Transmission of Gamma-Rays through Large Thicknesses of Heavy Materials

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A study has been made of the feasibility of accurate numerical determinations of the transmission of gamma-rays through large thicknesses of materials. The first procedure investigated consists in regarding the total probability of photon transmission, N_t, as the sum of the probabilities N_n, where N_n is the probability of photon transmission with exactly n scatterings. The total expected transmitted energy, E_t, is similarly considered to be given by ΣE_n. A numerical calculation of N_n and E_n has been made for $n=0, 1, 2, 3$ for a slab of uranium 20 cm thick, upon which photons are incident normally with energy $\alpha=10\ mc^2$. The maximum value of N_n/N_0 occurs at $n=2$ and of E_n/E_0 at $n=1$. These calculations are also adapted to a slab of lead 35 cm thick. Consideration has been given to the behavior of N_n and E_n for large n, and estimates are thereby made for N_t and E_t. The second procedure consists in deriving the transmission through a thick slab from a succession of transmissions through thin slabs. The transformation of an incident photon distribution into the distribution transmitted through a thin slab is conveniently expressed as a matrix, and the total transmission is then given by the iteration of the matrix on the successive transmitted distributions. Numerical results obtained by this procedure for particular incident photon distributions are presented.

I. INTRODUCTION

THE purpose of the present investigation has been a study of the feasibility of straightforward numerical determinations of the transmission of gamma-rays through large thicknesses of materials containing heavy elements. The limitation of the discussion to heavy elements makes possible some simplifications essentially because of the large probability of absorption by the photoelectric effect of photons which are degraded to low energy. The present approach to the gamma-ray transmission problem is based on the notion that one need consider only those transmitted gamma-rays which have suffered relatively few scatterings even for thicknesses of materials of approximately 20 mean free paths. The validity of this view is demonstrated by the results which are presented in the following sections. Specific calculations have been made of the attenuation of gamma-rays with incident energy 10 mc^2 through thicknesses of uranium up to 20 cm. These calculations are adjusted to give the attenuation through thicknesses of lead up to 35 cm.

The elementary processes of gamma-ray interaction with matter that are taken into account are the photoelectric effect, Compton scattering, and pair-production. Since the numerical results for gamma-ray transmission will depend upon the values taken for the absorption coefficients for these processes, a partial table of the values for uranium and lead used in the present calculations is given in Table I. If a minor alteration is made in these values, the effect on the transmission values given here could be determined without great difficulty. The probability of Compton scattering is assumed to be given by the Klein-Nishina formula, in which the effects of polarization of the radiation have been averaged out. Thus, the partial polarization which arises upon scattering of originally unpolarized radiation and the alteration in the probability of further Compton scattering has been disregarded. The gamma-ray energies of present concern are sufficiently high so that these polarization effects are believed to be unimportant. The possible contribution to the transmitted radiation from bremsstrahlung produced by Compton recoil electrons is not included in the present calculations; the range of gamma-ray energies considered here is known to be too low for this contribution to be significant.

Two methods for the determination of the probability of transmission of a photon through a slab of material of thickness a will be considered. In the first method, the total probability of transmission is taken as the sum of the probability of transmission with no scattering, plus the probability of transmission with one scattering, plus the probability of transmission with two scatterings, etc. In the second method, the slab of thickness a is divided into a series of thin slabs and the transmission through the total thickness is determined from the transmission through the series of thin slabs in succession.

II. SUCCESSIVE SCATTERED CONTRIBUTIONS TO THE TRANSMITTED RADIATION

Consider a homogeneous slab of material which has infinite extent in the y- and z-directions and which has thickness a in the x-direction, $0\leq x\leq a$. One can readily give an integral expression for the probability that a gamma-ray incident on the face $x=0$ is not absorbed in the slab and emerges after exactly n collisions from the face $x=a$. Let the gamma-ray enter with the energy α_0 in a direction making an angle ψ_0 with the normal to the face of the slab, travel a distance s_0 within the slab

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TABLE I. Values for the absorption coefficients for lead and uranium which were used in the calculations: α=photon energy in units of mc^2; μ_U=total absorption coefficient for uranium; μ_{Pb}=total absorption coefficient for lead.

α	μ_U	μ_{Pb}
0.2	68.8	57.0
0.5	12.0	6.10
1.0	3.07	1.686
2.0	1.339	0.768
4.0	0.890	0.514
6.0	0.824	0.477
8.0	0.847	0.479
10.0	0.868	0.494

to the first collision, scatter (if it is not absorbed) into a direction making an angle θ_1 with the original direction and an angle ψ_1 with the normal to the slab face, travel a distance s_1 to a second collision, scatter into a direction specified by θ_2 and ψ_2, and so on (see Fig. 1). After n collisions, the path angles are θ_n and ψ_n, and s_n is the distance traveled from the $(n-1)$th collision to the exit face of the slab at $x=a$. The path line after the kth collision is characterized by an azimuthal angle ϕ_k in addition to the angles θ_k and ψ_k; these angles are connected by the familiar relation,

$$\cos\psi_{k+1}=\cos\psi_k\cos\theta_{k+1}+\sin\psi_k\sin\theta_{k+1}\cos\phi_{k+1};$$

$k=0, 1, 2, \cdots, (n-1)$. The energy of the photon between the kth and the $(k+1)$th collision is α_k in units of mc^2, and the value of the total absorption coefficient for this energy α_k will be denoted by μ_k. The successive values of the energy α_k are related by the Compton formula:

$$\alpha_k=\alpha_{k-1}/[1-\alpha_{k-1}(1-\cos\theta_k)].$$

The probability, N_n, that the photon will be transmitted after exactly n collisions by any possible path is

$$N_n=\left(\frac{\nu r^2}{2}\right)^n\int\int_S\cdots\int e^{-\mu_n s_n}\prod_{k=0}^{n-1}e^{-\mu_k s_k}\times q_{k+1}ds_k d\phi_{k+1}d\theta_{k-1}, \quad (1)$$

where ν is the number of electrons per cm^3 in the slab material, r is the electron radius, e^2/mc^2, and q_{k+1} is obtained from the Klein-Nishina differential cross section as

$$q_{k+1}=\frac{\sin\theta_{k+1}}{[1+\alpha_k(1-\cos\theta_{k+1})]^2}\times\left[1+\cos^2\theta_{k+1}+\frac{\alpha_k^2(1-\cos\theta_{k+1})^2}{1+\alpha_k(1-\cos\theta_{k+1})}\right].$$

