rfloor$. It is easy to check that this is true if $n_z \geq 2n_x n_y - n_y$. \square

Definition 1. For relatively prime integers n_x and n_y let $\mathcal{L}(n_x, n_y) = \bigcup_{n_z \in \mathbb{N}} \mathcal{L}(n_x, n_y, n_z)$.

Theorem 8. Let n_x, n_y be relatively prime integers. Then

$$|\mathcal{L}(n_x, n_y)| \leq 4n_x n_y (n_x - 1)(n_y - 1).$$

If furthermore $n_x = 2$, then

$$|\mathcal{L}(2, n_y)| \leq 2(n_y - 1)(2n_y + 1).$$

Proof. For fixed n_x, n_y we need only consider $2n_x n_y$ consecutive values of n_z . To count the number of values that are relatively prime to both n_x and n_y we first subtract $2n_y$ multiples of n_x that lie in that range as well as $2n_x$ multiples of n_y and then add back in the 2 multiples of $n_x n_y$. Thus the number of possible values of n_z is bounded above by $2n_x n_y - 2n_x - 2n_y + 2 = 2(n_x - 1)(n_y - 1)$ and applying Theorem 1 yields the result. \square

3 The Phase Torus—Fourier-(1, 1, 2) Knots

The phase torus of a Fourier-(1, 1, k) is, in general, $k + 2$ dimensional although we may set any one phase shift equal to zero and drop to a $k + 1$ dimension space. If $k = 2$, $\phi_x = 0$ and we fix ϕ_y , then we may again think of the 2-dimensional phase torus associated to the pair $(\phi_{z,1}, \phi_{z,2})$. The singular curves are now much more complicated than in the Lissajous case, but can still be carefully described.

Suppose K is a Fourier-(1, 1, 2) knot with parameterization

$$\begin{aligned} x(t) &= \cos(n_x t) \\ y(t) &= \cos(n_y t + \phi_y) \\ z(t) &= \cos(n_{z,1} t + \phi_{z,1}) + A_{z,2} \cos(n_{z,2} t + \phi_{z,2}). \end{aligned} \tag{8}$$

Note that by rescaling we may assume that three of the four amplitudes are 1.

In the Lissajous case, we require that the three frequencies be pairwise relatively prime. The same proof (see [1]) can be used now to conclude that the three integers n_x, n_y and $\gcd(n_{z,1}, n_{z,2})$ must be pairwise relatively prime. This rules out several of the 16 cases that arise by considering all possible parities for the frequencies. Some of the remaining cases still give rise to highly symmetric knots, such as when all the frequencies are odd. In this case the knot is strongly plus amphicheiral just as in the Lissajous setting. But some of the parity cases produce knots with no apparent symmetry, suggesting that the set of Fourier-(1, 1, 2) is much richer than the set of Lissajous knots.

We will not undertake an exhaustive analysis of the phase torus of Fourier-(1, 1, 2) knots. Instead we offer a glimpse of the situation in the following Proposition which could be stated much more precisely. In particular, the constants in the statement of the proposition all depend on the pair of indices (k, j) associated to either a Type I or II crossing. The interested reader can easily determine the constants by going through the details of the proof. Results analogous to Propositions 5 and 7 seem much harder.

Proposition 9. *Let K be a Fourier-(1, 1, 2) knot with parameterization as given in 8. Then the singular curves on the phase torus are of four possible types:*

1. Lines of the form $\phi_{z,2} = c$,
2. Lines of the form $\phi_{z,1} = c$,
3. Lines of the form $\phi_{z,2} = \pm\phi_{z,1} + c$,
4. Curves with the shape of $\sin(\phi_{z,2}) = c \sin(\phi_{z,1})$

where c is a constant that, in the last case, is neither 0 nor ± 1 .

Proof. Suppose that (t_1, t_2) are a pair of times that produce a double point in the xy -projection of K . Using the identity $\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$ we obtain

$$z(t_1) - z(t_2) = -2 \sin\left(n_{z,1} \frac{t_1 + t_2}{2} + \phi_{z,1}\right) \sin\left(n_{z,1} \frac{t_1 - t_2}{2}\right) - 2A \sin\left(n_{z,2} \frac{t_1 + t_2}{2} + \phi_{z,2}\right) \sin\left(n_{z,2} \frac{t_1 - t_2}{2}\right).$$

We are interested in those values of $\phi_{z,1}$ and $\phi_{z,2}$ that make this difference zero.

Suppose now that (t_1, t_2) define a Type II crossing with indices (k, j) . Then

$$\begin{aligned} \frac{t_1 + t_2}{2} &= \frac{j\pi}{n_x} \\ \frac{t_1 - t_2}{2} &= -\frac{k\pi}{n_y} \end{aligned}$$

and the crossing is singular if

$$\sin\left(\frac{n_{z,1}j\pi}{n_x} + \phi_{z,1}\right) \sin\left(\frac{n_{z,1}k\pi}{n_y}\right) = -A \sin\left(\frac{n_{z,2}j\pi}{n_x} + \phi_{z,2}\right) \sin\left(\frac{n_{z,2}k\pi}{n_y}\right). \quad (9)$$

We are now led to several cases.

Case I: $n_y | n_{z,1}k$

If n_y divides $n_{z,1}$ then we must have that $\sin\left(\frac{n_{z,2}j\pi}{n_x} + \phi_{z,2}\right) = 0$ since $k < n_y$ and $n_y, n_{z,1}$ and $n_{z,2}$ cannot have a common factor. This means that

$$\phi_{z,2} = m\pi - \frac{n_{z,2}j\pi}{n_x}$$

for some integer m .

Case II: $n_y | n_{z,2}k$

This is similar to Case I leading to

$$\phi_{z,1} = m\pi - \frac{n_{z,1}j\pi}{n_x}$$

for some integer m .

If the first two cases do not occur, then we may rewrite Equation 9 as

$$\sin\left(\frac{n_{z,1}j\pi}{n_x} + \phi_{z,1}\right) = C \sin\left(\frac{n_{z,2}j\pi}{n_x} + \phi_{z,2}\right)$$

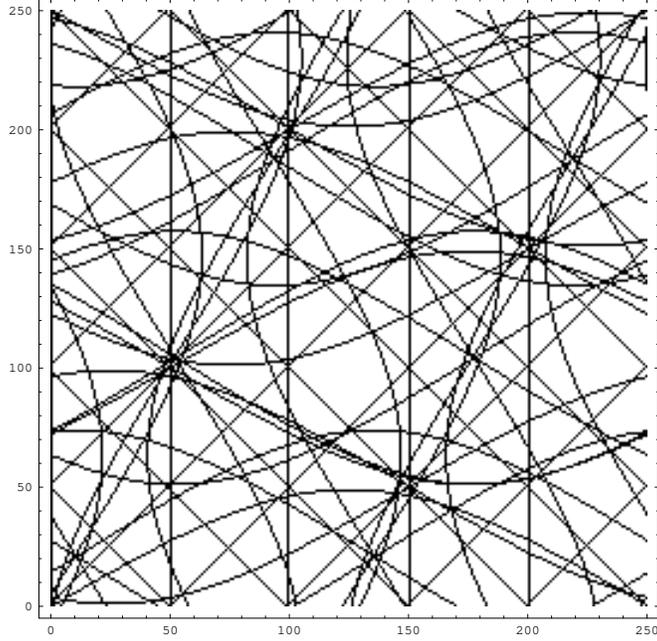


Figure 3: The phase torus for the Fourier-(1, 1, 2) knot with $n_x = 5, n_y = 6, n_{z,1} = 1, n_{z,2} = 2, \phi_x = 0, \phi_y = \pi/4$ and $A_{z,1} = 1$, shown for $0 \leq \phi_{z,1} \leq \pi$ and $0 \leq \phi_{z,2} \leq \pi$.

where

$$C = -A \sin\left(\frac{n_{z,2}k\pi}{n_y}\right) / \sin\left(\frac{n_{z,1}k\pi}{n_y}\right).$$

Case III: $|C| = 1$

In this case we must have

$$\left(\frac{n_{z,1}j\pi}{n_x} + \phi_{z,1}\right) \pm \left(\frac{n_{z,2}j\pi}{n_x} + \phi_{z,2}\right) = m\pi \quad (10)$$

for some integer m , where the parity of m depends on the sign of C and whether we are forming a sum or difference in Equation 10. Thus $\phi_{z,2} = \pm\phi_{z,1} + c$ for some constant c .

Case IV: $|C| \neq 1$

In this case we are left with a translate of the curve

$$\sin(\phi_{z,1}) = C \sin(\phi_{z,2}).$$

This is an interesting curve which, at first glance, appears much like a sine curve. It is oriented either vertically or horizontally depending on the value of $|C|$.

The analysis of a Type I crossing is similar and is left to the reader. □

In Figure 3 we give an example showing a 250 pixel by 250 pixel bitmap image of the phase torus for a specific set of parameters. Even with relatively small frequencies, one can begin to appreciate the difficulty of systematically sampling each region of the phase torus for an arbitrary Fourier-(1, 1, 2) knot.

4 2-Bridge Knots

Every 2-bridge knot can be classified by a pair of relatively prime integers (p, q) such that p is odd and $0 < q < p$. We will often write the pair (p, q) as the fraction p/q . If $K_{p/q}$ and $K_{p'/q'}$ are two 2-bridge knots with corresponding fractions p/q and p'/q' then they are equivalent knots if and only if $p = p'$ and $\pm q'q^{\pm 1} \equiv 1 \pmod{p}$. The reader is referred to [2] for details.

