Pole Assignment With Improved Control Performance by Means of Periodic Feedback

Javad Lavaei, Somayeh Sojoudi, and Amir G. Aghdam

Abstract—**This technical note is concerned with the pole placement of continuous-time linear time-invariant (LTI) systems by means of LQ sub-optimal periodic feedback. It is well-known that there exist infinitely many generalized sampled-data hold functions (GSHF) for any controllable LTI system to place the modes of its discrete-time equivalent model at prescribed locations. Among all such GSHFs, this technical note aims to find the one which also minimizes a given LQ performance index. To this end, the GSHF being sought is written as the sum of a particular GSHF and a homogeneous one. The particular GSHF can be readily obtained using the conventional pole-placement techniques. The homogeneous GSHF, on the other hand, is expressed as a linear combination of a finite number of functions such as polynomials, sinusoidals, etc. The problem of finding the optimal coefficients of this linear combination is then formulated as a linear matrix inequality (LMI) optimization. The procedure is illustrated by a numerical example.**

Index Terms—**Generalized sampled-data hold functions (GSHF), linear matrix inequality (LMI), linear time-invariant (LTI), linear time-varying (LTV), zero-order hold (ZOH).**

I. Introduction

Sampled-data control system design has been the subject of ongoing research activity in the literature in the past few decades. Discrete-time controllers are used in a broad range of applications such as robotics, autopilot, radar systems, etc., due mainly to their simple implementation, good performance and high accuracy. Various methods have been proposed in the literature for the analysis and synthesis of discrete-time control systems [1]–[5].

A typical discrete-time controller for a continuous-time system consists of a sampler, a digital processing unit, and a zero-order hold (ZOH). The processing unit's function may include, for instance, a Luenberger observer together with a linear time-invariant (LTI) discrete-time state feedback. The overall control operator (including sampler, processor and ZOH), however, acts as a linear time-varying (LTV) law for the original continuous-time system. It is noteworthy that the problem of output feedback stabilization by means of periodic controllers has been extensively studied for both discrete-time systems [6] and continuous-time systems [7], [8].

The idea of employing generalized sampled-data hold functions (GSHF) in lieu of ordinary ZOHs can be traced back to the papers [9], [10]. Several properties of GSHFs such as robustness and noise rejection were studied in [11], where it was shown that not only does a GSHF have a simple structure, but it also acts as a state feedback control without requiring a state estimator. Furthermore, it is known that GSHFs are very effective in simultaneous stabilization of LTI systems [12], [13]. The implementation of a GSHF is rather simple, requiring only a memory device (for storing the GSHF over one period) and a simple processor (for multiplication). Nonetheless, the design of a desirable GSHF may be a formidable job. To be more precise, the design of an LQ optimal GSHF constitutes a two-boundary point differential equation, for which no tractable systematic method is known to date [14]. To circumvent this problem, a novel technique was proposed in [15] to attain a near-optimal GSHF, by converting the infinite-dimensional working space to a finite-dimensional one. This was carried out by expressing the GSHF as a linear combination of some prescribed functions. The technique provided in [15] was further developed in [13] to find a locally optimal GSHF by solving a linear matrix inequality (LMI) problem, iteratively.

GSHFs are also known to be very effective in decentralized control of interconnected systems. In fact, a system that is not stabilizable with respect to LTI decentralized controllers may be stabilized via a proper decentralized GSHF [16]. Furthermore, it was shown in [17] that GSHFs outperform conventional LTI controllers, for a certain class of decentralized control systems.

This technical note deals with pole placement for continuous-time LTI systems using LQ suboptimal GSHFs. In general, there exist infinitely many GSHFs satisfying the pole-placement requirement in the discrete-time domain [11]; each one of them, however, has a distinct intersample ripple effect. Thus, it is desired to find a GSHF which minimizes the intersample ripple, while it places the modes of the closed-loop discrete-time equivalent model at prescribed locations. To this end, a continuous-time quadratic cost function is defined to evaluate the performance of the closed-loop system. The technique provided in [15] is then adopted to map a given set of functions, referred to as the characterizing functions, into a new set. Given a particular GSHF achieving the pole-placement property, it is shown that adding any linear combination of the functions in the new set to this GSHF would not change the location of the closed-loop modes in the discrete-time domain. However, proper adjustment of the coefficients of this linear combination can improve the overall performance of the closed-loop system in the continuous-time domain. The problem of finding the optimal set of coefficients is formulated as an LMI optimization, the solution of which leads to the globally optimal GSHF with respect to the given set of characterizing functions. The results obtained in this work can be generalized to design a structurally constrained (e.g., decentralized) GSHF.

