Eigenvalues of the Derangement Graph

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Abstract

We consider the Cayley graph on the symmetric group S_n generated by derangements. It is well known that the eigenvalues of this graph are indexed by partitions of n. We investigate how these eigenvalues are determined by the shape of their corresponding partitions. In particular, we show that the sign of an eigenvalue is the parity of the number of cells below the first row of the corresponding Ferrers diagram. We also provide some lower and upper bounds for the absolute values of these eigenvalues.

KEYWORDS: derangement, Cayley graph, eigenvalue, symmetric group

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1 Introduction

Let G be a finite group and let S be a nonempty subset of G satisfying the condition that $s \in S \implies s^{-1} \in S$ and $1 \notin S$. The *Cayley graph* $\Gamma(G, S)$ has the elements of G as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $uv^{-1} \in S$. In this paper, we shall be interested in the graph Γ_n which is $\Gamma(S_n, \mathcal{D}_n)$ where S_n is the symmetric group of permutations of the integers $1, \ldots, n$, denoted $[n]$, and $\mathcal{D}_n$ is the set of derangements of $[n]$ which are the permutations in S_n which fix no point, i. e. for which $g(x) \neq x$ for all $x \in [n]$. The graph Γ_n is called the *derangement graph* on $[n]$. Clearly, Γ_n is vertex-transitive and so it is D_n-regular, where $D_n = |\mathcal{D}_n|$. By a standard result in graph theory, D_n is the largest eigenvalue of Γ_n see [5].

For a graph Γ, let $\alpha(\Gamma)$ denote the *independence number* of Γ, i.e. the cardinality of an independent set of maximum size of Γ. For any k-regular graph Γ with N vertices, the independence number satisfies the Delsarte-Hoffman bound:

$$\alpha(\Gamma) \leq \frac{-\mu N}{k - \mu},$$

where μ is the smallest eigenvalue of Γ, see [6]. In particular, this implies that μ is negative .

Without applying the Delsarte–Hoffman bound, Deza and Frankl [2] first proved that $\alpha(\Gamma_n) = (n-1)!$ by purely combinatorial means. More recently, the structure of maximum-size independent sets of Γ_n has been determined by several authors ([1], [4], [10], [13]) using different methods, namely

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such a set must be a coset of the stabilizer of a point. Ku and Wong [9] conjectured that $-\frac{D_n}{n-1}$ is the smallest eigenvalue of Γ_n which would give equality to the Delsarte–Hoffman bound. This has been proved by Renteln [11]. It is immediate from this that $\alpha(\Gamma) = (n-1)!$ as the stabilizer of a point is an independent set.

The main theme of this work is to describe in more detail the properties of the eigenvalues of the derangement graph. Recall that a Cayley graph $\Gamma(G, S)$ is *normal* if S is closed under conjugation. Its spectra is described in the following lemma. See for example Lubotsky [8, Theorem 8.2.1].

Lemma 1.1 *The eigenvalues of a normal Cayley graph $\Gamma(G, S)$ are integers given by*

$$\eta_\chi \quad = \quad \frac{1}{\chi(1)} \sum_{s \in S} \chi(s),$$

where χ ranges over all the irreducible characters of G. Moreover, the multiplicity of η_χ is $\chi(1)^2$.

Since $\mathcal{D}_n$ is closed under conjugation, Γ_n is normal. It is well known that both the conjugacy classes of S_n and the irreducible characters of S_n are indexed by partitions λ of n [7]. Recall that a *partition* λ of n, denoted by $\lambda \vdash n$, is a weakly decreasing sequence $(\lambda_1, \ldots, \lambda_r)$ with $\lambda_r \geq 1$ such that $\sum_{i=1}^r \lambda_i = n$. Its *size* is $|\lambda|$, its *length* is r and each λ_i is the *i-th part* of the partition. We also adopt the notation $(\mu_1^{a_1}, \mu_2^{a_2}, \ldots, \mu_s^{a_s}) \vdash n$ where μ_i are the distinct non zero parts which occur with multiplicity a_i. For example,

$$(5, 4, 4, 3, 3, 3, 1) \longleftrightarrow (5, 4^2, 3^3, 1).$$

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions of n. Then $\lambda > \mu$ in *lexicographic order* if, for some index i,

$$\lambda_j = \mu_j \;\text{ for } j < i \;\text{ and } \lambda_i > \mu_i.$$

Note that for $\lambda_1 \geq \frac{n}{2}$, the partition $(\lambda_1, n - \lambda_1)$ is the largest partition in lexicographic order among the partitions with the same first part, λ_1.