If K is a Fourier $(1, 1, 2)$ knot with $n_x = 2$ then K is a 2-bridge knot. We may recover the fraction p/q from the Lissajous projection in the xy -plane as follows. This projection is a 4-plat diagram. As we move in the x -direction from left to right we see a single Type I crossings on the x -axis, then a pair of Type II crossings which are symmetric with respect to the x -axis, then another Type I crossing on the x -axis, and so on. Let η_1, η_2, \dots be the signs of the Type I crossings from left to right along the x -axis. Let $\{\varepsilon_1^1, \varepsilon_1^2\}, \{\varepsilon_2^1, \varepsilon_2^2\}, \dots$ be the signs of the pairs of Type II crossings from left to right. Proceeding in a fashion similar to that given on pages 300–303 in [14], we obtain that p/q is given by the continued fraction

$$p/q = [\eta_1, \varepsilon_1^1 + \varepsilon_1^2, \eta_2, \varepsilon_2^1 + \varepsilon_2^2, \dots, \eta_{n_y}] = \eta_1 + \frac{1}{\varepsilon_1^1 + \varepsilon_1^2 + \frac{1}{\eta_2 + \dots + \frac{1}{\eta_{n_y}}}} \quad (11)$$

Note that if K is Lissajous then it is rotationally symmetric with respect to the x -axis and each pair of Type II crossings has the same sign. In this case each $\varepsilon_i^1 + \varepsilon_i^2$ can be replaced with $2\varepsilon_i^1$. Using this formula, it is easy to determine the 2-bridge knot given by a Fourier $(1, 1, 2)$ representation with $n_x = 2$. Hence, when we sample Lissajous and Fourier $(1, 1, 2)$ knots with $n_x = 2$, even if we obtain knots with hundreds of crossings, it is a simple matter to distinguish them.

Since every Lissajous knot with $n_x = 2$ is 2-bridge, a good question is: What 2-bridge knots are Lissajous with $n_x = 2$? As mentioned in the Introduction, every Lissajous knot is either strongly plus amphicheiral, or 2-periodic and linking its axis of rotation once. It is known that a 2-bridge knot cannot be strongly plus amphicheiral [4]. It is also known (and will be shown below) that every 2-bridge knot is 2-periodic, but may or may not link its axis of rotation once. The following theorem makes it easy to identify which 2-bridge knots *might* be Lissajous.

Theorem 10. *Let K be a 2-bridge knot. Then the following are equivalent.*

- 1) K has a symmetry of period 2 with axis A such that A is disjoint from K and $|lk(A, K)| = 1$
- 2) $\Delta_K(t)$ is a square mod 2
- 3) $\Delta_K(t) \equiv 1 \pmod{2}$.

Proof. As already mentioned in the introduction, it follows from a result of Murasugi (see [12]) that 1) implies 2) and clearly 3) implies 2). We must show that 2) implies both 1) and 3).

Suppose K is a 2-bridge knot given by the pair of relatively prime integers (p, q) with p odd and $0 < q < p$. There is a unique continued fraction expansion

$$p/q = [2a_1, -2a_2, \dots, (-1)^{n+1}2a_n]$$

where each of the partial quotients $(-1)^{i+1}2a_i$ is even. Corresponding to this expansion is a Seifert surface made from plumbing together twisted bands as shown in Figure 4. Notice that the Seifert surface, and hence K is rotationally symmetric around the axis A . Thus every 2-bridge knot has a symmetry of period 2 with

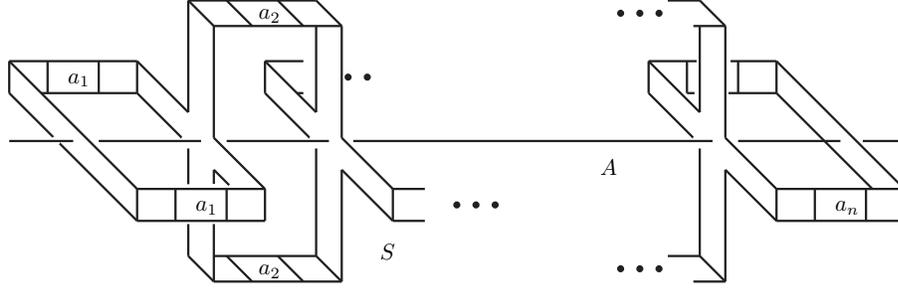


Figure 4: The Seifert surface S and axis A for the 2-bridge knot $K_{p/q}$. Each a_i represents a_i right handed half-twists in the band.

axis disjoint from the knot. However, the linking number of A and K need not be ± 1 in general. From the plumbing picture we also see that the number of bands, n , must be even in order to get a knot. If n is odd we obtain a 2-bridge link (of two components).

The axis A meets the Seifert surface S transversely in $n + 1$ points and we may compute the linking number of A and K by counting the signed intersection points. Let ϵ_i be the sign of the intersection point that occurs between band i and $i + 1$. Let ϵ_0 be the sign of the left-most intersection point in the figure and choose orientations so that $\epsilon_0 = 1$. It is easy to see that $\epsilon_{i+1} = \epsilon_i$ if a_i is odd and $\epsilon_{i+1} = -\epsilon_i$ if a_i is even. Thus the sequence $\{a_1, a_2, \dots, a_n\}$ determines the sequence $\{\epsilon_0, \epsilon_1, \dots, \epsilon_n\}$ which in turn determines the linking number between A and K .

From the Seifert surface we may obtain the Seifert matrix V and compute the Conway polynomial $\nabla(z) = \det(t^{-1/2}V - t^{1/2}V^T)$. It is a straightforward calculation to show that

$$\nabla_K(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a_1z & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -a_2z & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -a_nz & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(See page 207 of [3].)

If $n = 2$ the Conway polynomial is $\nabla(z) = 1 + a_1a_2z^2$ and the Alexander polynomial $\Delta(t) = \nabla(t^{1/2} - t^{1/2}) = a_1a_2t^{-1} + (1 - 2a_1a_2) + a_1a_2t$. In general, an Alexander polynomial of the form

$$\Delta(t) = b_0 + b_1(t + t^{-1}) + b_2(t^2 + t^{-2}) + \cdots + b_m(t^m + t^{-m})$$

is a square mod 2 if and only if $b_{2k+1} \equiv 0 \pmod{2}$ for $k = 0, 1, \dots$. Thus $a_1a_2t^{-1} + (1 - 2a_1a_2) + a_1a_2t$ is a square mod 2 if and only if at least one of a_1 or a_2 is even. But if this is the case it follows that $\sum \epsilon_i = \pm 1$. It is also true that $a_1a_2t^{-1} + (1 - 2a_1a_2) + a_1a_2t$ is a square mod 2 if and only if it is equal to 1 mod 2.

Suppose now that $n > 2$ and that K is a knot with $\Delta_K(t)$ a square mod 2. The first thing to show is that a_i is even for at least one value of i . If a_i were odd for every i , then replace each a_i with -1 . This does not change any $a_i \pmod{2}$, and hence does not change $\nabla_K(z) \pmod{2}$ or $\Delta_K(t) \pmod{2}$. But if $a_i = -1$ for all i , we can prove by induction on n that

$$\Delta(t) = 1 - t + t^2 - t^3 + \cdots + t^n$$

and it follows that $\Delta(t)$ is not a square mod 2. Thus at least one a_i is even. Replacing this a_i with zero transforms K into a knot J with the same Alexander polynomial mod 2 but with a Seifert surface having two fewer bands. Proceeding by induction on the number of bands, we have that $\text{lk}(J, A) = \pm 1$. But now because a_i is even, K must also link A once. If we begin instead with the assumption that $\Delta_K(t) \equiv 1 \pmod{2}$, the same argument will work. \square

n_y	$ \mathcal{L}(2, n_y) $						
3	3	29	645	55	1854	81	3761
5	11	31	737	57	1727	83	5805
7	28	33	533	59	2859	85	4654
9	37	35	684	61	3062	87	4195
11	78	37	1075	63	1946	89	6707
13	109	39	772	65	2639	91	5647
15	93	41	1339	67	3708	93	4805
17	203	43	1473	69	2593	95	5892
19	258	45	904	71	4191	97	7984
21	195	47	1782	73	4433	99	5208
23	390	49	1688	75	2584	101	8699
25	390	51	1365	77	3933	103	9036
27	387	53	2287	79	5248	105	4425

Table 1: The number of distinct Lissajous knots with $n_x = 2$ as a function of n_y .

5 Sampling Lissajous and Fourier Knots

Using the results of Sections 2 we are now in a position to efficiently sample Lissajous knots. In the case where $n_x = 2$ we obtain 2-bridge knots and can take advantage of this to compare knots in our sample. For the more general case of Fourier knots, we have not carried out a complete analysis of the phase torus, a task that seems much more difficult. Hence, we have not attempted to rigorously sample Fourier knots, but instead have relied on two methods, either random sampling or an algorithm which first “draws” a bitmap image of the phase torus (as in Figure 3) and then picks one point from each “white” region. This latter approach is fraught with difficulty since, for example, some white regions may be smaller than a single pixel and be missed. Our samples naturally fall into four cases which we describe in turn in this section.

5.1 Lissajous Knots with $2 = n_x < n_y < n_z$

We have determined all knots in $\mathcal{L}(2, n_y)$ for $3 \leq n_y \leq 105$. For a given value of n_y we let n_z run from $3n_y + 2$ to $7n_y$. These values of n_z are sufficient to guarantee that we obtain all possible knots in $\mathcal{L}(2, n_y)$. Since each of these knots is 2-bridge we were able to use Equation 11 to identify the associated pair (p, q) and thus compare knots in the output. The total number of knots in $\mathcal{L}(2, n_y)$ is given in Table 1 for each value of n_y . It is interesting to compare these numbers with the upper bound given by Theorem 8. Depending on n_y , the actual number of knots found is roughly between 5 and 10 per cent of the upper bound. The discrepancy is almost certainly due to the presence of huge numbers of unknots. The xy -projection of a Lissajous knot with $n_x = 2$ and $n_y = 99$ has $(2)(2)(99) - 2 - 99 = 295$ crossings, and knots in $\mathcal{L}(2, 99)$ have crossing numbers ranging from 5 to 293. Of course the bound of 78008 given by Theorem 8 for $n_x = 2$ and $n_y = 99$ is well below the upper bound of 2^{295} obtained by considering all possible crossing arrangements!

