# Time (in)dependence in general relativity

S. Deser*

*Department of Physics, Brandeis University,*

*Waltham, Massachusetts 02454 and*

*Lauritsen Laboratory, California Institute of Technology, Pasadena, California 91125*

J. Franklin†

*Department of Physics, Reed College, Portland, Oregon 97202*



## Abstract

We clarify the conditions for Birkhoff's theorem, that is, time-independence in general relativity. We work primarily at the linearized level where guidance from electrodynamics is particularly useful. As a bonus, we also derive the equivalence principle. The basic time-independent solutions due to Schwarzschild and Kerr provide concrete illustrations of the theorem. Only familiarity with Maxwell's equations and tensor analysis is required.

## I. INTRODUCTION

A major obstacle to teaching general relativity is the initially confusing mathematics underlying useful, physical simplifications. We focus in this paper on the conditions that lead to the simplest regime, time-independence. Because general relativity is coordinate-invariant, what does it mean to speak of a particular coordinate's independence? The answer is illuminating. Loosely, we expect that there exists a choice of coordinate frame in which the gravitational field does not depend on $t$. But is this a meaningful, that is, invariant criterion? The answer is yes: it means that the spacetime geometry allows the existence of a Killing vector field $f_\mu(x)$ that obeys the tensor equation

$$D_\nu f_\mu + D_\mu f_\nu \equiv \partial_\nu f_\mu + \partial_\mu f_\nu - g^{\sigma\rho}(\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}) f_\sigma = 0, \tag{1}$$

where $g_{\mu\nu}$ is the metric and $D_\mu$ is the covariant derivative with respect to it, as defined in Eq. (1). We use the signature $(- + ++)$ and units such that $c = 1$. If $f_\mu$ is also timelike ($f^2 < 0$), then the solution in the frame where $f_\mu = g_{0\mu}$ (more manifestly, the contravariant form $f^\mu$ of the vector is $f^\mu = \delta^\mu_0$) implies that

$$\partial_0 g_{\mu\nu} = 0, \tag{2}$$

and there is no time dependence. (A special property of time-independent geometries is that in (and only in) them, matter systems such as particles retain a conserved energy, just as in flat space.)

Our main point is that we have re-expressed the issue of when a given geometry is time-independent, that is, when there exists a frame where Eq. (2) holds, as a covariant (coordinate-independent) criterion: the existence of solutions to Eq. (1). All this transcription makes no reference to field equations. There exist many frames where $t$-dependence is present, but that is not the point. It is not true false that every geometry has a static frame – the Killing equation is a strong requirement.

## II. MAXWELL

We begin with electrodynamics whose field equations outside sources, unlike general relativity, can be written entirely in terms of gauge invariant field strengths,

$$\nabla \cdot \mathbf{E} = 0 \tag{3a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3b}$$

$$\dot{\mathbf{E}} = -\nabla \times \mathbf{B} \tag{3c}$$

$$\dot{\mathbf{B}} = \nabla \times \mathbf{E}. \tag{3d}$$

The $\dot{\mathbf{E}}$ equation's longitudinal part (see the following) implies that $\dot{\mathbf{E}}^L = 0$, which exhibits the fact that the "Coulomb" part of $\mathbf{E}$ is always time-independent, whatever the behavior of the interior charges. The remaining, dynamical transverse part $\mathbf{E}^T$ and its partner $\mathbf{B}$ (transverse by definition) cannot depend on time if they vanish identically, which is the case for spherically symmetric configurations: any $\mathbf{E}(r)$ is necessarily of the form $\nabla S(r)$ and is purely longitudinal. There is no monopole radiation; it is also the only guaranteed static case, as dipole and higher configurations define transverse vectors. Equation (3) does not therefore require time-dependence, or electro/magneto-statics would not exist.