The $3n$-dimensional space S over which the integration in Eq. (1) must be performed needs detailed description. A point $(s_0, \theta_1, \phi_1, s_1, \theta_2, \phi_2, \cdots, s_{n-1}, \theta_n, \phi_n)$ represents a path followed by the photon in reaching the nth collision and is a possible path or a point in S if all these collisions are within, or at the boundary of, the slab. Any value of ϕ_k and θ_k is possible as long as

$$0\leq\phi_k\leq 2\pi,\quad 0\leq\theta_k\leq\pi.\quad k=1, 2, \cdots, n.$$

The first collision, however, will not be in the slab unless

$$0\leq s_0\cos\psi_0\leq a.$$

Further, the value of the angle ψ_1 determines whether the first collision has given forward or backward scattering. If the scattering is forward,

$$0\leq\psi_1\leq\pi/2,$$

then the second collision will be possible only if

$$0\leq s_1\cos\psi_1\leq a-s_0\cos\psi_0.$$

If the scattering is backward,

$$\pi/2<\psi_1\leq\pi,$$

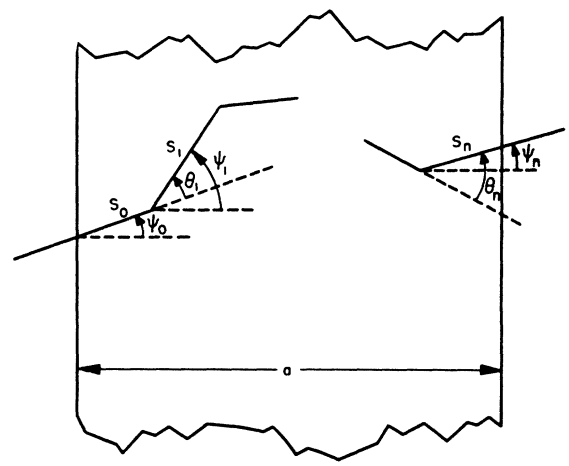

FIG. 1. Geometric parameters of a typical photon path through a slab.

then the inequality

$$0\geq s_1\cos\psi_1\geq -s_0\cos\psi_0$$

must hold. Continuing in this way, one sees that the space S is defined by the inequalities:

$$0\leq\phi_k\leq 2\pi;\quad 0\leq\theta_k\leq\pi; \quad (2a)$$

$$0\leq s_{k-1}\cos\psi_{k-1}\leq a-\sum_{i=0}^{k-2}s_i\cos\psi_i,\quad \text{if } 0\leq\psi_{k-1}\leq\tfrac{1}{2}\pi; \quad (2b)$$

$$0\geq s_{k-1}\cos\psi_{k-1}\geq -\sum_{i=0}^{k-2}s_i\cos\psi_i,\quad \text{if } \tfrac{1}{2}\pi<\psi_{k-1}\leq\pi. \quad (2c)$$

For each inequality of Eq. (2), $k=1, 2, \cdots, n$. It is to be noted that for $k=1$, one replaces

$$\sum_{i=0}^{k-2}s_i\cos\psi_i$$

by zero; the last inequality (2c), which gives the range

of $s_{k-1}\cos\psi_{k-1}$, is trivial; and the inequality defining the range of ψ_{k-1} loses significance.

The space S divides naturally into 2^{n-1} subspaces, where each subspace represents a particular sequence of forward and backward scatterings. The scatter forward or backward on the last, or nth, collision does not figure in these inequalities, but does enter by way of the integrand. For transmission, s_n is defined by the relation,

$$s_n\cos\psi_n = a - \sum_{i=0}^{n-1} s_i\cos\psi_i, \quad \text{if } 0\leq\psi_n\leq\tfrac{1}{2}\pi, \tag{3a}$$

and for reflection by

$$s_n\cos\psi_n = -\sum_{i=0}^{n-1} s_i\cos\psi_i, \quad \text{if } \tfrac{1}{2}\pi<\psi_n\leq\pi. \tag{3b}$$

Hence, the number of subspaces is increased to 2^n, and one may expect that the calculation of N_n as a practical matter will require separate treatment for each subspace.

The integral formula for N_n does not appear to be tractable to analytic treatment of the integration over the angle variables, but it will be noted that the integration with respect to the s_k's may be readily performed. In order not to single out a particular subspace, one may combine Eqs. (3a) and (3b) into the single relation

$$s_n\cos\psi_n = a_n - \sum_{i=0}^{n-1} s_i\cos\psi_i,$$

where a_n is assigned the value a or 0 according as (3a) or (3b) applies. Similarly, the integration limits on the variable s_k as given by (2b) or (2c) will be taken to be 0 and

$$a_k - \sum_{i=0}^{k-2} s_i\cos\psi_i,$$

where a_k has the value a or 0 according as the subspace defined by (2b) or (2c) applies. Then Eq. (1) requires the integration of the function,

$$\exp\left[\frac{\mu_n}{\cos\psi_n}\left(a_n - \sum_{i=0}^{n-1} s_i\cos\psi_i\right)\right]\cdot\exp\left[-\sum_{i=0}^{n-1}\mu_i s_i\right],$$

first with respect to s_{n-1} over the interval

$$\left(0,\, a_{n-1} - \sum_{i=0}^{n-2} s_i\cos\psi_i\right),$$

then with respect to s_{n-2} over the interval

$$\left(0,\, a_{n-2} - \sum_{i=0}^{n-3} s_i\cos\psi_i\right), \text{ etc.}$$

This series of operations has a recursive nature which can be made evident by the following device. Let

$$F_{j,k} = \begin{cases} \exp\left[-\lambda_j\left(a_j - \sum_{i=0}^{k} s_i\cos\psi_i\right)\right]\cdot\exp\left[-\sum_{i=0}^{k}\mu_i s_i\right], & \text{for } k\geq 0; \\ \exp[-\lambda_j a_j], & \text{for } k=-1. \end{cases}$$

Then, if J_k represents the operation of integrating with respect to s_k between the limits 0 and

$$a_k - \sum_{i=0}^{k-1} s_i\cos\psi_i,$$

one has

$$J_k(F_{j,k}) = [F_{j,k-1} - e^{-\lambda_j(a_j-a_k)}F_{k,k-1}]/(\lambda_k-\lambda_j)\cos\psi_k, \tag{4a}$$

$$J_{k-1}[J_k(F_{j,k})] = \frac{[F_{j,k-2} - e^{-\lambda_j(a_j-a_{k-1})}F_{k-1,k-2}]/(\lambda_{k-1}-\lambda_j)\cos\psi_{k-1} - e^{-\lambda_j(a_j-a_k)}[F_{k,k-2} - e^{-\lambda_k(a_k-a_{k-1})}F_{k-1,k-2}]/(\lambda_{k-1}-\lambda_k)\cos\psi_{k-1}}{(\lambda_k-\lambda_j)\cos\psi_k}. \tag{4b}$$

Further iterations proceed similarly. It is evident that the s-integrations in Eq. (1) are the end of a sequence of these operations of which (4a) and (4b) are the first two; the sequence in N_n is obtained from Eq. (4) by these substitutions: $k=n-1$, $j=n$, $\lambda_n=\mu_n/\cos\psi_n$. For $n=1$, one has

$$N_1(a_1) = \tfrac{1}{2}\nu r^2 \iint_{S_1} [e^{-\lambda_1 a_1} - e^{-\lambda_1(a_1-a)}e^{-\lambda_0 a}]q_1 d\theta_1 d\phi_1/(\lambda_0-\lambda_1)\cos\psi_0, \tag{5}$$

and for $n=2$,

$$N_2(a_1,a_2) = \left(\frac{\nu r^2}{2}\right)^2 \iiiint_{S_2} \frac{\dfrac{e^{-\lambda_2 a_2} - e^{-\lambda_2(a_2-a)}e^{-\lambda_0 a}}{(\lambda_0-\lambda_2)\cos\psi_0} - e^{-\lambda_2(a_2-a_1)}\dfrac{e^{-\lambda_1 a_1} - e^{-\lambda_1(a_1-a)}e^{-\lambda_0 a}}{(\lambda_0-\lambda_1)\cos\psi_0}}{(\lambda_1-\lambda_2)\cos\psi_1} q_1 q_2 d\phi_1 d\theta_1 d\phi_2 d\theta_2. \tag{6}$$