The total number of knots in Table 1 is 135061, far too many to describe one by one. However, in Tables 4–7 we list all knots in $\mathcal{L}(2, n_y)$, grouped by crossing number, for $3 \leq n_y \leq 15$. Several interesting things can be seen in these tables. The same knot often appears in $\mathcal{L}(2, n_y)$ for many different values of n_y . For example $K_{7/2}$ (which is the twist knot 5_2 in [14]) appears in every column of Table 4. In fact, $K_{7/2} \in \mathcal{L}(2, n_y)$ for $3 \leq n_y \leq 105$. This is also true for $K_{9/2}$. The knot $K_{15/4}$ first appears for $n_y = 3$, misses a few values of n_y , and then is contained in $\mathcal{L}(2, n_y)$ for $23 \leq n_y \leq 105$. Similar patterns hold for the other small-crossing knots suggesting that if $K \in \mathcal{L}(2, n_y)$ for some n_y then there exists N such that $K \in \mathcal{L}(2, n_y)$ for all $n_y \geq N$. A second observation is that several small crossings knots are already conspicuously absent. In particular,

there are exactly four 8-crossings knots with Alexander polynomial congruent to 1 mod 2 (and hence possibly Lissajous). These are $K_{17/4}$, $K_{23/7}$, $K_{25/9}$ and $K_{31/12}$, only one of which, $K_{31/12}$, appears to be Lissajous. While Tables 4–7 display only a small fraction of our total sample, it is in fact true that the other three 8-crossing knots do not appear for any n_y up to 105.

Question 1. *Does there exist a 2-bridge knot K with $\Delta_K(t) \equiv 1 \pmod{2}$ that is not Lissajous (with or without one frequency equal to 2)? In particular, are any of the 8-crossing 2-bridge knots $K_{17/4}$, $K_{23/7}$ or $K_{25/9}$ Lissajous?*

In Table 2 we list the numbers of 2-bridge knots, 2-bridge knots with Alexander polynomial congruent to 1 mod 2, and finally, the number of these that are Lissajous knots with $n_x = 2$ and $3 \leq n_y \leq 105$. The table has entries for each crossing number from 3 to 16. Very quickly we see that many 2-bridge knots with the required symmetry are not Lissajous, at least not with $n_x = 2$ and $3 \leq n_y \leq 105$. It seems unlikely that choosing $n_y > 105$ will yield more 2-bridge knots in the 3–16 crossing range. On the other hand, perhaps letting the even frequency be more than 2 will yield more 2-bridge knots with small crossing number. We examine this further in Section 5.2.

	crossing number													
	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2-bridge	1	1	2	3	7	12	24	45	91	176	352	693	1387	2752
$\Delta(t) \equiv 1$	0	0	1	1	2	4	8	13	26	51	97	185	365	705
$\mathcal{L}(2, n_y)$	0	0	1	1	2	1	3	4	8	5	9	7	15	15

Table 2: The number of 2-bridge knots, 2-bridge knots with Alexander polynomial congruent to 1 mod 2, and the number of these that are Lissajous with $n_x = 2$ and $3 \leq n_y \leq 105$, as a function of crossing number.

In Tables 8–9 we list all 2-bridge knots with crossings from 3 to 16 which are Lissajous knots with $n_x = 2$ and $3 \leq n_y \leq 105$. Here the knot name, as defined in [6] and used in *Knotscape* [5] appears in the first column. Following that we give the 2-bridge defining fraction p/q and then the frequency and phase shift data. For each knot, the given value of n_y is minimal. However, since our search let n_z run from $3n_y + 2$ to $7n_y$, it might be possible for a given knot to be represented with a smaller value of n_z .

As a check against errors, we took all the 2-bridge knots in the data set that have Lissajous diagrams with less than 50 crossings (the built-in limit for *Knotscape*) and crossing number less than 17, and looked them up in the *Knotscape* table of knots in two different ways. First we converted their Lissajous diagrams to Dowker-Thistlethwaite code (the input format for *Knotscape*) and then used the “Locate in Table” feature. Next we converted the defining fraction p/q into DT code and again used the “Locate in Table” routine. Happily, the results matched.

5.2 Lissajous Knots with $2 < n_x < n_y < n_z$

Our goal in this section is to simply find as many Lissajous knots in the 3–16 crossing range as we can. We may still use the results of Section 2 to efficiently sample Lissajous knots with all frequencies greater than 2, but it is more difficult to tabulate the output. This is because even with relatively small frequencies, very large crossing number knots can result, and we can no longer use the classification of 2-bridge knots to sort them out. Therefore, we limited ourselves to producing diagrams with at most 49 crossings, the limit of what can be input to *Knotscape*. Assuming that $2 < n_x < n_y$, and that $\gcd(n_x, n_y) = 1$, we are left with the following (n_x, n_y) pairs:

$$\{(3, 4), (3, 5), (3, 7), (3, 8), (3, 10), (4, 5), (4, 7), (5, 6)\}.$$

For each of these pairs we let n_z run from $2n_xn_y - n_x - n_y$ to $4n_xn_y - n_x - n_y - 1$, a range sufficient to produce all possible Lissajous knots. We obtained a total of 6352 knots of which *Knotscape* identified 1428 as unknots. The remaining 4924 knots fell into four categories:

1. knots identified as composites by *Knotscape*,
2. knots which *Knotscape* located in the Hoste-Thistlethwaite-Weeks table,
3. knots which *Knotscape* simplified to alternating projections with more than 16 crossings, and
4. knots which *Knotscape* simplified to nonalternating projections with more than 16 crossings.

In Table 3 and Tables 10–11 we list all knots in the first two categories. We note that while *Knotscape* can identify a knot as a composite, it identifies the summands only up to mirror image. In order to properly identify the composites in Table 3 we compared their Jones polynomials to the Jones polynomials of all possible composites using the given summands or their mirror images in all possible ways.

knot	n_x	n_y	n_z	ϕ_x	ϕ_y	ϕ_z
$3a1\#3a1$	3	4	23	0	0.25210	1.84229
$3a1\#\overline{3a1}$	3	5	29	0	0.23099	2.91059
$5a1\#5a1$	3	7	50	0	0.50522	1.58916
$5a1\#\overline{5a1}$	3	5	29	0	0.26179	1.83259
$6a1\#6a1$	4	5	37	0	0.18699	2.95459
$6a3\#6a3$	3	8	47	0	0.23799	0.80919
$6a3\#\overline{6a3}$	3	5	29	0	0.29259	0.75459
$3a1\#3a1\#5a1$	4	5	39	0	0.16064	2.19554
$3a1\#3a1\#\overline{5a1}$	4	7	55	0	0.13934	2.21684
$3a1\#3a1\#8a2$	5	6	59	0	0.11116	2.40211
$5a1\#5a1\#\overline{5a1}$	4	7	55	0	0.15201	1.41878
$6a3\#6a3\#\overline{6a3}$	4	7	55	0	0.16468	0.62071
$3a1\#3a1\#3a1\#3a1$	5	6	59	0	0.10149	1.78345

Table 3: Small-crossing composite Lissajous knots. A bar over a knot name indicates mirror image. Knot names are as in *Knotscape*.

The third category cannot include knots in the Hoste-Thistlethwaite-Weeks table and we make no attempt to list them here. The fourth category *might* have included knots with 16 or less crossings that *Knotscape* simply failed to simplify correctly. To investigate this we first computed the Jones polynomial of each knot and eliminated knots whose Jones polynomial had a span of 17 or more. (Recall that the crossing number of a knot is bounded below by the span of the Jones polynomial.) This left a total of 78 knots. Of these, only 5 shared the same Jones polynomial with prime knots having less than 17 crossings and furthermore having an Alexander polynomial that is a square mod 2. In each of these five cases either the Alexander polynomial or the Kauffman 2-variable polynomial was sufficient to show that the knots did indeed have crossing numbers of 17 or more.

Thus, barring clerical errors, Table 3 and Tables 10–11 provide a complete list of all Lissajous knots with x and y frequencies of $(3, 4)$, $(3, 5)$, $(3, 7)$, $(3, 8)$, $(3, 10)$, $(4, 5)$, $(4, 7)$ or $(5, 6)$ which are either composite, or prime with 16 or less crossings.

As mentioned in the introduction, there are exactly three prime knots with 12 or less crossings that are strongly plus amphicheiral: 10a103 (10₉₉), 10a121 (10₁₂₃), and 12a427. The knots 10a103 and 12a427 are Lissajous and are listed in Table 10. A natural question is,

Question 2. *Is the strongly plus amphicheiral knot 10a121 Lissajous?*

The knot 10a121 is one member of a family of knots known as *Turks Head* knots. These knots are conjectured to not be Lissajous by Przytycki. See [13].