For future use we recall that the transverse/longitudinal division of any vector field $\mathbf{V}$ is a decomposition of unity,

$$V_i = \left[(\delta_{ij} - \hat{k}_i\hat{k}_j) + \hat{k}_i\hat{k}_j\right]V_j, \tag{4}$$

along some arbitrary unit vector direction $\hat{k}$. Its more familiar Fourier transform is

$$\mathbf{V} = \mathbf{V}^T + \mathbf{V}^L, \tag{5}$$

where $\nabla \cdot \mathbf{V}^T = \nabla \times \mathbf{V}^L = 0$.

Our discussion has been couched in terms of the gauge invariant field strengths $\mathbf{E}$ and $\mathbf{B}$, whose time (in-)dependence is unaffected by the choice of gauge. The underlying potentials $(A_0, \mathbf{A})$ are another story: even if $(\mathbf{E}, \mathbf{B})$ are static, there exist gauge choices for which the potentials do depend on $t$ by adding gauge terms $\partial_\mu \Lambda(r,t)$ that do not affect $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In any case the transverse vector potentials are unaffected, being gauge invariant. Only $(A_0, \mathbf{A}^L)$ can be altered, keeping $\mathbf{E}^L$ unchanged.

It is instructive to analyze the equations in terms of the $A_\mu$ in parallel with the general relativity discussion in Sec. III where potentials are unavoidable.

## III. GENERAL RELATIVITY

For our purposes the gravitational field is a glorified tensor version of the vector Maxwell field $A_\mu$, and we expect similar properties of the results there to apply. At the linearized level, the Einstein equations outside sources are

$$2G_{\mu\nu} \equiv \Box h_{\mu\nu} - \big(\partial_\mu\partial_\alpha h^\alpha{}_\nu + \partial_\nu\partial_\alpha h^\alpha{}_\mu\big) + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\big(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}\big) = 0 \tag{6}$$

for the field $h_{\mu\nu}$ with $h \equiv h^\alpha{}_\alpha$; all indices are moved by the Minkowski metric $\eta_{\mu\nu}$. As for Maxwell's equations, we decompose Eq. (6) into space and time components, with the simplifying notation $h_{0i} \equiv N_i$ and $h_{00} \equiv N$. The theory is invariant under linearized gauge/coordinate transformations $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, that is, $G_{\mu\nu}(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) = 0$, an invariance that is useful to exploit.

The component form (the linearized version of a decomposition used long ago to analyze the full theory[1]) of Eq. (6) is

$$2G_{00} = \nabla^2\tilde{h} - \partial_i\partial_j h_{ij} \tag{7a}$$

$$2G_{0i} = \nabla^2 N_i - \partial_j\dot{h}_{ji} - \partial_i\partial_j N_j + \partial_i\dot{\tilde{h}} \tag{7b}$$

$$\begin{aligned} 2G_{ij} = {}& \Box h_{ij} + \partial_i\dot{N}_j + \partial_j\dot{N}_i - (\partial_i\partial_k h_{kj} + \partial_j\partial_k h_{ki}) \\ & + \big(\delta_{ij}\nabla^2 - \partial_i\partial_j\big)(N - \tilde{h}) + \ddot{\tilde{h}}\delta_{ij} + \delta_{ij}\big(\partial_m\partial_n h_{mn} - 2\partial_k\dot{N}_k\big), \end{aligned} \tag{7c}$$

with $\tilde{h} \equiv h^i{}_i$ the trace of the spatial part of the field. This slightly complicated set of equations simplifies when we decompose the spatial tensors $h_{ij}$ and the vectors $N_i$, the latter into transverse/longitudinal parts via Eq. (4), the former by the following partition of unity:

$$h_{ij} = h^{TT}_{ij} + h^T_{ij} + \partial_i h_j + \partial_j h_i, \tag{8a}$$

$$\partial_i h^{TT}_{ij} = \partial_i h^T_{ij} = 0 = h^{TT}_{ii} \tag{8b}$$

$$h^T_{ij} = \frac{1}{2}\big(\delta_{ij} - \nabla^{-2}\partial_i\partial_j\big)h^T. \tag{8c}$$

