INCREASING THE NUMBER OF FIBERED FACES OF ARITHMETIC HYPERBOLIC 3-MANIFOLDS

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Abstract. We exhibit a closed hyperbolic 3-manifold which satisfies a very strong form of Thurston's Virtual Fibration Conjecture. In particular, this manifold has finite covers which fiber over the circle in arbitrarily many ways. More precisely, it has a tower of finite covers where the number of fibered faces of the Thurston norm ball goes to infinity, in fact faster than any power of the logarithm of the degree of the cover, and we give a more precise quantitative lower bound. The example manifold M is arithmetic, and the proof uses detailed number-theoretic information, at the level of the Hecke eigenvalues, to drive a geometric argument based on Fried's dynamical characterization of the fibered faces. The origin of the basic fibration $M \to S^1$ is the modular elliptic curve $E = X_0(49)$, which admits multiplication by the ring of integers of $\mathbb{Q}[\sqrt{-7}]$. We first base change the holomorphic differential on E to a cusp form on GL(2) over $K = \mathbb{Q}[\sqrt{-3}]$, and then transfer over to a quaternion algebra D/K ramified only at the primes above 7; the fundamental group of M is a quotient of the principal congruence subgroup of $\mathscr{O}_D^*$ of level 7. To analyze the topological properties of M, we use a new practical method for computing the Thurston norm, which is of independent interest. We also give a noncompact finite-volume hyperbolic 3-manifold with the same properties by using a direct topological argument.

Contents.

1. Introduction. The most mysterious variant of the circle of questions surrounding Waldhausen's Virtual Haken Conjecture [Wal1] is:

1.1 *Virtual Fibration Conjecture*. (Thurston) If M is a finite-volume hyperbolic 3-manifold, then M has a finite cover which fibers over the circle, i.e., is a surface bundle over the circle.

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This is a very natural question, equivalent to asking whether $\pi_1(M)$ contains a *geometrically infinite* surface group. However, compared to the other forms of the Virtual Haken Conjecture, there are relatively few nontrivial examples where it is known to hold, especially in the case of closed manifolds (but see [Rei], [Lei], [Wal2], [But] and especially [Ago1]). Moreover, there are indications that fibering over the circle is, in suitable senses, a rare property compared, for example, to simply having nontrivial first cohomology [DT], [Mas1].

Despite this, we show here that certain manifolds satisfy Conjecture 1.1 in a very strong way, in that they have finite covers which fiber over the circle in many distinct ways. For a 3-manifold M, the set of classes in $H^1(M;\mathbb{Z})$ which can be represented by fibrations over the circle are organized by the Thurston norm on $H^1(M;\mathbb{R})$. The unit ball in this norm is a finite polytope where certain top-dimensional faces, called the fibered faces, correspond to those cohomology classes coming from fibrations (see Section 2 for details). The number of fibered faces thus measures the number of fundamentally different ways that M can fiber over the circle. We will sometimes abusively refer to these faces of the Thurston norm ball as "the fibered faces of M".

If $N \to M$ is a finite covering map, the induced map $H^1(M;\mathbb{R}) \to H^1(N;\mathbb{R})$ takes each fibered face of M to one in N; if $H^1(N;\mathbb{R})$ is strictly larger than $H^1(M;\mathbb{R})$, then it may (but need not) have additional fibered faces. The qualitative form of our main result is:

THEOREM 1.2. *There exists a closed hyperbolic 3-manifold M which has a sequence of finite covers M_n so that the number of fibered faces of the Thurston norm ball of M_n goes to infinity.*

Moreover, we prove a quantitative refinement of this result (Theorem 1.4) which bounds from below the number of fibered faces of M_n in terms of the degree of the cover. While it is the closed case of Conjecture 1.1 that is most interesting, we also give an example of a *noncompact* finite-volume hyperbolic 3-manifold with the same property (Theorem 8.3) using a simple topological argument.

The example manifold M of Theorem 1.2 is arithmetic, and the proof uses detailed number-theoretic information about it, at the level of the Hecke eigenvalues, to drive a geometric argument based on Fried's dynamical characterization of the fibered faces. To state the geometric part of the theorem, we need to introduce the Hecke operators (see Section 3.1 for details). Suppose M is a closed hyperbolic 3-manifold, and we have a pair of finite covering maps $p, q \colon N \to M$; when M is arithmetic there are many such pairs of covering maps because the commensurator of $\pi_1(M)$ in $\mathrm{Isom}\,(\mathscr{H}^3)$ is very large. The associated *Hecke operator* is the endomorphism of $H^1(M)$ defined by $T_{p,q} = q_* \circ p^*$, where $q_* \colon H^1(N) \to H^1(M)$ is the transfer map. The simplest form of our main geometric lemma is the following:

LEMMA 1.3. *Let M be a closed hyperbolic 3-manifold, and $p, q \colon N \to M$ a pair of finite covering maps. If $T_{p,q}(\omega) = 0$ for some $\omega \in H^1(M;\mathbb{Z})$ coming from a fibration over the circle, then $p^*(\omega)$ and $q^*(\omega)$ lie in distinct fibered faces.*

We prove this lemma in Section 3. Then in Section 4, we give a manifold with an infinite tower of covers to which Lemma 1.3 applies at each step, thus proving Theorem 1.2. In fact, we show the following refined quantitative version:

THEOREM 1.4. *There is a closed hyperbolic 3-manifold M of arithmetic type, with an infinite family of finite covers $\{M_n\}$ of degree d_n, where the number ν_n of fibered faces of M_n satisfies*

$$\nu_n \geq \exp\left(0.3\frac{\log d_n}{\log\log d_n}\right) \quad \textit{as } d_n \to \infty.$$

In particular, for any $t < 1$, there is a constant $c_t > 0$ such that

$$\nu_n \geq c_t e^{(\log d_n)^t}.$$

Note that this bound for ν_n is slower than any positive power of d_n, but is faster than any (positive) power of $\log d_n$. For context, the Betti number $b_1(M_n) = \dim H^1(M_n; \mathbb{R})$ is bounded above by (a constant times) the degree d_n, and bounded below (see Proposition 7.8), for any $\varepsilon > 0$, by (a constant times) $d_n^{1/2-\varepsilon}$ for n large enough (relative to ε), while ν_n is at least as large as (a constant times) $e^{(\log d_n)^{0.99}}$. In our noncompact example of Theorem 8.3, the number of faces grows exponentially in the degree while b_1 grows linearly.

We now describe the basic ideas behind the construction of the manifolds in Theorem 1.4. For an arithmetic hyperbolic 3-manifold M, one has a Hecke operator as above associated to each prime ideal of the field of definition. The key to applying Lemma 1.3 repeatedly is to have a fibered class $\omega \in H^1(M)$ which is killed by infinitely many such Hecke operators. One can produce cohomology classes which are killed by infinitely many Hecke operators using the special class of CM forms. Fortunately, there is a manifold of manageable size whose cohomology contains a CM form coming from base change of an automorphic form associated to a certain elliptic curve with complex multiplication, and that class turns out to fiber over the circle!

1.5. Outline of the arithmetic construction. Here is a sketch of how the manifolds $\{M_n\}$ of Theorem 1.4 are built from arithmetic; for details, see Section 4. Let E be the elliptic curve over $\mathbb{Q}$ defined by $y^2+xy = x^3-x^2-2x-1$, which has conductor 49 and admits complex multiplication by the ring of integers of the imaginary quadratic field $\mathbb{Q}[\sqrt{-7}]$. It corresponds to a holomorphic cusp form of weight 2 for the congruence subgroup $\Gamma_0(49)$ of $\mathrm{SL}(2,\mathbb{Z})$ acting on the upper half plane $\mathscr{H}$, given by $f(z) = \sum_{n\geq 1} a_n q^n$ with $q = e^{2\pi i z}$, where for every prime $p \neq 7$, the eigenvalue of f under the Hecke operator T_p is $a_p = p+1-|E(\mathbb{F}_p)|$. The differential $f(z)dz$ is invariant under $\Gamma_0(49)$, and hence defines a holomorphic 1-form on $Y_0(49) := \Gamma_0(49)\backslash\mathscr{H}$, and by the cuspidality of f, this differential extends to the natural (cusp) compactification of $Y_0(49)$. Put $K = \mathbb{Q}[\sqrt{-3}]$, in which the prime 7 splits as $Q\overline{Q}$, with $Q = \left(2 + (1 + i\sqrt{3})/2\right)$. Let f_K denote the base change of f to K, which is a cusp form on the hyperbolic 3-space $\mathscr{H}^3$ of "weight 2" and level $Q^2\overline{Q}^2$ for the group $\mathrm{GL}(2, \mathscr{O}_K)$. One can associate to f_K a cuspidal automorphic representation π' of $\mathrm{GL}(2, \mathbb{A}_K)$ with trivial central character and conductor $Q^2\overline{Q}^2$; here $\mathbb{A}_K$ denotes the adèle ring of K. By a basic property of base change, we have, for every degree 1 prime P of $\mathscr{O}_K$ above a rational prime $p \neq 7$ unramified in K, the P^{th} Hecke eigenvalue $a_P(f_K)$ of f_K equals the p^{th} Hecke eigenvalue of f, namely $a_p = p + 1 - |E(\mathbb{F}_p)|$.

Let D be the quaternion division algebra over K which is ramified exactly at the primes Q and $\overline{Q}$. Fortunately for us, the local components of π' at Q and $\overline{Q}$ are both supercuspidal (see Lemma 5.3); this has to do with the fact that E does not acquire good reduction over an abelian extension of $\mathbb{Q}_7$, this fact being controlled, thanks to a useful criterion of D. Rohrlich [Roh], by the valuation (at Q and $\overline{Q}$) of the discriminant Δ of E. (After writing this paper, we learned of an earlier proof of the supercuspidality at 7 in [GL], where the argument is somewhat different.) Therefore, by the Jacquet-Langlands correspondence, there is an associated cusp form h of weight 2 on $\mathscr{H}^3$ relative to a congruence subgroup Γ of the units in a maximal order $\mathscr{O}_D$, such that for every unramified prime $P \neq Q, \overline{Q}$, the P^{th} Hecke eigenvalues of f_K and of h coincide. Moreover, for $P \in \{Q, \overline{Q}\}$, we can read off the conductor and the dimension of the associated representation of $(D \otimes_K K_P)^*$ from the local correspondence (see Lemma 5.5). It follows that Γ, which is co-compact, is the principal congruence subgroup of level $7 = Q\overline{Q}$. Our base manifold in Theorem 1.4 is $M = X(7) = \Gamma\backslash\mathscr{H}^3$. While $H^1(M)$ is 3-dimensional, we show that the new subspace of $H^1(M)$ is 2-dimensional and, as a module under the Hecke algebra of correspondences, is isotypic of type $\eta = \eta_h$, the cohomology class defined by h. A difficult computation shows that M fibers over the circle, and moreover, this can be associated with a class of type η.

Now consider the set $\mathscr{P}$ of rational primes $p \neq 7$ which are inert in $\mathbb{Q}[\sqrt{-7}]$, but are split in K as $P\overline{P}$; it has density $1/4$. Then, if we put

$$\mathscr{P}_K = \left\{ P \;\middle|\; \mathscr{N}_{K/\mathbb{Q}}(P) = p \in \mathscr{P} \right\},$$

the Hecke operators T_P act by zero on η. Each P in $\mathscr{P}_K$ gives a covering $M(P)$ of M of degree $\mathscr{N}(P) + 1$ which defines the associated Hecke operator. Lemma 1.3 allows us, since $T_P\eta = 0$, to conclude that the two natural transforms η_1, η_P of η define cohomology classes of $M(P)$ which lie on two different fibered faces. If we order $\mathscr{P}_K$ according to the rational primes p defining them, then we inductively build a covering $M_n = M(P_1 \ldots P_n)$ of degree $d_n := \prod_{j=1}^n (1 + p_j)$ such that there are at least 2^n distinct fibered faces on M_n. Using the density of $\mathscr{P}$, we get the lower bound for $\nu_n \geq 2^n$ given in Theorem 1.4 in terms of d_n.

To understand why we chose to look at this particular example, it may be helpful to note the following. Suppose D_0 is an indefinite quaternion algebra over $\mathbb{Q}$, and Δ a congruence subgroup of D_0^* with associated *Shimura curve* $S = S_\Delta$

over $\mathbb{C}$, which is a compact Riemann surface. For any imaginary quadratic field K such that $D = D_0 \otimes_{\mathbb{Q}} K$ is still a division algebra, we may consider the hyperbolic 3-manifold $M = \Gamma\backslash\mathscr{H}_3$, for a congruence subgroup Γ of D^*. When $\Gamma \cap D_0^* = \Delta$, the surface S embeds in M, and since it is totally geodesic, the cohomology class defined by S cannot give rise to any fibering of M over the circle [Thu2]. This suggests that we will fail to construct cohomology classes on M with the desired fibering property if we transfer to D^* a cusp form on $\mathrm{GL}(2)/K$ which is the base change f_K of an elliptic modular cusp form f of weight 2 which will transfer to such a D_0^*. It is not an accident that we chose our example above where the f of interest does not transfer to any indefinite quaternion algebra over $\mathbb{Q}$.

Finally, it should be noted that if one starts with a non-CM elliptic curve E over $\mathbb{Q}$, then one knows, by a theorem of Elkies, that there are infinitely many primes p for which a_p is zero [Elk]. But for our method to work we would also need an example where this property holds for an infinite set $\mathscr{P}$ of primes p which split in a suitable imaginary quadratic field K. Even then it would only give a qualitative result as $\mathscr{P}$ would have density zero. Our quantitative result (Theorem 1.4) depends on the density of the corresponding $\mathscr{P}$ in the CM case being $1/4$.

1.6. Moral. For an *arithmetic* hyperbolic 3-manifold, the commensurator of its fundamental group is very large, in fact dense in $\mathrm{Isom}^+(\mathscr{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$. Recently, there has been much important work which exploits this density *geometrically*, see [LLR], [CLR2], [Ven], [Ago2]. On the number-theoretic side, the theory of automorphic forms tells us a great deal about the cohomology of arithmetic hyperbolic 3-manifolds as it relates to virtual Haken type questions (see e.g. [Clo], [CD]). To the best of our knowledge, our work here is the first time that *more refined* automorphic information has been combined with geometric arguments and yields, for example, a geometric/topological interpretation of the vanishing of the Hecke eigenvalues. Thus we hope for deeper connections between these two areas in the future. In particular, it would be very interesting to answer the following:

Question 1.7. Is there an automorphic criterion which implies that certain cohomology classes of arithmetic hyperbolic 3-manifold give fibrations over the circle?

1.8. Practical methods for computing the Thurston norm. The example manifold of Theorem 1.4 is quite complicated from a topological point of view; its hyperbolic volume is about 100 and triangulations of it need some 130 tetrahedra. Despite this, we were able to compute its Thurston norm and check that it fibers over the circle, which is necessary for the proof of Theorem 1.4. To do this, we used new methods for both these tasks. While loosely based on normal surfaces, these techniques eschew guaranteed termination in favor of quick results. The basic idea is to consider only normal surfaces representing elements of

$H^1(M)$ that are "obvious" with respect to the triangulation, and then randomize the triangulation until a minimal norm surface is found. These same techniques have been useful in many other examples and are of independent interest. See Sections 6.7 and 6.11 for a complete description of our method, which can often determine whether a manifold made up of several hundred tetrahedra fibers. Computing the Thurston norm is more subtle, and in the case of our particular M, we had to heavily exploit its symmetries, but we also suggest a general method, as of yet untested, for attacking this.

1.9. Improvements. In subsequent work, Long and Reid have considerably strengthened Lemma 1.3 and its extension Theorem 3.12 by showing that the hypothesis on the Hecke operators can be dispensed with:

THEOREM 1.10. (Long and Reid [LR]) *Let M be a closed arithmetic hyperbolic 3-manifold. If M fibers over the circle, then M has finite covers whose Thurston norm balls have arbitrarily many fibered faces.*

In addition to the work of Fried [Fri] on which Lemma 1.3 hinges, the proof of Theorem 1.10 uses work of Cooper, Long, and Reid on suspension pseudo-Anosov flows [CLR1], as strengthened by Masters [Mas2]. In Section 9 we give a simplified proof of Theorem 1.10 using only Fried's theorem; a different, but equally concise, simplification was given by Agol [Ago1]. In any case, the soft nature of its proof mean that Theorem 1.10 cannot be used to prove quantitative results such as Theorem 1.4. However, Theorem 1.10 does have the advantage that it is much easier to apply.

In a major breakthrough, Agol has just shown there are infinitely many commensurability classes of arithmetic hyperbolic 3-manifolds which fiber over the circle [Ago1]. Combining with Theorem 1.10, this means the qualitative behavior of Theorem 1.2 actually occurs in an infinite number of examples, providing further evidence for Conjecture 1.1.

In Theorems 1.4 and 1.10, the distinct fibered faces are all equivalent under the isometry group of the cover manifolds; indeed this is intrinsic to the method. A natural question is whether one can find a tower of covers where the fibered faces fall into arbitrarily many classes modulo isometries. In the case of manifolds with cusps, Theorem 8.3 gives such examples since the number of faces grows exponentially in the degree of the cover, whereas the size of the isometry group of a hyperbolic manifold is bounded linearly in the volume.

1.11. Paper outline. In Section 2, we review the basics of the Thurston norm and Fried's dynamical characterization of the fibered faces. In Section 3, we discuss Hecke operators and congruence covers in the context of arithmetic hyperbolic 3-manifolds, and then prove Lemma 1.3 and its generalization Theorem 3.12 which underpins Theorem 1.4. We give the precise description of the manifold used in Theorem 1.4 in Section 4. We then analyze the automorphic and topological properties of this manifold in Sections 5 and 6 respectively. In Section 7 we assemble the pieces and prove Theorem 1.4. Section 8 gives our example of this phenomenon in the case of hyperbolic 3-manifolds with cusps. Finally, Section 9 gives our simplified proof of Theorem 1.10.