It is easy to see that every composite knot of the form $K\#\overline{K}$ is strongly plus amphicheiral while composites of the form $K\#K$ are 2-periodic and link their axis of rotation once. Several knots of this form appear in Table 3. Thus another good question is,

Question 3. *Is every composite knot of the form $K\#K$ or $K\#\overline{K}$ Lissajous?*

5.3 Fourier-(1, 1, 2) Knots with $2 = n_x < n_y$

Rather than trying to algorithmically choose one point in each region of the phase torus for a Fourier-(1, 1, 2) knot, we chose instead to randomly sample points from the phase torus. Fixing $n_x = 2$, $\phi_x = 0$ and $A_{z,1} = 1$, we then let n_y take on odd values from 3 to 99. For each value of n_y the remaining parameters were then chosen at random such that:

$$\begin{aligned}\phi_y &= \frac{k}{7}\pi, k \in \{1, 2, 3, 4, 5, 6\} \\ 0 &< n_{z,1} < n_{z,2} < 301 \\ 0 &\leq \phi_{z,1} \leq \pi \\ 0 &\leq \phi_{z,2} \leq 2\pi \\ 0 &\leq A_{z,2} \leq 2\end{aligned}$$

For each value of n_y , random sampling in batches of 10000 took place until no new knots were found. If a knot was produced that had already been found, the one with the lexicographically smallest set $\{n_x, n_y, n_{z,1}, n_{z,2}\}$ was kept. This tended to produce knots with fairly small values of $\{n_x, n_y, n_{z,1}\}$ but with $n_{z,2}$ often in the hundreds. Furthermore, only knots with less than 17 crossings were kept in the sample.

After a modest amount of searching, we turned up all 2-bridge knots with 14 or less crossings, and nearly all 15 and 16-crossing ones as well. (We found 1386 out of 1387 15-crossing knots and 2731 out of 2752 16-crossing knots.) We believe the following conjecture is reasonable.

Conjecture 11. *Every 2-bridge knot can be expressed as a Fourier-(1, 1, k) knot with $n_x = 2$ and $k \leq 2$.*

Additional evidence for this conjecture is provided by the twist knots. The twist knot T_m , which is the 2-bridge knot $K_{\frac{2m+1}{2}}$, is shown in Figure 5. The mirror image of T_m is the twist knot T_{-1-m} . Thus it suffices to consider $m > 1$. It is shown in [7] that T_m is Lissajous if and only if $m \equiv 0 \pmod{4}$ or $m \equiv 3 \pmod{4}$. If this is not the case, the knot does not have the required symmetry to be Lissajous. However, in these cases, the following examples show that K_m is a Fourier-(1, 1, 2) knot. Thus all twist knots are Fourier-(1, 1, k) knots with $k \leq 2$.

Theorem 12. *Twist knots which are not Lissajous may be expressed as Fourier knots as follows.*

1. *The twist knot T_{4n+1} can be expressed as the Fourier-(1, 1, 2) knot with $n_x = 2$, $\phi_x = 0$, $n_y = 8n+3$, $\phi_y = 1/2$, $n_{z,1} = 2$, $\phi_{z,1} = \pi/4$, $n_{z,2} = 8n + 1$, $\phi_{z,2} = \frac{8n+1+(8n+5)\pi}{2(8n+3)}$ and $A_{z,2} = 1$ for all $n \geq 1$.*
2. *The twist knot T_{2n} can be expressed as the Fourier-(1, 1, 2) knot with $n_x = 2$, $\phi_x = 0$, $n_y = 2n + 1$, $\phi_y = 1/2$, $n_{z,1} = 2$, $\phi_{z,1} = \pi/4$, $n_{z,2} = 2n + 3$, $\phi_{z,2} = \frac{2n+3-3\pi}{2(2n+1)}$ and $A_{z,2} = 1$ for all $n \geq 1$.*

The proof is similar to the proof of Theorem 4 given in [7] and relies on very carefully determining the sign of each crossing in the diagram. The details are quite long and not particularly insightful. We leave this as a rather complicated exercise for the reader.

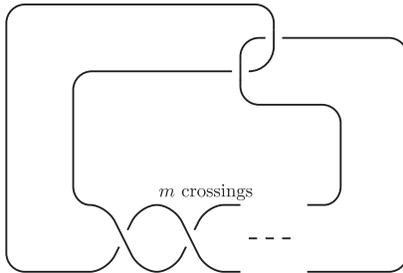


Figure 5: The twist knot T_m .

Our sample of all 2-bridge knots to 16 crossings expressed as Fourier-(1, 1, 2) knots is too large to reproduce here. Instead, in Tables 12–13 we list all 2-bridge knots to 10 crossings with associated Fourier data. To generate this table we again undertook a random sample but this time sharply reduced the range of the parameters. In particular, we kept all amplitudes equal to one, set $\phi_y = \pi/4$, and only allowed z -frequencies as large as 10. An interesting variation on Conjecture 11 would be to require that all amplitudes are 1. Knots appearing in Tables 12–13 which are known to be Lissajous are shown in boldface, while those that have Alexander polynomials congruent to 1 mod 2, and hence *might* be Lissajous, are shown in italics.

5.4 Fourier-(1, 1, 2) Knots with $2 < n_x < n_y$

We made only a modest attempt to sample Fourier-(1, 1, 2) knots with x and y frequencies greater than two. Rather than sampling at random as in Section 5.3, we now chose one sampling point from each region of the phase torus by first creating a bitmap image as in Figure 3 and then taking the centroid of each white region. Sometimes the centroid fell outside of the region and in this case an arbitrary point of the region was selected. Because of the large crossing numbers that result, and the consequent difficulty in identifying these knots, we again restricted our sample to x and y frequencies of (3, 4), (3, 5), (3, 7), (3, 8), (3, 10), (4, 5), (4, 7) or (5, 6). We further restricted the z -frequencies to be less than 15 and somewhat arbitrarily fixed all amplitudes at 1. Using *Knotscape* to identify the resulting knots, and keeping only knots with 16 or less crossings, we found several thousand prime knots. In Tables 14–16 we list all of these with 10 or less crossings which are not 2-bridge and thus listed in Tables 12–13. All knots through 9 crossings were found, and all but 20 alternating 10-crossings knots were found. We suspect that limiting the z -frequencies to less than 15 is a severe restriction.

We did however, find all torus knots up to 16 crossings. It is shown by Kauffman in [9] that every torus knot is a Fourier-(1, 3, 3) knot. Interestingly, we found that, up to 16 crossings, the torus knot $T_{p,q}$ can be represented by a Fourier-(1, 1, 2) knot with $n_x = p$ and $n_y = q$. Table 18 lists these results. The data suggests the following conjectures (which we have verified for a large number of (p, q) pairs, and hope to prove in a later paper).

Conjecture 13. *The torus knot $T_{2,q}$ can be represented as a Fourier-(1, 1, 2) knot with frequencies $n_x = 2, n_y = q, n_{z,1} = 2$ and $n_{z,2} = q - 2$ and phase shifts $\phi_x = 0, \phi_y = \pi/4, \phi_{z,1} = \pi/2$ and $\phi_{z,2} = \pi/4$.*

Conjecture 14. *The torus knot $T_{p,q}$, with $0 < p < q$, can be represented as a Fourier-(1, 1, 2) knot with frequencies $n_x = p, n_y = q, n_{z,1} = p$ and $n_{z,2} = q - p$.*

It would be interesting to undertake a large-scale sampling of Fourier-(1, 1, 2) knots with x and y frequencies greater than two to see if *every* knot with 16 or less crossings turns up. Such a study might shed light on the following question,

Question 4. *Is there a knot which cannot be expressed as a Fourier-(1, 1, k) knot for $k \leq 2$?*

References

- [1] M. G. V. Bogle, J. E. Hearst, V. F. R. Jones, and L. Stoilov. Lissajous knots. *J. Knot Theory Ramifications*, 3(2):121–140, 1994.
- [2] Gerhard Burde and Heiner Zieschang. *Knots*, volume 5 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2003.
- [3] Peter Cromwell. *Knots and Links*. Cambridge University Press, 2004.
- [4] Richard Hartley and Akio Kawachi. Polynomials of amphicheiral knots. *Math. Ann.*, 243(1):63–70, 1979.
- [5] Jim Hoste and Morwen Thistlethwaite. Knotscape. <http://www.math.utk.edu/~morwen>, 1998.
- [6] Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks. The first 1,701,936 knots. *Math. Intelligencer*, 20(4):33–48, 1998.
- [7] Jim Hoste and Laura Zirbel. Lissajous knots with lissajous projections. *arXiv: math.GT/0605632*, 2006.
- [8] Vaughan F. R. Jones and Józef H. Przytycki. Lissajous knots and billiard knots. *Banach Center Publications*, (42):145–163, 1998.
- [9] Louis H. Kauffman. Fourier Knots. *arXiv: q-alg/9711013*
- [10] Christoph Lamm. Fourier knots. *Preprint*.
- [11] Christoph Lamm. There are infinitely many lissajous knots. *Manuscripta Math.*, 93:29–37, 1996.
- [12] Kunio Murasugi. On periodic knots. *Comment. Math. Helv.*, 46:162–174, 1971.
- [13] Jozef Przytycki. Symmetric knots and billiard knots. *arXiv: math.GT/0405151*
- [14] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

6 Tables of Lissajous and Fourier Knots

	$\mathcal{L}(2, 3)$	$\mathcal{L}(2, 5)$	$\mathcal{L}(2, 7)$	$\mathcal{L}(2, 9)$	$\mathcal{L}(2, 11)$	$\mathcal{L}(2, 13)$	$\mathcal{L}(2, 15)$
cr	p/q	p/q	p/q	p/q	p/q	p/q	p/q
5	7/2	7/2	7/2	7/2	7/2	7/2	7/2
6	9/2	9/2	9/2	9/2	9/2	9/2	9/2
7	15/4	17/5	15/4	17/5	15/4	15/4	17/5
7			17/5		17/5	17/5	
8					31/12		
9			15/2	31/7	15/2	15/2	15/2
9			31/7		31/7	31/7	31/7
9					31/11		
10		17/2	17/2	17/2	55/12	17/2	55/12
10		49/20	49/20	57/16	57/16	49/20	57/16
10		57/16	55/12			57/16	
10			57/16				
11		65/14	73/16	49/9	49/9	41/13	49/9
11		73/16		65/14	73/16	49/9	73/16
11		97/26		73/16		71/20	
11						73/16	
11						97/26	
12		121/32	169/50	167/46	121/32	167/46	169/50
12					169/50		
13		209/56	239/71		23/2	71/11	23/2
13					71/11	209/56	71/11
14			25/2	25/2	407/119	25/2	407/119
14			89/36	89/36		409/121	
14			289/118	289/118			
14				409/121			
15			151/20	441/101	151/20	97/13	97/13
15			319/144	463/130	319/144	361/78	463/130
15			359/82		359/82	463/130	
15			463/130		433/122		
15					447/98		
15					463/130		
16			529/114	593/130	593/130	529/114	593/130
16			593/130		817/239	559/122	817/239
16			777/208			593/130	
16						815/237	
17			975/274		703/131	1353/380	31/2
17			983/260		1321/288		127/15
17			1351/362				975/274
17							983/260

Table 4: The sets $L(2, n_y)$ for $3 \leq n_y \leq 15$ given by crossing number and 2-bridge fraction p/q .