Based on the remarks above, we write η_λ to denote the eigenvalue η_{χ_λ} of Γ_n, where χ_λ is the irreducible character indexed by the partition $\lambda \vdash n$. We shall investigate how these eigenvalues are determined by the shape of their corresponding partitions. We used GAP [3] to list the eigenvalues for many values of n. Some values for small n are tabulated in Section 11. A glance at the table shows many striking properties about the values. We prove the following main results and offer a conjecture.

Theorem 1.2 (The Alternating Sign Property (ASP)) *For any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$,*

$$\begin{aligned} \operatorname{sign}(\eta_\lambda) &= (-1)^{n-\lambda_1} \\ &= (-1)^{\#\text{ cells under the first row of } \lambda} \end{aligned} \tag{1}$$

where $\operatorname{sign}(\eta_\lambda)$ *is* 1 *if* η_λ *is positive or* -1 *if* η_λ *is negative.*

Theorem 1.3 *Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$.*

(i) *If $\lambda_1 \geq \lfloor \frac{n}{2} \rfloor$ then*

$$|\eta_{(\lambda_1, 1^{n-\lambda_1})}| \leq |\eta_\lambda| \leq |\eta_{(\lambda_1+1, 1^{n-\lambda_1-1})}|.$$

(ii) *If $\lambda_1 < \lfloor \frac{n}{2} \rfloor$ then*

$$|\eta_\lambda| \leq |\eta_{\left(\lfloor \frac{n}{2} \rfloor + 1, 1^{n - \lfloor \frac{n}{2} \rfloor - 1}\right)}|,$$

with strict inequality if $n \geq 6$. Moreover, $|\eta_{(\lambda_1,1^{n-\lambda_1})}| = D_{\lambda_1} + (n-\lambda_1)D_{\lambda_1-1}$ for any λ_1. Here as usual $\lfloor \frac{n}{2} \rfloor$ is the greatest integer less than or equal to $\frac{n}{2}$.

Theorem 1.4 *Let λ_1 be $n-1$, $n-2$, $n-3$ or $n-4$ with n being at least 2, 4, 6, and 8 respectively. Then the absolute values of the eigenvalues which correspond to the partitions of n with λ_1 as their first part decrease in lexicographic order, i.e. $|\eta_\mu| \leq |\eta_\lambda|$ if and only if $\mu < \lambda$ in lexicographic order. The decrease is strict for $n \geq 7$.*

Remark 1.1 Because of Theorem 1.2 there are no eigenvaulues 0 for $n \geq 2$ and so the adjacency matrix is nonsingular. Note, however, that there is always an eigenvalue $(-1)^n$ from the partition $(2, 1^{n-2})$ by Lemma 2.2.

Remark 1.2 Theorem 1.3 implies that for $\lambda_1 \geq \lfloor \frac{n}{2} \rfloor$, $|\eta_\lambda| \leq |\eta_\mu|$ whenever $\lambda_1 < \mu_1$. In fact, we shall prove that the upper bound in part (i) of Theorem 1.3 is strict in most cases, namely when $\lambda \neq (\frac{n}{2}, \frac{n}{2})$ or $\lambda \neq (\frac{n-1}{2}, c, d)$ for any $c \leq \frac{n-1}{2}$, $d \geq 1$ (see Proposition 5.2). Our approach does not yield a good upper bound for $|\eta_\lambda|$ when λ_1 is small, i.e. $\lambda_1 < \lfloor \frac{n}{2} \rfloor$. The condition $\lfloor \frac{n}{2} \rfloor \leq \lambda_1$ for the upper bound in part (i) of Theorem 1.3 cannot be weakened for otherwise there are many counterexamples, e.g. for $n = 9$, we have $|\eta_{(3^3)}| = 32 > 19 = |\eta_{(4,1^5)}|$. Nevertheless, part (ii) of Theorem 1.3 implies that $|\eta_{(3^3)}| < |\eta_{(5,1^4)}| = 80$. For $\lambda_1 < \frac{n}{2}$, it is generally not true that $|\eta_{\lambda_*}| < |\eta_{(\lambda_1+1,1^{n-\lambda_1-1})}|$ where λ^* is the largest partition in lexicographic order among the partitions with λ_1 as their first part.