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2. Fibered faces of the Thurston norm ball. Let M be a closed orientable hyperbolic 3-manifold. When a cohomology class $\omega \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ can be represented by a fibration $M \to S^1$, we say that it *fibers*. In this section, we review the work of Thurston and Fried on the structure of the set of fibered classes in $H^1(M;\mathbb{Z})$. It is not hard to see that $\omega \in H^1(M;\mathbb{Z})$ fibers if and only if it can be represented by a nowhere vanishing 1-form: a fibration gives rise to such a form by pulling back the standard orientation 1-form on S^1, and conversely such a form can be integrated (since its periods are integers) into a map to S^1 which is a fibration. If we pass to real coefficients, the latter condition clearly defines an *open* subset U of $H^1(M;\mathbb{R}) \setminus \{0\}$; a fundamental result of Thurston shows that U must have the following very restricted form. This set is prescribed by the Thurston norm, which measures the simplest surface that represents the Poincaré dual of a cohomology class.

More precisely, for $\omega \in H^1(M;\mathbb{Z})$ define its *Thurston norm* by

$$\|\omega\| = \min\{-\chi(\Sigma) \mid \Sigma \text{ is an embedded orientable surface dual to } \omega\},$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, and we further require that Σ has no 2-sphere components. Thurston showed that this gives a norm on $H^1(M;\mathbb{Z})$ which extends continuously to one on $H^1(M;\mathbb{R})$; see [Thu1] for details of the assertions made in this paragraph. Moreover, the unit ball B of this norm is a bounded convex polytope, i.e. the convex hull of finitely many points. Moreover, there are top-dimensional faces of B, called the *fibered faces*, so that $\omega \in H^1(M;\mathbb{Z})$ fibers if and only if it lies in the cone over the interior of a fibered face, that is, the ray from the origin through ω intersects the interior of such a face. We will say that such a fibered ω *lies in* the corresponding fibered face. Finally, as the map $\omega \mapsto -\omega$ preserves fibering, the fibered faces come in pairs interchanged by this symmetry of the Thurston norm ball. When determining the number of fibered faces, we often count in terms of these fibered face pairs, and say that fibered classes α and β lie in *genuinely distinct* fibered faces if both α and $-\alpha$ are not in the fibered face of β.

2.1. Behavior of fibered faces under covers. Now suppose $p \colon N \to M$ is a finite covering map. The natural map $p^* \colon H^1(M; \mathbb{R}) \to H^1(N; \mathbb{R})$ is an embedding, and a deep theorem of Gabai shows that p^* preserves the Thurston norm [Gab1, Cor. 6.18]. Equivalently, if we denote the unit Thurston norm balls of M and N by B_M and B_N respectively, we have $p^*(B_M) = B_N \cap p^* \left(H^1(M; \mathbb{R})\right)$. By work of Stallings, a class $\omega \in H^1(M; \mathbb{Z})$ represents a fibration if and only if $p^*(\omega)$ does [Sta1]. In particular, if N fibers but M does not, then $H^1(N)$ is larger than $H^1(M)$. Thus each fibered face of B_M gives rise to a distinct fibered face of B_N; if N has additional cohomology, we can hope that B_N has new fibered faces, but the interiors of these must be disjoint from $p^* \left(H^1(M)\right)$.

2.2. Fried's work. We now turn to Fried's dynamical characterization of when two fibrations lie in the same fibered face, which is in terms of a certain flow that is transverse to the fibers of the fibration $M \to S^1$. Suppose ϕ is a self-homeomorphism of a closed surface Σ of genus at least 2, and consider the mapping torus with monodromy ϕ:

$$M_\phi = {\Sigma \times [0,1]} \Big/ {(s,1) \sim (\phi(s), 0)}$$

Thurston proved that M_ϕ is hyperbolic if and only if ϕ is what is called pseudo-Anosov [Thu2], [Ota]. The latter means that ϕ is isotopic to a homeomorphism which preserves a pair of foliations of Σ in a controlled way. Henceforth, we always assume that ϕ has been isotoped to such a preferred representative. Now, M_ϕ has a natural suspension flow $\mathscr{F}_\phi$ which is transverse to the circle fibers, where a point moves at unit speed in the $[0,1]$–direction. We will call $\mathscr{F}_\phi$ the *transverse pseudo-Anosov flow*.

Conversely, given $\omega \in H^1(M; \mathbb{Z})$ coming from a fibration, the monodromy ϕ of the bundle structure is well-defined up to isotopy. Thus there is a corresponding transverse pseudo-Anosov flow $\mathscr{F}_\omega$, which is well-defined up to isotopy. Fried's first result is:

THEOREM 2.3. (Fried [Fri]) *Let M be a closed orientable hyperbolic 3-manifold. Then two fibrations of M over the circle lie over the same fibered face if and only if the corresponding transverse pseudo-Anosov flows are isotopic.*

Fried also provided the following characterization of those ω lying over a fibered face. Let $\mathscr{F}$ be the flow associated to a particular fibered face F, and let $D \subset$

$H_1(M;\mathbb{R})$ be the set of *homology directions* of $\mathscr{F}$, namely the set of all accumulation points of the homology classes of long, nearly closed orbits of $\mathscr{F}$. Fried showed that the dual cone to D,

$$C = \left\{\omega \in H^1(M;\mathbb{R}) \;\mid \omega(v) > 0 \text{ for all } v \in D\right\},$$

is precisely the cone on the interior of the fibered face F [Fri, Thm. 7].

One kind of element of D is the homology class of a closed flowline of $\mathscr{F}$. There are always closed flowlines, for instance coming from the orbits of the finitely many singular points of the invariant foliations, which are permuted among themselves by the monodromy. The particular consequence of Fried's work that we will use here, which is immediate from our discussion, is:

LEMMA 2.4. (Fried) *Let M be a closed orientable hyperbolic 3-manifold, with fibered classes $\alpha, \beta \in H^1(M;\mathbb{Z})$. Let c be a closed orbit of the transverse pseudo-Anosov flow for α. If $\beta(c) = 0$, then α and β lie over genuinely distinct fibered faces.*

3. The geometric theorem. We begin this section by proving Lemma 1.3, which contains the key geometric idea of this paper: a fibered cohomology class which is annihilated by a Hecke operator gives rise to two genuinely distinct fibered faces in the corresponding cover. We then give a detailed review of Hecke operators for arithmetic hyperbolic 3-manifolds, and end by proving the complete version of our geometric result (Theorem 3.12) which is needed to prove Theorem 1.4.

3.1. Hecke operators. From a geometric point of view, a Hecke operator is the map on cohomology induced from the following setup. Suppose M is a topological space, and we have a pair of *finite* covering maps $p,q\colon N \to M$. The map on singular chains $C_*(M) \to C_*(N)$ which takes a singular simplex to the sum of its inverse images under q induces transfer homomorphisms $H_*(M) \to H_*(N)$ and $H^*(N) \to H^*(M)$ which run in opposite directions to the usual maps that q induces on (co)homology (see e.g. [Hat, §3.G] for details). The Hecke operator $T_{p,q}$ of this pair of covering maps is the endomorphism of $H^*(M)$ defined by $q_* \circ p^*$, where $p^*\colon H^*(M) \to H^*(N)$ the pullback map and $q_*\colon H^*(N) \to H^*(M)$ is the transfer map.

Remark 3.2. In this paper, M will always be a 3-manifold and we will be interested in the Hecke operator on $H^1(M)$. While the argument below is given purely in terms of cohomology, the geometrically minded reader may prefer to contemplate the Poincaré dual group $H_2(M)$. There, the Hecke operator $T_{p,q}$ on $H_*(M)$ is the composite $q_* \circ p^*$, where p^* is the transfer map. Such Hecke operators commute with the Poincaré duality isomorphism $H^1(M) \cong H_2(M)$, and so it makes no difference whether one takes the homological or cohomological point of view. The action of $T_{p,q}$ on a class $\omega \in H_2(M)$ is particularly simple to think of geometrically: If an embedded surface $\Sigma \subset M$ represents ω, then the immersed surface $q(p^{-1}(\Sigma))$ represents $T_{p,q}(\omega)$.

3.3. Main geometric idea. When M is arithmetic, there are many manifolds N which cover it in distinct ways. Before getting into this, let us give the central topological idea of Theorem 1.2 in its simplest form from the introduction:

LEMMA 1.3. *Let M be a closed hyperbolic 3-manifold, and suppose $\omega \in H^1(M;\mathbb{Z})$ comes from a fibration of M over the circle. Further assume that $p, q\colon N \to M$ are a pair of finite covering maps. If $T_{p,q}(\omega) = 0$ then $p^*(\omega)$ and $q^*(\omega)$ lie in genuinely distinct fibered faces.*

Proof. We will apply Lemma 2.4 to justify our claim. Downstairs in M, let c be a closed orbit of the pseudo-Anosov flow associated to the fibration ω. Then $q^{-1}(c)$ is a closed orbit of the flow associated to the fibration $q^*(\omega)$. To calculate $p^*(\omega)(q^{-1}(c))$, note that for any $\alpha \in H^1(N)$ one has $\alpha(q^*(c)) = (q_*(\alpha))(c)$ and hence

$$p^*(\omega)\left(q^{-1}(c)\right) = p^*(\omega)\left(q^*(c)\right) = q_*\left(p^*(\omega)\right)(c) = T_{p,q}(\omega)(c) = 0$$

as required to apply the lemma. □

In the case of a tower of covers of M, the following strengthening of the previous lemma will be needed to work inductively:

LEMMA 3.4. *Let M be a closed hyperbolic 3-manifold, and $p, q\colon N \to M$ be a pair of finite covering maps. Suppose $\omega_1, \omega_2, \ldots, \omega_n \in H^1(M;\mathbb{Z})$ lie in genuinely distinct fibered faces. If $T_{p,q}(\omega_i) = 0$ for all i, then N has at least $2n$ pairs of fibered faces. More precisely, $\{p^*(\omega_1), \ldots, p^*(\omega_n), q^*(\omega_1), \ldots, q^*(\omega_n)\}$ lie in genuinely distinct fibered faces.*

Proof. By the discussion in Section 2.1, it is clear that the $p^*(\omega_i)$ lie in distinct fibered faces, as do the $q^*(\omega_j)$. Thus it remains to distinguish the face of $p^*(\omega_i)$ from that of $q^*(\omega_j)$. As before, let c be a closed orbit of the pseudo-Anosov flow associated to ω_j, so that $q^{-1}(c)$ is an orbit of the flow associated to $q^*(\omega_j)$. Then

$$p^*(\omega_i)\left(q^{-1}(c)\right) = p^*(\omega_i)\left(q^*(c)\right) = q_*\left(p^*(\omega_i)\right)(c) = T_{p,q}(\omega_i)(c) = 0$$

and so Lemma 2.4 applies as needed. □

3.5. Sources of Hecke operators. Now we turn to the source of such multiple covering maps. For a hyperbolic 3-manifold M, let Γ be its fundamental group, thought of as a lattice in $\mathrm{PSL}(2,\mathbb{C})$. The commensurator of Γ is the subgroup

$$\mathrm{Comm}(\Gamma) = \left\{ g \in \mathrm{PSL}(2,\mathbb{C}) \;\middle|\; g^{-1}\Gamma g \cap \Gamma \text{ is finite index in both } \Gamma \text{ and } g^{-1}\Gamma g \right\}.$$

When M is arithmetic, $\mathrm{Comm}(\Gamma)$ is dense in $\mathrm{PSL}(2,\mathbb{C})$, and Margulis showed that the converse is true as well; indeed, if M is not arithmetic, then Γ has finite index in $\mathrm{Comm}(\Gamma)$.

Regardless, $g \in \mathrm{Comm}(\Gamma)$ can be associated to a Hecke operator as follows. Let $\Gamma_g = g^{-1}\Gamma g \cap \Gamma$, and $M_g = \Gamma_g \backslash \mathscr{H}^3$ be the corresponding closed hyperbolic 3-manifold. Consider the finite covering map $p_g \colon M_g \to M$ induced by the inclusion $\Gamma_g \to \Gamma$. We will define a second such covering map by considering $\Gamma_{g^{-1}} = g\Gamma g^{-1} \cap \Gamma$, and analogously $p_{g^{-1}} \colon M_{g^{-1}} \to M$. Now, as $g\Gamma_g g^{-1} = \Gamma_{g^{-1}}$, the action of g on $\mathscr{H}^3$ induces an isometry $\tau_g \colon M_g \to M_{g^{-1}}$ giving us the picture

$$\begin{tikzcd} M_g \arrow[rr, "\tau_g"] \arrow[dr, "p_g"'] & & M_{g^{-1}} \arrow[dl, "p_{g^{-1}}"] \\ & M. & \end{tikzcd}$$

Combining, we get a covering map $q_g \colon M_g \to M$ by taking $q_g = p_{g^{-1}} \circ \tau_g$. On the level of groups, the covering map q_g corresponds to the homomorphism $\Gamma_g \to \Gamma$ given by $\gamma \mapsto g\gamma g^{-1}$. By definition, the Hecke operator associated to g is T_{p_g,q_g} in the notation of Section 3.1.

3.6. Standard congruence covers. In number theory, one usually restricts to a subset of all the Hecke operators described above which have nice collective properties. For those readers who are unfamiliar with this, and to explain how it interacts with Lemma 3.4, we give a detailed review of the basic setup in Sections 3.6–3.10. This material is standard, and those with a background in number theory may wish to skip ahead to Section 3.11.

To describe these, we first review the general arithmetic construction of a lattice in $\mathrm{Isom}^+(\mathscr{H}^3)$; for details see e.g. [Vig, MR]. Begin with a number field K with exactly one complex place, and choose a quaternion algebra D over K which is ramified at all real places of K. Now at the complex place of K, the algebra $D \otimes_K \mathbb{C}$ must be isomorphic to $M_2(\mathbb{C})$, the algebra of 2×2 matrices. Taking the units gives a homomorphism $D^* \hookrightarrow \mathrm{GL}(2,\mathbb{C})$; dividing out by the respective centers embeds D^*/K^* as a dense subgroup of $\mathrm{PGL}(2,\mathbb{C}) \cong \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathscr{H}^3)$. Now if $\mathscr{O}_D$ is a maximal order of D, then the image Γ of the units $\mathscr{O}_D^*$ in $\mathrm{PGL}(2,\mathbb{C})$ is a lattice, which is cocompact provided D is a division algebra rather than $M_2(K)$. We will denote the quotient hyperbolic 3-orbifold as $X = \Gamma\backslash\mathscr{H}^3$.

For each ideal A of $\mathscr{O}_K$, we can define a corresponding congruence orbifold $X(A)$ covering X as follows. Since we will only need that case, we restrict to when A is square-free. The key case is that of a prime ideal P. Let K_P be the local completion of K at the place P; its valuation ring is denoted $\mathscr{O}_P$ with maximal ideal $\mathscr{P}$ and residue field $\mathbb{F} = \mathscr{O}_P/\mathscr{P}$. The cover $X(P)$ is constructed using the local algebra $D_P = D \otimes_K K_P$.

First, suppose D does not ramify at P, i.e. $D_P \cong M_2(K_P)$. The cover $X(P)$ is the congruence cover of "Γ_0-type" built as follows. The order $\mathscr{O}_{D_P} := \mathscr{O}_D \otimes_{\mathscr{O}_K} \mathscr{O}_P$ is maximal in D_P, and so is conjugate to $M_2(\mathscr{O}_P)$. Thus we get

$$\mathscr{O}_D^* \hookrightarrow \mathscr{O}_{D_P}^* \cong \mathrm{GL}(2, \mathscr{O}_p) \to \mathscr{O}_{D_P}^* \big/ 1 + \mathscr{P}\mathscr{O}_{D_P} \cong \mathrm{GL}(2, \mathbb{F})$$

which induces a homomorphism $\Gamma \to \mathrm{PGL}(2, \mathbb{F})$. By strong approximation, the image of Γ acts transitively on $P^1(\mathbb{F})$. By definition, $X(P) = \Gamma_0(P)\backslash\mathscr{H}^3$ where $\Gamma_0(P)$ is the Γ-stabilizer of a point in $P^1(\mathbb{F})$. This gives a cover $X(P) \to X$ of degree $\left|P^1(\mathbb{F})\right| = N(P) + 1$, where $N(P) = |\mathbb{F}|$ is the norm of P.

Suppose instead P is one of the finitely many primes where D ramifies. Then D_P is the unique quaternion division algebra over K_P, and so is a local skew-field whose valuation ring is $\mathscr{O}_{D_P}$ with a unique maximal bi-ideal $\mathscr{Q}$. If $q = |\mathbb{F}|$ we have

$$\mathscr{O}_D^* \hookrightarrow \mathscr{O}_{D_P}^* \to \mathscr{O}_{D_P}^* \big/ 1 + \mathscr{Q} \cong \mathbb{F}_{q^2}^*$$

See e.g. [MR, §6.4] for details. Set $\Gamma(P) \lhd \Gamma$ to be the projectivization of the kernel of $\mathscr{O}_D^* \to \mathbb{F}_{q^2}^*$, and let $X(P) = \Gamma(P)\backslash\mathscr{H}^3$. The precise degree of $X(P) \to X$ depends on K and D, as for instance $\mathscr{O}_D^* \to \mathbb{F}_{q^2}$ is rarely onto.

