Inelastic Scattering of Quanta with Production of Pairs

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The problem of accounting for the anomalous scattering of gamma-rays suggests the importance of investigating the probability of processes in which an incoming quantum produces an electron-positron pair in the field of a nucleus, going on in a new direction with diminished energy. To determine the cross section in the general case is difficult, but an estimate of the total magnitude of the effect in the energy range of interest is obtained by a calculation of the cross section as a function of the energies of the incident and scattered quanta and the angle between them in the limit where the electron-positron pair is produced with small kinetic energy.

While there exists a possibility of observing the process under suitable experimental conditions, the cross section is found to be too small to contribute appreciably to the production of the hard component in the radiation from heavy elements exposed to penetrating gamma-rays.

I

THE inelastic scattering of gamma-rays at a nucleus with the production of an electron-positron pair has an interest as a possible contribution to the anomalous scattering which is observed in heavy elements. In the effect considered here, the incident quantum produces a pair in the field of the nucleus and goes on in a new direction with diminished energy. This process involves a second order interaction with the radiation field, and is closely related to the type of radiative effect in which an electron emits two quanta on colliding with a nucleus. The quantitative relation between the two processes is given below. The connection between the inelastic scattering and the related double radiation is of the same kind as that between the photoelectric production of pairs and the production of a single quantum by an electron in the field of a nucleus.[1]

The calculation here uses the Born approximation for the wave functions of the electron and the positron in the Coulomb field of the nucleus. In the Born expansion of the wave functions in plane waves, only the terms up to the first order in Z are needed to give the inelastic scattering. Since the process is of the second order in the interaction with the radiation field, it requires a perturbation calculation altogether of the third order. For this reason the cross section involves many more terms than that for the photoelectric production of pairs, which is likewise first order in Z, but only of the first order in the interaction with the radiation field.

In the region of the high quantum energies it is in theory possible to treat the problem by the method (Williams[2] and Weizsäcker[3]) of replacing the nuclear field by an equivalent radiation field, and considering the interaction of the virtual quanta in this field with the incident quantum. Knowledge of the cross section for the collision of two quanta in field-free space with the production of a pair and a new quantum would then make possible a calculation in the region of high energies of the probability of the process considered here. As this cross section has not yet been calculated, it is simpler, as well as more accurate, to proceed directly as below.

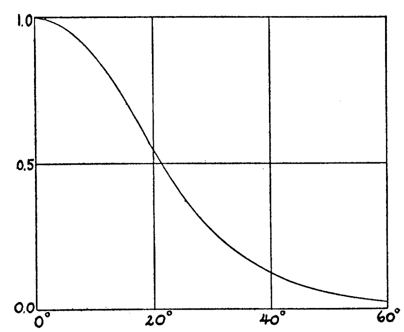

FIG. 1. Distribution in angle of the scattered quanta for $\nu=5$, $\sigma=3$.

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[1] H. Bethe and W. Heitler, Proc. Roy. Soc. **A146**, 83 (1934).

[2] E. J. Williams, Kgl. Danske Vid. Math.-fys. Medd. **13**, 4 (1935).

[3] C. V. Weizsäcker, Zeits. f. Physik **88**, 612 (1934).

II

Calculations concerned with electrons of relativistic energies take a simple and convenient form when the quantities entering are expressed in natural units.[4] These units are: for length, $\hbar/mc$; time, $\hbar/mc^2$; action, $\hbar$; energy, mc^2; momentum, mc; charge, e; potential, $emc/\hbar$. In these units, the potential energy of two electrons at a distance r is α/r, where $\alpha = e^2/\hbar c$, and the energy of the electromagnetic field is $(\alpha/8\pi)\int(E^2+H^2)dv$. The operators p_x and H go over into $-i\partial/\partial x$ and $i\partial/\partial t$, and Dirac's equation, including the electromagnetic field, becomes (following the usual derivation)

$$\{i\partial/\partial t+\boldsymbol{\alpha}\cdot\mathbf{p}+\alpha V+\beta-\sum_k N\nu+\sum_k(2\pi\alpha/\nu V)^{\frac{1}{2}}(b\boldsymbol{\alpha}\cdot\mathbf{f}\exp i\boldsymbol{\nu}\cdot\mathbf{x}+b^*\boldsymbol{\alpha}\cdot\mathbf{f}\exp -i\boldsymbol{\nu}\cdot\mathbf{x})\}\varphi=0. \quad (1)$$