The plan of the technical note is as follows. Some preliminary results are presented in Section II, followed by the problem formulation. The main results of this work are given in Section III. A practical example is then provided in Section IV. Finally, some concluding remarks are drawn in Section V.

Notation: Throughout this technical note, the sets of real, integer and natural numbers are denoted by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively.

II. Preliminaries

Consider an LTI system $\mathcal{S}$ with the following state-space representation:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$ are the state, input and output of the system, respectively. Assume that the pair (A, B) is controllable and that the pair (A, C) is observable. Suppose also that the initial state $x(0)$ is a random variable with zero mean and the covariance matrix X_0. Consider a desirable set of modes $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ in the Laplace domain, along with a sampling period h. Let these modes be mapped to the locations $e^{\lambda_1 h}, e^{\lambda_2 h}, \ldots, e^{\lambda_n h}$ in the z-plane. The goal is to design a discrete-time controller in the form of a generalized sampled-data hold function (GSHF) to place the modes of the discrete-time equivalent model at $e^{\lambda_1 h}, e^{\lambda_2 h}, \ldots, e^{\lambda_n h}$, while the intersample ripple effect in the original system is minimized. The control law being designed has the following form:

$$u(t) = F(t - \kappa h)y[\kappa], \quad \kappa h \le t < (\kappa+1)h, \quad \kappa \in \mathbb{Z} \tag{2}$$

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J. Lavaei and S. Sojoudi are with the Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125 USA (e-mail: lavaei@cds.caltech.edu; sojoudi@caltech.edu).

A. G. Aghdam is with the Department of Electrical and Computer Engineering, Concordia University, Montreal, QC H3G 2W1 Canada (e-mail: aghdam@ece.concordia.ca).

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where $F(t)$ is the GSHF to be obtained, which is periodic with the period h. Note that the discrete argument corresponding to the samples of any continuous-time signal is enclosed in brackets throughout this technical note; e.g., $y[\kappa] := y(\kappa h)$. In the special case when $F(t)$ is equal to 1, the controller (2) turns out to be a simple zero-order hold (ZOH).

It is well-known that the state of the system $\mathcal{S}$ under the controller (2) is governed by the equation given below

$$x(t) = \left(e^{(t-\kappa h)A} + \int_{\kappa h}^{t} e^{(t-\tau)A} BF(\tau - \kappa h) C d\tau \right) x[\kappa] \tag{3}$$

for all κ and t such that

$$\kappa h \leq t \leq (\kappa + 1)h, \quad \kappa \in \mathbb{Z}. \tag{4}$$

Consequently, one can write:

$$x[\kappa + 1] = A_d x[\kappa], \quad \kappa \in \mathbb{Z} \tag{5}$$

where

$$A_d = \left(e^{hA} + \int_0^h e^{(h-\tau)A} BF(\tau) C d\tau \right). \tag{6}$$

If $F(t)$ is designed in such a way that the eigenvalues of the matrix A_d are placed at $e^{\lambda_1 h}, e^{\lambda_2 h}, \ldots, e^{\lambda_n h}$, then the system response at the sampling instants $0, h, 2h, \ldots$ decays as if the modes of the original continuous-time system were placed at $\lambda_1, \lambda_2, \ldots, \lambda_n$. This signifies that for an appropriate choice of the sampling period h, the pole-placement problem for the continuous-time system $\mathcal{S}$ can be translated analogously to that for the discrete-time equivalent model with the matrix A_d, as noted above.