More generally, suppose $A = P_1 P_2 \cdots P_n$ is a square-free ideal of $\mathscr{O}_K$. Then $X(A)$ is defined as the common cover of the $X(P_i)$. If D does not ramify at any P_i, strong approximation implies that the degree of $X(A) \to X$ is the product of the degrees of $X(P_i) \to X$.

3.7. Standard Hecke operators. The Hecke operators for these covers are defined as follows. Let $\mathrm{ram}\,(D)$ denote the set of primes where D is ramified. Suppose A divides a square-free ideal B. Then we have the natural covering map

$$\phi_1 = \phi_{1,B/A} \colon X(B) \to X(A).$$

Moreover, if B/A is prime to $\mathrm{ram}(D)$, then for each ideal J dividing B/A, there is a certain covering map:

$$\phi_J = \phi_{J,B/A} \colon X(B) \to X(A),$$

which is just ϕ_1 when J is the unit ideal. These maps are defined below. In particular, for a prime $P \notin \mathrm{ram}\,(D)$ not dividing A, we have two covering maps:

$$\phi_1, \phi_P \colon X(AP) \to X(A)$$

Then the Hecke operator for P on $H^*(X(A))$ is defined as $T_P = T_{\phi_1,\phi_P}$ in the notation of Section 3.1. These have the following key properties:

PROPOSITION 3.8. *Let A be a square-free ideal of $\mathscr{O}_K$.*

(1) *The standard Hecke algebra for $X(A)$ is generated by the T_P for $p \nmid A$ and $p \notin \operatorname{ram}(D)$. For prime ideals P and Q, the Hecke operators T_P and T_Q commute.*

(2) *Consider a cover $\phi_J \colon X(B) \to X(A)$, with B square-free, and the associated* degeneracy map

$$\phi_J^* \colon H^i(X(A)) \to H^i(X(B)).$$

If P is a prime ideal not dividing B, then

$$\phi_J^* \circ T_P^A = T_P^B \circ \phi_J^*$$

where the superscripts on the Hecke operators indicate which orifolds' cohomology is being acted upon.

We will now give a definition of the maps ϕ_J, and indicate how to deduce the properties above. For this it is convenient to work in the adèlic setup.

Let $\mathbb{A}_K = K_\infty \times \mathbb{A}_{K,f}$ be the addle ring of K, with K_∞ denoting the product of the archimedean completions of K and $\mathbb{A}_{K,f}$ being the ring of finite adèles. Put $G = D^*$, which is a reductive algebraic group over K, with center Z. Then $G(K)$ is a discrete subgroup of the locally compact group $G(\mathbb{A}_K) = G_\infty \times G(\mathbb{A}_{K,f})$. Since K has a unique complex place and D is a quaternion division algebra ramified at all the real places, we have $G_\infty \simeq \mathrm{GL}(2, \mathbb{C}) \times \mathbb{H}^{[K:\mathbb{Q}]-2}$, where $\mathbb{H}$ is Hamilton's quaternion algebra over $\mathbb{R}$; consequently, $Z_\infty \backslash G_\infty$ is $\mathrm{PGL}(2, \mathbb{C}) \times \mathrm{SU}(2)^{[K:\mathbb{Q}]-2}$. Moreover, since G is anisotropic, Godement's compactness criterion implies that the adèlic quotient

$$X_{\mathbb{A}} := {}_{G(K)Z(\mathbb{A}_K)} \backslash \mathscr{H}^3 \times G\left(\mathbb{A}_{K,f}\right)$$

is a compact space, equipped with a right action by $G(\mathbb{A}_{K,f})$. We may view $X_{\mathbb{A}}$ as the projective limit of closed hyperbolic 3-orbifolds

$$X_U := X_{\mathbb{A}}/U,$$

as U varies over a cofinal system of compact open subgroups of $G(\mathbb{A}_{K,f})$. These orbifolds are in fact manifolds for deep enough U. The reduced norm on D^* induces a map $Nrd : G(\mathbb{A}_{K,f}) \to \mathbb{A}_{K,f}^*$, and if U is any compact open subgroup of $G(\mathbb{A}_{K,f})$, we can write $G(\mathbb{A}_K)$ as a finite union

$$G(\mathbb{A}_K) = \bigcup_{j=1}^{h(U)} G(K) G_\infty x_j U,$$

for any elements $x_1 = 1, x_2, \ldots, x_{h(U)}$ in $G(\mathbb{A}_{K,f})$ such that $\{Nrd(x_j)\}$ is a complete set of representatives for $K^* Nrd(U) \backslash \mathbb{A}_{K,f}^*$. We may choose these x_j to have components 1 at the primes in ram (D), and such that for finite $v \notin$ ram (D),

$$x_{j,v} = \begin{pmatrix} a_{j,v} & 0 \\ 0 & 1 \end{pmatrix},$$

for some $a_{j,v}$ in K_v^*, of course with $a_{j,v} \in \mathscr{O}_{K,v}^*$ for almost all v. When U is the maximal compact subgroup $G(\mathscr{O}_{K,f}) = \prod_P G(\mathscr{O}_{K,P})$, the number of components $h(U)$ is just the class number h of K.

We will use the following cofinal system of compact open subgroups U_A for ideals $A = \prod_P P^{m(P)}$, given by:

$$U_A = \left(\prod_{P \in \mathrm{ram}(D)} U_P(P^{m_P}) \right) \times \left(\prod_{P \notin \mathrm{ram}(D)} U_{0,P}(P^{m_P}) \right),$$

where for $P \in$ ram (D), the subgroup $U_P(P^{m_P})$ is the principal congruence subgroup of $G(K_P)$ of level P^{m_P}, while for $P \notin$ ram (D),

$$U_{0,P}(P^{m_P}) = \left\{ g \in G(\mathscr{O}_{K,P}) \;\middle|\; g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod P^{m_P} \right\}.$$

Note that $Nrd(U_{0,P}(P^{m_P}))$ is $\mathscr{O}_{K,P}^*$, implying that

$$(A, \mathrm{ram}(D)) = 1 \Rightarrow h(A) := h(U_A) = h.$$

For the main example of this paper $K = \mathbb{Q}[\sqrt{-3}]$ for which $h = 1$.

The space X_{U_A} has $h(A)$ connected components. More precisely, if we put

$$\Gamma_j(A) = G(K) \cap x_j U_A x_j^{-1},$$

we get

$$X_{U_A} = \bigcup_{j=1}^{h(A)} \Gamma_j(A) \backslash \mathscr{H}^3.$$

Here $\Gamma_1(A) \backslash \mathscr{H}^3$ is the orbifold $X(A)$ that we defined in Section 3.6.

3.9. Degeneracy maps. We now turn to the degeneracy maps discussed in Proposition 3.8(2). If P is a prime ideal not in ram (D), choose a uniformizer ϖ at P and define $g(P) \in G(\mathbb{A}_K)$ to be $\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$ at P and 1 at all other places. We then define a map via right multiplication:

$$X_{\mathbb{A}} \to X_{\mathbb{A}} \quad \text{given by } \xi \mapsto \xi g(P).$$

Note that $g(P)$ has only one nontrivial component, namely at P, and that component is diagonal. At the finite levels we get the induced maps

$$\phi'_P\colon X_{U_A \cap g(P) U_A g(P)^{-1}} \to X_{U_A}.$$

It is immediate that

$$P \nmid A \Rightarrow U_A \cap g(P) U_A g(P)^{-1} = U_{AP}.$$

We let ϕ'_P denote again the restriction to the connected component $X(AP)$. It is easy to extend this to square-free ideals J away from A and $\operatorname{ram}(D)$, and if J divides B/A for another square-free ideal B, then we may compose the natural map $\phi_1\colon X(B) \to X(AJ)$ with $\phi_J\colon X(AJ) \to X(A)$ to obtain the map ϕ_J discussed above.

For each prime $P \nmid A$, $P \notin \operatorname{ram}(D)$, the Hecke correspondence $T_P : X(A) \to X(A)$, defined above by pulling back along ϕ_1 and then pushing down by ϕ_P, also acts on the cohomology of $X(A)$ in the obvious way. This Hecke operator does not depend on our choice of uniformizer, because if we replace ϖ by another uniformizer ϖ' at P, then $u := \varpi'\varpi^{-1}$ is a unit at P, and right multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ acts trivially on $X(A)$. So the action of T_P on $H^*(X(A))$ is invariantly defined for any such P, and is denoted as T_P^A if we want to keep track of the level A.

Note that right multiplication by $g(P)$ on the adèlic space $X_{\mathbb{A}}$ induces a family of maps

$$g(P)^A : X_{U_P} \to X_U,$$

as $U = U_A$ runs over congruence subgroups of level A prime to P. Also, the right multiplication by $g(P)$ and $g(Q)$ on $X_{\mathbb{A}}$ evidently commute for two distinct primes P, Q, since the matrices $g(P)$ and $g(Q)$ themselves commute. One gets the identities, for any compact open subgroup U of level A prime to PQ,

$$\begin{aligned} U_{PQ} &= U \cap \left(g(P)g(Q)Ug(Q)^{-1}g(P)^{-1}\right) \\ &= U \cap \left(g(Q)g(P)Ug(P)^{-1}g(Q)^{-1}\right) = U_{QP}. \end{aligned}$$

Moreover

$$U_P \cap \left(g(Q)U_P g(Q)^{-1}\right) = U_{PQ} \quad \text{and} \quad U_Q \cap \left(g(P)U_Q g(P)^{-1}\right) = U_{PQ}.$$

and one also gets a commutative diagram

$$\begin{tikzcd} & X(APQ) \arrow[dl, "g(Q)^{AP}"'] \arrow[dr, "g(P)^{AQ}"] & \\ X(AP) \arrow[dr, "g(P)^{AQ}"'] & & X(AQ) \arrow[dl, "g(Q)^{A}"] \\ & X(A) & \end{tikzcd}$$

From this, and the definition of T_P^A, we get assertion (1) of Proposition 3.9. For assertion (2) we note that T_P descends compatibly to levels A and B with $A \mid B$, as long as $P \nmid B$. Compatibility with ϕ_J^* follows.

3.10. Newforms and oldforms. The cohomology of $X(A)$ decomposes naturally into oldforms and newforms. The former, denoted $H^i(X(A))^{\mathrm{old}}$ is by definition the subspace of $H^i(X(A))$ generated by the images of $\varphi^*_{J,B/A}$, as B varies over all the ideals properly containing A and J varies over divisors of B/A which are prime to $\mathrm{ram}(D)$. Note that an oldform is therefore fixed by a conjugate of a congruence subgroup of smaller level.

As we saw above, the adèlic quotient $X(U_A)$ is not connected, and we have

$$H^3(X(U_A)) = H^3(X_{\mathbb{A}})^{U_A} \simeq \mathbb{Q}^{h(A)},$$

and this holds even when $X(U_A)$ is an orbifold. As $X(A)$ is the connected component, it follows that

$$H^3(X(A)) \simeq \mathbb{Q}.$$

Let us focus on H^1. There is a complementary *new subspace* $H^1(X(A))^{\mathrm{new}}$, which consists of cohomology classes which are genuine to the level at hand, and it can be defined as the annihilator of $H^2(X(A))^{\mathrm{old}}$ under the cup product pairing

$$\cup\colon H^1(X(A)) \times H^2(X(A)) \to H^3(X(A)) \simeq \mathbb{Q}.$$

Since the elements of the Hecke algebra act as correspondences on $X(A)$, this pairing is also functorial for the Hecke action.

3.11. The main geometric theorem. Now we turn to:

THEOREM 3.12. *Let $M = X(A)$ be an arithmetic hyperbolic 3-manifold defined as above from a quaternion algebra D/K and an ideal A of $\mathscr{O}_K$. Let $P_1, P_2, \ldots, P_n$ be prime ideals of $\mathscr{O}_K$ coprime to A at which D does not ramify. Consider the corresponding congruence cover $M(P_1 P_2 \cdots P_n) \to M$, which has degree*

$$\prod \left(1 + N_{K/\mathbb{Q}}(P_i)\right).$$

Suppose $\omega \in H^1(M)$ comes from a fibration over the circle, and that $T_{P_i}(\omega) = 0$ for each P_i. Then $M(P_1 P_2 \cdots P_n)$ has at least 2^n pairs of genuinely distinct fibered faces.

Theorem 1.2 follows from Theorem 3.12 once we exhibit in Section 4 an M and an ω, together with an *infinite* set of primes $\{P_i\}$ such that $T_{P_i}(\omega) = 0$, by considering all the covers $M(P_1 P_2 \cdots P_m)$.

Proof. We claim inductively that $M_m = M(P_1 P_2 \cdots P_m)$ has 2^m fibered classes ω_i lying in genuinely distinct fibered faces which are killed by T_{P_k} for all $k > m$. To simplify notation, set $P = P_{m+1}$. By the discussion above, we have two covering maps

$$\phi_1, \phi_P \colon M_{m+1} \to M_m$$

and let $\widetilde{\omega}_j$ be the 2^{m+1} classes which are the pull-backs of the ω_i by ϕ_1 and ϕ_P. As T_P kills all the ω_i, Lemma 3.4 implies that the $\widetilde{\omega}_j$ lie in genuinely distinct fibered faces. The commutatively property of Hecke operators and degeneracy maps implies that $T_{p_k}(\widetilde{\omega}_j) = 0$ for all j and any $k > m + 1$, completing the induction. $\square$

4. The base manifold *M*. Here is the arithmetic description of our example in the framework of Section 3.6. Consider the imaginary quadratic field $K = \mathbb{Q}[\sqrt{-3}]$, with ring of integers $\mathscr{O}_K$. The nontrivial automorphism of K can be identified with complex conjugation via embedding K into $\mathbb{C}$, and will be denoted as such. The rational prime 7, or rather the ideal $7\mathscr{O}_K$, splits in K as $Q\overline{Q}$. Let D be the unique quaternion division algebra over K which ramifies exactly at Q and $\overline{Q}$. Let $\mathscr{O}_D$ be a maximal order of D; this order is unique up to conjugation as K is quadratic and has restricted class number $h_\infty = 1$. The corresponding hyperbolic 3-orbifold is $X = \mathscr{O}_D^\times \backslash \mathscr{H}^3$. Our example manifold M is the (principal) congruence cover of X of level $Q\overline{Q}$. To state its key properties, we need one more piece of notation. Let $\mathscr{P}$ be the set of rational primes $p \neq 7$ which are inert in $\mathbb{Q}[\sqrt{-7}]$ but are split in K, and consider the set of prime ideals of $\mathscr{O}_K$ given by $\mathscr{P}_K = \left\{ P \;\middle|\; N_{K/\mathbb{Q}}(P) = p \in \mathscr{P} \right\}$. We will show:

THEOREM 4.1. *Let M be the arithmetic hyperbolic 3-manifold described above.*

(1) *The new subspace $V = H^1(M; \mathbb{Q})^{\mathrm{new}}$ is 2-dimensional and isotypic under the Hecke action. Moreover, for each prime P in the set $\mathscr{P}_K$ described above, V is in the kernel of the Hecke operator T_P.*

(2) *There is an $\omega \in V$ coming from a fibration of M over the circle.*

The proofs of parts (1) and (2) are essentially independent, and we tackle them separately in the next two sections. The main result of this paper, Theorem 1.4, follows from considering the congruence covers $M(P_1P_2\cdots P_n) \to M$ where $P_j \in \mathscr{P}_K$. Theorem 4.1 implies that $M(P_1P_2\cdots P_n)$ satisfies the hypotheses of Theorem 3.12, and hence has at least 2^{2n} fibered faces. Using that $\mathscr{P}$ consists of $1/4$ of all rational primes, we then calculate a lower bound on the number of fibered faces in terms of the degree of the cover. The details of this proof of Theorem 1.4 are given in Section 7.

The manifold M is the only one we could find with a CM form coming from a fibration over the circle. Because CM forms are fairly rare, there are very few potential examples where one can computationally examine the topology in order to check fibering, even with the improved methods we introduce here. We were very fortunate that M does indeed have the desired properties, since there are probably only one or two more potential examples that are within reach. The next example to look at would have been starting with the elliptic curve $y^2+y = x^3$ over $\mathbb{Q}$ which has conductor 27 and is CM by $\mathbb{Q}(\sqrt{-3})$ and then base-changing to $\mathbb{Q}(\sqrt{-2})$.

5. The automorphic structure of the cohomology of *M*. In this section, we give the arithmetic construction of the needed cohomology class on M; the reader is urged to refer back to Section 1.5 for an outline before proceeding (the basic construction here has was introduced in [LS]). We show in particular why the underlying automorphic form π_K on $\mathrm{GL}(2)/K$, $K = \mathbb{Q}[\sqrt{-3}]$, transfers to the quaternion division algebra D/K ramified only at the primes above 7, and moreover determine exactly its level, which is crucial. The final result of the section gives a specific (closed) arithmetic hyperbolic 3-manifold M (of level 7) such that the new subspace of $H^1(M, \mathbb{R})$ contains a plane defined by a CM form π_K^D which is killed by half of the Hecke operators T_P. Later in Section 6, we will show that this new subspace is at most 2-dimensional and contains a vector coming from an S^1-fibration. Taken together, this will prove Theorem 4.1.

The proofs are given in detail, especially when there are no precisely quotable references, even though the experts may well be aware of the various steps. While there are an abundance of CM forms which define similar cohomology classes on corresponding manifolds, so far we have managed to find only this one particular form at the confluence of the required arithmetic *and* topological properties. Thus we wish to take extra care in checking the details.