Remark 1.3 Notice that Theorem 1.4 is a strengthening of Theorem 1.3 for partitions with $\lambda_1 \geq n-4$ and $n \geq 8$.

Remark 1.4 It seems at first sight from our computations that the absolute values of eigenvalues should decrease in lexicographic order among the partitions with the same first part λ_1 for all λ_1. However, this is not true in general for both large and small values of λ_1 with respect to n. For example when $n = 15$, $|\eta_{(7,4,1^4)}| = 5558 < 5566 = |\eta_{(7,3^2,2)}|$ but in lexicographic order $(7, 4, 1) > (7, 3, 2^2)$. In fact, the smallest n for which this occurs is $n = 11$ with $|\eta_{(4,3,1^4)}| = 37 < 38 = |\eta_{(4,2^3,1)}|$. Also, when $n = 17$ we have $|\eta_{(9,5,1^3)}| = 347104 < 349624 = |\eta_{(9,4^2)}|$. Notice here $\lambda_1 > \frac{n}{2}$. These values have been computed in GAP but are not in the tables in Section 11.

In view of Theorem 1.3, Theorem 1.4, and values for small n we make the following conjecture:

Conjecture 1.1 *Suppose $\lambda^* \vdash n$ is the largest partition in lexicographic order among all the partitions with λ_1 as their first part. Then, for every $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$,*

$$|\eta_{(\lambda_1,1^{n-\lambda_1})}| \leq |\eta_\lambda| \leq |\eta_{\lambda^*}|.$$

Notice this follows for $n \geq 8$ and $\lambda_1 \geq n-4$ by Theorem 1.4. For $\lambda_1 \geq \lfloor \frac{n}{2} \rfloor$ the lower bound holds by Theorem 1.3. The upper bound obtained in Theorem 1.3 is a different upper bound.

Our main results can be regarded as a strengthening of the following result of Renteln:

Theorem 1.5 (Renteln, [11]) *The smallest eigenvalue of* Γ_n *is*

$$\eta_{(n-1,1)} = -\frac{D_n}{n-1} = -(D_{n-1} + D_{n-2}).$$

Moreover, if $\lambda \neq (n), (n-1, 1)$ *then* $|\eta_\lambda| < |\eta_{(n-1,1)}| < |\eta_{(n)}|$.

Notice Theorem 1.4 is an extension of this result.

Our methods rely heavily on the following remarkable recurrence formula for the eigenvalues of Γ_n proved by Renteln [11]. To describe this result, we require some terminology. To the Ferrers diagram of a partition λ, we assign xy-coordinates to each of its boxes by defining the upper-left-most box to be $(1, 1)$, with the x axis increasing to the right and the y axis increasing downwards. Then the *hook* of λ is the union of the boxes $(x', 1)$ and $(1, y')$ of the Ferrers diagram of λ, where $x' \geq 1$, $y' \geq 1$. Let $\widehat{h}_\lambda$ denote the hook of λ and let h_λ denote the size of $\widehat{h}_\lambda$. Similarly, let $\widehat{c}_\lambda$ and c_λ denote the first column of λ and the size of $\widehat{c}_\lambda$ respectively. Note that c_λ is equal to the number of rows of λ. When λ is clear from the context, we replace $\widehat{h}_\lambda$, h_λ, $\widehat{c}_\lambda$ and c_λ by $\widehat{h}$, h, $\widehat{c}$ and c respectively. Let $\lambda - \widehat{h} \vdash n - h$ denote the partition obtained from λ by removing its hook. Also, let $\lambda - \widehat{c}$ denote the partition obtained from λ by removing the first column of its Ferrers diagram, i.e. $(\lambda_1, \ldots, \lambda_r) - \widehat{c} = (\lambda_1 - 1, \ldots, \lambda_r - 1) \vdash n - r$.