Assume that the sampling period h is not pathological [1] (it is to be noted that in any compact interval in the one-dimensional space, only a finite number of sampling periods are pathological). Since the pair (A, C) is observable, it follows from [1] that the pair (e^{hA}, C) is observable as well. Find a gain L for which the eigenvalues of the matrix $e^{hA} + LC$ are equal to the desirable values $e^{\lambda_1 h}, \ldots, e^{\lambda_n h}$. Now, solve the equation

$$\int_0^h e^{(h-\tau)A} BF(\tau) d\tau = L \tag{7}$$

for the variable $F(t)$. One solution of this equation, denoted by $F_0(t)$, can be obtained as follows [11]:

$$F_0(t) = B^T e^{(h-t)A^T} W_c^{-1} L \tag{8}$$

where W_c is the controllability Gramian associated with the controllable pair (A, B) over the interval $[0, h)$, i.e.:

$$W_c = \int_0^h e^{(h-\tau)A} BB^T e^{(h-\tau)A^T} d\tau. \tag{9}$$

It is easy to argue that (7) has infinitely many solutions (recall that even the classical pole-assignment via conventional state-feedback control does not necessarily have a unique solution for multi-input multi-output systems). This question will be addressed in the next section.

III. Main Results

Corresponding to each initial state $x(0)$, define the following quadratic performance index:

$$J(x(0)) = \int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt \tag{10}$$

where Q and R are given positive semi-definite and positive definite matrices, respectively. Since the initial state of the system is a random variable, define $\bar{J}$ as $\mathcal{E}\{J(x(0))\}$, in which $\mathcal{E}\{\cdot\}$ denotes the expectation operator. The objective here is to find a GSHF $F(t)$ to achieve the following:

i) the pole-placement requirement as given by (7);
ii) the optimal performance index $\bar{J}$ for the system $\mathcal{S}$.

As substantiated in [14], finding a GSHF $F(t)$ to address only criterion (ii) involves solving a two-boundary point differential equation. Since the corresponding problem is computationally cumbersome, a method is proposed in [13] to design a GSHF which is only (locally) optimal with respect to a given set of basis functions, using an LMI formulation. This technique will be exploited in the present work to address both criteria discussed above.

Consider a given set of scalar real-valued functions $\{\zeta_i(t)\}_{i=1}^{\infty}$, referred to as characterizing functions, which are linearly independent over the field of real numbers. Denote the distinct eigenvalues of A with $\sigma_1, \sigma_2, \ldots, \sigma_p$, and assume that the multiplicity of σ_i as an eigenvalue of A is equal to α_i, $i \in \{1, \ldots, p\}$. Create a column vector $V(t)$ consisting of n scalar functions $t^{j-1} e^{-\sigma_i t}$, $\forall i \in \{1, \ldots, p\}$, $\forall j \in \{1, \ldots, \alpha_i\}$. Define now

$$\Gamma = \int_0^h V(t) \begin{bmatrix} \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_n(t) \end{bmatrix} dt,$$

$$\Gamma_l = \int_0^h V(t) \zeta_{n+l}(t) dt, \quad \forall l \in \mathbb{N}. \tag{11}$$

Assumption 1: The matrix Γ is nonsingular.

Note that the above assumption can always be met by a proper choice of the characterizing functions $\{\zeta_i(t)\}_{i=1}^{n}$. Introduce the following functions:

$$\xi_i(t) := - \begin{bmatrix} \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_n(t) \end{bmatrix} \Gamma^{-1} \Gamma_i + \zeta_{n+i}(t) \tag{12}$$

where $i \in \mathbb{N}$.

Lemma 1: The functions $\{\xi_i(t)\}_{i=1}^{\infty}$ are linearly independent over the field of real numbers.

Proof: The proof is an immediate consequence of the observations given below:

- For every integer $i > 1$, the function $\xi_i(t)$ contains a term $\zeta_{n+i}(t)$ which does not exist in any of its prior functions $\xi_1(t), \xi_2(t), \ldots, \xi_{i-1}(t)$.
- $\{\zeta_i(t)\}_{i=1}^{\infty}$ are linearly independent over reals. ■

Theorem 1: Let $\bar{f}(t)$ be a linear combination of the functions $\{\xi_i(t)\}_{i=1}^{\infty}$. For any arbitrary matrix $N \in \mathbb{R}^{m \times r}$, the GSHF $F(t) = F_0(t) + \bar{f}(t)N$ places the modes of the system $\mathcal{S}$ at $\{e^{\lambda_1 h}, \ldots, e^{\lambda_n h}\}$ in the discrete-time domain.