5.1. A cusp form on $\mathrm{GL}(2,\mathbb{Q})$**.** Let E be the elliptic curve over $\mathbb{Q}$ defined by

$$y^2 + xy = x^3 - x^2 - 2x - 1.$$

It has conductor 49, admitting complex multiplication by an order in the imaginary quadratic field $L := \mathbb{Q}[\sqrt{-7}]$. For any rational prime p, we set, as usual,

$$a_p(E) := p + 1 - N_p(E),$$

where $N_p(E)$ is the number of points of E modulo p. The L-function of E is given by

$$L(s,E) := \prod_p \left(1 - a_p(E)p^{-s} + p^{1-2s}\right)^{-1}$$

which, by a theorem of Deuring (see e.g. [Gro]), equals

$$L(s,\Psi) := \prod_P \left(1 - \Psi(P)\mathscr{N}_{L/\mathbb{Q}}(P)^{-s}\right)^{-1} = \prod_p \left(\prod_{P|p} \left(1 - \Psi(P)\mathscr{N}_{L/\mathbb{Q}}(P)^{-s}\right)\right)^{-1}$$

where Ψ is a Hecke character of L of weight 1, and $\mathscr{N}_{L/\mathbb{Q}}$ denotes the norm from L to $\mathbb{Q}$. Equating, for each p, the corresponding Euler p-factors of $L(s,E)$ and $L(s,\Psi)$, we see that

$$p\mathscr{O}_L \quad \text{prime} \Rightarrow a_p(E) = 0.$$

One knows by Hecke, or by using the converse theorem, that Ψ defines a holomorphic cusp form of weight 2 for the congruence subgroup $\Gamma_0(49)$ of $\mathrm{SL}(2,\mathbb{Z})$, given by $f(z) = \sum_{n\geq 1} a_n q^n$ with $q = e^{2\pi i z}$ which is new for that level. Here, we normalize f so that $a_1(f) = 1$ and $a_p(f)$ is the eigenvalue of the p^{th} Hecke operator T_p. One obtains, for every prime $p \neq 7$, that $a_p(f) = a_p(E)$. Thus $a_p(f)$ is zero whenever p is inert in L.

5.2. Ramification at 7. The cusp form f is ramified at 7, but to make this precise, and to see what happens when we base change to $K = \mathbb{Q}[\sqrt{-3}]$, it is necessary for us to consider the situation adèlically.

For any number field k, let $\mathbb{A}_k$ denote the topological ring of adèles of k, which is a restricted direct product $\prod'_v \mathbb{Q}_v$, as v runs over the places of k and k_v denotes the completion of k at v. When k has only one archimedean place up to complex conjugation, e.g. $k = \mathbb{Q}$ or k imaginary quadratic, this place is denoted by ∞. Given any algebraic group over k, it makes sense to consider the locally compact topological group $G(\mathbb{A}_k) = \prod'_v G(k_v)$.

One knows that the cusp form f generates a unitary, cuspidal automorphic representation $\pi = \otimes'_v \pi_v$ of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ with trivial central character, with π_∞ being the lowest discrete series representation of $\mathrm{GL}(2, \mathbb{R})$ ([Gel]), such that $L(s - 1/2, \pi) = L(s, f)$. By construction, π_p is unramified at every prime $p \neq 7$, which translates to the elliptic curve E having good reduction at p. More precisely, the *conductor* $c(\pi)$ of π satisfies

$$c(\pi) = c(\pi_7) = 49.$$

We need to identify the local representation at 7, and in fact to make sure that it is not a principal series representation.

LEMMA 5.3. *Let f, π be as above. Then π_7 is a supercuspidal representation of* $\mathrm{GL}(2, \mathbb{Q}_7)$ *of conductor* 49.

After this paper was written, we learnt from B. Gross of an earlier, different proof of this Lemma in [GL]. Moreover, the referee has remarked that such a result should by now be well known. We will nevertheless give a proof, as it is explicit, not long, and uses an explicit criterion of Rohrlich involving the discriminant.

Proof. Let us first note a few basic facts. Since π is defined by a Hecke character of a quadratic extension, or equivalently since E has complex multiplication, the local representation π_p at any prime p is either in the principal series or is supercuspidal. In either case, we claim that there is a finite extension $F/\mathbb{Q}_p$ over which (the base change of) π_p becomes an unramified principal series representation. Indeed, by the local Langlands correspondence for GL(2) [Kut], at any p the local representation π_p is attached to a semisimple 2-dimensional $\mathbb{C}$-representation σ_p of the local Weil group $W_{\mathbb{Q}_p}$, whose determinant corresponds to the central character of π_p (by local class field theory), which is trivial in our case. Recall that $W_{\mathbb{Q}_p}$ is the dense subgroup of the absolute Galois group $G_p = \mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ generated by the inertia subgroup and the integral powers of a lift to G_p of the Frobenius automorphism $x \to x^p$ of $\overline{\mathbb{F}}_p$. One knows that π_p is in the principal series if and only if σ_p is reducible, necessarily of the form $\nu_p \oplus \nu_p^{-1}$. We may write ν_p as an unramified character times a finite order character λ_p, since the inertia subgroup I_p acts by a finite quotient, and so σ_p becomes unramified over the finite extension $\mathbb{Q}_p(\lambda_p)$, which is the cyclic extension of $\mathbb{Q}_p$ cut out by λ_p by class field theory. By looking at the ℓ-adic representation of G_p attached to σ_p, and using the Néron-Ogg-Shafarevich criterion [Sil], we see that E also acquires good reduction over $\mathbb{Q}_p(\lambda_p)$. Suppose now σ_p is irreducible, in which case π_p is supercuspidal. Again, σ_p is acted on by I_p through a finite quotient H_p, because I_p is profinite and σ_p takes values in $\mathrm{GL}(2, \mathbb{C})$. Consequently, there exists a finite Galois extension $F/\mathbb{Q}_p$ with Galois group H_p such that over F, the representation σ_p becomes unramified and E attains good reduction. Hence the claim.

We argue further that this line of reasoning shows that π_p is in the principal series if and only if E acquires good reduction over an *abelian* extension of $\mathbb{Q}_p$. Indeed, this is clear if π_p is in the principal series, so we may assume that π_p is not of this kind. Then σ_p is irreducible and becomes unramified only over a more complicated extension F due to irreducibility; F is dihedral for $p \geq 3$, but can be wilder for $p = 2$.

For $p \geq 5$, a theorem of Rohrlich [Roh] asserts that E acquires good reduction over an abelian extension of $\mathbb{Q}_p$ if and only if the following field is abelian over $\mathbb{Q}_p$:

$$F := \mathbb{Q}_p(\Delta^{1/e}),$$

where Δ is the discriminant of E and

$$e = \frac{12}{(v_p(\Delta), 12)},$$

where the denominator on the right signifies the gcd of $v_p(\Delta)$ and 12.

In the present case, $p = 7$ and $\Delta = -7^3$, so that $e = 12/3 = 4$. Consequently, F is generated over $\mathbb{Q}_7$ by a fourth root of 343. The only way F can be abelian over $\mathbb{Q}_7$ is for $\mathbb{Q}_7$ to contain the fourth roots of unity, i.e., for -1 to be a square in $\mathbb{Q}_7$, and hence in $\mathbb{F}_7$. But this does not happen because the Legendre symbol $\left(\frac{-1}{7}\right)$ is $(-1)^{(7-1)/2} = -1$. Hence π_7 is supercuspidal.

Finally, the conductor $c(\pi)$ of π factors as

$$c(\pi) = \prod_p c(\pi_p),$$

and since π is associated to E, the conductor $c(\pi)$ coincides with that of E, which is 49. Moreover, as π_p is unramified at every prime outside 7, we have $c(\pi_p) = 1$ for every $p \neq 7$. It follows that $c(\pi_7) = 49$. □

5.4. Local transfer to the quaternion algebra over $\mathbb{Q}_7$. Denote by B the unique quaternion division algebra over $\mathbb{Q}_7$. Then B^* is an inner form of GL(2) over $\mathbb{Q}_7$. By the local Jacquet-Langlands correspondence [JL, §16], there is a finite-dimensional, irreducible representation π_7^B of B^* functorially associated to π_7. Let $\mathscr{O}_B$ denote the maximal order of B, and for any $m \geq 0$, let $\Gamma_B(7^m)$ denote the principal congruence subgroup of level 7^m, with $\Gamma_B(1)$ denoting $\mathscr{O}_B^*$.

LEMMA 5.5. *The dimension of π_7^B is 2, and its conductor is 49. This representation is nontrivial when restricted to $\mathscr{O}_B^*$, and the kernel contains $\Gamma_B(7)$.*

Some people would be tempted to say that π_7^B has conductor 7, but it would not be in agreement with the convention used in the theory of automorphic forms, where the trivial representation of B^* is said to have (normalized) conductor 7 (see [Tun, §3]).

Proof. Two of the basic properties of the correspondence $\pi_7 \to \pi_7^B$ are the following:

(1) The dimension of π_7^B is the formal dimension of π_7.

(2) $c(\pi_7^B) = c(\pi_7)$.

To elaborate, under the Jacquet-Langlands correspondence (see [GJ], Theorem 8.1, for example), the character of π_7^B is the negative of the (generalized) character $ch(\pi_7)$ of (the discrete series representation) π_7 on the respective sets of regular elliptic elements (which are in bijection); $ch(\pi_7)$ is *à priori* a distribution, but is represented by a function on the regular elliptic set which is locally constant. For a proof of the identity (1), which follows from this character relation, see for example [Rog, Prop. 5.9]. The point is that $ch(\pi_7)$ on regular elliptic elements close to 1 equals, up to sign, its formal dimension, and similarly for the character of of π_7^B. For a discussion of the identity (2), see e.g. [Pra, page 21]. One can also refer to [BH, §56], once one observes that (in the notation of that book) $c(\pi_7^B) = 7^{\ell(\pi_7^B)+1}$ and $c(\pi_7) = 7^{2\ell(\pi_7)+1}$.

Thanks to (1), it suffices to check that the formal dimension of π_7 is 2. Let us first make the following:

Claim 5.6. The 2-dimensional representation σ_7 of $W_{\mathbb{Q}_7}$ associated to π_7 is induced by a character χ of W_F, where F is the unique unramified quadratic extension of $\mathbb{Q}_7$.

To begin, as π is associated to a Hecke character Ψ of $\mathbb{Q}[\sqrt{-7}]$, the representation σ_7 is induced by a character ψ, of W_k, where k is the ramified quadratic extension of $\mathbb{Q}_7$ obtained by completing L at the prime $(\sqrt{-7})$. (The character ψ is the local component at $\sqrt{-7}$ of the global unitary idèle class character of L defined by Ψ.) Then we have

$$c(\sigma_7) = \mathcal{N}_{k/\mathbb{Q}_7}(c(\psi))disc_{k/\mathbb{Q}_7},$$

where $\mathcal{N}$ denotes the norm and *disc* the discriminant. Since $-7 \equiv 1 \pmod 4$, $disc_{k/\mathbb{Q}_7}$ is -7, which forces $c(\psi)$ to be $P := (\sqrt{-7})\mathcal{O}_k$, because $c(\sigma_7) = 49$ and $\mathcal{N}_{k/\mathbb{Q}_7}(\sqrt{-7}) = 7$. Moreover, we have

$$\det(\sigma_7) = Ver_{k/\mathbb{Q}_7}(\psi)\eta_k,$$

where *Ver* denotes the *transfer map*, and η_k is the quadratic character $W_{\mathbb{Q}_7}$ attached to $k/\mathbb{Q}_7$. As the determinant of σ_7 is 1, the restriction of ψ to $\mathbb{Q}_7^*$ must be the quadratic character η_k. If τ denotes the nontrivial automorphism of $k/\mathbb{Q}_7$, we must have

$$\psi^\tau = \psi\nu,$$

where ν is the unique quadratic unramified character of W_k. (As usual, $\psi^\tau(x) = \psi(x^\tau)$, for all $x \in W_k$.) Indeed, with $\{U_k^j\}$ denting the usual filtration of the unit group U_k, ψ and ψ^τ are trivial on U_k^1, but not on U_k, since $c(\psi) = P$. Also, ψ/ψ^τ is evidently trivial on $\mathbb{Q}_7^*$, implying that ψ/ψ^τ is even trivial on the full unit group U_k, and thus must be an unramified character ν. Also, since $(\psi^\tau)^\tau = \psi$, we get $\nu^2 = 1$. On the other hand, since $k = \mathbb{Q}_7(\sqrt{-7})$, the automorphism τ sends the uniformizer $\sqrt{-7}$ to its negative, and so ψ^τ differs from ψ; in other words, ν must be the unique unramified quadratic character of W_k. Next observe that since $k/\mathbb{Q}_7$ is ramified, the unique unramified quadratic extension of k is the compositum Fk, where F is the unique unramified quadratic extension of $\mathbb{Q}_7$. This forces ν to be of the form $\eta_F \circ \mathscr{N}_{K/k}$, and so we have

$$\sigma_7 \otimes \eta_F \simeq I_k^{\mathbb{Q}_7}(\psi\nu) \simeq I_k^{\mathbb{Q}_7}(\psi^\tau) \simeq \sigma_7,$$

as σ_7 is induced equally by ψ and ψ^τ. Thus σ_7 admits a self-twist under η_F, or equivalently, that there is a $W_{\mathbb{Q}_7}$-homomorphism

$$\eta_F \hookrightarrow \sigma_7 \otimes \sigma_7 \simeq 1 \oplus \mathrm{sym}^2(\sigma_7).$$

Here we have used the triviality of $\det(\sigma_7)$, and $\mathrm{sym}^2(\sigma_7)$ denotes the (3-dimensional) symmetric square representation. Combining this with $\sigma_7 \simeq \sigma_7^\vee$, we get a $W_{\mathbb{Q}_7}$-isomorphism

$$\mathrm{End}(\sigma_7) \simeq \sigma_7^{\otimes 2} \simeq \eta_F \oplus \underline{1} \oplus \beta,$$

for a 2-dimensional representation β ($\subset \mathrm{sym}^2(\sigma_7)$). Since η_F is trivial on W_F, we get

$$\dim_{\mathbb{C}} \mathrm{Hom}_{W_F}(\underline{1}, \mathrm{End}(\sigma_7)) \geq 2.$$

So by Schur, the restriction of σ_7 to W_F is reducible, and if χ is a character of W_F appearing in $\sigma_{7|_F}$, Frobenius reciprocity implies that σ_7 is induced by χ. This proves Claim 5.6.

By the local correspondence [JL], the representation π_7 is associated to the character χ of F^* ($\simeq W_F^{\mathrm{ab}}$), with $F/\mathbb{Q}_7$ unramified. Now we may apply Lemma 5 of [PR], which summarizes the results we want on the formal degree, etc. In that lemma, cond($-$) means the *exponent* of $c(-)$, and the quadratic extension K there corresponds to our (unramified) F here. (It should be noted that the K in that lemma is taken to be unramified if the representation is attached to more than one quadratic extension.) Since cond(χ) = 1 for us, we see that π_7^B has dimension 2. Also, $c(\pi_7^B) = 1$. This implies the last part of the lemma by the discussion in [Tun], cf. the two paragraphs in Section 3 before Theorem 3.6. □

5.7. Base change to $\mathbb{Q}[\sqrt{-3}]$ and the global transfer. Put $K = \mathbb{Q}[\sqrt{-3}]$, in which the prime 7 splits as follows:

$$7\mathcal{O}_K = Q\overline{Q}, \quad Q = \left(2 + (1 + i\sqrt{3})/2\right).$$

Let $\pi_K = \otimes'_v \pi_{K,v}$ denote the base change [Lan] of π to K, which has conductor $Q^2\overline{Q}^2$, the reason being that 7 splits in K as above, and consequently,

$$\pi_{K,Q} \simeq \pi_{K,\overline{Q}} \simeq \pi_7.$$

Also, π_K still has trivial central character. More importantly, since π_K has a supercuspidal component, it must be cuspidal. One can also see this from Deuring's theory since K and L are disjoint.

Let D be, once again, the quaternion division algebra over K, which is ramified only at Q and $\overline{Q}$. Then one has, by the Jacquet-Langlands correspondence, a corresponding cuspidal automorphic representation π_K^D of $D(\mathbb{A}_K)^*$ such that

$$v \notin \{Q, \overline{Q}\} \Rightarrow \pi_{K,v}^D = \pi_{K,v},$$

which makes sense because D_V and $\mathrm{GL}(2, K_v)$ are the same at any place $v \neq Q, \overline{Q}$. More importantly,

$$D_Q \simeq D_{\overline{Q}} \simeq B,$$

where B is the quaternion division algebra considered in the previous section, and by the naturality of the Jacquet-Langlands correspondence,

$$\pi_{K_Q}^D \simeq \pi_{K_{\overline{Q}}}^D \simeq \pi_7^B. \tag{5.8}$$

LEMMA 5.9. *The conductor of* π_K^D *is* 49. *The corresponding principal congruence subgroup* U *of* $D^*(\mathbb{A}_{K,f})$ *is of level* $7\mathcal{O}_K = Q\overline{Q}$. *Moreover, the space of vectors fixed by* U *is of dimension* 4.

