RELATIVISTIC WAVE MECHANICS OF ELECTRONS DEFLECTED BY A MAGNETIC FIELD

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Abstract

It is shown that the relativistic wave equation for electrons in a uniform magnetic field leads to the same wave function as that already deduced by Page from the non-relativistic equation. As in the latter case the motion at right angles to the field is quantized.

An expression is found for the current density from the relativistic wave equation. The relativistic expression differs from the non-relativistic only by a constant factor which does not affect the calculation of the mean radii of curvature of the electron current.

Hence, for the relativistic case, as for the non-relativistic, the mean radius of curvature is less than that expected on the classical theory. It follows that the classical relativistic relation between ϵ/μ and the mean radius of curvature upon deflection gives a value of ϵ/μ which is too large.

THE non-relativistic Schrödinger wave equation which governs the motion of electrons in a uniform magnetic field has been solved by Page.[1] The solution which he obtains shows that the mean radius of curvature of the electron current is less than that given by the classical theory. An appreciable error is thus introduced in applying the classical relation to the determination of ϵ/μ from deflection experiments.

Since the energies of the electrons used in deflection experiments are so large, it was thought that a treatment of the problem by the relativistic wave mechanics would be of interest.

The Relativistic Wave Equation

The relativistic wave equation[2] for electrons in a magnetic field is

$$\nabla\cdot\nabla\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{4\pi i\epsilon}{hc}A\cdot\nabla\psi - \frac{4\pi^2}{h^2}\left(\mu^2c^2 + \frac{\epsilon^2}{c^2}A^2\right)\psi = 0 \tag{1}$$

where A is the vector potential and μ is the rest mass of the electron. For a uniform field H in the X direction

$$A = -j\frac{H}{2}z + k\frac{H}{2}y.$$

Eq. (1) then becomes in cylindrical coordinates r, θ, x,

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[1] Leigh Page, Phys. Rev. **36,** 444 (1930).

[2] L. Brillouin, J. de Phys. et Le Radium **8,** 74 (1927).

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{2\pi i\epsilon H}{hc}\frac{\partial\psi}{\partial\theta}$$
$$-\frac{4\pi^2}{h^2}\left(\mu^2c^2+\frac{\epsilon^2H^2r^2}{4c^2}\right)\psi = 0. \tag{2}$$

We put

$$\psi = R(r)X(x)e^{-im\theta}e^{-2\pi i(\epsilon/h)t} \tag{3}$$

where $\mathcal{E}$ is the total energy, and m must be an integer for ψ to be single-valued. Eq. (2) then leads to the ordinary differential equations

$$\frac{d^2X}{dx^2}+\alpha X = 0, \tag{4}$$

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left\{\frac{4\pi^2\mathcal{E}^2}{h^2c^2}-\frac{4\pi^2\mu^2c^2}{h^2}-\frac{2\pi\epsilon Hm}{hc}\right.$$
$$\left.-\frac{\pi^2\epsilon^2H^2r^2}{h^2c^2}-\frac{m^2}{r^2}-\alpha\right\}R = 0, \tag{5}$$

in which α is the constant of separation. Eq. (4) states that the electrons moving parallel to the field act like free particles. If we introduce the quanity

$$W \equiv \frac{1}{2\mu c^2}(\mathcal{E}^2-\mu^2c^4) \tag{6}$$

into Eq. (5), the differential equation for R becomes identical in form with that deduced by Page from the non-relativistic wave equation. We have therefore, as in Page's development,

$$W = \left(s+\frac{1}{2}\right)h\left(\frac{\epsilon H}{2\pi\mu c}\right),\quad s = m+k = 0, 1, 2, \cdots. \tag{7}$$

The energy $\mathcal{E}$ is thus completely quantized.[3] We have also for the radius of curvature r of the deflected electrons[4]

$$r^2 = \frac{2\mu c^2W}{\epsilon^2H^2}\frac{x}{2s+1} \tag{8}$$

in which $x\equiv(2\pi\epsilon H/hc)r^2$. Relativistic electrodynamics gives for the radius of curvature in terms of W as defined in Eq. (6)

$$r^2 = \frac{2\mu c^2W}{\epsilon^2H^2} \tag{9}$$

so that the same correction is introduced here by the wave mechanics treatment as in the non-relativistic case.