Proof: One can write

$$\begin{aligned}
&\int_0^h V(t) \xi_\eta(\tau) d\tau \\
&= \int_0^h V(t) \zeta_{n+\eta}(t) dt \\
&\quad - \left(\int_0^h V(t) \begin{bmatrix} \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_n(t) \end{bmatrix} dt \right) \Gamma^{-1} \Gamma_\eta \\
&= -\Gamma_\eta + \int_0^h V(t) \zeta_{n+\eta}(t) dt = 0
\end{aligned} \tag{13}$$

or equivalently

$$\int_0^h \tau^{j-1} e^{-\sigma_i \tau} \xi_\eta(\tau) d\tau = 0 \tag{14}$$

for all integers η, i and j satisfying

$$\eta \in \mathbb{N}, \quad i \in \{1,\ldots,p\}, \quad j \in \{1,\ldots,\alpha_i\}. \tag{15}$$

Using the Cayley–Hamilton theorem, one can conclude from the above equation that

$$\int_0^h e^{(h-\tau)A} B\xi_\eta(\tau)d\tau = 0, \quad \forall \eta \in \mathbb{N}. \tag{16}$$

Thus:

$$\int_0^h e^{(h-\tau)A} B\bar{f}(t)d\tau = 0. \tag{17}$$

Given the above relation, the proof results from the discussion following (7). ∎

Consider an arbitrary positive integer q. According to Theorem 1, the GSHF

$$F_0(t) + G_1\xi_1(t) + G_2\xi_2(t) + \cdots + G_q\xi_q(t) \tag{18}$$

places the modes of the discrete-time equivalent model at the desired locations $e^{\lambda_1 h}, e^{\lambda_2 h}, \ldots, e^{\lambda_n h}$, for any arbitrary $m \times r$ real matrices $G_1, G_2, \ldots, G_q$. An important implication of this result is that no constraints have been imposed on the matrices $G_1, G_2, \ldots, G_q$. Define now

$$\begin{aligned}\Pi(t) &:= [\xi_1(t)I_m \quad \xi_2(t)I_m \quad \cdots \quad \xi_q(t)I_m],\\ G &:= \begin{bmatrix} G_1^T & G_2^T & \cdots & G_q^T\end{bmatrix}^T\end{aligned} \tag{19}$$

where I_m is the $m \times m$ identity matrix. The objective is to find the matrices $G_1, \ldots, G_q$ such that the performance index $\bar{J}$ corresponding to the system $\mathcal{S}$ under the GSHF $F_0(t) + \Pi(t)G$ is minimized. To this end, a number of matrices are defined in the sequel

$$\Phi_1(t) := e^{tA} + \int_0^t \left(e^{(t-\tau)A} BF_0(\tau)C\right) d\tau, \tag{20a}$$

$$\Phi_2(t) := \int_0^t \left(e^{(t-\tau)A} B\Pi(\tau)\right) d\tau, \tag{20b}$$

$$\Phi_3 := \int_0^h \left(\Phi_1(t)^T Q\Phi_1(t)\right) dt + \int_0^h \left(C^T F_0(t)^T RF_0(t)C\right) dt, \tag{20c}$$

$$\Phi_4 := \int_0^h \left(\Phi_1(t)^T Q\Phi_2(t)\right) dt + \int_0^h \left(C^T F_0(t)^T R\Pi(t)\right) dt, \tag{20d}$$

$$\Phi_5 := \int_0^h \left(\Phi_2(t)^T Q\Phi_2(t)\right) dt + \int_0^h \left(\Pi(t)^T R\Pi(t)\right) dt, \tag{20e}$$

$$\Phi := e^{hA} + LC, \tag{20f}$$

$$\bar{\Phi} := \Phi_3 + \Phi_4 GC + (\Phi_4 GC)^T + C^T G^T \Phi_5 GC. \tag{20g}$$