The six components of $h_{ij}$ are decomposed linearly, orthogonally, and uniquely into two TT (transverse traceless), one T (traceless), and three $h_i$ parts. Any spatial tensor equation thus consists of three independent sets. The four quantities $(h_i, N^L_i)$ are pure gauges (variables that can be arbitrarily changed by using the gauge freedom of the theory) that cry out to be set to zero, leaving the gauge invariant set $(h_{ij}^{TT}, h^T, N_i^T, N)$ once we use the available gauge invariance. Now Eq. (7) reduces to

$$2G_{00} = \nabla^2 h^T = 0 \tag{9a}$$

$$2G_{0i} = \nabla^2 N_i^T + \partial_i \dot{h}^T = 0 \tag{9b}$$

$$\begin{aligned} 2G_{ij} &= \Box h_{ij}^{TT} + \big(\partial_i \dot{N}_j^T + \partial_j \dot{N}_i^T\big) + \big(\delta_{ij}\nabla^2 - \partial_i\partial_j\big)\big(N - \frac{1}{2}h^T\big) \\ &\quad + \frac{1}{2}(\delta_{ij} + \nabla^{-2}\partial_i\partial_j)\ddot{h}^T = 0. \end{aligned} \tag{9c}$$

The time-independence of $h^T$ follows from the longitudinal part of Eq. (9b), and the relation $N = \frac{1}{2}h^T$ follows from Eq. (9c). This seemingly innocuous equality is none other than the expression of Einstein's principle of equivalence. This expression of the equivalence principle even applies to full GR.[1] The latter states that (in suitable units) the inertial and gravitational masses of every physical system are equal. Inertial mass/energy is the conserved quantity that (in the linear regime) sums over the $T_{00}$ contributions of the interior sources. This sum is the monopole moment of the Poisson equation (9a) (if we restore $T_{00}$ as its right-hand side); hence it is the coefficient of the leading $1/r$ term in $h^T$. In contrast, gravitational mass is a very different quantity that determines the system's gravitational pull, the "Newtonian" force, on slow particles. (Einstein implicitly assumed the existence of static frames, as we have also established here.) This force is the gradient of the leading $1/r$ part of $h_{00}$. Thus, in general relativity the field equation (9c) enforces the universal equality of the desired $1/r$ coefficients.

The time-independence of $N_i^T$ results from the transverse vector part of Eq. (9c): The four "Newtonian" components of the field are time-independent outside sources. Time dependence can reside only in the remaining $\Box h_{ij}^{TT}$ dynamical modes, namely those field components unaffected by the choice of gauge and undetermined by the interior sources. Hence $t$-independence is forced whenever TT tensors are forbidden. Spherical symmetry is one such case, because all spherically symmetric tensors have the form

$$S_{ij}(r) = \delta_{ij}A(r) + \partial_i\partial_j B(r), \tag{10}$$

and so, by Eq. (8a) have no TT parts. This result is the basis of the Birkhoff theorem:[2] all spherically symmetric configurations are also time-independent, a result valid also in full general relativity.

Unlike Maxwell, there is another category of fields lacking a TT part, namely those with dipole character. As we saw there, dipoles permit a transverse vector, but their single direction is not generic enough to construct a TT tensor. Axial symmetry does permit TT, for example via the tensor harmonic $P_2(\cos\theta)$. To summarize at this point, both Maxwell and linearized general relativity gauge fields only allow time-dependence of their true dynamical excitations, and only when those modes can be present, which always excludes spherical symmetry and also dipole symmetry for the general relativity case.