Proof. If v is any finite place of K outside $\{Q, \overline{Q}\}$, the local representation $\pi_{K,v}^D \simeq \pi_{K,v}$ is unramified and hence has trivial conductor. Thus we have, by applying Lemma 5.5,

$$c\left(\pi_K^D\right) = c\left(\pi_{K,Q}^D\right) c\left(\pi_{K,\overline{Q}}^D\right) = c\left(\pi_7^B\right)^2 = 7^2 = \left(Q\overline{Q}\right)^2.$$

By our convention on the conductor, it follows the corresponding principal congruence subgroup is of level $7\mathcal{O}_K$. The final claim of the lemma follows from Lemma 5.5, in view of (5.8) above. □

5.10. The π^D-isotypic subspace. Put

$$\mathscr{V}(D) := L^2 \left({}_{Z(\mathbb{A}_K)D^*}\backslash^{D(\mathbb{A}_K)^*} \right),$$

where Z is the center ($\simeq K^*$) of the algebraic group D^* over K. Then $\mathscr{V}(D)$ is a unitary representation of $D(\mathbb{A}_K)^*$ under the right action. The coset space $Z(\mathbb{A}_K)D^*\backslash D(\mathbb{A}_K)^*$ is compact, which follows by Godement's compactness criterion as D^* contains no unipotents $\neq 1$. Consequently, one has a (Hilbert) direct sum decomposition (as unitary $D(\mathbb{A}_K)^*$-modules)

$$\mathscr{V}(D) \simeq \oplus_\eta m(\eta)\mathscr{V}_\eta,$$

where $(\eta, \mathscr{V}_\eta)$ runs over irreducible unitary, admissible representations of the group $D(\mathbb{A}_K)^*$, with multiplicities $m(\eta) \geq 0$.

The Jacquet-Langlands correspondence which embeds $\mathscr{V}(D)$ into the discrete spectrum of the corresponding space for $\mathrm{GL}(2, \mathbb{A}_K)$, and since the multiplicities are 1 on the $\mathrm{GL}(2)/K$-side [JL], we get

$$m(\eta) \leq 1, \quad \text{for all } \eta.$$

Since $\pi_K^D = \pi_{K,\infty} \otimes \pi_{K,f}^D$ is by construction a cuspidal automorphic representation of $D(\mathbb{A}_K)^*$ of trivial central character, it gives rise to a nontrivial summand in $\mathscr{V}(D)$. In particular,

$$m(\pi_K^D) = 1.$$

The π_K^D-isotypic subspace of $\mathscr{V}(D)$, identifiable with $\mathscr{V}_{\pi_K^D}$, is infinite-dimensional. However, due to the admissibility of $\pi_{K,f}^D$, given any compact open subgroup U of $D(\mathbb{A}_{K,f})^*$, the space U-invariants in $\pi_{K,f}^D$ is finite dimensional. Applying Lemma 5.9, we obtain the following:

LEMMA 5.11.

$$\dim_{\mathbb{C}} \mathscr{V}_{\pi_{K,f}^D}^{U_D(7)} = 4.$$

5.12. Connected components of the adèlic double coset space. Now put $G = D^*/K$ and consider the adèlic quotient space

$$X_{\mathbb{A}} := {}_{G(K)Z(\mathbb{A}_K)}\backslash^{\mathscr{H}^3 \times G\left(\mathbb{A}_{K,f}\right)},$$

introduced in the middle of Section 3.7, which admits a right action by $G(\mathbb{A}_{K,f})$. Also set

$$A = Q\overline{Q}, \quad \text{and} \quad X_{U_A} = X_{\mathbb{A}}/U_A,$$

where

$$U_A = U_Q(Q) \times U_{\overline{Q}}(\overline{Q}) \times \left(\prod_{P \neq Q, \overline{Q}} G(\mathcal{O}_{K_P}) \right),$$

where $U_Q(Q)$ is the principal congruence subgroup of $G(K_Q)$ of level 1, i.e., the kernel of the map to $G(\mathcal{F}_7)$, and the same for $U_{\overline{Q}}(\overline{Q}) \leq G(K_{\overline{Q}})$. We will write $U_D(7)$ instead of U_A to refer to the specific choice of A here.

LEMMA 5.13. *X_{U_A} has two connected components, both diffeomorphic to the manifold M defined in Section 4.*

Proof. The disconnectedness comes because we are working with the group D^* and not its semisimple part

$$D^1 := \operatorname{Ker}\left(Nrd : D^* \to K^*\right),$$

where Nrd denotes the reduced norm. In our case, we have

$$X_{U_A} = D^* \backslash \mathcal{H}^3 \times \left(Z(\mathbb{A}_{K,f}) \backslash D(\mathbb{A}_{K,f})^* / U_D(7) \right),$$

where D^* acts diagonally. Thus the number of connected components of $M^{\mathbb{A}}$ is

$$\pi_0(M^{\mathbb{A}}) = D^* Z(\mathbb{A}_{K,f}) \backslash D(\mathbb{A}_{K,f})^* / U_D(7). \tag{5.14}$$

Were we working with D^1, we would have

$$\pi_0 = D^1 \backslash D(\mathbb{A}_{K,f})^1 / U,$$

where the right-hand quotient is a single element as D^1 is dense in $D(\mathbb{A}_{K,f})^1$ by strong approximation.

Returning to the case at hand, it is straightforward to check that we can evaluate the right-hand side of (5.14) by taking its image under Nrd. Since Nrd surjects D^* onto K^*, as well as $D(\mathbb{A}_K)^*$ onto $\mathbb{A}_K^*$, and $Z(\mathbb{A}_K)$ onto ${\mathbb{A}_K^*}^2$, we get

$$\pi_0(M^{\mathbb{A}}) = {\mathbb{A}_{K,f}^*}^2 K^* \backslash \mathbb{A}_{K,f}^* / Nrd(U_D(7)).$$

Note that since the class number of $K = \mathbb{Q}[\sqrt{-3}]$ is 1, we have by strong approximation,

$$\mathbb{A}_K^* = K^* K_\infty^* \prod_P U_P,$$

where P runs over all the (finite) primes of $\mathscr{O}_K$, and $U_P = \mathscr{O}_{K_P}^*$. Moreover,

$$Nrd(U_D(7)) = \left(\prod_{P \neq Q, \overline{Q}} U_P \right) \times U_Q^1 \times U_{\overline{Q}}^1,$$

where $U_Q^1 = 1 + Q\mathscr{O}_{K_Q}$.

As 7 splits into $Q, \overline{Q}$ in K, we have $\mathscr{O}_{K_Q} \simeq \mathscr{O}_{K_{\overline{Q}}} \simeq \mathbb{Z}_7$, and moreover, $\mathbb{Z}_7^*/\mathbb{Z}_7^{*2}$ identifies with $\mathbb{F}_7^*/\mathbb{F}_7^{*2} \simeq \{\pm 1\}$. It follows that

$$\pi_0(M^{\mathbb{A}}) = \mathrm{Coker}\left(\mathscr{O}_K^* \to \left(\mathscr{O}_{K_Q}^* / \mathscr{O}_{K_Q}^{*\,2} \right) \times \left(\mathscr{O}_{K_{\overline{Q}}}^* / \mathscr{O}_{K_{\overline{Q}}}^{*\,2} \right) \simeq \left(\mathbb{F}_7^* / \mathbb{F}_7^{*2} \right)^2 \right).$$

The map $\mathscr{O}_K^* \to \mathbb{F}_7^* \times \mathbb{F}_7^*$ is onto the diagonal, and thus $M^{\mathbb{A}}$ has two connected components.

To complete the lemma, here is an explicit description of the two components. Let $x = (x_v)$ be an element of $\mathbb{A}_K^*$ with $x_v = 1$ for all $v \neq Q$, such that the Q-component x_Q is an element of $\mathscr{O}_{K_Q}^*$ ($\simeq \mathbb{Z}_7^*$) mapping onto a nonsquare of $\mathbb{F}_7^*$. By the preceding, we get an identification

$$M^{\mathbb{A}} = M_\Gamma \bigcup M_{\Gamma'},$$

with connected components

$$M_\Gamma = \Gamma \backslash \mathscr{H}^3, \quad M_{\Gamma'} = \Gamma' \backslash \mathscr{H}^3,$$

where

$$\Gamma = \Gamma(Q\overline{Q}) = D^* \cap U_D(7), \quad \Gamma' = D^* \cap \left(x U_D(7) x^{-1} \right).$$

Now M_Γ is precisely the manifold described in Section 4, as claimed, and $M_{\Gamma'}$ is diffeomorphic to M_Γ since $U_D(7)$ is a normal subgroup of U_D. □

5.15. Cohomology on the compact side. Put $G = D^*/Z$. Then $\Gamma = \Gamma(Q\overline{Q})$ defines a torsion-free lattice in $G_\infty = \mathrm{PGL}(2, \mathbb{C})$. Since $\mathscr{H}^3$ is contractible, M_Γ is an Eilenberg-MacLane space for Γ, and thus

$$H^*(M_\Gamma, \mathbb{Q}) \simeq H^*(\Gamma, \mathbb{Q}).$$

Moreover, since Γ is cocompact, by a refinement of Shapiro's lemma [Bla], we have

$$H^*(\Gamma, \mathbb{C}) \simeq H^*_{\mathrm{cont}} \left(G_\infty, L^2 \left(\Gamma \backslash G_\infty \right)^\infty \right),$$

where H^*_{cont} denotes continuous cohomology, and the superscript ∞ on (the right regular representation) $L^2(\Gamma\backslash G_\infty)$, which is the unitary analog of the group algebra of a finite group, denotes taking the smooth vectors. Consequently, if $L^2(\Gamma\backslash G_\infty)$ decomposes as a unitary G_∞-module as $\oplus m_\infty(\beta)W_\beta$,

$$H^*(M_\Gamma,\mathbb{C}) \simeq H^*_{\mathrm{cont}}\left(G_\infty, L^2\left(\Gamma\backslash G_\infty\right)^\infty\right) \simeq \oplus_{(\beta,W_\beta)} m_\infty(\beta) H^*_{\mathrm{cont}}(G_\infty, W_\beta^\infty).$$

One knows that there is a unique irreducible, unitary representation (β_0, W_{β_0}) of G_∞ for which

$$H^i_{\mathrm{cont}}(G_\infty, W^\infty_{\beta_0}) \neq 0, \quad \text{for } i = 1, 2,$$

and the dimension is 1. In fact (see [Clo], for example), the 2-dimensional representation of $W_{\mathbb{C}} = \mathbb{C}^*$ attached to this β_0 (by the local archimedean correspondence) is given by

$$\sigma_\infty : z \to \left(\frac{z}{|z|}\right) \oplus \left(\frac{\bar{z}}{|z|}\right).$$

By construction, the parameter of our $\pi_{K,\infty}$ ($\simeq \pi^D_{K,\infty}$) is also of this form. Hence we have

$$\pi^D_{K,\infty} \simeq W^\infty_{\beta_0}.$$

If $L^2(G(K)\backslash G(\mathbb{A}_K)/U_D(7)) \simeq \oplus_\Pi m(\Pi)\Pi$ (as unitary $G(\mathbb{A}_K)$-modules), a straightforward adèlic refinement of the above yields the following identification:

$$\begin{aligned} H^*(M^{\mathbb{A}},\mathbb{C}) &\simeq H^*_{\mathrm{cont}}\left(G_\infty, L^2\left(G(K)\backslash G(\mathbb{A}_K)/U_D(7)\right)\right) \\ &\simeq \oplus_\Pi H^*_{\mathrm{cont}}(G_\infty, \mathscr{V}_{\Pi_\infty}) \otimes \Pi_f^{U_D(7)}, \end{aligned}$$

where $\mathscr{V}_\Pi = \mathscr{V}_{\Pi_\infty} \otimes \mathscr{V}_{\Pi_f}$ denotes the admissible subspace of (the space of) $\Pi = \Pi_\infty \otimes \Pi_f$.

Next recall that m_Π is 1, and that $H^i_{\mathrm{cont}}(G_\infty, W^\infty_{\beta_0})$ is 1-dimensional for $i = 1, 2$. So we get an isotypic decomposition (for $i \in \{1, 2\}$),

$$H^i(M^{\mathbb{A}},\mathbb{C}) \simeq \oplus_{\{\Pi:\ \Pi_\infty \simeq \pi^D_{K,\infty}\}} H^i(M^{\mathbb{A}}, \Pi_f),$$

where

$$\dim_{\mathbb{C}} H^i(M^{\mathbb{A}}, \Pi_f) = \dim_{\mathbb{C}} \mathscr{V}^{U_D(7)}_{\Pi_f}.$$

In particular, since our π^D_K is one such Π, and since the space of $U_D(7)$-invariants of $\pi^D_{K,f}$ is 4-dimensional, we obtain:

LEMMA 5.16. *For $i = 1, 2$, we have* $\dim_{\mathbb{C}} H^i(M^{\mathbb{A}}, \pi^D_{K,f}) = 4$.

Combining this with the discussion of the connected components of $M^{\mathbb{A}}$ in the previous section, since $M^{\mathbb{A}}$ has two isomorphic components, we get the following, where $\mathscr{P}_K$ is the set of primes defined in Section 4:

THEOREM 5.17. *For $i = 1, 2$, we have* $\dim_{\mathbb{C}} H^i(M_\Gamma; \mathbb{C})^{\text{new}} \geq 2$. *More precisely, $H^i(M_\Gamma; \mathbb{C})^{\text{new}}$ contains a plane (defined by $\pi^D_{K,f}$) on which the Hecke operators T_P act by zero for all $P \in \mathscr{P}_K$.*

6. The Thurston norm and fibered faces of M. In this section, we study the topological properties of M and show:

THEOREM 6.1. *Let $M = X(Q\overline{Q})$ be the hyperbolic 3-manifold described in Section 4. Then $H^1(M; \mathbb{Q})$ is 3-dimensional and*

(1) *The Thurston norm ball $B_T \subset H^1(M; \mathbb{R})$ is a parallelepiped, i.e. an affine cube.*

(2) *Exactly four of the six faces of B_T are are fibered.*

(3) *The subspace of oldforms $H^1(M; \mathbb{Q})^{\text{old}}$ contains the line passing the barycenters of the nonfibered faces.*

Here, the *barycenter* of a face is the average of its vertices. We will now show that if Theorem 6.1 holds, then so does Theorem 4.1.

Proof of Theorem 4.1 modulo Theorem 6.1. By Theorem 5.17, the subspace $V = H^1(M; \mathbb{Q})^{\text{new}}$ contains a 2-dimensional subspace V_0 on which the Hecke operators T_P act by zero for $P \in \mathscr{P}_K$. By Theorem 6.1, we know$H^1(M; \mathbb{Q})$ is 3-dimensional and $H^1(M; \mathbb{Q})^{\text{old}}$ is nonzero; this forces $V = V_0$. It remains to show that V intersects the interiors of one of the four fibered faces. We can change coordinates on $H^1(M; \mathbb{Q})$ by an element of $\mathrm{GL}(3, \mathbb{Q})$ so that B_T is the standard cube with vertices $(\pm 1, \pm 1, \pm 1)$, and the nonfibered faces are the ones intersecting the z-axis. By part (3), the set of oldforms W is then the z-axis itself. Now $\dim V = 2$, and among all 2-dimensional subspaces, only two miss the interiors of the fibered faces, namely the ones containing a vertical edge of B_T where two fibered faces meet. But both of these subspaces contain the z-axis of oldforms W and so can't be V. So there is a fibered class in V as desired. □

The rest of this section is devoted to *proving* Theorem 6.1. We will produce an explicit triangulation of M and use a series of tricks to compute the Thurston norm. We believe these tricks are new; they are remarkably effective in many examples, and (we hope) are of independent interest. Our computations used an extensive variety of software: [W], [G], [CBFS], [CD], [SAGE]. Source code, data files, and complete computational details are available at [DR].

6.3. Finding a topological description. First, we give a concrete topological description of M. Let's review the setup of Section 4. Let $K = \mathbb{Q}(\sqrt{-3})$ and let

Q and $\overline{Q}$ be the prime ideals sitting over 7. Let D be the quaternion algebra over K ramified precisely at Q and $\overline{Q}$. Let $\mathcal{O}_D$ be a maximal order of D. Let ϕ be the embedding of D^*/K^* into $\mathrm{PGL}(2,\mathbb{C})$, and set $\Gamma = \phi(\mathcal{O}_D^*)$ and $X = \Gamma\backslash\mathcal{H}^3$. Also, let $\mathcal{O}_D^1$ denote the elements of norm 1, and set $\Gamma' = \phi(\mathcal{O}_D^1)$ and $X' = \Gamma'\backslash\mathcal{H}^3$. We are interested in the congruence covers $X(Q)$, $X(\overline{Q})$, and $M = X(Q\overline{Q})$.

6.4. The topology of X'. It is often easier to find X' than X because of the following characterization of the former. For a lattice Λ in $\mathrm{PSL}(2,\mathbb{C})$, let $\widetilde{\Lambda}$ denote the preimage lattice in $\mathrm{SL}(2,\mathbb{C})$. Let $k\Lambda$ be the number field generated by the traces of elements of $\widetilde{\Lambda}$, and $A\Lambda$ be the quaternion algebra generated by the elements of $\widetilde{\Lambda}$ (see e.g. [MR, Ch 3] for details). Then Corollary 8.3.6 and Theorem 11.1.3 of [MR] imply that $\Lambda\backslash\mathcal{H}^3$ is X' if:

(1) $k\Lambda = K$ and $A\Lambda = D$.

(2) The trace of every element in $\widetilde{\Lambda}$ is an algebraic integer.

(3) The volume of $\Lambda\backslash\mathcal{H}^3$ is $3^{3/2}36\zeta_K(2)/(4\pi^2) \approx 6.0896496384579219$.

Here, the degree of K is small, as is the volume, so we can expect these conditions to be verified by the program Snap [CGHN], [G] if it is given a topological description of the orbifold X'.