	$\mathcal{L}(2, 3)$	$\mathcal{L}(2, 5)$	$\mathcal{L}(2, 7)$	$\mathcal{L}(2, 9)$	$\mathcal{L}(2, 11)$	$\mathcal{L}(2, 13)$	$\mathcal{L}(2, 15)$
cr	p/q	p/q	p/q	p/q	p/q	p/q	p/q
18			1681/450	33/2	33/2	1489/337	241/32
18				129/52	129/52		1487/335
18				529/214	241/32		
18				1681/696	529/214		
18				2321/622	1681/696		
19			2911/780	273/32	4063/1202	273/32	
19				673/78		673/78	
19				1961/800		1025/161	
19				2001/898		1961/800	
19				3329/989		2001/898	
19						4015/1106	
20				3761/1056	1279/282	3409/744	2481/559
20					3535/996	3521/992	3631/796
20					3761/1056	3761/1056	3761/1056
20					5681/1661		
21				4297/926	4817/1056	4817/1056	1407/191
21				4305/944	6143/1407	9833/2909	4817/1056
21				4817/1056			
22				7921/2224	41/2	41/2	10737/2354
22				7985/2112	169/68	169/68	
22				10865/2912	769/310	769/310	
22					3073/1272	3073/1272	
22					9801/4058	9801/4058	
22					11257/3102	10471/3098	
22						11441/3327	
23				18817/5042	415/36	2129/282	415/36
23					1223/146	9793/1825	1223/146
23					3487/1424		3487/1424
23					11759/4802		11759/4802
23					11999/4900		11999/4900
23					18663/4996		
24				23409/6272	2425/322	33433/9892	21009/4733
24					6041/800		31759/6924
24					12769/5762		
24					14369/3282		
24					32593/8738		
25				40545/10864	10359/2284	10377/2288	14279/2243
25					16511/3554	28743/8098	30551/8578
25					18583/4000	30551/8578	
25					23543/5086	55801/14954	
25					30551/8578		
25					46367/13775		

Table 5: The sets $L(2, n_y)$ for $3 \leq n_y \leq 15$ given by crossing number and 2-bridge fraction p/q .

	$\mathcal{L}(2, 3)$	$\mathcal{L}(2, 5)$	$\mathcal{L}(2, 7)$	$\mathcal{L}(2, 9)$	$\mathcal{L}(2, 11)$	$\mathcal{L}(2, 13)$	$\mathcal{L}(2, 15)$
cr	p/q	p/q	p/q	p/q	p/q	p/q	p/q
26					34905/7522	49/2	49/2
26					34969/7666	209/84	209/84
26					39129/8578	1009/406	1009/406
26					51193/14384	4465/1848	4465/1848
26						17921/7418	17921/7418
26						39129/8578	39129/8578
26						57121/23660	57121/23660
26						80305/23857	97681/28898
27					64343/18066	601/48	3871/514
27					64351/18068	1793/142	
27					64863/17156	6409/2592	
27					87031/23298	21401/8738	
27						69121/28222	
27						69361/28320	
27						85561/19597	
27						91193/25584	
28					108241/28622	4385/514	162521/48279
28					108657/28732	10817/1268	
28					151313/40544	26937/3122	
28						78489/32020	
28						80089/35940	
28						151697/44882	
29					188175/50374	21441/2840	136399/25419
29					188287/50404	50969/11642	
29					262087/70226	81025/36576	
29						91193/20830	
29						115561/26396	
29						259969/69676	
30					326041/87362	84041/18530	57/2
30						134689/28992	249/100
30						151553/32622	1249/502
30						191561/41968	5857/2424
30						248169/69680	26041/10778
30						453929/121630	104441/43260
30							248169/69680
30							332929/137902
30							448689/120112

Table 6: The sets $L(2, n_y)$ for $3 \leq n_y \leq 15$ given by crossing number and 2-bridge fraction p/q .

	$\mathcal{L}(2, 3)$	$\mathcal{L}(2, 5)$	$\mathcal{L}(2, 7)$	$\mathcal{L}(2, 9)$	$\mathcal{L}(2, 11)$	$\mathcal{L}(2, 13)$	$\mathcal{L}(2, 15)$
cr	p/q	p/q	p/q	p/q	p/q	p/q	p/q
31					564719/151316	283537/61102	807/52
31						284049/62270	2599/210
31						284057/62272	9215/3728
31						317849/69680	39159/15842
31						415929/116866	127399/51540
31						645809/191861	317849/69680
31							400895/166464
31							402287/164258
32						522665/146752	1118489/332261
32						522729/146768	
32						526889/139360	
32						697745/186784	
33						864945/228716	38951/5172
33						872153/230622	97015/12882
33						882809/233440	241791/32020
33						1216977/326170	511079/230624
33							511119/230642
33							575119/131362
33							1191711/272951
34						1509537/404098	2090425/559602
34						1515361/407460	
34						2107561/564720	
35						2621905/702714	2015903/566018
35						2622017/702520	
35						3650401/978122	
36						4541161/1216800	2303201/496338
36							2307361/505826
36							2307425/505842
36							2581921/566018
36							6322681/1694162
37						7865521/2107560	4245663/1192082
37							4246175/1192226
37							4246183/1192212
37							4279975/1132036
37							8994959/2672279
38							9748249/2609584
40							21077281/5642338
40							21089825/5645698
40							29354521/7865520
41							50843527/13623482
42							63250209/16947842
43							109552575/29354524

Table 7: The sets $L(2, n_y)$ for $3 \leq n_y \leq 15$ given by crossing number and 2-bridge fraction p/q .

knot	p/q	n_x	n_y	n_z	ϕ_x	ϕ_y	ϕ_z
5a1	7/2	2	3	11	0	0.56099	2.58059
6a3	9/2	2	3	11	0	0.67319	0.89759
7a3	17/5	2	5	17	0	0.49979	2.64179
7a6	15/4	2	3	11	0	0.78539	2.35619
8a1	31/12	2	11	41	0	0.39269	2.74889
9a8	31/11	2	11	41	0	0.48332	1.08747
9a27	15/2	2	7	25	0	0.49087	1.07992
9a33	31/7	2	7	23	0	0.47123	2.67035
10a23	49/20	2	5	17	0	0.71399	0.85679
10a63	55/12	2	7	25	0	0.44178	2.69980
10a69	57/16	2	5	19	0	0.58904	2.55254
10a75	17/2	2	5	17	0	0.57119	0.99959
11a91	129/49	2	41	153	0	0.34816	2.79342
11a140	65/17	2	17	63	0	0.47123	1.09955
11a192	97/26	2	5	17	0	0.64259	2.49899
11a210	73/16	2	5	19	0	0.65449	0.91629
11a226	71/20	2	13	49	0	0.45603	1.11475
11a246	41/13	2	13	47	0	0.47123	1.09955
11a333	65/14	2	5	19	0	0.78539	0.78539
11a334	49/9	2	9	29	0	0.45470	2.68688
12a38	71/28	2	25	89	0	0.41336	1.15742
12a257	191/74	2	17	63	0	0.37306	2.76852
12a715	169/50	2	7	25	0	0.53996	2.60163
12a729	167/46	2	9	31	0	0.43196	2.70962
12a1034	121/32	2	5	19	0	0.71994	2.42164
13a640	55/19	2	19	65	0	0.44879	1.12199
13a1884	289/80	2	25	93	0	0.35941	2.78217
13a2683	287/79	2	63	235	0	0.34262	2.79896
13a2760	239/71	2	7	23	0	0.57595	2.56563
13a3143	23/2	2	11	37	0	0.45814	1.11264
13a3896	111/23	2	23	85	0	0.46542	1.10537
13a4304	209/56	2	5	17	0	0.78539	2.35619
13a4570	79/19	2	19	69	0	0.46409	1.10669
13a4822	71/11	2	11	35	0	0.44392	2.69767

Table 8: All Lissajous knots with frequencies $n_x = 2, 3 \leq n_y \leq 105$ and with less than 14 crossings. Knot names are as in *Knotscape*.