Theorem 1.6 (Renteln, [11]) *For any partition* $\lambda = (\lambda_1, \ldots, \lambda_r)$,

$$\eta_\lambda \;=\; (-1)^h (\eta_{\lambda - \widehat{h}} + (-1)^{\lambda_1} h \eta_{\lambda - \widehat{c}}) \tag{2}$$

with initial condition $\eta_\emptyset = 1$.

Since the above recurrence will be used extensively throughout the paper, we shall often refer to it as the *main recurrence*.

The rest of the paper is organized as follows. In Section 2, we give some useful formulae for the eigenvalues which correspond to partitions of simple shapes. Theorem 1.2 is proved in Section 3. In Sections 4 and 5 respectively, we provide a lower and an upper bound for the eigenvalues when the first part of the corresponding partition is large. In Section 6 we consider partitions with few parts. We prove Theorem 1.3 in Section 7. Sections 8, 9, and 10 are devoted to a proof of Theorem 1.4. As mentioned, some values for small n have been tabulated in Section 11.

2 Some preliminary results

In this section, we collect some basic formulae for some special types of partitions. Recall the following useful facts about the derangement numbers:

Lemma 2.1 *For* $n \geq 1$,

(1) $D_n = nD_{n-1} + (-1)^n$.

(2) $D_n = (n-1)(D_{n-1} + D_{n-2})$.

(3) *The first eleven derangement numbers are* $D_0 = 1$, $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, $D_6 = 265$, $D_7 = 1854$, $D_8 = 14833$, $D_9 = 133496$, *and* $D_{10} = 1334961$.

Proof. See [12], page 67 for (1) and (2). The values in (3) are tabulated in the first entries in the tables in section 11 as η_λ for $\lambda = (n)$. ■

Remark 2.1 Notice it follows from Lemma 2.1 that the values of D_n are stricIty increasing for $n \geq 1$.

Lemma 2.2 *Let $\lambda = (\lambda_1, 1^{n-\lambda_1})$ be a hook. Then*

$$\eta_{(\lambda_1,1^{n-\lambda_1})} = (-1)^n \left(1 + (-1)^{\lambda_1} n D_{\lambda_1 - 1}\right) = (-1)^{n-\lambda_1}\left(D_{\lambda_1} + (n-\lambda_1) D_{\lambda_1-1}\right).$$

In particular, $\eta_{(n-1,1)} = -\frac{D_n}{n-1} = -(D_{n-1} + D_{n-2})$.

Proof. The first equality is [11, Lemma 7.4].

For the second use the first.

$$\begin{aligned}
\eta_{(\lambda_1,1^{n-\lambda_1})} &= (-1)^n \left(1 + (-1)^{\lambda_1} n D_{\lambda_1-1}\right) \\
&= (-1)^{\lambda_1+(n-\lambda_1)} \left(1 + (-1)^{\lambda_1}\left(\lambda_1 D_{\lambda_1-1} + (n-\lambda_1) D_{\lambda_1-1}\right)\right) \\
&= (-1)^{n-\lambda_1} \left((-1)^{\lambda_1} + \lambda_1 D_{\lambda_1-1} + (n-\lambda_1) D_{\lambda_1-1}\right) \\
&= (-1)^{n-\lambda_1} \left((-1)^{\lambda_1} + D_{\lambda_1} - (-1)^{\lambda_1} + (n-\lambda_1) D_{\lambda_1-1}\right) \\
&= (-1)^{n-\lambda_1} \left(D_{\lambda_1} + (n-\lambda_1) D_{\lambda_1-1}\right).
\end{aligned}$$

Applying this when $\lambda_1 = n-1$ gives the last statement. ■

We define a partition $\lambda = (\lambda_1, 2, 1^{n-\lambda_1-2})$ to be a *near hook*.