In our case, a brute-force search finds the following description of X'. Let B be the orbifold shown in Figure 6.2. From this topological description, SnapPea [W] derives the following presentation for $\pi_1(B)$:

$$\left\langle a, b \;\middle|\; b^2 = aBABaba^4baBAbabABA^4BAb = aBABaba^3baBABaba^3b = 1 \right\rangle,$$

where $A = a^{-1}$ and $B = b^{-1}$. Hence $H_1(B;\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/8$, and so B has a unique regular cover C with covering group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. SnapPea can explicitly build C as a cover of B given the action of $\pi_1(B)$ on the cosets of $\pi_1(C)$. Snap checks that C is X'. This is a little subtle as the reasonable spun ideal triangulations of

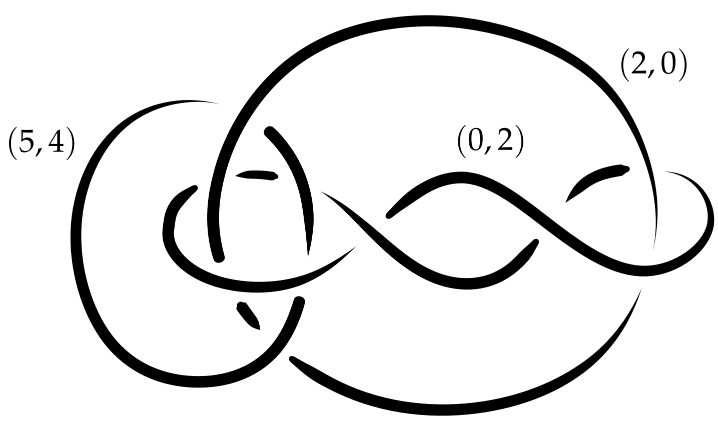

Figure 6.2. A Dehn surgery description of the hyperbolic 3-orbifold B. Our orientation conventions are given by , and match those of SnapPea [W].

C have some flat ideal tetrahedra (though no negatively oriented ones), and it is necessary to invoke [PW], in slightly modified form, to justify that we have constructed the hyperbolic structure. For details, see [DR].

Though we did not check this rigorously, and will not use it in this paper, the manifold X is almost certainly the 2-fold cover of B whose homology is $\mathbb{Z}/8 \oplus \mathbb{Z}/8$; this cover is the SnapPea orbifold $s883(2,-2)(2,-2)$.

Remark 6.5. Prior to 2007/9/12, the version of the SnapPea kernel [W] that came with SnapPeaPython [WCD] contained a bug where it sometimes returned the wrong orbifold when the cover it was trying to construct was a proper orbifold. The underlying topology was correct but the orbifold loci are sometimes mislabeled by proper divisors of their real orders. This bug also affects the MacOS Classic versions of SnapPea, e.g., SnapPea 2.6 and the derived Windows port. The calculations here were, of course, done with a corrected version of the kernel.

6.6. Constructing $X(Q)$. In order to build $X(Q)$, we begin by claiming that it covers X' and that this cover $X(Q) \to X'$ is a cyclic cover of degree 4. First, we work out the structure of the local quaternion algebra $D_Q = K_Q \otimes_K D$ by following [MR, §6.4]. We have $K_Q \cong \mathbb{Q}_7$ by Hensel's lemma since -3 has a square-root mod 7. Let F be the unique unramified quadratic extension of K_Q, concretely $F = \mathbb{Q}_7(i)$. Then we have $D_Q \cong F \oplus Fj$ where i and j anti-commute and $j^2 = 7$. Then $\mathcal{O}_{D_Q} \cong \mathcal{O}_F \oplus \mathcal{O}_F j$. Consider the homomorphism

$$\mathcal{O}^*_{D_Q} \to \overline{F}^* \quad \text{defined by } x + yj \mapsto \overline{x}, \text{ where } \overline{F} \cong \mathbb{F}_{49} \text{ is the residue field of } F.$$

If Λ is the kernel of the restricted homomorphism $\mathcal{O}^*_D \to \overline{F}^*$, then by definition $X(Q) = \Lambda \backslash \mathcal{H}^3$.

Now the norm map $Nrd\colon D_Q \to K_Q$ is given by $Nrd(x{+}yj) = Nrd(x){-}7Nrd(y)$. Thus if α is in the kernel of $\mathcal{O}^*_D \to \overline{F}^*$, we have $Nrd(\alpha) = 1$ in $\overline{K}_Q$. As the units of $\mathcal{O}_K$, the sixth roots of unity, all map to distinct elements in $\overline{K}^*_Q$, it follows that α is in fact in $\mathcal{O}^1_D$. Thus we have $\Lambda \subset \mathcal{O}^1_D$ and so $X'(Q) = X(Q)$; henceforth we focus on the former description.

The image of $\mathcal{O}^1_{D_Q} \to \overline{F}^\times$ is just the elements of $\mathbb{F}_{49}/\mathbb{F}_7$-norm 1; these elements form a (cyclic) subgroup C of order 8. By strong approximation, the image of $\mathcal{O}^1 \to \overline{F}^\times$ is the same. Now $-1 \notin \Lambda$, and so $\Lambda \cong \pi_1(X'(Q))$. On the other hand, -1 maps into C nontrivially, and so we conclude that $\pi_1(X'(Q))$ is a normal subgroup of $\pi_1(X')$ with quotient $C/\{\pm 1\} \cong C_4$.

Further, we can characterize Λ as exactly those $\alpha \in \mathcal{O}^1_D$ such that $\operatorname{tr}(\alpha) \equiv 2 \bmod Q$. This is because if $x \in \overline{F}$ has norm 1 and $\operatorname{tr}(x) = 2$ then $x = 1$ because x satisfies $x^2 - \operatorname{tr}(x)x + Nrd(x)$. Searching through the normal subgroups of $\pi_1(X')$ with quotient C_4, one quickly finds a unique subgroup for which all tested elements have trace which is $\pm 2 \pmod{Q}$ in the topologically given $\mathrm{PSL}(2,\mathbb{C})$-

representation; this subgroup must be Λ. In fact, $X(Q)$ and $X(\overline{Q})$ are exactly those 4-fold cyclic covers of X' whose homology is $\mathbb{Z}/28 \oplus \mathbb{Z}/28 \oplus \mathbb{Z}$. Repeating the same procedure for $\overline{Q}$ builds $\pi_1(X(\overline{Q}))$ as a subgroup of $\pi_1(X') \leq \pi_1(B)$. Taking their intersection yields $\pi_1\big(X(Q\overline{Q})\big)$. From the associated permutation action on $\pi_1(B)/\pi_1\big(X(Q\overline{Q})\big)$, SnapPea can build an explicit triangulation for $M = X(Q\overline{Q})$.

6.7. The Thurston norm of $M = X(Q\overline{Q})$. For this manifold M, we calculate $H_1(M;\mathbb{Z}) = \mathbb{Z}^3 \oplus (\mathbb{Z}/14)^4 \oplus (\mathbb{Z}/2)^2$. Our next task is to compute the Thurston norm ball B_T in $H^1(M;\mathbb{R}) = \mathbb{R}^3$. In general, there is an algorithm using normal surfaces for computing the Thurston norm. Normal surfaces are those which cut through each simplex in the triangulation like a union of affine planes, and have been the bedrock algorithmic tool in 3-dimensional topology since Haken first used them to detect the unknot [Hak]. For the Thurston norm, the running time of this normal surface algorithm is "only" simply exponential in the size of the triangulation [CT], but any triangulation of M must have at least 95 tetrahedra as its volume is ≈ 97.434394, and the triangulations we constructed had 130 or more. At that complexity, normal surface methods appear to be hopeless. Instead, we take a different approach, which involves producing normal surfaces representing certain classes in $H_2(M)$ quite cheaply.

The other key tool will be the Alexander polynomial Δ_M of M which lies in the group ring $\mathbb{Z}\left[H_1(M;\mathbb{Z})/(\text{torsion})\right]$. Following McMullen [McM], if $\Delta_M = \sum a_i g_i$ for $a_i \in \mathbb{Z}$ and $g_i \in H_1(M;\mathbb{Z})$ then we define the *Alexander norm* on $H^1(M;\mathbb{R})$ by

$$\|\omega\|_A = \sup_{i,j} \omega(g_i - g_j).$$

The unit ball B_A of this norm is the dual polyhedron to the Newton polytope of Δ_M inside $H_1(M;\mathbb{R})$, which is the convex hull of the g_i. Since $b_1(M) = \dim H^1(M;\mathbb{R}) > 1$, we have $\|\omega\|_A \leq \|\omega\|_T$ for all $\omega \in H^1(M;\mathbb{R})$, or equivalently $B_T \subset B_A$ [McM]. For a general 3-manifold, these norms do not always coincide, but it is actually quite common for them to do so, and we will show

LEMMA 6.8. *The Alexander and Thurston norms agree for $M = X(Q\overline{Q})$.*

Computing directly from a presentation of $\pi_1(M)$ yields:

$$\Delta_M = (16xyz - xy - xz - y - z + 16)^4 \tag{6.9}$$

Taking powers of a polynomial just dilates its Newton polytope, so the Newton polytope of Δ_M is, up the change of basis, an octahedron. Dually, this means that B_A is a cube. Since B_T is a convex subset of B_A, to prove Lemma 6.8 it suffices to check that the two norms agree on the vertices of the cube. Moreover, M is very symmetric as $\pi_1(M) \triangleleft \pi_1(B)$, and any symmetry of M preserves both norms, as does the map $\iota\colon \omega \to -\omega$ on $H^1(M;\mathbb{R})$. One calculates that the image of

$$\langle \iota, \pi_1(B)\rangle \to \operatorname{Aut}\left(H^1(M;\mathbb{R})\right)$$

has order 16. Now the full symmetry group of the cube has order 48, and any subgroup of order 16 acts transitively on the vertices, since the full vertex stabilizer has order 6. Thus, it suffices to check $\|\omega\|_A = \|\omega\|_T$ for a single vertex of the cube.

For computing the Thurston norm of a single element of $\omega \in H^1(M;\mathbb{Z})$ we used the following method. Choose a (nonclassical) triangulation $\mathscr{T}$ of M which has only one vertex, à la Jaco-Rubinstein [JR]. Then there is a unique simplicial 1-cocycle representing the given class ω. As observed in [Cal], any 1-cocycle gives a canonical map $M \to S^1 = \mathbb{R}/\mathbb{Z}$ which is affine on each simplex of $\mathscr{T}$. More precisely, the integer ω assigns to an edge specifies how many times to wrap it around the circle, and this extends over each simplex because of the cocycle condition. The inverse image of a generic point in S^1 is a normal surface which is Poincaré dual to ω. By randomizing the triangulation $\mathscr{T}$ a few times using Pachner moves, we were able to find a genus 3 surface representing one of the vertices of B_A, which shows the two norms agree for that class. This proves Lemma 6.8, and thus part (1) of Theorem 6.1.

Remark 6.10. In practice, this seems to be a very effective method for computing the Thurston norm when the first betti number b_1 is 1, even when the triangulation has a couple hundred tetrahedra. When $b_1 > 1$, things are more difficult, as it seems one cannot expect to find a triangulation where these special dual surfaces realize the Thurston norm for all ω. While here we evaded this issue by using symmetry, a general approach would be to carry along cohomology isomorphisms between the new triangulations and the original one while changing the triangulation using Pachner moves. This should enable probing the whole Thurston norm ball using this method.

6.11. Fibering of *M*. Next, we determine which faces of B_T are fibered. Again, there is a normal surface based algorithm to decide this [Sch], [TW], [JT]. However, instead of using it, we build on the method described above. First, we use the symmetries of M to simplify the problem. We know each face of $B_T = B_A$ is associated to a vertex of the Newton polytope of Δ_M, and hence has a coefficient of Δ_M associated to it. A generalization of the classical Alexander polynomial obstruction to fibering says that if a face of B_T is fibered, then this coefficient must be ± 1 (see e.g. [Dun, Thm. 5.1]). In our case, these vertex coefficients are $\{16^4, 16^4, 1, 1, 1, 1\}$; recall from (6.9) that Δ_M is a power of a simpler polynomial f, and so the coefficients are the corresponding powers of the vertex coefficients of the Newton polytope of f. Thus there is one pair of faces which are definitely not fibered (since their coefficients are 16^4), and two pairs of faces which *likely* fiber. Recall we have a group S of order 16 acting on $H^1(M;\mathbb{R})$ which preserves $B_T = B_A$, including the coefficient labels on the faces (up to sign). Thus S preserves the two faces with labels 16^4 and has a subgroup of order 8 which leaves each of them invariant; this subgroup acts there by the full symmetry group of the square, the dihedral group of order 8. In particular, S acts transitively on the remaining four faces of B_A. Thus to show that all these faces fiber, and so prove part (2) of Theorem 6.1, it suffices to show:

LEMMA 6.12. *The manifold $M = X(Q\overline{Q})$ fibers over the circle.*

To check this, we consider the 8-fold cover N of the orbifold B of Figure 6.2 given by

$$\pi_1(B) \to S_8 \quad \text{where} \quad a \mapsto (1,2,3,6,8,5,7,4),\, b \mapsto (1,3)(2,5)(4,6)(7,8),$$

which is a manifold. Now $\pi_1(N)$ is not a subgroup of $\pi_1(M)$, but we can consider their minimal common cover Y with fundamental group $\pi_1(N) \cap \pi_1(M)$ as shown

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ N \arrow[dr] & & M \arrow[dl] \\ & B & \end{tikzcd}$$

A key fact is that $\dim H^1(Y;\mathbb{R}) = 3$. Thus we may apply the theorem of Stallings mentioned in Section 2.1 and conclude, as Y and M have the same first Betti number, that M fibers if and only if Y does. Thus if N fibers, so does M. Now $H_1(N;\mathbb{Z}) = \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}$, and its Alexander polynomial is $\Delta_N = t^4 + 30t^2 + 1$, suggesting that N is fibered by genus 2 surfaces. The method of Section 6.7 finds a genus 2 normal surface Σ representing the generator of $H_2(N;\mathbb{Z})$ (the particular triangulation of N we used, and hence an explicit description of Σ, is available at [DR]). To show that $N \setminus \Sigma$ is $\Sigma \times I$ and hence N fibers, it suffices to check that $\pi_1(N \setminus \Sigma)$ is *abstractly* isomorphic to $\pi_1(\Sigma)$ (see e.g. [Hem, Thm. 10.6]). To find an initial presentation of $\pi_1(N \setminus \Sigma)$, we can exploit the fact that in a suitable triangulation $\mathscr{T}$ of N, the normal surface Σ is very simple; in each tetrahedron it looks like one of the three possibilities shown in Figure 6.13(a). Part (b) of the same figure describes a 2-complex P which is a spine for $N \setminus \Sigma$, i.e. $N \setminus \Sigma$ deformation retracts to P. It is easy to read off a presentation for $\pi_1(N \setminus \Sigma)$ from this spine; simplifying this presentation using Tietze transformations [MKS] and relabeling the generators yields

$$\pi_1(N \setminus \Sigma) = \left\langle a, b, c, d \;\middle|\; aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \right\rangle$$

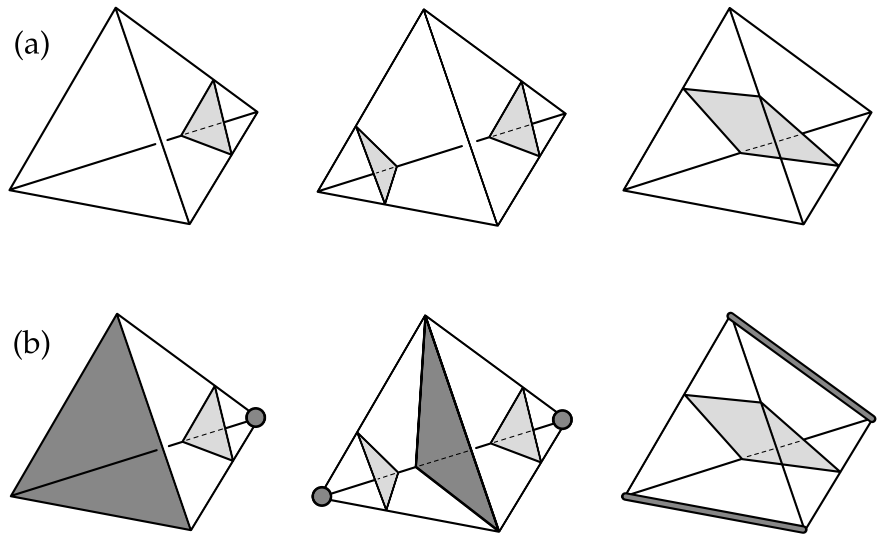

Figure 6.13. At the top, the three possibilities, besides the empty set, for how Σ intersects a tetrahedron of $\mathscr{T}$. The pictures at the bottom describe, in dark grey, a 2-complex P to which $N \setminus \Sigma$ deformation retracts.

which is the standard presentation for the fundamental group of a genus 2 surface. So N fibers, and hence so does M.

Remark 6.14. This procedure for checking fibering is applicable much more generally. For any normal surface, there is a natural spine for its complement, introduced by Casson in his study of 0-efficient triangulations (paralleling [JR]). Writing down a presentation of $\pi_1(N \setminus \Sigma)$ from this is straightforward. In more complicated cases, simplifying the presentation of $\pi_1(N \setminus \Sigma)$ in the hopes of recognizing it as a surface group, it is important to employ not just Tietze transformations, but Nielsen transformations as well, i.e. the application of elements in $\mathrm{Aut}(F_n)$ to the generating set of the presentation in order to shorten the relators. If one can reduce down to a presentation with only one relator, then checking it is a surface group is often easy: just glue up the edges of the relator polygon, and if there is only one vertex in what results, then the group is indeed a surface group.