The summations are extended over all quantum frequencies $\boldsymbol{\nu}_K$ and polarizations $\mathbf{f}_K$; in (1) b_K and N_K have the matrix elements $b_{ij}=\sqrt{j}\delta_{i+1,\ j}$, $N_{ij}=j\delta_{ij}$. N_K is the operator for the number of quanta of type K.

The term in (1) involving the b's represents the interaction between the electron and the radiation field. To the first order in this interaction, scattering does not occur. A perturbation calculation to the second order gives for the general scattering cross section the expression

$$d\varphi_i=\alpha^2(\sigma/\nu)\left|\sum_j\{B_{ij}A_{j0}/(E_0-E_j+\nu)+A_{ij}B_{j0}/(E_0-E_j-\sigma)\}\right|^2 d\Omega. \quad (2)$$

$d\varphi_i/d\Omega$ is the cross section per unit solid angle of emission of the scattered quantum $\boldsymbol{\sigma}$ for a transition in which the incident quantum $\boldsymbol{\nu}$ disappears and the electron is raised from the initial state ψ_0 of energy E_0 to the final state ψ_i of energy $E_i=E_0+\nu-\sigma$. The summation Σ_j is to be extended over all electron states ψ_j. The A's and B's are the matrix elements for the interaction with the radiation field:

$$A_{ij}=\int\psi_i{}^*\mathbf{a}\cdot\boldsymbol{\alpha}\exp i(\boldsymbol{\nu}\mathbf{x})\psi_j dv; \quad B_{ij}=\int\psi_i{}^*\mathbf{b}\cdot\boldsymbol{\alpha}\exp(-i\boldsymbol{\sigma}\mathbf{x})\psi_j dv. \quad (3)$$

$\mathbf{a}=\mathbf{f}_\nu$ and $\mathbf{b}=\mathbf{f}_\sigma$ are the polarization vectors of the incident and scattered quanta.

In the present problem of the inelastic scattering by the negative energy electrons in a nuclear field, ψ_0 is a state of negative energy and ψ_i one of positive energy, these two states, respectively, corresponding to the positron and electron produced in the process. To determine the total cross section for the production of a scattered quantum in the energy range $d\sigma$ and in the solid angle $d\Omega$, summation of $d\varphi_i$ is made first over all states ψ_i of energy between $E_i-dE_i=E_0+\nu-(\sigma+d\sigma)$ and $E_i=E_0+\nu-\sigma$, keeping E_0 fixed; a sum is then carried out over all negative energy states ψ_0 between the two limits $E_0=-(\nu-\sigma-1)$ and $E_0=-1$. Finally the result is summed over both polarization directions of the scattered quantum and averaged over both polarizations of the incident quantum to give the cross sections desired:

$$\frac{d^2\varphi}{d\Omega d\sigma}=\alpha^2\frac{\sigma}{\nu}\frac{1}{2}\sum_a\sum_b\int dE_0\sum_{S_0}\int d\Omega_0\left(\frac{L}{2\pi}\right)^3 p_0|E_0|\sum_{S_i}\int d\Omega_i\left(\frac{L}{2\pi}\right)^3 p_iE_i\left|\sum_j\{\cdots\}\right|^2. \quad (4)$$

In (4) the wave functions are considered normalized over a cube of volume L^3, so that the number of states, having a given spin s_0 and a momentum at a large distance from the nucleus in the range dp_0 and the solid angle $d\Omega_0$, is $(L/2\pi)^3p_0{}^2dp_0d\Omega_0=(L/2\pi)^3p_0E_0dE_0d\Omega_0$.