Lemma 2.3 *Let $\lambda = (\lambda_1, 2, 1^{n-\lambda_1-2})$ be a near hook. Then*

$$\begin{aligned}
\eta_{(\lambda_1,2,1^{n-\lambda_1-2})} &= (-1)^{n+\lambda_1}(n-1)\frac{D_{\lambda_1}}{\lambda_1 - 1} \\
&= (n-1)\left((-1)^{n-1} + (-1)^{n+\lambda_1}\lambda_1 D_{\lambda_1-2}\right).
\end{aligned}$$

Proof. Use the main recurrence and the properties above or [11, Lemma 8.3] . ■

Lemma 2.4 *The values of $|\eta_\lambda|$ for hooks are given by Lemma 2.2. The absolute values are as follows.*

$$\begin{aligned}
|\eta_{(1^n)}| &= n-1 \\
|\eta_{(2,1^{n-2})}| &= 1 \\
|\eta_{(3,1^{n-3})}| &= n-1
\end{aligned}$$

For $2 \leq \lambda_1 \leq n-1$ we have $|\eta_{(\lambda_1,1^{n-\lambda_1})}| < |\eta_{(\lambda_1+1,1^{n-\lambda_1-1})}|$.

Proof. By Lemma 2.2 we see $|\eta_{(\lambda_1,1^{n-\lambda_1})}| = D_{\lambda_1} + (n-\lambda_1)D_{\lambda_1-1}$. The values for $\lambda_1 \leq 3$ are as given. We need only show for $3 \leq \lambda_1 \leq n-1$ that $|\eta_{(\lambda_1,1^{n-\lambda_1})}| < |\eta_{(\lambda_1+1,1^{n-\lambda_1-1})}|$. This means we need to show $D_{\lambda_1} + (n-\lambda_1)D_{\lambda_1-1} < D_{\lambda_1+1} + (n-\lambda_1-1)D_{\lambda_1}$ for these values of λ_1. Using $(n-\lambda_1)D_{\lambda_1-1} < (n-\lambda_1)D_{\lambda_1}$ we need only show $0 < D_{\lambda_1+1} - 2D_{\lambda_1}$ which is true using $D_{\lambda_1+1} = (\lambda_1+1)D_{\lambda_1} \pm 1$ and $\lambda_1 \geq 3$. ■

3 Proof of the Alternating Sign Property

Recall that the Alternating Sign Property (ASP) is the assertion that for any partition $\lambda \vdash n$,

$$\operatorname{sign}(\eta_\lambda) = (-1)^{n-\lambda_1} = (-1)^{\#\text{ cells under the first row of }\lambda}.$$

Proposition 3.1 *Let* $\lambda \vdash n$*. Suppose* ASP holds for partitions of smaller size. Then

- if $\lambda_1 \equiv \lambda_2 \pmod 2$, then $|\eta_\lambda| = |h|\eta_{\lambda-\widehat{c}}| - |\eta_{\lambda-\widehat{h}}||$;
- if $\lambda_1 \not\equiv \lambda_2 \pmod 2$, then $|\eta_\lambda| = h|\eta_{\lambda-\widehat{c}}| + |\eta_{\lambda-\widehat{h}}|$.

Moreover, ASP holds for λ if $\lambda_1 \not\equiv \lambda_2 \pmod 2$. If $\lambda_1 \equiv \lambda_2 \pmod 2$, then ASP for λ is equivalent to $|\eta_{\lambda-\widehat{h}}| < h|\eta_{\lambda-\widehat{c}}|$.

Proof. In view of the main recurrence $\eta_\lambda = (-1)^h \left((-1)^{\lambda_1} h\eta_{\lambda-\widehat{c}} + \eta_{\lambda-\widehat{h}}\right)$, the absolute value of η_λ depends on the values and the signs of $(-1)^{\lambda_1}\eta_{\lambda-\widehat{c}}$ and $\eta_{\lambda-\widehat{h}}$. Since ASP holds for $\lambda-\widehat{c}$ and $\lambda-\widehat{h}$, we have

$$\begin{aligned}\operatorname{sign}((-1)^{\lambda_1}\eta_{\lambda-\widehat{c}}) &= (-1)^{\lambda_1+(\#\text{cells under the first row of }\lambda-\widehat{c})}\\ &= (-1)^{\lambda_1+\lambda_2-1+(\#\text{cells under the first row of }\lambda-\widehat{h})}\\ &= (-1)^{\lambda_1+\lambda_2-1}\operatorname{sign}(\eta_{\lambda-\widehat{h}}).\end{aligned} \tag{3}$$