Thus, the probability of a photon passing through the slab with exactly one collision is given by Eq. (5) as $N_1(a)$, and the probability of being reflected out through the incident face of the slab with one collision is $N_1(0)$. The probability of transmission with exactly two collisions is given by Eq. (6) as the sum of the two probabilities $N_2(a,a)$ and $N_2(0,a)$, and the probability of reflection out through the incident face with two collisions is the sum

of the two probabilities $N_2(a,0)$ and $N_2(0,0)$. If one defines $v_i=\lambda_i a$, then these probabilities may be written as follows:

$$N_1(a)=\tfrac{1}{2}\nu r^2 a e^{-v_0}\int\int_{S_1(a)}[1-e^{-(v_0-v_1)}]q_1 d\phi_1 d\theta_1/(v_0-v_1)\cos\psi_0; \tag{7}$$

$$N_1(0)=\tfrac{1}{2}\nu r^2 a\int\int_{S_1(0)}[1-e^{-(v_0-v_1)}]q_1 d\phi_1 d\theta_1/(v_0-v_1)\cos\psi_0; \tag{8}$$

$$N_2(a,a)=\left(\frac{\nu r^2 a}{2}\right)^2 e^{-v_0}\int\int\int\int_{S_2(a,a)}\frac{[1-e^{-(v_2-v_0)}]/(v_2-v_0)\cos\psi_0-[1-e^{-(v_1-v_0)}]/(v_1-v_0)\cos\psi_0}{(v_1-v_2)\cos\psi_1}q_1q_2 d\phi_1 d\theta_1 d\phi_2 d\theta_2; \tag{9a}$$

$$N_2(0,a)=\left(\frac{\nu r^2 a}{2}\right)^2 e^{-v_0}\int\int\int\int_{S_2(0,a)}\frac{[1-e^{-(v_2-v_1)}]/(v_2-v_1)\cos\psi_0-[1-e^{-(v_2-v_0)}]/(v_2-v_0)\cos\psi_0}{(v_1-v_0)\cos\psi_1}q_1q_2 d\phi_1 d\theta_1 d\phi_2 d\theta_2; \tag{9b}$$

$$N_2(a,0)=\left(\frac{\nu r^2 a}{2}\right)^2 \int\int\int\int_{S_2(a,0)}\frac{[1-e^{-(v_1-v_2)}]/(v_1-v_2)\cos\psi_0-[1-e^{-(v_0-v_2)}]/(v_0-v_2)\cos\psi_0}{(v_0-v_1)\cos\psi_1}q_1q_2 d\phi_1 d\theta_1 d\phi_2 d\theta_2; \tag{10a}$$

$$N_2(0,0)=\left(\frac{\nu r^2 a}{2}\right)^2 \int\int\int\int_{S_2(0,0)}\frac{[1-e^{-(v_0-v_2)}]/(v_0-v_2)\cos\psi_0-[1-e^{-(v_0-v_1)}]/(v_0-v_1)\cos\psi_0}{(v_2-v_1)\cos\psi_1}q_1q_2 d\phi_1 d\theta_1 d\phi_2 d\theta_2. \tag{10b}$$

One notes that e^{-v_0} appears as a factor in all the transmission probabilities; e^{-v_0} is, of course, the probability that a photon is transmitted through the slab without absorption or scattering.

In heavy materials, the probability of photoelectric absorption increases very rapidly as the photon energy decreases. As a consequence, the transmission probability for a path which includes two or more backward scatterings will be unimportant compared with the transmission probability for a path with the same number of collisions all of which are in the forward direction; e.g., $N_2(0,a)\ll N_2(a,a)$. In terms of the integration spaces, the subspace corresponding to $a_1=a_2=\cdots=a_n=a$ is the important one that need be considered of all the types N_n. This particular sequence of probabilities has the form

$$N_n(a,a,\cdots,a)=\left(\frac{\nu r^2 a}{2}\right)^n\int\int\cdots\int_{S_n(a,a,\cdots a)}f_n(v_n,v_{n-1},\cdots,v_0)\times\prod_{k=0}^{n-1}q_{k+1}d\phi_{k+1}d\theta_{k+1}, \tag{11}$$

where

$$f_0(v_0)=e^{-v_0},$$
$$f_1(v_1,v_0)=[f_0(v_1)-f_0(v_0)]/(v_0-v_1)\cos\psi_0,$$
$$f_2(v_2,v_1,v_0)=[f_1(v_2,v_0)-f_1(v_1,v_0)]/(v_1-v_2)\cos\psi_1,$$
$$f_3(v_3,v_2,v_1,v_0)=[f_2(v_3,v_1,v_0)-f_2(v_2,v_1,v_0)]/(v_2-v_3)\cos\psi_2,$$

etc.

The expected energy, E_n, say, which is carried by a photon transmitted or reflected is obtained if the integrand in Eq. (1) is multiplied by α_n, where

$$\alpha_n=\alpha_{n-1}/[1-(1-\cos\theta_n)\alpha_{n-1}]=\alpha_0/\prod_{k=0}^{n-1}[1-(1-\cos\theta_{k+1})\alpha_k],$$

so that

$$E_n=\alpha_0\left(\frac{\nu r^2}{2}\right)^n\int\int\cdots\int_{S_n}e^{-\mu_n s_n}\times\prod_{k=0}^{n-1}e^{-\mu_k s_k}p_{k+1}ds_k d\phi_{k+1}d\theta_{k+1}, \tag{12}$$

In Eq. (12)

$$p_k=q_k/[1-(1-\cos\theta_{k+1})\alpha_k].$$

Every formula developed for N_n has its counterpart for E_n.

The formulas summarized in Eqs. (11) and (12) do not appear at all amenable to analytic evaluation.

TABLE II. Transmission values for photons incident normally with energy 10 mc^2 ($\alpha_0=10$, $\psi_0=0$). The values for uranium apply to a plane slab of thickness $a=20$ cm; the values for lead apply to a slab of thickness $a=35$ cm. N_n is the number of photons transmitted with exactly n scatterings. E_n is the expected energy transmitted with exactly n scatterings.

	Transmission values for U: $\alpha_0=10$, $\psi_0=0$, $a=20$ cm		Transmission values for Pb: $\alpha_0=10$, $\psi_0=0$, $a=35$ cm	
n	N_n/N_0	E_n/E_0	N_n/N_0	E_n/E_0
0	1.0	1.0	1.0	1.0
1	2.6	1.7	2.8	1.85
2	3.0	1.5	3.6	1.8
3	2.2	0.88	2.85	1.15

Numerical integration, on the other hand, cannot be carried out for large values of n in a reasonable length of time. It would be of great interest, however, to have a quantitative notion of how many collisions contribute significantly to the number of photons and the energy transmitted through a large thickness of a heavy material. A numerical calculation was therefore carried out for a slab of uranium 20 cm thick upon which photons are incident normally ($\psi_0=0$) with energy $\alpha_0=10$. The results through the third scattered beam are given in Table II. It is estimated that these values may be in error by as much as 10 percent. The error is greatest, of course, for N_3 and E_3; and for these the error may be pessimistically estimated to be as large as 20 percent.