knot	p/q	n_x	n_y	n_z	ϕ_x	ϕ_y	ϕ_z
14a2651	89/36	2	7	23	0	0.62831	0.94247
14a6166	289/118	2	7	23	0	0.73303	0.83775
14a12186	407/119	2	11	37	0	0.42542	2.71616
14a12212	409/121	2	9	31	0	0.51050	2.63108
14a12308	103/12	2	57	215	0	0.36382	2.77776
14a12652	127/28	2	29	103	0	0.40459	1.16619
14a12741	25/2	2	7	23	0	0.52359	1.04719
15a21965	113/29	2	29	99	0	0.44178	1.12900
15a25723	745/288	2	19	71	0	0.40142	2.74016
15a32142	319/144	2	7	25	0	0.78539	0.78539
15a44612	359/82	2	7	25	0	0.68722	0.88357
15a46260	361/78	2	13	49	0	0.50670	1.06408
15a50643	447/98	2	11	39	0	0.40840	2.73318
15a50772	433/122	2	11	41	0	0.45311	2.68847
15a51438	463/130	2	7	27	0	0.60059	2.54099
15a52567	151/20	2	7	25	0	0.58904	0.98174
15a54893	65/21	2	21	71	0	0.44392	1.12687
15a71359	505/109	2	85	317	0	0.33994	2.80164
15a71603	169/29	2	29	107	0	0.46199	1.10879
15a76044	441/101	2	9	29	0	0.53737	2.60421
15a78853	129/25	2	25	91	0	0.46040	1.11039
15a84772	97/13	2	13	41	0	0.43633	2.70526
16a7016	111/44	2	39	131	0	0.40655	1.16423
16a57423	431/170	2	39	139	0	0.40593	1.16486
16a135506	1103/456	2	23	85	0	0.36361	2.77798
16a219884	961/208	2	93	347	0	0.33914	2.80244
16a221291	777/208	2	7	25	0	0.63813	2.50345
16a221836	783/220	2	41	155	0	0.37667	2.76492
16a224238	791/212	2	17	65	0	0.44058	2.70100
16a225074	593/130	2	7	27	0	0.64679	0.92399
16a228722	559/122	2	13	47	0	0.39269	2.74889
16a229409	577/162	2	21	79	0	0.43982	1.13097
16a249132	817/239	2	11	37	0	0.49087	2.65071
16a249195	815/237	2	13	43	0	0.42074	2.72084
16a252419	385/114	2	43	153	0	0.40071	1.17008
16a252465	399/110	2	47	177	0	0.37166	2.76993
16a333209	529/114	2	7	27	0	0.73919	0.83159

Table 9: All Lissajous knots with frequencies $n_x = 2$, $3 \leq n_y \leq 105$ and with 14–16 crossings. Knot names are as in *Knotscape*.

knot	n_x	n_y	n_z	ϕ_x	ϕ_y	ϕ_z
5a1	3	4	29	0	0.49186	1.60252
6a3	3	4	29	0	0.39666	1.69772
7a6	3	7	68	0	0.52359	2.61799
8a2	3	4	23	0	0.29088	2.85070
8n2	3	4	37	0	0.49805	2.64353
9a25	3	5	28	0	0.26973	1.82466
10a20	3	4	23	0	0.32967	0.71752
10a73	3	7	40	0	0.23394	2.90764
10a89	3	4	29	0	0.36493	0.68226
10a103	3	5	29	0	0.41579	2.72579
10n28	3	7	40	0	0.21166	1.88272
11n50	4	5	57	0	0.36736	1.98883
11n151	4	7	69	0	0.24802	1.32277
12a426	3	10	83	0	0.45603	2.68555
12a427	3	5	29	0	0.32339	2.81819
12a448	3	5	28	0	0.30146	0.74573
12a868	3	5	26	0	0.28713	1.80726
12a1164	4	5	61	0	0.34509	2.01109
12n133	3	4	23	0	0.36845	1.72593
12n293	3	10	59	0	0.28077	1.81362
12n322	4	5	37	0	0.16829	2.18789
12n483	5	6	113	0	0.35111	2.16215
13a2233	3	8	67	0	0.48171	2.65988
13a4774	3	4	23	0	0.40724	2.73434
13n1405	4	5	51	0	0.35062	2.00557
13n1734	3	10	103	0	0.56066	1.53372
13n3594	3	4	23	0	0.44602	0.60116

Table 10: Small-crossing Lissajous knots with all frequencies greater than 2. Only three knots, 5a1, 6a3 and 7a6 are 2-bridge and appear in Table 8. Knot names are as in *Knotscape*.

knot	n_x	n_y	n_z	ϕ_x	ϕ_y	ϕ_z
14a1491	4	5	39	0	0.19634	0.58904
14a6398	3	7	59	0	0.49979	2.64179
14a6912	3	8	67	0	0.42586	0.62133
14a8662	3	7	53	0	0.35779	0.68940
14a13089	4	7	53	0	0.15707	1.41371
14a15296	3	4	29	0	0.46013	0.58706
14a16309	3	7	53	0	0.47996	1.61442
14a16437	3	5	29	0	0.35419	1.74019
14a18187	3	5	37	0	0.38646	0.66073
14n6560	3	8	47	0	0.25703	2.88455
14n9732	3	7	53	0	0.42760	1.66678
14n13886	3	4	29	0	0.42839	2.71319
14n14189	4	5	63	0	0.39269	2.74889
14n15552	3	5	28	0	0.33319	2.80839
14n18513	3	8	83	0	0.58113	1.51325
14n22071	3	7	64	0	0.50884	2.63274
14n22073	3	5	29	0	0.50819	2.63339
14n23738	3	7	50	0	0.39499	1.69939
14n24494	3	8	61	0	0.35665	1.73774
14n25903	3	7	53	0	0.37524	1.71914
15a80928	4	5	37	0	0.20569	0.57969
15n77228	4	5	69	0	0.42453	1.14625
15n92508	3	8	61	0	0.43253	2.70905
15n103019	5	6	103	0	0.30263	2.83896
15n116110	4	7	53	0	0.14398	2.21220
16a128851	3	7	40	0	0.30079	2.84080
16a151023	3	7	38	0	0.26761	0.77958
16a168328	3	7	41	0	0.29452	2.84706
16a202258	3	7	40	0	0.27850	1.81588
16a295212	3	5	26	0	0.38847	1.70591
16a312423	3	5	29	0	0.38499	0.66219
16a340770	3	7	41	0	0.38179	0.66540
16n42863	4	5	39	0	0.21419	1.35659
16n228473	3	7	38	0	0.29088	1.80350
16n390014	3	5	26	0	0.35469	2.78689
16n507235	3	5	28	0	0.39666	0.65053
16n562396	3	5	28	0	0.36493	1.72946
16n768985	3	4	23	0	0.48481	1.60958
16n982564	4	5	73	0	0.44304	1.12775
16n988939	3	7	41	0	0.51269	0.53450
16n1008347	3	7	50	0	0.32150	0.72568

Table 11: Small-crossing Lissajous knots with all frequencies greater than 2. Knot names are as in *Knotscape*.

knot	p/q	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
3a1	3/1	2	3	1	2	0	$\pi/4$	0.39269	1.66017
4a1	5/2	2	3	1	3	0	$\pi/4$	1.62773	5.79254
5a1	7/2	2	3	1	7	0	$\pi/4$	0.10580	2.49320
5a2	5/1	2	5	2	3	0	$\pi/4$	0.96046	4.09767
6a1	13/5	2	5	1	5	0	$\pi/4$	0.03573	2.53353
6a2	11/3	2	7	1	7	0	$\pi/4$	1.90655	5.01637
6a3	9/2	2	3	1	5	0	$\pi/4$	0.18165	1.75945
7a1	21/8	2	5	1	5	0	$\pi/4$	1.60021	5.52412
7a2	19/7	2	7	3	7	0	$\pi/4$	1.66835	6.11271
7a3	17/5	2	7	5	9	0	$\pi/4$	0.08774	5.55745
7a4	11/2	2	7	3	7	0	$\pi/4$	1.60853	6.27384
7a5	13/3	2	9	4	7	0	$\pi/4$	1.57817	4.41032
7a6	15/4	2	3	1	7	0	$\pi/4$	1.87792	4.64352
7a7	7/1	2	7	2	5	0	$\pi/4$	0.93991	0.93104
8a1	31/12	2	11	1	5	0	$\pi/4$	0.37204	1.78795
<i>8a4</i>	<i>25/9</i>	2	7	1	5	0	$\pi/4$	2.04720	5.29197
8a5	29/12	2	5	3	5	0	$\pi/4$	1.59453	2.05821
8a6	23/5	2	9	1	9	0	$\pi/4$	0.35397	2.65710
8a7	29/8	2	9	1	5	0	$\pi/4$	1.47451	2.10447
8a8	17/3	2	11	3	10	0	$\pi/4$	0.39241	5.09182
8a9	27/8	2	9	1	5	0	$\pi/4$	2.03830	2.05668
<i>8a10</i>	<i>23/7</i>	2	7	1	9	0	$\pi/4$	0.48400	5.18915
8a11	13/2	2	5	3	5	0	$\pi/4$	1.58524	0.24531
8a16	25/7	2	7	3	7	0	$\pi/4$	0.04412	2.25248
8a17	19/4	2	9	3	7	0	$\pi/4$	1.92077	6.06457
<i>8a18</i>	<i>17/4</i>	2	7	1	5	0	$\pi/4$	1.42912	1.98797
<i>9a3</i>	<i>41/16</i>	2	7	3	5	0	$\pi/4$	1.88608	4.98854
9a8	31/11	2	5	7	9	0	$\pi/4$	0.09293	5.50516
<i>9a10</i>	<i>39/16</i>	2	5	1	7	0	$\pi/4$	2.08636	5.17367
9a12	49/18	2	17	1	9	0	$\pi/4$	0.60289	4.56332
9a13	55/21	2	7	1	7	0	$\pi/4$	0.00613	5.43134
9a14	39/14	2	9	1	9	0	$\pi/4$	2.13083	5.89035
9a15	47/13	2	9	1	9	0	$\pi/4$	1.74912	1.92322
9a16	45/19	2	9	1	5	0	$\pi/4$	1.93719	4.94328
9a17	37/8	2	5	2	9	0	$\pi/4$	0.19367	2.96295
9a19	41/11	2	15	3	10	0	$\pi/4$	2.62089	0.60844
9a20	33/7	2	17	5	7	0	$\pi/4$	0.10752	5.13086
9a21	43/12	2	5	3	9	0	$\pi/4$	2.14045	5.61205
9a22	35/8	2	5	7	9	0	$\pi/4$	1.67486	1.65979
9a23	27/5	2	13	4	5	0	$\pi/4$	1.36285	5.38881
<i>9a24</i>	<i>41/12</i>	2	11	1	7	0	$\pi/4$	0.44828	2.24339
9a26	29/9	2	11	8	9	0	$\pi/4$	1.42268	3.50098
9a27	15/2	2	5	1	7	0	$\pi/4$	0.00135	6.21184
9a33	31/7	2	11	4	9	0	$\pi/4$	0.93417	3.86038
9a34	37/10	2	9	3	5	0	$\pi/4$	1.98367	5.56618
9a35	21/4	2	11	9	10	0	$\pi/4$	0.35932	5.18305
<i>9a36</i>	<i>23/4</i>	2	9	3	5	0	$\pi/4$	1.65102	3.04593
9a38	19/3	2	13	1	7	0	$\pi/4$	1.93386	2.02910
<i>9a39</i>	<i>33/10</i>	2	11	3	7	0	$\pi/4$	2.16159	2.03213
9a41	9/1	2	9	2	7	0	$\pi/4$	0.86114	0.74621