Therefore, if $\lambda_1 \not\equiv \lambda_2 \pmod 2$, then $\operatorname{sign}((-1)^{\lambda_1}\eta_{\lambda-\widehat{c}}) = \operatorname{sign}(\eta_{\lambda-\widehat{h}})$. This implies that $\operatorname{sign}(\eta_\lambda) = \operatorname{sign}((-1)^h(-1)^{\lambda_1}\eta_{\lambda-\widehat{c}}) = (-1)^{\lambda_1+r-1+\lambda_1+(n-r)-(\lambda_1-1)} = (-1)^{n-\lambda_1}$, i.e. ASP holds for λ and $|\eta_\lambda| = h|\eta_{\lambda-\widehat{c}}| + |\eta_{\lambda-\widehat{h}}|$. Otherwise, $|\eta_\lambda| = |h|\eta_{\lambda-\widehat{c}}| - |\eta_{\lambda-\widehat{h}}||$. Here ASP is equivalent to $\operatorname{sign}(\eta_\lambda) = (-1)^{n-\lambda_1}$ which is equivalent to $|\eta_{\lambda-\widehat{h}}| < h|\eta_{\lambda-\widehat{c}}|$. ■

Proposition 3.2 *Let* $\lambda \vdash n$*,* $n \geq 3$*. Assume* $\lambda = (n)$ *or* $\lambda_1 \geq \lambda_2 + 2$*. Suppose* ASP holds for partitions of smaller size. Then

$$|\eta_\lambda| > |\eta_{(\lambda_1-2,\lambda_2,\ldots,\lambda_r)}|.$$

Moreover, ASP holds for λ.

Proof. Let $\lambda' = (\lambda'_1, \lambda_2, \ldots, \lambda_r)$ where $\lambda'_1 = \lambda_1 - 2$, i.e. λ' is the partition of $n-2$ obtained from λ by deleting the first two cells of λ_1 from the right. Let $\widehat{h'}$ and $\widehat{c'}$ denote the hook and the first column of λ' respectively. Also, let h' denote the size of $\widehat{h'}$. We shall prove by induction on $n = |\lambda|$ that $|\eta_\lambda| > |\eta_{\lambda'}|$.

When $n = 3$, the only partition which satisfies the conditions of the theorem is $\lambda = (3)$. So $|\eta_\lambda| = 2 > |\eta_{\lambda'}| = 0$. Indeed if $\lambda = (n)$ the statement follows as $D_n > D_{n-2}$. This means we can assume $r \geq 2$. Let $n > 3$. As ASP holds for $\lambda - \widehat{c}$, $\lambda' - \widehat{c'}$, $\lambda - \widehat{h}$, $\lambda' - \widehat{h'}$, we have

$$\begin{aligned}\operatorname{sign}((-1)^{\lambda_1}\eta_{\lambda-\widehat{c}}) &= \operatorname{sign}((-1)^{\lambda_1-2}\eta_{\lambda-\widehat{c}})\\ &= \operatorname{sign}((-1)^{\lambda'_1}\eta_{\lambda'-\widehat{c'}})\\ \operatorname{sign}(\eta_{\lambda-\widehat{h}}) &= \operatorname{sign}(\eta_{\lambda'-\widehat{h'}}).\end{aligned}$$

This means the signs in the main recurrence for η_λ and $\eta_{\lambda'}$ are the same. In particular, $|\eta_\lambda| = |h|\eta_{\lambda-\widehat{c}}| + |\eta_{\lambda-\widehat{h}}||$ if and only if $|\eta_{\lambda'}| = |h'|\eta_{\lambda'-\widehat{c'}}| + |\eta_{\lambda'-\widehat{h'}}||$ and the same when the signs are negative.

We can use induction unless $\lambda = (3, 1^{n-3})$. By Lemma 2.2, $|\eta_{(3,1^{n-3})}| = D_3 + (n-3)D_2 = n-1$. However, $|\eta_{(1^{n-2})}| = D_1 + n - 2 - 1 = n-3$ and so the result holds.