These results can be used to obtain transmission values through a slab of lead for photons incident under the same conditions: $\psi_0=0$, $\alpha_0=10$. It may be noted (see Table I) that the total absorption coefficient, $\mu(\alpha)$, for uranium is larger than that for lead in the energy range from 1 to 10 mc^2 by an almost constant factor of approximately 1.75. Consequently, integrals (7), (8), (9), (10), etc. evaluated for 20 cm of uranium should have about the same values for 1.75×20 cm $=35$ cm of lead except for the factor $(\nu r^2 a/2)^n$. The quantity in this factor which varies in going from uranium to lead is νa, and the ratio of νa for lead to its value for uranium is 1.09. In this way, one gets the transmission values for lead shown in Table II.†

These results point to the interesting conclusion that the major portion of the photons and of the energy transmitted through a slab of heavy material of the order of 20 mean free paths thick is contained in the contributions from the first four or five scattered beams. If an extrapolation is made in a reasonable geometric fashion to higher values of n, one gets for the total energy and total number of photons transmitted through a slab of uranium

$$E_t\approx 5.5E_0, \tag{13a}$$

$$N_t\approx 10.8N_0, \tag{13b}$$

with the transmitted photons having the average energy

$$E_t/N_t\approx 5mc^2. \tag{13c}$$

Corresponding values for the slab of lead are

$$E_t\approx 6.4E_0, \tag{14a}$$

$$N_t\approx 12.7N_0, \tag{14b}$$

$$E_t/N_t\approx 5mc^2. \tag{14c}$$

This extrapolation is not so daring as might appear. One would expect that once $E_n(a)$ and $I_n(a)$ begin to fall, they would decrease ever more rapidly as n increases. This inference is strengthened by a qualitative examination of the integrals for E_n and I_n, which indicates for large a a trend for a behavior of the following type:

$$E_n(a)/E_0\approx (ka)^n/n!, \tag{15a}$$

$$I_n(a)/I_0\approx (ca)^n/n!, \tag{15b}$$

where k and c do not vary strongly with n. The first four values of E_n/E_0 given in Table II agree fairly well with Eq. (15a) with the value $k=0.087$ cm^{-1}; the value $c=0.12$ cm^{-1} in Eq. (15b) gives a rough fit to the tabulated values for N_n/N_0. Corresponding values for the slab of lead are $k=0.054$ cm^{-1}; $c=0.075$ cm^{-1}. It should be noted that Eqs. (15a) and (15b) are of the form which would be obtained from a one-dimensional calculation of the transmission probabilities. Although the values for k and c derived from a one-dimensional calculation would depend on the kind of approximation made, one would not expect that the one-dimensional form would underestimate the general trend of the contributions to the transmission from large values of n. One may, therefore, argue that one gets rapid convergence in E_n and I_n, and that an extrapolation made with k and c chosen for the best fit over the first few values of n which go beyond the maxima in E_n and I_n would be reasonably safe. Such an extrapolation, which uses Eqs. (15) to get the contributions for n beyond those calculated, gives essentially the same results for E_t and N_t as does the geometric extrapolation mentioned above.

No effort was made to improve the accuracy of the computed values of E_n and I_n presented in Table II because of the difficulty involved in this straightforward approach to the calculation. The functions involved have a behavior unsuited to numerical methods; and, furthermore, the number of numerical quadratures which must be performed, for example, in the case $n=3$, prohibits the use of more than a few points in each integration. The present results were obtained by using from three to five points for each integration. So few points would indicate a poorer accuracy than stated if one were to assume the use of an unadorned Simpson's rule. The worst function behavior, however, is found with the functions $p_k(\theta_k)$ and $q_k(\theta_k)$, which are all of one class, so that they may be considered together. Further, the functions of θ_k and ψ_k arising from $f_n(v_n,v_{n-1},\cdots,v_0)$ are also more or less of a class; of these, the greatest difficulty for the numerical integrations comes from a variation in which the function rises to a maximum and then falls rapidly in roughly exponential manner.

A typical integral which must be evaluated is of the form

$$\int_0^{\theta_k^*} f(\theta_k)q_k(\theta_k)d\theta_k. \tag{16}$$

Now if the function $f(\theta_k)$ is approximated by a polynomial $c_0+c_1\theta_k+\cdots+c_n\theta_k^n$ which passes through n

† *Note added in proof:* A direct calculation for a 35-cm slab of Pb has been made for $\alpha_0=10$ with the following results: $N_1/N_0=2.6$, $N_2/N_0=3.4$; $E_1/E_0=1.85$, $E_2/E_0=1.83$. The agreement of these values with the Pb values in Table II is very good.

points of $f(\theta)_k$ and if the integrals

$$\int_{\theta_k'}^{\theta_k''} \theta_k{}^j q_k(\theta_k) d\theta_k, \quad j=0, 1, 2, \cdots, n,$$

are evaluated, then an approximate value of Eq. (16) is obtained in which the number of points required depends on the behavior of $f(\theta_k)$ and very little on that of $q_k(\theta_k)$. Such a procedure is useful since the functions q_k (and p_k) are amenable to the required analysis. A complete study of a few typical integrals of the type (16) showed that three points can be used to approximate $f(\theta_k)$ with sufficient accuracy to give an error of about 5 percent in the integrated value. It is essentially such a procedure which was followed with some care and control to give the results of Table II in a reasonable time.

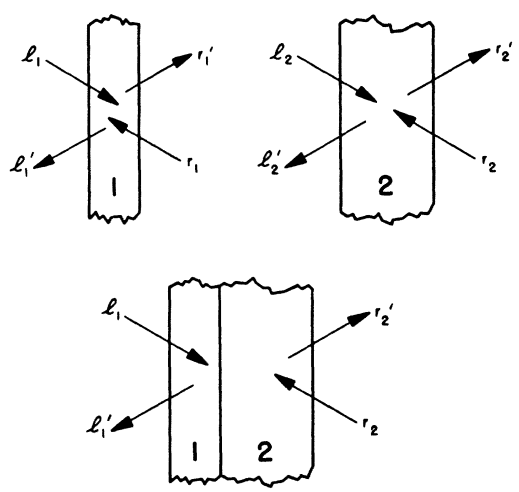

FIG. 2. A schematic representation of the incident, reflected, and transmitted photon distributions for two component slabs 1 and 2 and for the combined slab.

III. TOTAL TRANSMISSION AS SUCCESSIVE TRANSMISSION THROUGH A SERIES OF THIN SLABS

It is clear from the preceding results that the transmission through a slab one or two mean free paths thick should not require the evaluation of scattered contributions beyond the two-collision beam. Indeed, the twice scattered photons would then constitute only a small portion of the total transmission. The calculation of the unscattered and singly scattered transmission beams is a simple matter, and the calculation of the doubly scattered transmission beam is also not difficult if high accuracy in its numerical value is not required. With a practicable method of evaluating the transmission through a thin slab, one might propose to obtain the transmission through a thick slab by considering it to be composed of a number of thin slabs. This procedure may readily be examined in a formal manner.