## IV. KERR AND SCHWARZSCHILD

It is instructive, at the linearized level, to relate the exterior solution properties to explicit matter sources. In electrodynamics the current consists of two parts: the charge density $\rho$ and the longitudinal current $\mathbf{j}^L$, which obey the continuity equation $\dot{\rho} + \nabla \cdot \mathbf{j}^L = 0$, and the transverse current $\mathbf{j}^T$. The $(\rho, \mathbf{j}^L)$ subset couples only to the longitudinal electric field, which is equivalent to it, and as we saw, is time-independent away from sources. The transverse electric and magnetic fields are generated by the transverse current and can be time dependent if $\mathbf{j}^T$ is. Similar reasoning applies to general relativity: the source here is the tensor $T_{\mu\nu}$, whose $(T_{00}, T^L_{0i})$ components are like $(\rho, \mathbf{j}^L)$. They obey the same continuity equation and excite only the metric component $h^T$, which is also $t$-independent outside of source distributions. Because general relativity is a tensor theory, there is another "charge" associated with momentum like $T_{00}$ was with energy, namely $(T_{0i}, T^L_{ij})$, which also obeys continuity and is coupled to $N^T_i$. The remaining source part, $T^{TT}_{ij}$, which may, but need not, depend on time, excites the dynamical $h^{TT}_{ij}$ fields.

An important example of time-independence is furnished by the Kerr solution[3,4] of full general relativity, which we will reproduce in the following. In our linearized context, the static metric is generated by a time-independent spinning point mass with

$$T_{00} = m\delta^3(\mathbf{r}), \quad T_{0i} = am\epsilon_{ijk}s_j\partial_k\delta^3(\mathbf{r}), \tag{11}$$

where $s_j$ denotes the (constant) unit spin vector. As explained in Ref. 5 the space integral of $T_{00}$ is the total mass $m$, and that of $T_{0i}$ vanishes because there is no momentum. Its first moment, the angular momentum $\mathbf{J}$, is given by $\mathbf{J} = am\mathbf{s}$. The notation choice that expresses $J \sim am$ is historical, but has the virtue that $m = 0$ is actually just flat space (also in full general relativity) and the parameter $a$ reduces to that defining ellipsoidal coordinates in ordinary euclidean 3-space. The opposite limit, $a = 0$, defines the spherically symmetric static Schwarzschild solution.

We will not discuss in detail the full general relativity extensions of our linear results. Consider, without deriving it (there is no simple way to do so) the full Kerr interval

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi. \tag{12}$$

There are five functions of $(r, \theta)$ which are (in units of $c = 1 = 16\pi G$),

$$g_{tt} = -(1 - 2Mr/\rho^2) \tag{13a}$$

$$g_{rr} = \rho^2/\Delta \tag{13b}$$

$$g_{\theta\theta} = \rho^2 \tag{13c}$$

$$g_{\phi\phi} = \sin^2\theta\big[r^2 + a^2) + 2a^2 Mr\sin^2\theta/\rho^2\big] \tag{13d}$$

$$g_{t\phi} = -2aMr\sin^2\theta/\rho^2, \tag{13e}$$

with $\rho^2 \equiv r^2 + a^2\cos^2\theta$ and $\Delta \equiv a^2 - 2Mr + r^2$.

The linearized limit of Eqs. (12) and (13), or equivalently its asymptotic form, is a superposition of the (linearized) Schwarzschild solution and a spin term $h_{0\phi}$ corresponding to the source (11)

$$h_{00} = \frac{2m}{r} \tag{14a}$$

$$h_{0\phi} = -\frac{2am\sin^2\theta}{r} \tag{14b}$$

$$h_{ij} = \frac{2m}{r}\frac{x_i x_j}{r^2}. \tag{14c}$$

We emphasize that the time-independence here is derivable directly from the exterior equations, apart from details of the interior source, as we would expect for a spinning spherical ball of charge in E&M, its natural analogue.[6]

## V. CONCLUSIONS

By working primarily in the linearized limit, we have provided, using the Maxwell template, a framework for understanding the basis of time-independence in general relativity in terms of the underlying physics and source geometry. Our main conclusion is that the time-dependence of solutions of gauge theories such as Maxwell's or general relativity is a property of their radiation modes. If these are forbidden due to spherical (dipole) symmetry, then time-independence is guaranteed. In particular, the Kerr and Schwarzschild solutions illustrate the absence of dipole and monopole excitations. Although the full general relativity is unavoidably more complicated (and involves global issues we have bypassed here), our results capture at least its long distance properties.

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* Electronic address: deser@brandeis.edu

† Electronic address: jfrankli@reed.edu

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