In an accurate evaluation of (4) the wave functions satisfy the equation

$$(E-i\boldsymbol{\alpha}\cdot\nabla+\beta+\alpha Z(r)/r)\psi=0; \quad (5)$$

$Z(r)$ determines the potential for an electron in the field of a screened nucleus of charge $Z=Z(0)$. The normalized solution of (5) to the first Born approximation is

[4] A. E. Ruark, Phys. Rev. **38**, 2240 (1931).

$$\psi = L^{-\frac{3}{2}}\left\{\exp(i\mathbf{p}\cdot\mathbf{x}) - (\alpha/2\pi^2)\int \frac{Z-F(|\mathbf{p}-\mathbf{p}'|)}{|\mathbf{p}-\mathbf{p}'|^2}\,\frac{E-\boldsymbol{\alpha}\cdot\mathbf{p}'-\beta}{E^2-p'^2-1}\exp(i\mathbf{p}'\cdot\mathbf{x})dp_x'dp_y'dp_z'\right\}S. \tag{6}$$

F is the usual atomic form factor; S is the normalized spin vector for free electron waves of momentum $\mathbf{p}$ and energy E. The wave functions ψ_0, ψ_j and ψ_i entering in (4) are approximated by taking the solution (6) with appropriate values E and $\mathbf{p}$; with this form for the wave functions the integrations over the coordinates and momenta entering into the A's and B's are easily carried out. Owing to the appearance of δ functions in the matrix elements only four values of $\mathbf{p}_j$ contribute to the sum over intermediate states. The summation over spins is simplified by the relation holding for the two independent spin functions corresponding to the same energy, E, and momentum, $\mathbf{p}$:

$$\sum_{\text{Spin}} SS^* = (E-\boldsymbol{\alpha}\cdot\mathbf{p}-\beta)/2E.$$

The additional relation that

$$\sum_{S_0}\sum_{S_i}|M_{i0}|^2 = \text{Spur}\,\frac{E_i-\boldsymbol{\alpha}\cdot\mathbf{p}_i-\beta}{2E_i}M\frac{E_0-\boldsymbol{\alpha}\cdot\mathbf{p}_0-\beta}{2E_0}M^*,$$

which holds for any matrix element M_{i0} of the form $(S_i^*MS_0)$, reduces the expression (4) for the inelastic cross section to

$$\frac{d^2\varphi}{d\Omega d\sigma} = \frac{\alpha^4}{2^9\pi^4}\frac{\sigma}{\nu}\sum_a\sum_b\int dE_0\int d\Omega_0\int d\Omega_i\frac{p_0p_i}{\tau^4}(Z-F(\tau))^2\,\text{Spur}\,(E_i-\boldsymbol{\alpha}\cdot\mathbf{p}_i-\beta)N(E_0-\boldsymbol{\alpha}\cdot\mathbf{p}_0-\beta)N^*. \tag{7}$$

Here $\boldsymbol{\tau}$ is the momentum transfer to the nucleus, $\boldsymbol{\nu}-\boldsymbol{\sigma}+\mathbf{p}_0-\mathbf{p}_i$; and the matrix $N=N_1+N_2$, where