Theorem 2: The optimal matrix G minimizing the performance index $\bar{J}$ can be obtained by solving the following optimization problem:

$$\begin{aligned}&\min_{K,G} \operatorname{trace}(KX_0)\\ &\text{s.t.} \quad \begin{bmatrix} -K+\bar{\Phi} & \Phi^T K\\ K\Phi & -K\end{bmatrix} < 0.\end{aligned} \tag{21}$$

Proof: Using (3), one can show that the performance index corresponding to the system $\mathcal{S}$ under a *stabilizing* GSHF $F_0(t) + \Pi(t)G$ can be written as

$$\begin{aligned} J(x(0)) &= \sum_{\kappa=0}^{\infty} \left(\int_{\kappa h}^{(\kappa+1)h} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \right)\\ &= \sum_{\kappa=0}^{\infty} \left(x^T[\kappa]\bar{\Phi}x[\kappa]\right)\\ &= x^T(0) \sum_{\kappa=0}^{\infty} \left(\Phi^{\kappa T}\bar{\Phi}\Phi^{\kappa}\right) x(0). \end{aligned} \tag{22}$$

Therefore

$$\begin{aligned}\bar{J} &= \mathcal{E}\left\{J(x(0)\right\} = \mathcal{E}\left\{x(0)^T K x(0)\right\}\\ &= \mathcal{E}\left\{\operatorname{trace}\left(Kx(0)x(0)^T\right)\right\} = \operatorname{trace}(KX_0)\end{aligned} \tag{23}$$

where K is the solution of the following Lyapunov equation:

$$\Phi^T K\Phi + \bar{\Phi} - K = 0. \tag{24}$$

It is known that the closed-loop stability condition and the above equation can be concurrently replaced by the inequalities $K > 0$ and $\Phi^T K\Phi + \bar{\Phi} - K < 0$ in the underlying optimization problem (see Lemma 2 in [18] for a detailed discussion on this point). The proof results from applying the Schur complement formula to these two inequalities. ∎

Problem 1: Consider the following LMI optimization problem:

$$\min_{K,G} \operatorname{trace}(KX_0) \tag{25}$$

subject to :

$$\begin{bmatrix} -K+\Phi_3+\Phi_4GC+(\Phi_4GC)^T & \Phi^TK & C^TG^T\Phi_5^{\frac{1}{2}}\\ K\Phi & -K & 0\\ \Phi_5^{\frac{1}{2}}GC & 0 & -I\end{bmatrix} < 0. \tag{26}$$

Denote a minimizer of this convex optimization with (K^*, G^*).

Remark 1: It is important to note that the matrix Φ_5 in (20e) is positive semi-definite; therefore, its square root exists.

Theorem 3: Among the GSHFs of the form $F_0(t) + \Pi(t)G$ (which place the modes of the discrete-time equivalent model at $e^{\lambda_1 h}, e^{\lambda_2 h}, \ldots, e^{\lambda_n h}$), the one that minimizes the performance index $\bar{J}$ is $F_0(t) + \Pi(t)G^*$.

Proof: Observe that

$$\begin{aligned}\begin{bmatrix} -K+\bar{\Phi} & \Phi^TK\\ K\Phi & -K\end{bmatrix} &= \begin{bmatrix} C^TG^T\Phi_5^{\frac{1}{2}}\\ 0\end{bmatrix}\begin{bmatrix}\Phi_5^{\frac{1}{2}}GC & 0\end{bmatrix}\\ &+ \begin{bmatrix} -K+\Phi_3+\Phi_4GC+(\Phi_4GC)^T & \Phi^TK\\ K\Phi & -K\end{bmatrix}.\end{aligned} \tag{27}$$

The proof follows by applying the Schur complement formula to the above equation, and using the result of Theorem 2. ∎

Designing an LQ optimal GSHF in the form of a linear combination of a given set of functions is formulated in [13] as an iterative LMI, due to its nonconvexity in general. In contrast, it is shown in this technical note that if the elements of this set (i.e. ξ_i, $i \in \mathbb{N}$) are related to each other in such a way that a pole-placement condition is met, then the underlying optimization can be recast as an LMI problem.