Table 12: Fourier-(1, 1, 2) descriptions of all 2-bridge knots to 9 crossings. All amplitudes are 1. Boldface entries are known to be Lissajous while italic entries *might* be Lissajous. All others cannot be Lissajous. Knot names are as in *Knotscape*.

knot	p/q	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
10a5	51/20	2	11	3	7	0	$\pi/4$	0.18595	2.82470
10a19	37/13	2	13	3	7	0	$\pi/4$	1.85642	1.83386
10a23	49/20	2	5	1	7	0	$\pi/4$	0.36657	2.47281
10a25	89/34	2	7	1	7	0	$\pi/4$	1.62228	5.59775
10a26	61/22	2	17	3	8	0	$\pi/4$	0.19982	4.43387
10a29	59/25	2	9	1	5	0	$\pi/4$	1.82942	5.29828
10a30	75/29	2	15	5	7	0	$\pi/4$	1.66311	4.79250
10a31	81/31	2	17	3	7	0	$\pi/4$	1.36514	5.07457
10a32	79/29	2	17	3	7	0	$\pi/4$	2.51922	6.10032
<i>10a33</i>	<i>57/13</i>	2	17	1	9	0	$\pi/4$	1.22778	5.48698
10a34	67/18	2	13	1	5	0	$\pi/4$	1.94658	5.20996
10a35	71/26	2	17	5	7	0	$\pi/4$	2.17223	4.46442
<i>10a43</i>	<i>47/11</i>	2	17	5	8	0	$\pi/4$	2.17425	5.50211
10a44	53/14	2	9	3	5	0	$\pi/4$	2.12967	5.39832
10a49	53/23	2	13	1	7	0	$\pi/4$	1.00888	3.57446
10a52	73/27	2	17	3	7	0	$\pi/4$	2.90021	2.80121
10a53	63/17	2	17	1	9	0	$\pi/4$	0.46162	2.91082
10a54	53/12	2	5	5	7	0	$\pi/4$	0.00438	1.96518
10a55	69/19	2	13	3	7	0	$\pi/4$	2.09996	5.23165
10a56	33/5	2	19	5	6	0	$\pi/4$	0.37923	4.72239
10a57	59/18	2	21	3	8	0	$\pi/4$	0.38316	4.48117
<i>10a58</i>	<i>71/21</i>	2	13	6	9	0	$\pi/4$	1.04484	0.64343
10a59	23/3	2	23	2	7	0	$\pi/4$	0.57078	0.10280
10a60	45/14	2	17	1	9	0	$\pi/4$	1.72208	1.81144
<i>10a61</i>	<i>65/19</i>	2	17	2	5	0	$\pi/4$	1.05935	4.53128
10a63	55/12	2	11	1	7	0	$\pi/4$	2.07884	1.80942
10a64	45/8	2	17	5	9	0	$\pi/4$	1.41312	1.49892
10a65	43/8	2	19	1	7	0	$\pi/4$	1.64470	4.03256
10a68	43/9	2	13	9	10	0	$\pi/4$	0.30693	1.31007
10a69	57/16	2	9	3	5	0	$\pi/4$	0.02223	2.46173
10a70	37/7	2	19	5	7	0	$\pi/4$	0.36129	4.69287
<i>10a71</i>	<i>55/16</i>	2	13	3	5	0	$\pi/4$	1.15846	5.71558
10a74	35/11	2	13	5	10	0	$\pi/4$	0.31052	4.18157
10a75	17/2	2	5	2	7	0	$\pi/4$	0.00295	3.10669
10a107	41/9	2	9	5	9	0	$\pi/4$	0.11322	5.44496
10a108	51/11	2	17	2	7	0	$\pi/4$	1.01343	4.49878
<i>10a109</i>	<i>65/18</i>	2	11	2	5	0	$\pi/4$	0.22667	5.30925
10a110	39/7	2	19	4	5	0	$\pi/4$	1.10277	4.42691
10a111	61/17	2	19	4	5	0	$\pi/4$	1.35067	1.23088
<i>10a112</i>	<i>49/13</i>	2	17	5	7	0	$\pi/4$	0.06896	1.24086
10a113	27/4	2	17	4	7	0	$\pi/4$	1.05187	2.20380
10a114	29/5	2	19	5	7	0	$\pi/4$	0.18690	4.45167
<i>10a115</i>	<i>47/10</i>	2	11	7	10	0	$\pi/4$	0.50632	4.53216
10a116	43/10	2	15	7	9	0	$\pi/4$	0.09320	1.94013
<i>10a117</i>	<i>25/4</i>	2	7	1	5	0	$\pi/4$	2.15907	2.20357

Table 13: Fourier-(1, 1, 2) descriptions of all 2-bridge knots with 10 crossings. All amplitudes are 1. Boldface entries are known to be Lissajous while italic entries *might* be Lissajous. All others cannot be Lissajous. Knot names are as in *Knotscape*.

knot	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
8a2	3	4	1	7	0	$\pi/6$	1.63362	2.03575
8a3	3	5	6	13	0	$\pi/6$	0.56548	2.03575
8a12	3	4	3	5	0	$\pi/6$	1.04300	0.80424
8a13	3	4	1	9	0	$\pi/6$	0.26389	1.58336
8a14	3	4	7	14	0	$\pi/6$	1.28176	1.78442
8a15	3	7	1	10	0	$\pi/6$	1.64619	2.31221
8n1	3	4	1	14	0	$\pi/6$	1.94778	2.76460
8n2	3	4	1	5	0	$\pi/6$	0.05026	2.23681
8n3	3	4	1	3	0	$\pi/6$	0.26389	1.58336
9a1	3	5	7	10	0	$\pi/6$	2.29964	0.03769
9a2	3	7	1	6	0	$\pi/6$	0.35185	1.05557
9a4	3	4	1	14	0	$\pi/6$	1.33203	2.27451
9a5	3	8	1	8	0	$\pi/6$	0.65345	1.70902
9a6	3	7	1	10	0	$\pi/6$	1.88495	2.43787
9a7	3	4	1	14	0	$\pi/6$	2.03575	2.37504
9a9	3	7	4	15	0	$\pi/6$	1.15610	0.76654
9a11	3	5	9	14	0	$\pi/6$	1.04300	1.33203
9a18	3	4	2	7	0	$\pi/6$	2.37504	2.03575
9a25	3	7	2	14	0	$\pi/6$	0.98017	0.37699
9a28	3	7	4	5	0	$\pi/6$	1.28176	0.45238
9a29	4	7	2	13	0	$\pi/8$	0.26389	2.07345
9a30	3	7	8	9	0	$\pi/6$	0.23876	0.77911
9a31	3	4	10	11	0	$\pi/6$	1.38230	1.87238
9a32	3	7	4	13	0	$\pi/6$	0.46495	1.20637
9a37	3	4	2	11	0	$\pi/6$	1.04300	2.62637
9a40	3	7	4	13	0	$\pi/6$	1.06814	1.06814
9n1	3	4	1	14	0	$\pi/6$	0.05026	0.27646
9n2	3	5	4	7	0	$\pi/6$	0.15079	1.99805
9n3	3	8	1	6	0	$\pi/6$	0.76654	0.95504
9n4	3	4	2	11	0	$\pi/6$	0.35185	2.51327
9n5	3	4	1	4	0	$\pi/6$	1.33203	2.09858
9n6	3	4	2	13	0	$\pi/6$	0.08796	2.48814
9n7	3	7	2	9	0	$\pi/6$	2.62637	1.05557
9n8	3	7	4	13	0	$\pi/6$	0.05026	0.18849

Table 14: Fourier-(1, 1, 2) descriptions of non 2-bridge knots up to 9 crossings. All amplitudes are 1. Knot names are as in *Knotscape*.