$$N_1 = \frac{\mathbf{b}\cdot\boldsymbol{\alpha}[E_0+\nu-\boldsymbol{\alpha}\cdot(\mathbf{p}_i+\boldsymbol{\sigma})-\beta][E_0+\nu-\boldsymbol{\alpha}\cdot(\mathbf{p}_0+\boldsymbol{\nu})-\beta]\mathbf{a}\cdot\boldsymbol{\alpha}}{[E_i\sigma+\mathbf{p}_i\cdot\boldsymbol{\sigma}][E_0\nu-\mathbf{p}_0\cdot\boldsymbol{\nu}]}$$
$$+\frac{2\mathbf{b}\cdot\boldsymbol{\alpha}[E_0+\nu-\boldsymbol{\alpha}\cdot(\mathbf{p}_i+\boldsymbol{\sigma})-\beta]\mathbf{a}\cdot\boldsymbol{\alpha}[E_0-\boldsymbol{\alpha}\cdot(\mathbf{p}_i+\boldsymbol{\sigma}-\boldsymbol{\nu})-\beta]}{[E_i\sigma-\mathbf{p}_i\cdot\boldsymbol{\sigma}][p_0^2-(\mathbf{p}_i+\boldsymbol{\sigma}-\boldsymbol{\nu})^2]}$$
$$+\frac{2[E_i-\boldsymbol{\alpha}\cdot(\mathbf{p}_0+\boldsymbol{\nu}-\boldsymbol{\sigma})-\beta]\mathbf{b}\cdot\boldsymbol{\alpha}[E_0+\nu-\boldsymbol{\alpha}\cdot(\mathbf{p}_0+\boldsymbol{\nu})-\beta]\mathbf{a}\cdot\boldsymbol{\alpha}}{[E_0\nu-\mathbf{p}_0\cdot\boldsymbol{\nu}][p_i^2-(\mathbf{p}_0+\boldsymbol{\nu}-\boldsymbol{\sigma})^2]}; \tag{8}$$

and

$$N_2(\boldsymbol{\nu},\mathbf{a};\boldsymbol{\sigma},\mathbf{b}) = N_1(-\boldsymbol{\sigma},\mathbf{b};-\boldsymbol{\nu},\mathbf{a}). \tag{9}$$

The elementary cross section for the production of a scattered quantum in the solid angle $d\Omega$ with the appearance of a positron in the cube $dp_0dq_0dr_0$ of momentum space at $-\mathbf{p}_0$ and an electron in $dp_idq_idr_i$ at $\mathbf{p}_i$ is

$$d^7\varphi = (Z-F(\tau))^2\alpha^4 2^{-9}\pi^{-4}\sigma\nu^{-1}\tau^{-4}E_0^{-1}E_i^{-1}\sum_a\sum_b\text{Spur}\,(E_i-\boldsymbol{\alpha}\cdot\mathbf{p}_i-\beta)N(E_0-\boldsymbol{\alpha}\cdot\mathbf{p}_0-\beta)N^*$$
$$\times d\Omega dp_0dq_0dr_0dp_idq_idr_i. \tag{10}$$

The elementary cross section for the radiation of two quanta when an electron passes through the field of a nucleus is simply related to (10).

$$d^7\Phi = (Z-F(\tau))^2\alpha^4 2^{-9}\pi^{-4}\sigma^{-1}\nu^{-1}p_0p_i^{-1}\tau'^{-4}\sum_a\sum_b\text{Spur}\,(E_i-\boldsymbol{\alpha}\cdot\mathbf{p}_i-\beta)N'(E_0-\boldsymbol{\alpha}\cdot\mathbf{p}_0-\beta)N'^*$$
$$\times d\Omega_0 d\nu_x d\nu_y d\nu_z d\sigma_x d\sigma_y d\sigma_z \tag{11}$$

is the cross section for the emission of two quanta of energies ν and σ in the ranges $d\nu_xd\nu_yd\nu_z$ and

$d\sigma_x d\sigma_y d\sigma_z$ by an electron of initial momentum $\mathbf{p}_i$ and of final momentum $\mathbf{p}_0$ in the solid angle $d\Omega_0$. In (11), $\boldsymbol{\tau}'(\mathbf{v}, \boldsymbol{\sigma}) = \boldsymbol{\tau}(\mathbf{v}, -\boldsymbol{\sigma})$; and $N'(\mathbf{v}, \boldsymbol{\sigma}) = N(\mathbf{v}, -\boldsymbol{\sigma})$, where $\boldsymbol{\tau}$ and N are the quantities occurring in (10).