Remark 2: It is straightforward to show that the matrix A_d given in (6) is the same for both GSHFs $F_0(t)$ and $F_0(t) + \Pi(t)G$. Hence, (5) yields that the state of the system $\mathcal{S}$ under both GSHFs is the same at the sampling instants, but the corresponding inputs are different at the sampling times unless $\xi_1(0) = \cdots = \xi_q(0) = 0$.

Remark 3: Designing a structurally constrained GSHF to achieve the pole-placement objective is discussed in [12], and a formula analogous to (8) is presented therein to attain the desired GSHF. The method provided here can be employed to improve the performance of the corresponding structurally constrained GSHF while maintaining the desired locations for the closed-loop modes. It is to be noted that in the case when $F(t)$ is required to be a constrained matrix (e.g., a block-diagonal matrix representing a decentralized control structure), certain entries of G in Problem 1 must be set to zero.

IV. Numerical Example

Consider the problem of controlling the planar motion of two unmanned aerial vehicles (UAVs) in a leader-follower formation. Let the vehicles be labeled as UAV 1 (leader) and UAV 2 (follower). The goal is to fly the UAVs at a constant velocity $\mathbf{v}_0$ with the relative distance vector $\mathbf{d}_0$. To this end, one can write the following equation of motion for UAV i (i =1, 2):

$$\begin{bmatrix} \dot{X}_i(t) \\ \dot{Y}_i(t) \\ \dot{\psi}_i(t) \\ \dot{v}_i(t) \end{bmatrix} = \begin{bmatrix} v_i(t)\cos(\psi_i(t)) \\ v_i(t)\sin(\psi_i(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_i(t) \\ \omega_i(t) \end{bmatrix} \tag{28}$$

where $X_i(t), Y_i(t), \psi_i(t), v_i(t), a_i(t)$ and $\omega_i(t)$ denote the horizontal coordinates, vertical coordinates, heading angle, speed, acceleration and angular velocity of UAV i, respectively (see [19] or [20] for the detailed derivation of the equations presented in this example). So long as $v_i(t)$ is bounded away from zero, a proper change of variables can be used to rewrite the equation of motion in the following form:

$$\begin{bmatrix} \dot{z}_1^i(t) \\ \dot{z}_2^i(t) \end{bmatrix} = \begin{bmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{bmatrix} \begin{bmatrix} z_1^i(t) \\ z_2^i(t) \end{bmatrix} + \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix} u^i(t) \tag{29}$$

where 0_2 represents the 2×2 zero matrix, and

$$z_1^i(t) = \begin{bmatrix} X_i(t) \\ Y_i(t) \end{bmatrix}, \quad z_2^i(t) = \begin{bmatrix} v_i(t)\cos(\psi_i(t)) \\ v_i(t)\sin(\psi_i(t)) \end{bmatrix},$$
$$u^i(t) = \begin{bmatrix} a_i(t)\cos(\psi_i(t)) - v_i(t)\omega_i(t)\sin(\psi_i(t)) \\ a_i(t)\sin(\psi_i(t)) + v_i(t)\omega_i(t)\cos(\psi_i(t)) \end{bmatrix}. \tag{30}$$

In order to write the formation model in relative coordinates, define

$$x(t) := \begin{bmatrix} z_2^1(t) - \mathbf{v}_0 \\ z_1^1(t) - z_1^2(t) - \mathbf{d}_0 \\ z_2^2(t) - \mathbf{v}_0 \end{bmatrix}, \quad u(t) := \begin{bmatrix} u^1(t) \\ u^2(t) \end{bmatrix}. \tag{31}$$

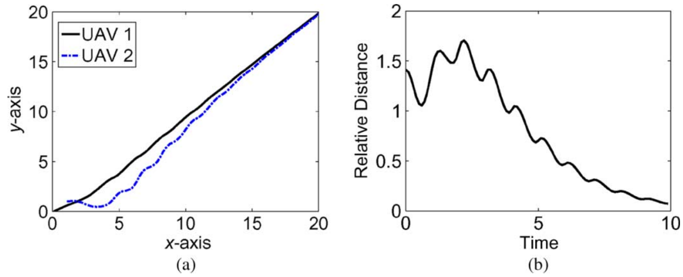

Fig. 1. (a) Trajectories of the UAVs under the initial GSHF; (b) the relative distance between the UAVs.