6.15. Oldforms for $M = X(Q\overline{Q})$. We turn now to the last assertion of Theorem 6.1, namely that the space of oldforms in $H^1(M;\mathbb{Q})$ is the line determined by the barycenters of the nonfibered faces. By Theorem 5.17, we know that $\dim H^1(M;\mathbb{Q})^{\mathrm{new}} \geq 2$, so as $\dim H^1(M;\mathbb{Q}) = 3$ it follows that $H^1(M;\mathbb{Q})^{\mathrm{old}}$ has dimension at most 1. Since $H^1(X(Q);\mathbb{Q})$ is nonzero, it follows that $H^1(M;\mathbb{Q})^{\mathrm{old}}$ is the image of $\phi^*\colon H^1(X(Q);\mathbb{Q}) \to H^1(M;\mathbb{Q})$, where $\phi = \phi_1\colon M \to X(Q)$ is the standard covering map. Since we have explicit presentations for both $\pi_1(X(Q))$ and $\pi_1(M)$ as subgroups of $\pi_1(B)$, computing the image of ϕ^* is straightforward; for details see [DR].

7. Proof of the main result.

7.1. Proof of Theorem 1.4. Continuing the notation of Section 4, let M be the arithmetic hyperbolic 3-manifold described there, and $\omega \in H^1(M;\mathbb{Z})$ be the class given by part (2) of Theorem 4.1. Then ω is fibered and for each prime P in the special set $\mathscr{P}_K$ we have $T_P(\omega) = 0$. Denote the primes of $\mathscr{P}$ as $p_1, p_2, p_3, \ldots$ in increasing order, and fix an ordering, once and for all, of the primes of $\mathscr{P}_K$ as $P_1, P_2, \ldots$ such that $i \leq j \Rightarrow N(P_i) \leq N(P_j)$.

Consider the congruence cover $M_n = M(P_1 \cdots P_{2n})$. By Theorem 3.12, the number ν_n of pairs of fibered faces of M_n is at least 2^{2n}, and the degree d_n of $M_n \to M$ is (see Section 3.6):

$$d_n = \prod_{i=1}^{2n} \left(1 + \mathscr{N}_{K/\mathbb{Q}}(P_i)\right) = \left(\prod_{i=1}^{n} (1 + p_i)\right)^2. \tag{7.2}$$

To prove Theorem 1.4 we will estimate d_n by using the fact that $\mathscr{P}$ consists of $1/4$ of all rational primes. More precisely, recall that the *natural density* of a set $\mathscr{Y}$ of primes is $\alpha \in [0,1]$ if the number of primes in $\mathscr{Y}$ which are $\leq x$ is asymptotic (for x large) to α times $\pi(x) = |\{p \leq x\}|$. In our case, $\mathscr{P}$ is the set of primes p which split in both $\mathbb{Q}[\sqrt{-3}]$ and $\mathbb{Q}[\sqrt{-7}]$, i.e., those which split completely in the biquadratic field $\mathbb{Q}[\sqrt{-3}, \sqrt{-7}]$, and so $\mathscr{P}$ has natural density $1/4$ by the Tchebotarev density theorem. Theorem 1.4 now follows directly from:

LEMMA 7.3. *Let $\mathscr{P} = \{p_1, p_2, p_3 \ldots\}$ be a set of rational primes of natural density $1/4$, and d_n the product defined in (7.2). Then*

$$n = \left(\frac{1}{2} + o(1)\right) \frac{\log d_n}{\log \log d_n}.$$

In particular,

$$2^{2n} = \exp\left(\left(\log 2 + o(1)\right) \frac{\log d_n}{\log \log d_n}\right) \leq c_t e^{(\log d_n)^t},$$

for any $t < 1$, for a suitable constant $c_t > 0$.

Some people prefer to work with weaker notions of density, like analytic density, but in those cases this lemma may not hold.

Proof. If f and g are functions of x on a subset of $\mathbb{R}_+$, we will write $f(x) \sim g(x)$ if their ratio tends to 1 as x goes to infinity. Now recall that the prime number theorem asserts that $\pi(x) \sim \frac{x}{\log x}$, and consequently the n^{th} prime is $\sim n \log n$. As the set $\mathscr{P}$ has natural density $1/4$, the n^{th} prime p_n in $\mathscr{P}$ satisfies

$$p_n \sim 4n \log n. \tag{7.4}$$

Moreover, we claim that

$$\log\left(\prod_{j=1}^{n}(1+p_j)\right) \sim \frac{1}{4}p_n. \tag{7.5}$$

In fact, the Riemann hypothesis implies that this holds with an additive error term $\rho(n)$ of the order of $n^{1/2+\varepsilon}$. Here, we only need that $\rho(n)$ is $o(n\log n)$, which is known; in fact one has (cf. cite[Ch. 6]MontgomeryVaughan):

$$|\rho(n)| \leq ne^{-c\sqrt{\log n}} \quad \text{for some } c > 0.$$

Combining the asymptotic statements (7.4) and (7.5), and noting the formula for d_n, we get

$$2n\log n \sim \log d_n.$$

This yields, upon taking logarithms,

$$\log n + \log(2\log n) \sim \log\log d_n,$$

and since the left-hand side is $(1+o(1))\log n$, we obtain

$$2n \sim \frac{\log d_n}{\log n} = (1+o(1))\,\frac{\log d_n}{\log\log d_n}.$$

Since $2^{2n} = e^{2n\log 2}$, this implies

$$2^{2n} = \exp\left(\left(\log 2 + o(1)\right)\frac{\log d_n}{\log\log d_n}\right). \tag{7.6}$$

Moreover, for any real number $t < 1$, the quantity $\frac{\log d_n}{\log\log d_n}$ dominates $(\log d_n)^t$ as $n \to \infty$. So the right-hand side of (7.6) is bounded below by a constant (depending on t) times $\exp\left((\log d_n)^t\right)$, as asserted by the lemma. This completes the proof of Theorem 1.4. $\square$

7.7. A lower bound for $b_1(M_n)$. Recall that the first Betti number of any of the manifolds M_n is the same as the dimension of space of the cohomological cusp forms on D^* of level $I_n = Q\overline{Q}\prod_{j=1}^{n} P_j\overline{P}_j$, with $P_j, \overline{P}_j$ being the prime ideals in $\mathcal{O}_K$ above the rational prime $p_j \in \mathcal{P}$. From the arithmetic point of view, the dimension of this space is a complete mystery. This is because the two standard tools, namely the Riemann-Roch theorem and the Selberg trace formula, which work so well in the case of Hilbert modular forms (of weight 2), are not applicable here; the former because the M_n are not algebraic varieties, and the latter because the relevant archimedean representation is not in the discrete series, making it impossible to separate it from all the other archimedean representations. Nevertheless, we can get a lower bound as follows.

First recall that for any $N \geq 1$, the dimension of the space $S_2(N)$ of classical, holomorphic cusp forms on the upper half-plane $\mathscr{H}$ of level N is the genus $g_0(N)$ of the standard (cusp) compactification of the Riemann surface $\Gamma_0(N)\backslash\mathscr{H}$, where $\Gamma_0(N)$ is the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $N \mid c$. It is classical that for N square-free, one has

$$g_0(N) = 1 + \frac{1}{12}\prod_{p|N}(p+1) - \frac{2^s}{4} - \frac{2^t}{3} - \frac{2^m}{2},$$

where m is the number of prime divisors of N, and s (resp. t) the number of such divisors which are 1 mod 4 (resp. 1 mod 3); the last three terms of the formula are the error terms corresponding to the elliptic points i and $(1+\sqrt{-3})/2$ on $\mathscr{H}$ and the cusp at infinity. Applying this with $N = N_n = 7\prod_{j=1}^{n} p_j$, with $p_j \neq 7$ being split in $\mathbb{Q}[\sqrt{-3}]$ and inert in $\mathbb{Q}[\sqrt{-7}]$, we see by the formula for d_n that

$$g_0(N_n) \geq \frac{\sqrt{d_n} - (13)2^n}{12}.$$

On the other hand, by Lemma 7.3, the term 2^n is bounded from above by any positive power of d_n. Moreover, the dimension of the subspace $S_2(N_n)^{7\text{-new}}$ of $S_2(N_n)$ consisting of cusp forms of level N which are new at 7 is $\frac{5}{7}g_0(N)$. Putting these together, we obtain, for any $\varepsilon > 0$,

$$\dim_{\mathbb{C}} S_2(N_n)^{7\text{-new}} \geq \frac{5}{84} d_n^{\frac{1}{2}-\varepsilon},$$

for all large enough n (depending on ε). It is an easy exercise to see that the dimension on the left is derived purely from the knowledge of the dimensions of the space $S_2(N_n')^{\mathrm{new}}$ of *newforms* of level N_n' with $7 \mid N_n' \mid N_n$, together with the factorization of N_n/N_n'.

Next consider the base change $g \to g_K$ from $\mathrm{GL}(2)/\mathbb{Q}$ to $\mathrm{GL}(2)/K$ ([Lan]). We claim that the image of each $S_2(N_n')^{\mathrm{new}}$ under this base change has the same dimension. Indeed, if $g_K = h_K$ for two newforms g, h in this space, the newform h must be a twist of g by the quadratic Dirichlet character δ corresponding to K. But since g, h have odd levels, this would imply that h has level divisible by 4, the level of δ, which is impossible. Hence the claim. Now by construction, every newform g_k, or rather the cuspidal automorphic representation β_K of $\mathrm{GL}(2,\mathbb{A}_K)$ attached to it, is Steinberg at each of the primes dividing 7, and so transfers to our D^* and defines a cusp form β_K^D relative to $\Gamma_D(N_n/7)$; it also lifts to level

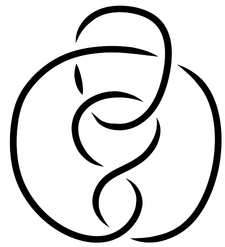

Figure 8.1. The Whitehead link.

$\Gamma_D(N_n)$ by the natural map. The cusp forms on $\mathrm{GL}(2)/K$ which are Steinberg at $\{Q, \overline{Q}\}$, and which are old away from 7, are seen to correspond in a one-to-one way to old forms on D^*, since $D^*_v \simeq \mathrm{GL}(2, K_v)$ for any $v \nmid 7$. Consequently, we get the following:

PROPOSITION 7.8. *Let $\{M_n\}$ be the arithmetic tower of closed hyperbolic 3-manifolds we have constructed above, with M_n a nonnormal cover of M of degree d_n. Then for every $\varepsilon > 0$, we can find an integer L_ε such that*

$$n \geq L_\varepsilon \Rightarrow b_1(M_n) \geq \frac{5}{84} d_n^{\frac{1}{2}-\varepsilon}.$$

8. The Whitehead link complement. In this section, we give a very concrete example of covers of a finite volume hyperbolic 3-manifold with cusps where the number of fibered faces increases exponentially in the degree. Consider the Whitehead link shown in Figure 8.1, and let W be its exterior. The interior of W admits a hyperbolic metric of finite volume; indeed, W is arithmetic with $\pi_1(W)$ a subgroup of $\mathrm{PSL}(2, \mathbb{Z}[i])$ of index 12. We consider n-fold cyclic covers dual to the thrice punctured sphere bounded by one of the components. More precisely, let ω be an element of the basis of $H^1(W; \mathbb{Z})$ which is dual to a basis of $H_1(W; \mathbb{Z}) \cong \mathbb{Z}^2$ consisting of meridians. (This link is quite symmetric, so it doesn't matter which basis element we pick.) Then let W_n be the cover corresponding to the kernel of the composition $\pi_1(W) \xrightarrow{\omega} \mathbb{Z} \to \mathbb{Z}/n$. The manifold W_n is also the complement of the link L_n pictured in Figure 8.2. Following that figure, let $\mu_i \in H_1(W_n; \mathbb{Z})$ be a meridian for the component C_i of L_n. Let $\{\omega_i\}$ be the dual basis for $H^1(W_n; \mathbb{Z})$. This section is devoted to proving:

THEOREM 8.3. *Let W_n be the n-fold cyclic cover of the exterior of the Whitehead link described above. Then the number of fibered faces of the Thurston norm ball B_T of W_n is 2^{n+1}. More precisely, B_T is the convex hull of the ω_i and all of its top-dimensional faces are fibered.*

The polytope B_T is called an n-orthoplex or a cross polytope; it is dual to the unit cube in $H_1(W_n; \mathbb{Z})$ spanned by the μ_i. Note that this form for B_T is plausible as each component of L_n bounds an obvious twice-punctured disc, and

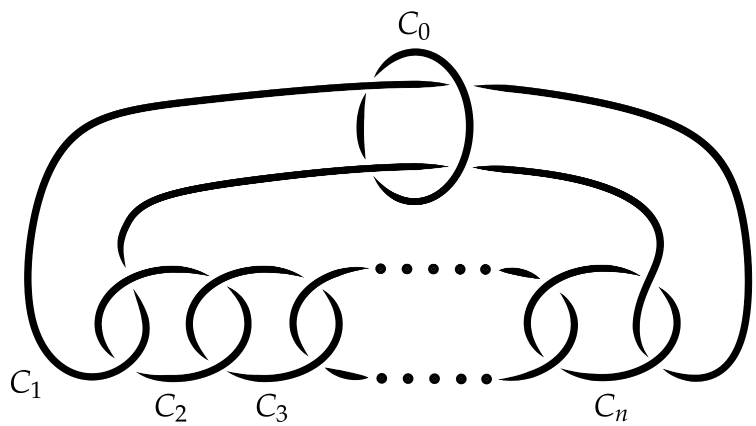

Figure 8.2. The covering manifold W_n is the exterior of the shown link L_n.

hence $\|\omega_i\|_T = 1$ for every i as hyperbolic manifolds contain no simpler essential surfaces. The theorem will follow easily from:

LEMMA 8.4. *Any class of $H^1(W_n; \mathbb{Z})$ of the form*

$$\omega = \epsilon_0\omega_0 + \epsilon_1\omega_1 + \cdots + \epsilon_n\omega_n \quad \textit{for } \epsilon_i \in \{-1, 1\}$$

fibers and $\|\omega\|_T = n + 1$.

Let us first derive the theorem from the lemma.

Proof of Theorem 8.3. We need to show that for any choices of $\epsilon_i \in \{-1, 1\}$, the simplex Δ spanned by $\eta_i = \epsilon_i\omega_i$ is a face of B_T. We know $\|\eta_i\|_T = 1$, and so $\Delta \subset B_T$. By Lemma 8.4, the class $\eta = \frac{1}{n+1}\sum \eta_i$ has norm 1; this gives another point in Δ which we know lies in ∂B_T. Now consider $\alpha \in \Delta$, which necessarily has the form

$$\alpha = \sum_{i=0}^{n} a_i\eta_i \quad \text{for } a_i \geq 0 \text{ and } \sum a_i = 1.$$

We need to show $1 \leq \|\alpha\|_T$, as then α lies in ∂B_T; knowing this for all α implies Δ is a face of B_T.

After permuting the coordinates, we can assume $a_0 \geq a_i$ for all i. As a consequence, $a_0 \geq 1/(n+1)$ and

$$\eta = b_0\alpha + b_1\eta_1 + b_2\eta_2 + \cdots + b_n\eta_n \quad \text{where } b_i \geq 0 \text{ and } \textstyle\sum_{i=0}^{n} b_i = 1.$$

Now the sublinearity of the Thurston norm gives

$$\|\eta\|_T \leq b_0\|\alpha\|_T + b_1\|\eta_1\|_T + b_2\|\eta_2\|_T + \cdots + b_n\|\eta_n\|_T.$$

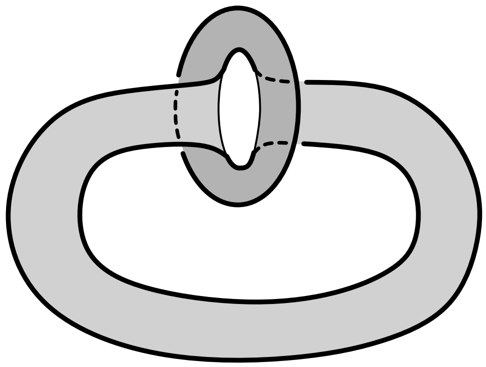

Figure 8.5. The base link/surface for the Murasugi sum shown in Figure 8.6.