For given ν, in the case of the inelastic scattering of quanta with pair production, the momentum transfer, $\boldsymbol{\tau}$, to the nucleus has its minimum value when $\sigma = 0$ and the pair particles divide the energy ν equally and go in the forward direction. In this case, $\tau = \nu - (\nu^2 - 4)^{\frac{1}{2}}$. The screening effect of the atomic electrons enters in $F(\tau)$, and is negligible for values of $\tau > Z^{\frac{1}{3}}/137 (= 0.032$ for Pb). Thus for incident quantum energies less than $50mc^2$ screening can be neglected for Pb and all lighter elements, and $Z - F$ in (7) may be replaced by Z. With this limitation on the energy range, the cross section may be written in ordinary units of length as $d^2\varphi/d\Omega d\sigma = \alpha^2 Z^2 (e^2/mc^2)^2 f(\nu, \sigma, \theta)$ where f is a function only of the energies of the incident and scattered quanta and the angle between them.

The large number of terms entering into the spur when the spin reductions are carried out greatly complicates the evaluation of the cross section (7) in the general case. It is sufficient, however, to obtain an estimate and upper limit for the total cross section in order to determine how much the inelastic scattering with pair production can contribute to the intensity of the radiation observed experimentally from heavy elements exposed to hard gamma-rays. The calculations are much simplified in the limiting case in which the pair particles have small kinetic energy. This fact suggests the procedure of evaluating the cross section accurately in this limit, and using the result to estimate the total cross section. In this limit the integrations over the directions of the pair particles may be directly performed, and the cross section reduces to

$$\frac{d^2\varphi}{d\Omega d\sigma} = \frac{Z^2\alpha^4}{2^7\pi} \frac{\sigma}{\nu} \frac{(\nu-\sigma-2)^2}{(\mathbf{v}-\boldsymbol{\sigma})^4} \sum_a \sum_b \text{Spur } (1-\beta)N(1+\beta)N^*. \tag{12}$$

N here is taken from Eqs. (8) and (9) with $p_0 = 0$, $E_0 = -1$; $p_i = 0$, $E_i = 1$. In general the spur of $(1-\beta)M(1+\beta)M^*$ has the value $8\{(\mathbf{B}+\mathbf{E})\cdot(\mathbf{B}^*+\mathbf{E}^*) + (G+H)(G^*+H^*)$ for any spin matrix M,

$$M = A + \mathbf{B}\cdot\boldsymbol{\alpha} + C\beta + \mathbf{D}\cdot\mathbf{S} + \mathbf{E}\cdot\boldsymbol{\alpha}\beta + \mathbf{F}\cdot\mathbf{S}\beta + G\alpha_1\alpha_2\alpha_3 + H\alpha_1\alpha_2\alpha_3\beta.$$

Accordingly the cross section depends only on the coefficients of $\boldsymbol{\alpha}$ and β in $N(1+\beta)$. In this way the cross section for inelastic scattering with pair production near the limit $\sigma = \nu - 2$ is found to be

$$\frac{d^2\varphi}{d\Omega d\sigma} = \frac{Z^2\alpha^4}{\pi} \frac{\sigma}{\nu} \frac{(\nu-\sigma-2)^2}{(\mathbf{v}-\boldsymbol{\sigma})^8} [(1+\mu^2) + \nu\sigma(1-\mu^3) + \nu^2\sigma^2(1+\mu)(1-\mu)^2(3-\mu)/8], \tag{13}$$

where $\mu = \mathbf{v}\cdot\boldsymbol{\sigma}/\nu\sigma$.