It is easy to verify that

$$\dot{x}(t) = \begin{bmatrix} 0_2 & 0_2 & 0_2 \\ I_2 & 0_2 & -I_2 \\ 0_2 & 0_2 & 0_2 \end{bmatrix} x(t) + \begin{bmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \\ 0_2 & I_2 \end{bmatrix} u(t). \tag{32}$$

Let the output of the above system be

$$y(t) = \begin{bmatrix} 0_2 & I_2 & 0_2 \\ I_2 & 0_2 & I_2 \end{bmatrix} x(t). \tag{33}$$

Denote the i-th entry of $x(t)$ and the j-th entry of $u(t)$ with $x_i(t)$ and $u_j(t)$, respectively, for every $i \in \{1, \ldots, 6\}$ and $j \in \{1, \ldots, 4\}$. It is worth noting that:

- $(x_1(t), x_2(t))$ is the velocity vector of UAV 1 minus the desired velocity $\mathbf{v}_0$.
- $(x_3(t), x_4(t))$ is the distance between UAV 1 and UAV 2 minus the desired relative distance $\mathbf{d}_0$.
- $(x_5(t), x_6(t))$ is the velocity vector of UAV 2 minus the desired velocity $\mathbf{v}_0$.

Using the above formulation, the formation flying problem posed in this example reduces to designing a stabilizing controller for the system governed by (32) and (33). A periodic feedback will be contrived in the sequel to achieve this end. Assume that $h = 1$ sec, $\mathbf{d}_0 = 0$, $\mathbf{v}_0 = [2 \;\; 2]^T$, and that the modes of the discrete-time equivalent model are desired to be placed at {0.550, 0.551, 0.552, 0.553, 0.554, 0.555} (this is, in fact, a simple rendezvous problem). A particular GSHF satisfying the pole-placement constraint can be obtained using (8) as the one given in (34), shown at the bottom of the page.

The trajectories of the UAVs under the initial GSHF are plotted in the 2-D plane in Fig. 1(a) for the first 10 s of the flight. Moreover, the relative distance between UAV 1 and UAV 2 is depicted in Fig. 1(b). These figures demonstrate that the distance between the UAVs gradually decreases. To improve the performance of this control system, it is desired to design a cheap control optimal GSHF. Suppose that the initial state $x(0)$ is a random variable with zero mean and identity covariance. Choose $R = 10^{-4} I_4$ and $Q = \mathrm{diag}(0, 0, 4, 4, 0, 0)$, which imply that there is almost no constraint on the control effort and only the relative distances in the x and y directions are to be regulated.

Consider the basis functions $\zeta_k(t) = e^{-(t/k^5)}$, $k \in \mathbb{N}$. The function $\xi_1(t)$ can be computed using (12), on noting that

$$\Gamma^{-1}\Gamma_1 = [1.849 \quad -8.126 \times 10^{-1} \quad -2.051 \times 10^{-2} \quad -1.714 \times 10^{3} \quad 1.714 \times 10^{3} \quad 1.187 \times 10^{-1}]^T. \tag{35}$$

$$F_0(t) = \begin{bmatrix} -\frac{2477}{1000} + \frac{951}{200}t & 0 & -\frac{2671}{10000} + \frac{201}{2500}t & 0 \\ 0 & -\frac{6231}{2500} + \frac{5979}{1250}t & 0 & -\frac{789}{2500} + \frac{831}{5000}t \\ \frac{2477}{1000} - \frac{951}{200}t & 0 & -\frac{1789}{10000} - \frac{201}{2500}t & 0 \\ 0 & \frac{6231}{2500} - \frac{5979}{1250}t & 0 & -\frac{667}{5000} - \frac{831}{5000}t \end{bmatrix} \tag{34}$$