[3] If the restriction of m to integral values is removed, the energy is no longer quantized. However, k remains integral in order that the series for R shall terminate, and the validity of subsequent calculations in both the non-relativistic and the relativistic case is not affected.

[4] Page, Eq. (33).

The Current

It is necessary to deduce from Eq. (1) an expression for the current density. If the differential equation satisfied by $\bar{\psi}$ is multiplied by ψ and the result subtracted from the product of Eq. (1) and $\bar{\psi}$, we get

$$\bar{\psi}\nabla\cdot\nabla\psi - \psi\nabla\cdot\nabla\bar{\psi} - \frac{1}{c^2}\left(\bar{\psi}\frac{\partial^2\psi}{\partial t^2} - \psi\frac{\partial^2\bar{\psi}}{\partial t^2}\right) - \frac{4\pi i\epsilon}{hc}A\cdot(\bar{\psi}\nabla\psi + \psi\nabla\bar{\psi}) = 0. \quad (10)$$

Also

$$\bar{\psi}\nabla\cdot\nabla\psi - \psi\nabla\cdot\nabla\bar{\psi} = \nabla\cdot(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}),$$

and

$$A\cdot(\bar{\psi}\nabla\psi + \psi\nabla\bar{\psi}) = \nabla\cdot(A\psi\bar{\psi}).$$

In addition we have

$$\bar{\psi}\frac{\partial^2\psi}{\partial t^2} - \psi\frac{\partial^2\bar{\psi}}{\partial t^2} = \frac{\partial}{\partial t}\left\{\psi\bar{\psi}\frac{\partial}{\partial t}\left(\log\frac{\psi}{\bar{\psi}}\right)\right\}$$
$$= -4\pi i\frac{\mathcal{E}}{h}\frac{\partial}{\partial t}(\psi\bar{\psi})$$

using Eq. (3). Eq. (10) may now be written

$$\frac{\epsilon hc^2}{4\pi i\mathcal{E}}\nabla\cdot\left(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi} - \frac{4\pi i\epsilon}{hc}A\psi\bar{\psi}\right) + \frac{\partial}{\partial t}(\epsilon\psi\bar{\psi}) = 0. \quad (11)$$

We identify Eq. (11) with the equation of continuity so that the current density is

$$j = \frac{\epsilon hc^2}{4\pi i\mathcal{E}}\left(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi} - \frac{4\pi i\epsilon}{hc}A\psi\bar{\psi}\right). \quad (12)$$

This result is to be compared with the non-relativistic expression[5]

$$j = \frac{\epsilon h}{4\pi\mu i}\left(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi} - \frac{4\pi i\epsilon}{hc}A\psi\bar{\psi}\right). \quad (13)$$

The factor $\epsilon hc^2/4\pi i\mathcal{E}$ appearing in Eq. (12) is, like that in Eq. (13), constant in the coordinates so that it does not affect the calculation of the mean radii of curvature averaged with respect to the current density. Hence the relativistic wave equation gives the same values for the mean radii of curvature as those calculated by Page.[6]

Conclusion

The relativistic wave equation leads to the same wave function for electrons in a uniform magnetic field as that determined from the Schrödinger equation.

[5] Condon and Morse, Quantum Mechanics, p. 30.

[6] Page, Eqs. (45) and (48).

The electron motion at right angles to the field is found to be completely quantized as before. The relativistic expression for the current density differs from the non-relativistic only in that $\mathcal{E}/c^2$ replaces μ; this difference does not affect the calculation of the mean radii of curvature. We are led to a conclusion similar to that reached by the non-relativistic treatment: the classical relativistic relation between ϵ/μ and the mean radius of curvature upon deflection gives a value for ϵ/μ which is too large.

In conclusion the writer wishes to acknowledge his indebtedness to Professor Page for suggesting this problem and for his helpful criticism during its consideration.