Let $\mathbf{O}$ be the operator which transforms a distribution of photons incident upon one face of a slab into the distribution of photons leaving the second face, and let $\mathbf{S}$ be the operator which transforms the incident distribution into the distribution leaving the incident face of the slab. It is plausible, in view of the physical situation, to assume for any distributions l and r that the operational relations

$$\mathbf{O}(l+r)=\mathbf{O}l+\mathbf{O}r,$$
$$\mathbf{O}l+\mathbf{S}l=(\mathbf{O}+\mathbf{S})l, \quad \mathbf{O}(\mathbf{S}l)=\mathbf{O}\mathbf{S}l,$$

are valid. Consider two slabs in contact (Fig. 2) and let l_1 be the distribution entering the left face of the first slab, r_1 the distribution entering the right face, and l_1' and r_1' the distributions leaving the left and right faces, respectively. Let l_2, r_2, l_2', r_2' be the corresponding distributions for the second slab. The two slabs, for the general discussion, need not be identical either in material or thickness, so that the first slab has operators $\mathbf{O}_1$, $\mathbf{S}_1$ and the second slab has corresponding operators $\mathbf{O}_2$, $\mathbf{S}_2$. Then the following relations hold:

$$r_1'=\mathbf{O}_1l_1+\mathbf{S}_1r_1, \quad l_1'=\mathbf{S}_1l_1+\mathbf{O}_1r_1,$$
$$r_2'=\mathbf{O}_2l_2+\mathbf{S}_2r_2, \quad l_2'=\mathbf{S}_2l_2+\mathbf{O}_2r_2,$$

and, since the distributions which leave one inner face must enter the other,

$$r_1'=l_2, \quad r_1=l_2'.$$

These six equations reduce to

$$\begin{aligned} r_1'&=\mathbf{O}_1l_1+\mathbf{S}_1r_1, \quad l_1'=\mathbf{S}_1l_1+\mathbf{O}_1r_1,\\ r_2'&=\mathbf{O}_2r_1'+\mathbf{S}_2r_2, \quad r_1=\mathbf{S}_2r_1'+\mathbf{O}_2r_2. \end{aligned} \tag{17}$$

Substitution of the fourth of these equations into the first leads to

$$r_1'=\mathbf{O}_1l_1+\mathbf{S}_1\mathbf{O}_2r_2+\mathbf{S}_1\mathbf{S}_2r_1',$$

and this relation when applied to itself gives

$$r_1'=(\mathbf{O}_1+\mathbf{S}_1\mathbf{S}_2\mathbf{O}_1)l_1+(\mathbf{S}_1\mathbf{O}_2+\mathbf{S}_1\mathbf{S}_2\mathbf{S}_1\mathbf{O}_2)r_2+(\mathbf{S}_1\mathbf{S}_2)^2r_1';$$

repetition of this process k times yields the relation

$$\begin{aligned} r_1'=&[\mathbf{O}_1+\mathbf{S}_1\mathbf{S}_2\mathbf{O}_1+(\mathbf{S}_1\mathbf{S}_2)^2\mathbf{O}_1+\cdots+(\mathbf{S}_1\mathbf{S}_2)^k\mathbf{O}_1]l_1\\ &+[\mathbf{S}_1\mathbf{O}_2+\mathbf{S}_1\mathbf{S}_2\mathbf{S}_1\mathbf{O}_2+(\mathbf{S}_1\mathbf{S}_2)^2\mathbf{S}_1\mathbf{O}_2+\cdots\\ &+(\mathbf{S}_1\mathbf{S}_2)^k\mathbf{S}_1\mathbf{O}_2]r_2+(\mathbf{S}_1\mathbf{S}_2)^{k+1}r_1'. \end{aligned} \tag{18}$$

If one substitutes the first of Eqs. (17) into the fourth, and iterates in a similar way, one finds

$$\begin{aligned} r_1=&[\mathbf{S}_2\mathbf{O}_1+\mathbf{S}_2\mathbf{S}_1\mathbf{S}_2\mathbf{O}_1+(\mathbf{S}_2\mathbf{S}_1)^2\mathbf{S}_2\mathbf{O}_1+\cdots\\ &+(\mathbf{S}_2\mathbf{S}_1)^k\mathbf{S}_2\mathbf{O}_1]l_1+[\mathbf{O}_2+\mathbf{S}_2\mathbf{S}_1\mathbf{O}_2+\cdots\\ &+(\mathbf{S}_2\mathbf{S}_1)^k\mathbf{O}_2]r_2+(\mathbf{S}_2\mathbf{S}_1)^{k+1}r_1. \end{aligned} \tag{19}$$

The quantities $(\mathbf{S}_1\mathbf{S}_2)^{k+1}r_1'$ and $(\mathbf{S}_2\mathbf{S}_1)^{k+1}r_1$ represent a distribution of photons which have been reflected back and forth between the slabs $k+1$ times. For k sufficiently large, these terms may be neglected. If Eqs. (18) and (19) so simplified are substituted into the second and

third of Eqs. (17), one finds

$$l_1' = [\mathbf{S}_1 + \mathbf{O}_1\mathbf{S}_2\mathbf{O}_1 + \mathbf{O}_1\mathbf{S}_2\mathbf{S}_1\mathbf{S}_2\mathbf{O}_1 + \mathbf{O}_1(\mathbf{S}_2\mathbf{S}_1)^2\mathbf{S}_2\mathbf{O}_1 + \cdots]l_1 + [\mathbf{O}_1\mathbf{O}_2 + \mathbf{O}_1\mathbf{S}_2\mathbf{S}_1\mathbf{O}_2 + \mathbf{O}_1(\mathbf{S}_2\mathbf{S}_1)^2\mathbf{O}_2 + \cdots]r_2; \quad (20)$$

$$r_1' = [\mathbf{O}_2\mathbf{O}_1 + \mathbf{O}_2\mathbf{S}_1\mathbf{S}_2\mathbf{O}_1 + \mathbf{O}_2(\mathbf{S}_1\mathbf{S}_2)^2\mathbf{O}_1 + \cdots]l_1 + [\mathbf{S}_2 + \mathbf{O}_2\mathbf{S}_1\mathbf{O}_2 + \mathbf{O}_2\mathbf{S}_1\mathbf{S}_2\mathbf{S}_1\mathbf{O}_2 + \mathbf{O}_2(\mathbf{S}_1\mathbf{S}_2)^2\mathbf{S}_1\mathbf{O}_2 + \cdots]r_2. \quad (21)$$

Equations (20) and (21) give the two distributions emerging from the two outer faces of the combined slabs in terms of the two entering distributions. Each sum of operator products in the brackets of these equations represents an operator (**S** or **O**) for the composite slab in terms of the operators for the two constituent slabs. It is clear that, if a third slab is brought into the system, the foregoing process could be repeated to obtain the operators for the combination of three slabs, and this process may be extended for any number of component slabs. In any such extension, the operator coefficient of l_1 will either be the transformation of the distribution incident on the left face of the composite slab into the distribution transmitted out of the last face on the right, or it will be the transformation which reflects this distribution back from the incident face. The behavior in a general case of any number of component slabs is illustrated well enough by Eqs. (20) and (21). The coefficient of l_1 in Eq. (21) consists of the following operators: $\mathbf{O}_2\mathbf{O}_1$, which represents the operator for direct transmission through the two slabs; $\mathbf{O}_2\mathbf{S}_1\mathbf{S}_2\mathbf{O}_1$, which represents transmission through the first slab, a reflection from the second slab, a reflection from the first slab, and finally a transmission through the second slab; etc. As would be expected, this operator sum represents all possible successions of transmissions and reflections which end finally in transmission. The other operator brackets have similar interpretations.