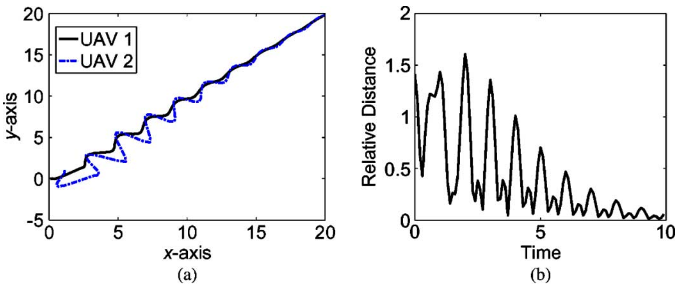

Fig. 2. (a) Trajectories of the UAVs under the suboptimal GSHF; (b) the relative distance between the UAVs.

According to Theorem 1, the GSHF $F(t) = F_0(t) + G_1\xi_1(t)$ satisfies the pole-placement requirement for every arbitrary matrix G_1 of appropriate dimension. One can solve Problem 1 in order to find a matrix that minimizes the performance index $\bar{J}$. This leads to

$$G_1^* = \begin{bmatrix} 199.137 & 0 & -1.372 & 0 \\ 0 & 198.763 & 0 & -2.837 \\ -199.137 & 0 & 1.3724 & 0 \\ 0 & -198.763 & 0 & 2.837 \end{bmatrix}. \quad (36)$$

The performance index $\bar{J}$ corresponding to the particular GSHF $F_0(t)$ and the suboptimal GSHF $F_0(t) + G_1\xi_1(t)$ is equal to 66.3 and 25.3, respectively. This means that adding only one function $(q = 1)$ to the initial GSHF $F_0(t)$ improves the performance of the system by about 62%. For the sake of simulation, consider the following deterministic initial state:

$$x(0) = [1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1]^T. \quad (37)$$

The trajectories of UAVs 1 and 2 under the suboptimal GSHF are sketched in the 2-D plane in Fig. 2(a) for the first 10 sec of the flight. In addition, the relative distance between the UAVs is provided in Fig. 2(b). By comparing Fig. 2(b) with Fig. 1(b), one can observe that under the suboptimal GSHF, the UAVs fly very close to each other most of the time, whereas the initial (particular) GSHF reduces the distance between the UAVs slowly (notice that the L_2 norm of the relative distance given in Fig. 2(b) is much smaller than that of the one in Fig. 1(b)). As shown in Figs. 1(a) and 2(a), this is achieved by forcing the UAVs to follow some oscillatory paths. Note that the input energy of the optimal control system in the transient period [0, 10] is about five times greater than that of the initial control system (because in the cheap control problem the input energy is not taken into account). Hence, the suboptimal GSHF designed using the method proposed in the present work reduces the relative distance between the UAVs by about 62% in average (in the LQ sense specified earlier), at the cost of exerting more input energy.

V. Conclusion

This work tackles the problem of designing a suboptimal generalized sampled-data hold function (GSHF) which places the modes of the discrete-time equivalent model of a given continuous-time linear time-invariant (LTI) system in the desired locations. Given a controllable continuous-time LTI system, there exist infinitely many GSHFs to assign the closed-loop poles of the discrete-time equivalent model in prescribed locations. In order to find a GSHF in the above-mentioned infinite set which yields an optimal LQ performance with respect to a certain subset of this infinite set, the underlying hold function is expressed as the sum of two functions: the particular GSHF and the homogeneous GSHF. It is straightforward to design the particular GSHF using existing techniques. The homogeneous GSHF, on the other hand, is parameterized systematically using a given set of so-called characterizing functions. A new set of functions is subsequently formed in terms of the characterizing functions, in such a way that any linear combination of the functions in the resulting set is a homogeneous GSHF. The problem of finding the optimal linear combination is translated into an LMI optimization. Simulations demonstrate the performance benefits of the proposed design approach.

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