Evaluating the norms we know gives:

$$1 \leq b_0 \|\alpha\|_T + b_1 + \cdots + b_n$$

Since $\sum b_i = 1$, this implies $1 \leq \|\alpha\|_T$, as required to prove the theorem. □

We conclude this section with:

Proof of Lemma 8.4. Since the Thurston norm is invariant under $\omega \mapsto -\omega$, we can assume $\epsilon_0 = 1$. We analyze this class using Murasugi sum, closely following the approach used in [Lei] to analyze chain links. Given $a_i \in \{-1, 1\}$, consider the link $L(a_1, a_2, \ldots, a_n)$ shown in Figure 8.6, which is described as the boundary

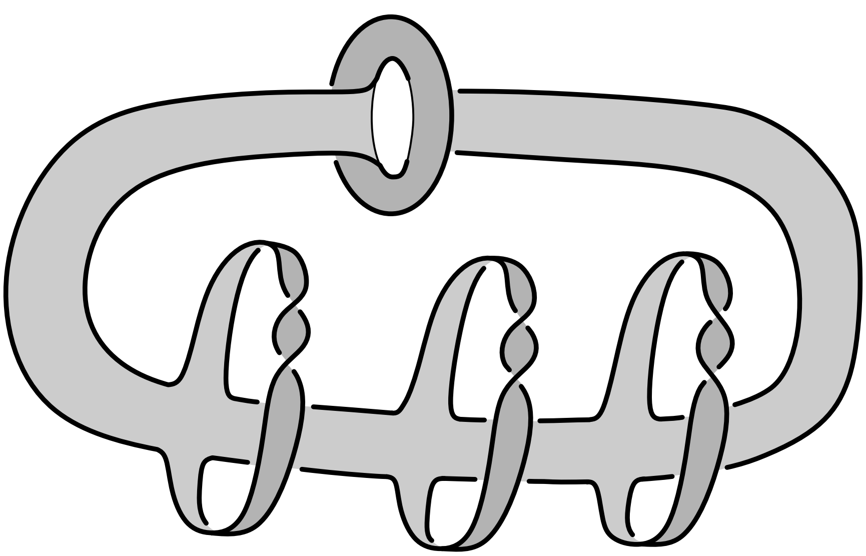

Figure 8.6. The link $L(a_1, a_2, \ldots, a_n)$ is the Murasugi sum of the link in Figure 8.5 with n Hopf bands with appropriately oriented twists. Shown is $L(1, 1, -1)$.

of a surface which is the Murasugi sum of the surface in Figure 8.5 with n Hopf bands, where the twist on the i^{th} Hopf band is right- or left-handed depending on a_i. Let $M(a_1, \ldots, a_n)$ be the exterior of $L(a_1, \ldots, a_n)$. Since all the surfaces in the Murasugi sum are fibers, the surface Σ pictured in Figure 8.5 is a fiber of a fibration of $M(a_1, \ldots, a_n)$ over the circle [Sta2], [Gab2]. Orient the components of $L(a_1, \ldots, a_n)$ as the boundary of the surface shown in Figure 8.6, and let ω_i be the corresponding dual basis of $H^1(M(a_1, \ldots, a_n); \mathbb{Z})$. Thus Σ is Poincaré dual to $\omega = \omega_1 + \cdots + \omega_n$ and $\|\omega\|_T = n + 1$. The key to the lemma is to show:

Claim 8.7. Fix $a_1, \ldots, a_n$ in $\{-1, 1\}$, and pick $k \in \{1, 2, \ldots, n\}$. Suppose $b_i = a_i$ except $b_{k-1} = -a_{k-1}$ and $b_k = -a_k$. (If $k = 1$ then here $k - 1$ should be interpreted as n.) Then $M(a_1, \ldots, a_n) \cong M(b_1, \ldots, b_n)$ via a homeomorphism that acts on H^1 by

$$\begin{aligned} \omega_i &\mapsto \omega_i \text{ for } i \neq k \text{ and} \\ \omega_k &\mapsto -\omega_k. \end{aligned}$$

The proof of this claim is given in Figure 8.8. Now to prove the lemma, note that $L(1, 1, \ldots, 1)$ is L_n and applying the claim repeatedly to a suitable starting $L(a_1, \ldots, a_n)$ allows us to see any that class of the form $\omega_0 + \epsilon_1\omega_1 + \cdots + \epsilon_n\omega_n$ is represented by a fiber surface with Euler characteristic $-(n + 1)$ as required. □

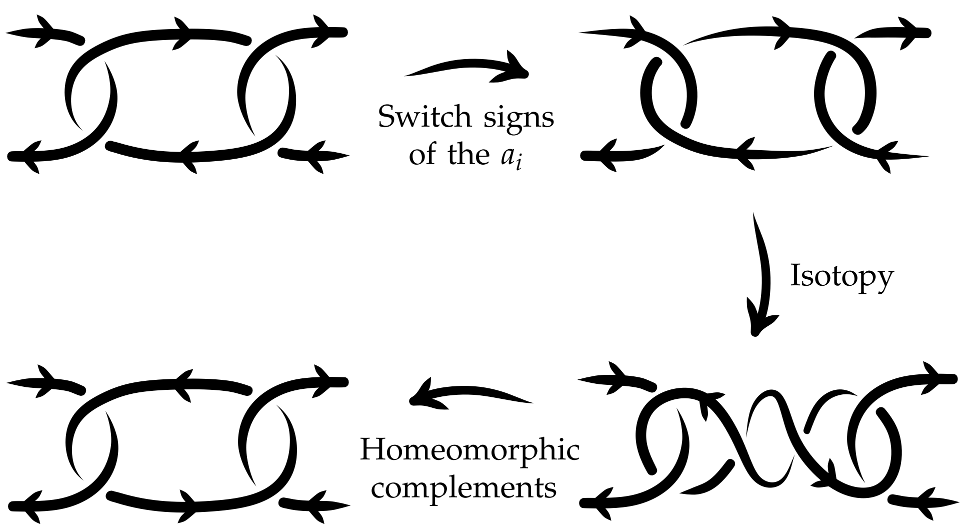

Figure 8.8. Switching the signs of two consecutive a_i can be partially reversed at the cost of adding a twist as shown in the bottom right picture. (Here both a_i are 1, but the other cases are similar.) We can undo said twist by a homeomorphism of the link complements by cutting along a twice punctured disc bounded by component C_0 of the link and doing a full twist. To prove Claim 8.7, just note the different orientations of the central loop in the two pictures at left.

9. Work of Long and Reid. In this section, we give a proof of Theorem 1.10 which is simpler than the original one in [LR]. A different simplification was given by Agol in [Ago1]. We begin by introducing an invariant of a fibered cohomology class ω of a hyperbolic manifold M. Let $\mathscr{F}_\omega$ be the associated transverse pseudo-Anosov flow described in Section 2.2. As discussed, this flow has a positive finite number of singular orbits, each of which is closed. From the closed geodesics $\gamma_1, \gamma_2, \ldots, \gamma_n$ homotopic to these orbits, define $c(\omega)$ to be the subset of the universal cover $\mathscr{H}^3$ of M consisting of the inverse images of the γ_i. By Theorem 2.3, any other class η in the fibered face of ω has isotopic $\mathscr{F}_\eta$, and hence c is really an invariant of a fibered face. In fact, we regard the geodesics in $c(\omega)$ as unoriented, so that c is an invariant of a fibered face pair. With this invariant in hand, we turn to the proof itself.

Proof of Theorem 1.10. Let ω be a fibered class for our arithmetic 3-manifold $M = \Gamma\backslash\mathscr{H}^3$. We will first find a cover of M with two pairs of fibered faces. Following Section 3.5, given $g \in \mathrm{Comm}(\Gamma)$ we have the associated manifold M_g with two covering maps $p_g, q_g \colon M_g \to M$. Then M_g has two fibered classes $p_g^*(\omega)$ and $q_g^*(\omega)$, with associated c invariants equal to $c(\omega)$ and $g^{-1} \cdot c(\omega)$ respectively. Thus these two fibered classes of M_g lie in genuinely distinct faces provided $c(\omega)$ is not setwise invariant under g. Now the setwise stabilizer of $c(\omega)$ in $\mathrm{PSL}(2,\mathbb{C})$ is a discrete subgroup which contains Γ as a subgroup of finite index; as $\mathrm{Comm}(\Gamma)$ is dense in $\mathrm{PSL}(2,\mathbb{C})$, there are plenty of $g \in \mathrm{Comm}(\Gamma)$ which do not fix $c(\omega)$, as desired.

To create any number of fibered faces, we simply repeat this process inductively, assuming at each stage we have constructed a cover M_n with fibered classes $\omega_1, \ldots, \omega_n$ where the $c(\omega_i)$ are all distinct and then building a cover of using a $g \in \mathrm{Comm}(\pi_1(M_n))$ so that all $c(\omega_i)$ and $g \cdot c(\omega_k)$ are distinct. □

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REFERENCES

[Ago1] I. Agol, Criteria for virtual fibering, preprint, 2007; arXiv:0707.4522.

[Ago2] ———, Virtual betti numbers of symmetric spaces, preprint, 2006; math.GT/0611828.

[Bla] P. Blanc, Sur la cohomologie continue des groupes localement compacts, *Ann. Sci. École Norm. Sup. (4)* **12** (1979), 137–168.

[BH] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2), *Grundlehren Math. Wiss. [Fundamental Principles of Mathematical Sciences]*, vol. 335, Springer-Verlag, Berlin, 2006.

[But] J. O. Button, Fibred and virtually fibred hyperbolic 3-manifolds in the censuses, *Experiment. Math.* **14** (2005), 231–255; arXiv:math.GT/0410321.

[Cal] D. Calegari, Foliations transverse to triangulations of 3-manifolds, *Comm. Anal. Geom.* **8** (2000), 133–158.

[CD] F. Calegari and N. M. Dunfield, Automorphic forms and rational homology 3-spheres, *Geom. Topol.* **10** (2006), 295–329; arXiv:math.GT/0508271.

[CBFS] J. Cannon, W. Bosma, C. Fieker, and A. Steel, eds., *Handbook of MAGMA functions*, Computational Algebra Group, University of Sydney, 2.14 ed., 2007, 4500 pages; http://magma.maths.usyd.edu.au/.

[Clo] L. Clozel, On the cuspidal cohomology of arithmetic subgroups of SL($2n$) and the first Betti number of arithmetic 3-manifolds, *Duke Math. J.* **55** (1987), 475–486.

[CLR1] D. Cooper, D. D. Long, and A. W. Reid, Bundles and finite foliations, *Invent. Math.* **118** (1994), 255–283.

[CLR2] ———, On the virtual Betti numbers of arithmetic hyperbolic 3-manifolds, *Geom. Topol.* **11** (2007), 2265–2276.

[CT] D. Cooper and S. Tillmann, The Thurston norm via Normal Surfaces, preprint, 2007; arXiv:0706.0673.

[CGHN] D. Coulson, O. A. Goodman, C. D. Hodgson, and W. D. Neumann, Computing arithmetic invariants of 3-manifolds, *Experiment. Math.* **9** (2000), 127–152.

[Cre] J. E. Cremona, Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields, *Compositio Math.* **51** (1984), 275–324.

[CD] M. Culler and N. M. Dunfield, t3m: a Python 3-manifold library, 2006; http://www.math.uic.edu/∼t3m.

[Dun] N. M. Dunfield, Alexander and Thurston norms of 3-manifolds fibering over the circle, *Pacific J. Math.* **200** (2001), 43–58; arXiv:math.GT/9908050.

[DR] N. M. Dunfield and D. Ramakrishnan, Source code and data files; http://www.computop.org/software/fibered-faces/.

[DT] N. M. Dunfield and D. P. Thurston, A random tunnel number one 3-manifold does not fiber over the circle, *Geom. Topol.* **10** (2006), 2431–2499; arXiv:math.GT/0510129.

[Elk] N. D. Elkies, The existence of infinitely many supersingular primes for every elliptic curve over $\mathbf{Q}$, *Invent. Math.* **89** (1987), 561–567.

[Fri] D. Fried, Fibrations over S^1 with pseudo-Anosov monodromy, *Travaux de Thurston sur les surfaces*, *Astérisque* **66** (1979), 251–266.

[Gab1] D. Gabai, Foliations and the topology of 3-manifolds, *J. Differential Geom.* **18** (1983), 445–503.

[Gab2] ———, The Murasugi sum is a natural geometric operation, II, *Combinatorial Methods in Topology and Algebraic Geometry (Rochester, NY, 1982)*, *Contemp. Math.*, vol. 44, Amer. Math. Soc., Providence, RI, 1985, pp. 93–100.

[Gel] S. S. Gelbart, *Automorphic Forms on Adèle Groups*, *Ann. Math. Stud.*, no. 83, Princeton, NJ, 1975.

[GJ] S. Gelbart and H. Jacquet, Forms of GL(2) from the analytic point of view, *Automorphic Forms, Representations and L-Functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, *Proc. Sympos. Pure Math.*, vol. XXXIII, Amer. Math. Soc., Providence, RI, 1979, pp. 213–251.

[G] O. Goodman, Snap, http://www.ms.unimelb.edu.au/∼snap/.

[Gro] B. H. Gross, *Arithmetic on Elliptic Curves with Complex Multiplication*, *Lecture Notes in Math.*, vol. 776, Springer-Verlag, Berlin, 1980, with an appendix by B. Mazur.

[GL] B. H. Gross and J. Lubin, The Eisenstein descent on $J_0(N)$, *Invent. Math.* **83** (1986), 303–319.

[Hak] W. Haken, Theorie der Normalflachen, *Acta Math.* **105** (1961), 245–375.

[Hat] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[Hem] J. Hempel, 3-*Manifolds*, Princeton University Press, Princeton, NJ, 1976; *Ann. of Math. Studies*, no. 86.

[JR] W. Jaco and J. H. Rubinstein, 0-efficient triangulations of 3-manifolds, *J. Differential Geom.* **65** (2003), 61–168; arXiv:math.GT/0207158.

[JT] W. Jaco and J. L. Tollefson, Algorithms for the complete decomposition of a closed 3-manifold, *Illinois J. Math.* **39** (1995), 358–406.

[JL] H. Jacquet and R. P. Langlands, *Automorphic Forms on* GL(2), *Lecture Notes in Math.*, vol. 114, Springer-Verlag, Berlin, 1970.

[Kut] P. Kutzko, The Langlands conjecture for Gl_2 of a local field, *Ann. of Math. (2)* **112** (1980), 381–412.

[LS] J.-P. Labesse and J. Schwermer, On liftings and cusp cohomology of arithmetic groups, *Invent. Math.* **83** (1986), 383–401.

[LLR] M. Lackenby, D. D. Long, and A. W. Reid, Covering spaces of arithmetic 3-orbifolds, preprint, 2006; math.GT/0601677.

[Lan] R. P. Langlands, *Base change for* GL(2), *Ann. Math. Stud.*, vol. 96, Princeton University Press, Princeton, NJ, 1980.

[Lei] C. J. Leininger, Surgeries on one component of the Whitehead link are virtually fibered, *Topology* **41** (2002), 307–320.

[LR] D. D. Long and A. W. Reid, Finding fibre faces in finite covers, preprint, 2007.

[MR] C. Maclachlan and A. W. Reid, *The Arithmetic of Hyperbolic 3-Manifolds*, *Grad. Texts in Math.*, vol. 219, Springer-Verlag, New York, 2003.

[MKS] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory*, Dover Publications Inc., NY, revised ed., 1976.

[Mas1] J. D. Masters, Counting immersed surfaces in hyperbolic 3-manifolds, *Algebr. Geom. Topol.* **5** (2005), 835–864; math.GT/0205250.

[Mas2] ———, Thick surfaces in hyperbolic 3-manifolds, *Geom. Dedicata* **119** (2006), 17–33; math.GT/0502395.

[McM] C. T. McMullen, The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology, *Ann. Sci. École Norm. Sup. (4)* **35** (2002), 153–171.

[MV] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, *Cambridge Stud. Adv. Math*, vol. 97, Cambridge University Press, Cambridge, England, 2006.

[Ota] J.-P. Otal, Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3, *Astérisque* (1996), x+159.

[PW] C. Petronio and J. R. Weeks, Partially flat ideal triangulations of cusped hyperbolic 3-manifolds, *Osaka J. Math.* **37** (2000), 453–466.

[Pra] D. Prasad, Trilinear forms for representations of GL(2) and local ϵ-factors, *Compositio Math.* **75** (1990), 1–46.

[PR] D. Prasad and D. Ramakrishnan, Lifting orthogonal representations to spin groups and local root numbers, *Proc. Indian Acad. Sci. Math. Sci.* **105** (1995), 259–267.

[Rei] A. W. Reid, A non-Haken hyperbolic 3-manifold covered by a surface bundle, *Pacific J. Math.* **167** (1995), 163–182.

[Rog] J. D. Rogawski, Representations of GL(n) and division algebras over a p-adic field, *Duke Math. J.* **50** (1983), 161–196.

[Roh] D. E. Rohrlich, Galois theory, elliptic curves, and root numbers, *Compositio Math.* **100** (1996), 311–349.

[Sch] S. Schleimer. *Almost Normal Heegaard Splittings*, Ph.D. thesis, Berkeley, 2001.

[Sil] J. H. Silverman, *The Arithmetic of Elliptic Curves*, *Grad. Texts in Math*, vol. 106, Springer-Verlag, New York, 1986.

[Sta1] J. Stallings, On fibering certain 3-manifolds, *Topology of 3-Manifolds and Related Topics (Proc. The Univ. of Georgia Institute, 1961)*, Prentice-Hall, Englewood Cliffs, NJ, 1962, pp. 95–100.

[Sta2] ———, Constructions of fibred knots and links, *Algebraic and Geometric Topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, *Proc. Sympos. Pure Math.*, vol. XXXII, Amer. Math. Soc., Providence, RI, 1978, pp. 55–60.

[SAGE] W. Stein, et al., *SAGE Mathematics Software (Version 2.8)*, 2007; http://www.sagemath.org.

[Thu1] W. P. Thurston, A norm for the homology of 3-manifolds, *Mem. Amer. Math. Soc.* **59** (1986), i–vi and 99–130.

[Thu2] ———, Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle; arXiv:math.GT/9801045.

[TW] J. L. Tollefson and N. Wang, Taut normal surfaces, *Topology* **35** (1996), 55–75.

[Tun] J. B. Tunnell, On the local Langlands conjecture for *GL*(2), *Invent. Math.* **46** (1978), 179–200.

[Ven] T. N. Venkataramana, Virtual Betti numbers of compact locally symmetric spaces, preprint, 2006; arXiv:math.GR/0611733.

[Vig] M.-F. Vignéras, *Arithmétique des algèbres de quaternions*, *Lecture Notes in Math.*, vol. 800, Springer-Verlag, Berlin, 1980.

[Wal1] F. Waldhausen, The word problem in fundamental groups of sufficiently large irreducible 3-manifolds, *Ann. of Math. (2)* **88** (1968), 272–280.

[Wal2] G. S. Walsh, Great circle links and virtually fibered knots, *Topology* **44** (2005), 947–958; math.GT/0407361.

[W] J. Weeks, SnapPea; http://www.geometrygames.org.

[WCD] J. Weeks, M. Culler, and N. Dunfield, SnapPeaPython; http://www.math.uic.edu/∼t3m.