The number of photons in the distributions $\mathbf{O}_2\mathbf{O}_1l_1$, $\mathbf{O}_2\mathbf{S}_1\mathbf{S}_2\mathbf{O}_1l_1$, $\cdots$ must be decreasing. Indeed, for a material with an appreciable absorption cross section, one would expect that the important contribution to the transmission would be accurately represented by $\mathbf{O}_2\mathbf{O}_1$. With this assumption, the transmission through k identical slabs is given by the simple operator $\mathbf{O}^k$.

This operational process is a promising approach and has the advantage of giving details about the photon distribution at a succession of thicknesses of material. In this way a rather complete history of the attenuated beam is obtained.

The formulation in Sec. II gives the necessary basis for the construction of the operator **O**. The distribution of photons incident upon a slab may be specified in terms of the initial energy, α_0, and the angle of incidence, ψ_0. The variables $\beta = 1/\alpha_0$ and $\gamma = \cos\psi_0$ are somewhat more convenient, so that the incident frequency distribution will be $I(\beta,\gamma)$. Then, for the transmitted distribution, one has

$$I'(\beta',\gamma') = \mathbf{O}I(\beta,\gamma).$$

Now **O** may be considered to be the sum of the operators $\mathbf{O}_0$, $\mathbf{O}_1$, $\mathbf{O}_2$, $\cdots$, where $\mathbf{O}_k$ transforms $I(\beta,\gamma)$ into $I_k'(\beta',\gamma')$, and $I_k'(\beta',\gamma')$ is the function for those photons which are transmitted with exactly k collisions. The first of these operators, $\mathbf{O}_0$, is found at once. Those photons in the incident distribution element $I(\beta,\gamma)d\beta d\gamma$, which are transmitted without collision, have unaltered energy and direction of travel and are diminished in number of the factor $\exp[-a\mu(\beta)/\gamma]$, where $\mu(\beta)$ is the total cross section of the material for photons of energy β and a is the slab thickness. Hence, $\mathbf{O}_0$ is defined exactly by the relation

$$I_0'(\beta',\gamma') = \exp[-a\mu(\beta)/\gamma]I(\beta,\gamma).$$

The operator $\mathbf{O}_1$ may be constructed as follows. The number of photons of initial energy $1/\beta$ and incident on a slab of thickness a with the angle $\cos^{-1}\gamma$ which are transmitted after exactly one collision with energy and angle within the intervals $(\beta',\beta'+d\beta')$ and $(\gamma',\gamma'+d\gamma')$ is

$$dN_1(a) = \nu r^2 a(e^{-v} - e^{-v'})q' \times [\partial(\theta,\phi)/\partial(\gamma',\beta')]d\gamma' d\beta'/2\gamma(v'-v).$$

The quantities v and v' in this expression are the v_0 and v_1 respectively of Sec. II. It follows that the number of transmitted photons in the intervals $(\beta',\beta'+d\beta')$ and $(\gamma',\gamma'+d\gamma')$, which initially were in the incident element $I(\beta,\gamma)d\beta d\gamma$, is $I(\beta,\gamma)d\beta d\gamma \cdot dN_1(a)$. Thus,

$$I_1'(\beta',\gamma') = \tfrac{1}{2}\nu r^2 a \int\int I(\beta,\gamma)\frac{e^{-v}-e^{-v'}}{\gamma(v'-v)}q'\frac{\partial(\theta,\phi)}{\partial(\gamma',\beta')}d\gamma d\beta$$

$$= \tfrac{1}{2}\nu r^2 a \int\int I(\beta,\gamma)\frac{e^{-v}-e^{-v'}}{v'-v}\frac{\beta}{\beta'^3} \times \frac{\beta'\beta^3 + (1+2\beta'-2\beta'^2)\beta^2 + (\beta'^3-2\beta'^2)\beta + \beta'^2}{\gamma(1-\cos^2\theta_1 - \gamma'^2 + 2\cos\theta_1\gamma'\gamma - \gamma^2)^{\frac{1}{2}}}d\gamma d\beta, \quad (22)$$

where $\cos\theta_1 = 1+\beta-\beta'$. It is evident that the operator $\mathbf{O}_1$ is a multiplication by a function followed by an integration over the proper space in the variables β, γ.

The operators $\mathbf{O}_2$, $\mathbf{O}_3$, etc. can be defined in the same way. There remains, however, the problem of performing the indicated transformations with sufficient accuracy and speed. The essential features of one approach to this problem will be illustrated with the operator $\mathbf{O}_1$. The incident distribution $I(\beta,\gamma)$ is represented in the range of interest or significance by a set of discrete points, $i\cdot k$ in number, taken from i values of β: $\beta_1 < \beta_2 < \cdots < \beta_i$, and k values of γ: $\gamma_1 < \gamma_2 < \cdots < \gamma_k$. These values of $I(\beta_s\gamma_t)$ are arranged in a column matrix $\|I_{st}\|$. Then $I'(\beta',\gamma')$ as determined by Eq. (22) can be evaluated at a given point (β',γ') by performing i numerical integrations over γ (β successively fixed at $\beta_1, \beta_2, \cdots, \beta_i$) followed by a numerical integration over β. This procedure can consist in multiplying each discrete value of $I(\beta,\gamma)$ in the column

matrix by a suitable coefficient and accumulating the products. It is, thus, readily suggested that one form a square matrix $\|\mathbf{O}_1\|$ of order $i\cdot k$ in which the elements in the first, second, third, etc. rows are respectively the coefficients for obtaining $I_1'(\beta_1,\gamma_1)$, $I_1'(\beta_1,\gamma_2)$, $I_1'(\beta_1,\gamma_3)$, etc. One has, consequently, a matrix representation for the transformation $\mathbf{O}_1$. A matrix representation $\|\mathbf{O}_2\|$ for $\mathbf{O}_2$ can also be found, and its derivation is entirely similar to the above. The operator $\mathbf{O}_0$, which is merely a multiplication by a function with no integrations, is represented by a diagonal matrix. For a slab which is sufficiently thin so that the higher operators are negligible, one has the total matrix

$$\|\mathbf{O}\| = \|\mathbf{O}_0\| + \|\mathbf{O}_1\| + \|\mathbf{O}_2\|,$$

such that the total transmitted distribution I' is given by

$$\|I'\| = \|\mathbf{O}\| \times \|I\|.$$

If an accurate calculation requires matrices of high order, the feasibility of this procedure is brought into question. The functions involved are not well behaved so that the order of the matrices tend to be large. It may be noted, for example, that the integrand in Eq. (22) becomes infinite at the boundary of the region of integration. This region is defined by the simultaneous inequalities:

$$\beta'-2\leq\beta\leq\beta',$$
$$\gamma'\cos\theta_1-[(1-\gamma'^2)(1-\cos^2\theta_1]^{\frac{1}{2}}\leq\gamma\leq\gamma'\cos\theta_1+[(1-\gamma'^2)(1-\cos^2\theta_1)]^{\frac{1}{2}},$$
$$\gamma\geq 0.$$

The area so defined is clearly not rectangular, so that there is the difficulty of ending the numerical integration with respect to γ (β fixed) on the boundary when no discrete point falls on the boundary. These difficulties, however, are not insuperable and can be so treated as to make the method feasible.