knot	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
10a1	3	4	5	9	0	$\pi/6$	1.04300	1.58336
10a2	3	8	4	13	0	$\pi/6$	0.70371	0.08796
10a3	3	8	7	14	0	$\pi/6$	2.70176	0.23876
10a4	3	5	4	13	0	$\pi/6$	2.34991	2.34991
10a7	3	5	8	11	0	$\pi/6$	0.01256	1.60849
10a9	3	8	5	6	0	$\pi/6$	0.27646	1.58336
10a10	3	10	2	11	0	$\pi/6$	2.48814	2.94053
10a11	3	8	3	14	0	$\pi/6$	0.05026	1.01787
10a12	3	8	2	5	0	$\pi/6$	1.40743	1.67132
10a14	3	8	2	7	0	$\pi/6$	0.33929	2.81486
10a15	4	7	1	12	0	$\pi/8$	1.01787	0.67858
10a16	4	7	1	12	0	$\pi/8$	0.20106	0.66601
10a17	3	8	5	15	0	$\pi/6$	0.22619	1.06814
10a18	3	8	4	9	0	$\pi/6$	0.17592	1.58336
10a20	3	4	1	7	0	$\pi/6$	0.22619	1.47026
10a21	3	8	1	10	0	$\pi/6$	0.82938	2.71433
10a22	3	8	1	10	0	$\pi/6$	2.46300	2.75203
10a28	4	7	1	14	0	$\pi/8$	0.43982	0.11309
10a36	4	5	1	14	0	$\pi/8$	0.82938	1.60849
10a37	3	5	7	14	0	$\pi/6$	0.07539	2.90283
10a38	3	7	2	11	0	$\pi/6$	0.66601	0.01256
10a42	3	10	5	8	0	$\pi/6$	2.61380	0.42725
10a47	3	4	5	14	0	$\pi/6$	2.19911	1.96035
10a48	3	8	5	10	0	$\pi/6$	2.02318	1.39486
10a50	3	7	1	8	0	$\pi/6$	1.60849	1.09327
10a51	3	7	2	4	0	$\pi/6$	2.56353	5.45380
10a62	5	6	4	11	0	$\pi/10$	0.27646	0.59061
10a66	3	5	2	11	0	$\pi/6$	1.52053	1.85982
10a67	3	8	2	13	0	$\pi/6$	1.40743	3.12902

Table 15: Fourier-(1, 1, 2) descriptions of alternating non 2-bridge knots with 10 crossings. All amplitudes are 1. Knot names are as in *Knotscape*.

knot	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
10a72	4	5	7	14	0	$\pi/8$	2.47557	1.73415
10a73	3	5	2	10	0	$\pi/6$	1.98548	0.16336
10a76	3	7	6	11	0	$\pi/6$	1.58336	1.05557
10a77	5	6	3	14	0	$\pi/10$	0.30159	1.75929
10a78	5	6	1	12	0	$\pi/10$	1.45769	1.26920
10a80	3	10	4	15	0	$\pi/6$	1.97292	0.31415
10a82	5	6	4	11	0	$\pi/10$	0.37699	2.86513
10a84	4	7	6	13	0	$\pi/8$	0.31415	1.73415
10a85	3	4	7	14	0	$\pi/6$	1.08070	2.04831
10a87	4	7	2	3	0	$\pi/8$	1.43256	2.14884
10a89	3	4	3	13	0	$\pi/6$	0.18849	1.48283
10a90	5	6	6	7	0	$\pi/10$	0.02513	1.53309
10a91	4	7	1	14	0	$\pi/8$	0.70371	0.05026
10a92	3	4	7	14	0	$\pi/6$	1.09327	2.46300
10a93	3	8	2	13	0	$\pi/6$	1.57079	0.69115
10a94	3	4	7	10	0	$\pi/6$	1.28176	1.36973
10a95	4	7	4	13	0	$\pi/8$	1.58336	0.06283
10a96	3	4	7	13	0	$\pi/6$	1.63362	5.17734
10a97	3	8	2	7	0	$\pi/6$	1.35716	2.36247
10a99	3	8	2	7	0	$\pi/6$	1.36973	2.63893
10a100	3	10	4	7	0	$\pi/6$	1.73415	1.41999
10a101	5	6	3	8	0	$\pi/10$	0.06283	2.57610
10a102	3	4	1	13	0	$\pi/6$	2.02318	1.24407
10a103	3	5	9	13	0	$\pi/6$	0.75398	2.57610
10a105	3	4	10	13	0	$\pi/6$	1.77185	2.86513
10a119	4	7	10	15	0	$\pi/8$	0.45238	1.25663
10a121	3	5	3	7	0	$\pi/6$	1.04300	0.85451
10a122	4	7	7	9	0	$\pi/8$	1.28176	3.99610
10a123	3	7	2	6	0	$\pi/6$	0.57805	4.31026

Table 16: Fourier-(1, 1, 2) descriptions of alternating non 2-bridge knots with 10 crossings. All amplitudes are 1. Knot names are as in *Knotscape*.

knot	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
10n1	3	4	3	7	0	$\pi/6$	1.04300	1.47026
10n2	3	4	5	14	0	$\pi/6$	0.99274	2.87769
10n3	3	4	1	5	0	$\pi/6$	1.94778	5.88106
10n4	3	4	6	11	0	$\pi/6$	0.35185	0.18849
10n5	3	4	1	10	0	$\pi/6$	1.11840	2.37504
10n6	5	6	1	4	0	$\pi/10$	1.60849	1.93522
10n7	3	7	3	4	0	$\pi/6$	1.04300	0.52778
10n8	3	5	4	13	0	$\pi/6$	0.75398	2.81486
10n9	3	8	2	7	0	$\pi/6$	0.47752	1.45769
10n10	3	8	2	5	0	$\pi/6$	1.99805	2.06088
10n11	4	7	3	8	0	$\pi/8$	1.26920	0.79168
10n12	3	8	2	7	0	$\pi/6$	1.39486	2.85256
10n13	3	5	2	11	0	$\pi/6$	1.53309	1.28176
10n14	3	5	2	9	0	$\pi/6$	0.33929	1.58336
10n15	3	8	2	3	0	$\pi/6$	0.52778	1.58336
10n16	3	8	6	7	0	$\pi/6$	0.76654	0.52778
10n17	3	10	1	4	0	$\pi/6$	0.47752	1.70902
10n18	3	4	1	10	0	$\pi/6$	2.02318	1.91008
10n19	3	5	2	7	0	$\pi/6$	1.25663	2.04831
10n20	3	4	5	10	0	$\pi/6$	0.62831	1.93522
10n21	3	5	2	3	0	$\pi/6$	0.31415	1.58336
10n22	3	10	4	13	0	$\pi/6$	0.02513	0.02513
10n23	3	7	8	11	0	$\pi/6$	0.05026	1.28176
10n24	3	4	7	14	0	$\pi/6$	1.74672	1.48283
10n25	3	7	1	12	0	$\pi/6$	2.09858	1.58336
10n26	3	7	5	6	0	$\pi/6$	0.69115	1.58336
10n27	4	5	1	5	0	$\pi/8$	0.03769	4.80035
10n28	3	4	7	11	0	$\pi/6$	0.05026	2.23681
10n29	3	5	2	11	0	$\pi/6$	1.92265	2.09858
10n30	3	7	2	4	0	$\pi/6$	0.38955	1.52053
10n31	3	7	4	15	0	$\pi/6$	0.52778	0.75398
10n32	3	5	7	10	0	$\pi/6$	0.41469	2.31221
10n33	3	5	2	9	0	$\pi/6$	1.45769	1.58336
10n34	3	8	1	12	0	$\pi/6$	0.66601	1.58336
10n35	3	4	10	13	0	$\pi/6$	0.07539	3.00336
10n36	4	5	1	4	0	$\pi/8$	0.18849	1.18123
10n37	3	8	1	4	0	$\pi/6$	2.22424	0.22619
10n38	3	4	5	10	0	$\pi/6$	2.55097	1.49539
10n39	3	5	3	10	0	$\pi/6$	1.58336	0.52778
10n40	3	5	7	14	0	$\pi/6$	1.00530	2.55097
10n41	3	10	2	7	0	$\pi/6$	2.57610	2.78973
10n42	3	10	1	8	0	$\pi/6$	1.43256	0.79168

Table 17: Fourier-(1, 1, 2) descriptions of nonalternating non 2-bridge knots with 10 crossings. All amplitudes are 1. Knot names are as in *Knotscape*.

torus knot	knot	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
$T_{2,3}$	3a1	2	3	1	2	0	$\pi/4$	$\pi/2$	$\pi/4$
$T_{2,5}$	5a2	2	5	2	3	0	$\pi/4$	$\pi/2$	$\pi/4$
$T_{2,7}$	7a7	2	7	2	5	0	$\pi/4$	$\pi/2$	$\pi/4$
$T_{2,9}$	9a41	2	9	2	7	0	$\pi/4$	$\pi/2$	$\pi/4$
$T_{2,11}$	11a367	2	11	2	9	0	$\pi/4$	$\pi/2$	$\pi/4$
$T_{2,13}$	13a4878	2	13	2	11	0	$\pi/4$	$\pi/2$	$\pi/4$
$T_{2,15}$	15a85263	2	15	2	13	0	$\pi/4$	$\pi/2$	$\pi/4$
$T_{3,4}$	8n3	3	4	1	3	0	$\pi/6$	0.26389	1.58336
$T_{3,5}$	10n21	3	5	2	3	0	$\pi/6$	0.31415	1.58336
$T_{3,7}$	14n21881	3	7	3	4	0	$\pi/6$	1.57079	0.37699
$T_{4,5}$	15n41185	4	5	1	4	0	$\pi/8$	0.40212	1.58336
$T_{3,8}$	16n783154	3	8	3	5	0	$\pi/6$	1.57079	0.40212

Table 18: Fourier- $(1, 1, 2)$ descriptions of all torus knots up to 16 crossings. All amplitudes are 1. Knot names are as in *Knotscape*.