Eq. (13) shows that, as the energy of the scattered quantum increases, the cross section falls off as the square of the energy difference, $(\nu-\sigma-2)^2$. If (13) is extrapolated from the limit $\sigma = \nu - 2$ toward the limit $\sigma = 0$, it gives a rapidly increasing cross section; since it is clear for physical reasons that the cross section remains finite for all σ, Eq. (13) may reasonably be used to give an upper bound for total cross section. In this way, the following upper limit is found for the total cross section in the special case $\nu = 5$ by integrating over all angles Ω and all scattered frequencies σ:

$$\varphi < 0.011\alpha^2 Z^2 (e^2/mc^2)^2. \quad (\nu = 5.) \tag{14}$$

The most important contribution to the total cross section (14) integrated over σ comes from values of σ not near the limit $\nu - 2$. Thus when $\nu = 5$, for most of the pairs produced, the kinetic energy $\nu - 2 - \sigma$ of the two particles is considerable. Analogy with calculations on the photoelectric production of pairs would indicate that also in the present problem the most probable situation is that in which the energy is divided fairly equally. Accordingly the main contribution to the total cross section comes from processes in which both the pair particles produced will have sufficiently large momenta for the Born approximation to be valid ($p_0, p_i > \alpha Z$). Eq. (7) is therefore accurate in the most important region of energies of the scattered quanta. Any errors due to failure of the Born approximation will be in the direction to make (14) an overestimate. The approximation to (7) by the use of (13) must be regarded as the first term in a purely analytical development not involving the question of the validity of the Born approximation in the limit $\sigma=\nu-2$.

For the Th C″ radiation with $\nu=5.15$, the inelastically scattered radiation ranges from $\sigma=0$ to $\sigma=3.15$, with the intensity falling rapidly towards the high energy side. The more energetic scattered quanta are strongly concentrated within a small angle $\sim 1/\nu$ (see Fig. 1), as is evident from the presence of the denominator $(\boldsymbol{\nu}-\boldsymbol{\sigma})^8$ in (13).

The estimate (14) shows that in pair production the ratio of the cross section with inelastic scattering to that with simple photoelectric absorption ($\varphi=0.67\alpha Z^2(e^2/mc^2)^2$) is less than 1.2×10^{-4} for Th C″ gamma-rays; the contribution of the inelastic scattering is clearly negligible, both to absorption, and to the production of the gamma-radiation observed to come from heavy elements exposed to high energy quanta. The only simple possibility for observing the effect discussed here would be in a cloud chamber containing a heavy gas, or a thin metallic foil, exposed to hard monochromatic gamma-rays. With this arrangement the appearance of pairs with total energies less than the incident energy would be attributable to the inelastic scattering.

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Carbon Radioactivity and Other Resonance Transmutations by Protons

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By observations with direct-current potentials of from 200 to 900 kv and currents of protons and deuterons up to 10 microamperes, it is shown that: (1) The induced radioactivity from the reaction ($C+H^1$) is produced by a resonance process with two resonance lines at about 400 and 480 kv indicated, whereas the efficiency of the corresponding process for ($C+D^2$) increases approximately exponentially throughout the voltage range covered; (2) the gamma-rays emitted in the reaction ($Li+H^1$) show resonances at 450 kv and 850 kv; (3) the gamma-rays from ($F+H^1$) suggest resonances at 320, 700 and 800 kv; (4) the gamma-rays from ($Be+H^1$) do not appear to be produced by a resonance process.

Introduction

THIS paper gives a more detailed report of the experiments on which recent Letters to the Editor[1] from the Department of Terrestrial Magnetism of the Carnegie Institution of Washington were based.

I. Experimental Technique

High voltage equipment

The concentric one- and two-meter Van de Graaff installation at the Department, shown schematically in Fig. 1, was used in this work.

A maximum of 1300 kv is available with this apparatus, but it was limited to about 900 kv in this work by sparking to ground along the paper charging belt which had been in use nearly eight months and was in bad condition. While replacement of the belt is a relatively minor operation, it was postponed until the present work was finished to avoid possible extraneous delays.

Charging currents

Under best conditions a charging current of 750 microamperes has been obtained with this department's equipment, but all the work reported here was done with reduced belt speed and with charging currents of about 300 microamperes. Some of the observations reported were made in August 1934 with the relative humidity as high as 71 percent.

Voltage calibration

At the higher voltages it was possible to obtain range measurements on the primary beam, but

[1] Hafstad and Tuve, Phys. Rev. **47**, 506 (1935).