The transmission of four different distributions incident on a slab of uranium was calculated by this iterative method. The results are presented in Tables III, IV, V, VI, and VII. Case A in these tables represents the first attempt with this procedure and is largely an exploratory calculation. The incident, or initial, distribution is constant in the reciprocal energy range $0.1\leq\beta\leq 0.15$ and is also constant over the range of cosine of incident angle $0.8\leq\gamma\leq 1.0$; the incident distribution is zero elsewhere. A net of 49 points was used in the computation and their values are given by $\beta=0.1$, 0.125, 0.15, 0.20, 0.25, 0.35, 0.45 and $\gamma=0$, 0.2, 0.4, 0.6, 0.8, 0.9, 1.0. As the calculations proceeded, it became clear that the accuracy could be improved by a different arrangement of points without a great increase in their number. The improvement consists in taking a closer spacing of γ-values for small β and a larger spacing of γ-values for large β. Since the distributions which are operated on vary rapidly for small β and slowly for large β, it is clear that this nonuniform net is advantageous. The improved net was incorporated in the matrix operator for the incident distributions, which are labelled Cases I, II, and III. For each of these, the incident beam is monoenergetic, $\beta=0.1$, and the distributions over γ are, respectively, 1, $5(\gamma-0.8)$, 25 $(\gamma-0.8)^2$ in the interval $0.8\leq\gamma\leq 1.0$, and zero elsewhere. This choice of angular distributions was motivated by several considerations. Some distribution of the incident beam over a finite range of γ is indicated for the practical reason that a monoangular entering beam adds greatly to the computational difficulties. On the other hand, an incident distribution with appreciable magnitude at or near $\gamma=0$ would tend to increase the importance of the matrix $\|\mathbf{S}\|$, which is neglected. The incident distribution of Case I is precisely the distribution from an isotropic point source out of which radiation in a cone of half-angle 36° 52′ is accepted. The successive distributions of Case I, II, and III may be regarded as the first three elements of an infinite sequence of incident distributions which converges to the monoenergetic and monoangular beam ($\gamma=1$, $\alpha=10$) considered in Sec. II.

The thickness of the slab element, i.e., the "thin" slab associated with $\|\mathbf{O}\|$, has always been held at 2 cm. This thickness is less than 2 mean free paths, and reduces the contribution of the twice scattered beam to a few percent of the total. The ratio of the total number of photons transmitted to the number incident

TABLE III. Ratios of the total number of photons transmitted, N_t, to the number incident, N_i, for various thicknesses, a, of plane slabs of uranium. The four cases, A, I, II, III, for which N_t/N_i is tabulated, represent various incident photon distributions in energy and angle and are specified in the text.

a (cm)	A	I	II	III
0	1	1	1	1
2	2.36×10^{-1}	2.22×10^{-1}	2.34×10^{-1}	2.42×10^{-1}
4	5.20×10^{-2}	4.68×10^{-2}	5.21×10^{-2}	5.50×10^{-2}
6	1.12×10^{-2}	9.80×10^{-3}	1.14×10^{-2}	1.23×10^{-2}
8	2.40×10^{-3}	2.05×10^{-3}	2.46×10^{-3}	2.69×10^{-3}
10	5.07×10^{-4}	4.29×10^{-4}	5.28×10^{-4}	5.85×10^{-4}
12	1.07×10^{-4}	8.97×10^{-5}	1.13×10^{-4}	1.26×10^{-4}
14	2.24×10^{-5}	1.87×10^{-5}	2.40×10^{-5}	2.71×10^{-5}
16	4.70×10^{-6}	3.91×10^{-6}	5.08×10^{-6}	5.79×10^{-6}
18	9.83×10^{-7}	8.14×10^{-7}	1.07×10^{-6}	1.23×10^{-6}
20	2.06×10^{-7}	1.70×10^{-7}	2.26×10^{-7}	2.61×10^{-7}

TABLE IV. Ratios of the total expected energy transmitted, E_t, to the incident energy, E_i, for the conditions of Table III.

a (cm)	A	I	II	III
0	1	1	1	1
2	2.05×10^{-1}	1.85×10^{-1}	1.97×10^{-1}	2.03×10^{-1}
4	4.14×10^{-2}	3.45×10^{-2}	3.86×10^{-2}	4.09×10^{-2}
6	8.34×10^{-3}	6.50×10^{-3}	7.63×10^{-3}	8.25×10^{-3}
8	1.69×10^{-3}	1.24×10^{-3}	1.52×10^{-3}	1.67×10^{-4}
10	3.42×10^{-4}	2.42×10^{-4}	3.04×10^{-4}	3.40×10^{-4}
12	6.95×10^{-5}	4.74×10^{-5}	6.10×10^{-5}	6.92×10^{-5}
14	1.42×10^{-5}	9.38×10^{-6}	1.23×10^{-5}	1.41×10^{-5}
16	2.89×10^{-6}	1.87×10^{-6}	2.50×10^{-6}	2.88×10^{-6}
18	5.93×10^{-7}	3.75×10^{-7}	5.08×10^{-7}	5.91×10^{-7}
20	1.22×10^{-7}	7.56×10^{-8}	1.04×10^{-7}	1.21×10^{-7}

TABLE V. Ratios of the total number of photons transmitted, N_t, to the number transmitted without scattering, N_u, for the conditions of Table III.

a (cm)	A	I	II	III
0	1.0	1.0	1.0	1.0
2	1.49	1.53	1.51	1.50
4	2.06	2.19	2.14	2.12
6	2.73	3.07	2.95	2.91
8	3.55	4.24	4.00	3.91
10	4.49	5.78	5.36	5.19
12	5.60	7.78	7.08	6.82
14	6.87	10.4	9.28	8.87
16	8.35	13.6	12.0	11.4
18	9.98	17.8	15.5	14.6
20	11.8	23.0	19.9	18.6

is given in Table III for thicknesses of uranium from 2 to 20 cm in 2 cm intervals. The expected energy transmitted is similarly tabulated in Table IV. It is clear that, for the sequence of Cases I, II, and III, the mean angle of the initial distribution with the normal to the slab decreases; and, therefore, the transmission should increase. One may examine the question as to how rapidly this sequence of cases is approaching the monoangular transmission values. For $\beta=0.1$, $\gamma=1.0$, and a total slab thickness of 20 cm of uranium, one has for the ratio of the number transmitted without scatter to the number incident, $N_0/N_i=\exp[-\mu_0 a]=2.89\times10^{-8}$. This ratio and Eqs. (13) give for the monoangular distribution

$$N_t/N_i=3.12\times10^{-7};\quad E_t/E_i=1.59\times10^{-7}.$$

These values compared with those of the sequences in Tables III and IV show slow convergence to the monoangular values. This same slow convergence is found in the unscattered transmission, which can be calculated simply and accurately. For 20 cm of uranium, the sequence of monoenergetic cases considered here gives for the unscattered transmitted number, N_u, and the unscattered transmitted energy, E_u, the successive values

$$N_u/N_i=E_u/E_i=7.36\times10^{-9};\quad 1.14\times10^{-8};\quad 1.40\times10^{-8}.$$

The limit value 2.89×10^{-8} indicates the convergence of the unscattered transmission is fully as slow as the convergence of the total transmission.

TABLE VI. Ratios of the total energy transmitted, E_t, to the energy transmitted without scattering, E_u, for the conditions of Table III.

a (cm)	A	I	II	III
0	1.0	1.0	1.0	1.0
2	1.31	1.28	1.27	1.26
4	1.65	1.61	1.59	1.58
6	2.05	2.03	1.98	1.96
8	2.53	2.57	2.47	2.43
10	3.08	3.25	3.08	3.01
12	3.72	4.11	3.84	3.73
14	4.45	5.18	4.78	4.61
16	5.24	6.53	5.94	5.69
18	6.16	8.20	7.36	7.02
20	7.18	10.3	9.12	8.63

TABLE VII. The mean energy in the transmitted beam, E_t/N_t, for the conditions of Table III.

a (cm)	A	I	II	III
0	8.10	10.0	10.0	10.0
2	7.05	8.36	8.39	8.39
4	6.45	7.36	7.42	7.43
6	6.03	6.63	6.72	6.73
8	5.71	6.07	6.18	6.21
10	5.46	5.63	5.74	5.80
12	5.27	5.28	5.42	5.47
14	5.12	5.01	5.15	5.20
16	4.99	4.79	4.92	4.98
18	4.88	4.61	4.74	4.80
20	4.80	4.46	4.59	4.64

Tables V and VI summarize the values for the ratio of the total transmission to the unscattered transmission. These ratios express the transmission in a convenient manner. The unscattered transmission N_u and E_u (or N_0 and E_0 for the monoenergetic, monoangular initial distributions) are obtained by an elementary calculation which is subject to no computational inaccuracy; further, N_u and E_u give a rough order of magnitude of the transmission. One may compare the behavior of the sequence of values of N_t/N_u, E_t/E_u for Cases I, II, and III with the monoangular values of Sec. II. The values in Tables V and VI stand in the proper relationship with each other and with the limit. Again, the slow convergence of the sequence of angular distributions toward the values for the monoangular case is evident. It is worthy of note that the effects of an initial angular distribution, even though it is fairly sharp, are very evident in the trans-

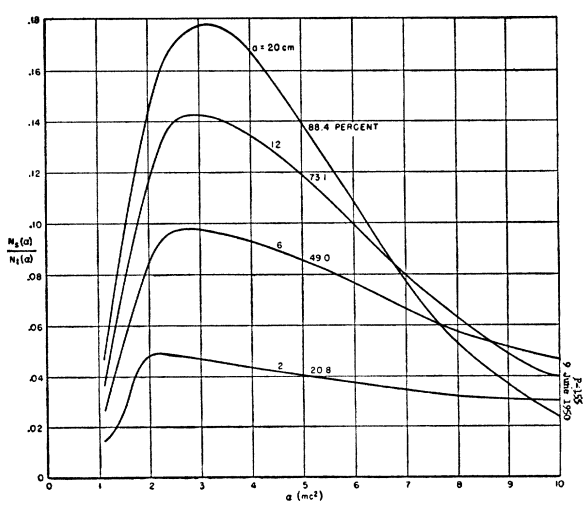

FIG. 3. The ratio of scattered transmitted photons $N_s(\alpha)$ to the total transmitted photons N_t is plotted as ordinate against the photon energy α in units of mc^2 for the incident distribution of Case III. The curves are given for various thicknesses, a, of plane slabs of uranium, and these curves are normalized so that the area under each curve equals the fraction of the total transmitted number which consists of scattered photons. These fractions are indicated as percentages on each curve.

mission values, and these effects persist as the angular distribution is sharpened. Another such effect is to be noted in the behavior of the average energy of the photons transmitted. These values are tabulated in Table VII. For 20 cm, these average photon energies for the sequence are in proper relationship with each other and with the limit value for 5 mc^2 for the mono-angular case.

The distributions for Case III of the scattered photons emerging from slabs of various thicknesses are shown in Fig. 3. For purposes of comparison, these distributions have been divided by the total number of photons transmitted for the given thickness. Therefore, the area under each distribution curve gives the percentage of the total transmission which comes from photons which suffered at least one collision. For slab thickneses of 2, 6, 12, and 20 cm, these percentages are 20.8, 49.0, 73.1, and 88.4, respectively.

TABLE VIII. Ratios of the total number of photons transmitted, N_t, to the number transmitted without scattering, N_u, for various thicknesses, a, of plane slabs of lead. The four cases, A, I, II, III, represent the same incident distributions as were used to obtain the results of Table III.

a (cm)	A	I	II	III
0	1.0	1.0	1.0	1.0
3.5	1.54	1.59	1.56	1.55
7.0	2.19	2.34	2.29	2.26
10.5	2.98	3.39	3.24	3.19
14.0	3.96	4.8	4.5	4.4
17.5	5.1	6.7	6.2	6.0
21.0	6.5	9.3	8.4	8.1
24.5	8.1	12.7	11.3	10.7
28.0	10.0	17.1	14.9	14.1
31.5	12.2	22.6	19.6	18.4
35.0	14.6	30.2	25.8	24.0

The results for uranium may be adjusted to give estimates for transmission through lead (Tables VIII and IX). The method follows closely the one used in Sec. II. It is clear from the discussion given in that section that one can write

$$\frac{N_t}{N_u}=\sum_{k=0}^{\infty}\frac{N_k}{N_u}=\sum_{k=0}^{\infty}(\nu a)^k I_k,$$

where I_k is the appropriate integral. The ratio of the absorption coefficient for uranium to that for lead, it will be remembered, is roughly 1.75 in the interval $1.0\leq\alpha\leq 10.0$. Therefore, I_k for a cm of uranium has approximately the same value for $1.75a$ cm of lead. However, because of the differences in the value of ν for lead and uranium, one must correct the kth term by multiplying it by the factor $(1.09)^k$. In Sec. II, where the first few terms in this sum were evaluated explicitly, this correction was straightforward; but here one is given only the sum. Suppose, however, one assumes that for a cm of uranium $N_k/N_u=K^k/k!$, as Eqs. (15a) and (15b) suggest. Then, if one selects K so that

$$\sum_{k=0}^{\infty}\frac{K^k}{k!}=e^K=\frac{N_t}{N_u},$$

one has for $1.75a$ cm of lead $N_t/N_u=e^{1.09K}$. The assumption basic to this easy transformation of results is probably not, in view of its purpose, too seriously in error, since the transformed result is not very sensitive to the manner in which N_t/N_u is distributed over the terms of the series. The greatest divergence from the assumption occurs for thin slabs; but for these, the first term, which needs none of this correction, gives most of the total transmission.

TABLE IX. Ratios of the total expected energy, E_t, to the energy transmitted without scattering, E_u, for the conditions of Table VIII.

a (cm)	A	I	II	III
0	1.0	1.0	1.0	1.0
3.5	1.34	1.31	1.30	1.29
7.0	1.72	1.68	1.66	1.64
10.5	2.18	2.16	2.10	2.08
14.0	2.74	2.79	2.67	2.63
17.5	3.40	3.60	3.40	3.31
21.0	4.2	4.6	4.3	4.2
24.5	5.1	6.0	5.5	5.3
28.0	6.0	7.7	6.9	6.6
31.5	7.2	9.8	8.8	8.3
35.0	8.5	12.6	11.0	10.4

As a general remark concerning the tabulated values, the use of three significant figures might be taken as indicative of over-optimism regarding the accuracy of the computations, because of their extent and complexity. However, it is estimated that a numerical result from one matrix iteration is accurate to 2 percent or better. Since the error may be cumulative, it might be implied that 10 iterations, which carry the transmission through 20 cm of uranium, could have an error of 20 percent. Such an accumulation of errors cannot be dismissed merely by saying that it is improbable. However, the results of the various calculations support each other with notable consistency; and this behavior has led the writers to the view that the results and the procedure have very acceptable accuracy.