We can now use induction, and so $|\eta_{\lambda-\widehat{c}}| > |\eta_{\lambda'-\widehat{c'}}|$. Since $h = h'+2$ and $\lambda - \widehat{h} = \lambda' - \widehat{h'}$, we deduce that

$$\begin{aligned} |\eta_\lambda| &= |(h'+2)|\eta_{\lambda-\widehat{c}}| \pm |\eta_{\lambda'-\widehat{h'}}|| \\ &> |h'|\eta_{\lambda'-\widehat{c'}}| \pm |\eta_{\lambda'-\widehat{h'}}|| \\ &= |\eta_{\lambda'}|. \end{aligned}$$

Here in the case with negative sign we have used $|\eta_{\lambda'-\widehat{h'}}| < h'|\eta_{\lambda'-\widehat{c'}}|$ since ASP holds for λ' using Proposition 3.1.

If $|\eta_\lambda| = h|\eta_{\lambda-\widehat{c}}| + |\eta_{\lambda-\widehat{h}}|$ then $\mathrm{sign}(\eta_\lambda) = \mathrm{sign}((-1)^h(-1)^{\lambda_1}\eta_{\lambda-\widehat{c}}) = (-1)^{n-\lambda_1}$. This follows as by assumption $\mathrm{sign}(\eta_{\lambda-\widehat{c}}) = (-1)^{n-r-(\lambda_1-1)}$ and so mod 2, $\lambda_1 + (r-1) + \lambda_1 + n - r - (\lambda_1 - 1)$ is $n - \lambda_1$. This means ASP holds for η_λ. Otherwise, $|\eta_\lambda| = |h|\eta_{\lambda-\widehat{c}}| - |\eta_{\lambda-\widehat{h}}||$ and $|\eta_{\lambda'}| = |h'|\eta_{\lambda'-\widehat{c'}}| - |\eta_{\lambda'-\widehat{h'}}||$. But, since ASP holds for λ', this means that $|\eta_{\lambda'}| = h'|\eta_{\lambda'-\widehat{c'}}| - |\eta_{\lambda'-\widehat{h'}}| > 0$. Therefore,

$$h|\eta_{\lambda-\widehat{c}}| > h'|\eta_{\lambda'-\widehat{c'}}| \geq |\eta_{\lambda'-\widehat{h'}}| = |\eta_{\lambda-\widehat{h}}|.$$

This means $\mathrm{sign}(\eta_\lambda)$ is $(-1)^h(-1)^{\lambda_1}\mathrm{sign}(\eta_{\lambda-\widehat{c}}) = (-1)^{n-\lambda_1}$ and so ASP holds for η_λ. ■

We want to prove ASP for λ by assuming that the assertion holds for partitions of smaller size than $|\lambda|$. By Proposition 3.1 and Proposition 3.2, it remains to consider the case $\lambda_1 = \lambda_2$.

To state our next results, we require some new terminology. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ and $0 \leq i \leq \lambda_1$, let $\lambda - \widehat{c}_i$ denote the partition obtained from λ by deleting the first i columns. In particular, $\lambda = \lambda - \widehat{c}_0$, $\lambda - \widehat{c} = \lambda - \widehat{c}_1$. Similarly, for $0 \leq i \leq r$, let $\lambda - \widehat{\rho}_i$ denote the partition obtained from λ by deleting the first i rows. When $i = 1$, we also write $\lambda - \widehat{\rho}$ instead of $\lambda - \widehat{\rho}_1$. Using these notations, note that $\lambda - \widehat{h} = (\lambda - \widehat{c}) - \widehat{\rho}$.

For the rest of this section, we let h_i denote the size of the hook of $\lambda - \widehat{c}_{i-1}$, where $1 \leq i \leq \lambda_1$. We have the following upper bound for $|\eta_\lambda|$ in terms of the h_i's:

Proposition 3.3 *Let $\lambda \vdash n$. Then*

$$|\eta_\lambda| \leq \prod_{i=1}^{\lambda_1}(h_i + 1).$$

Proof. As this is not needed in the sequel, we omit the proof which is a routine interation of Lemma ■

We shall be interested in partitions with $\lambda_1 = \lambda_2$ and $\lambda_3 < \lambda_1$ if $r \geq 3$ where as usual r is the number of rows. For this we denote λ_1 by t and assume $t \geq 2$. Note that the smallest partition satisfying these conditions is $\lambda = (1^2)$. For the definition of δ below we assume $\lambda_1 = \lambda_2$ with $\lambda_3 < \lambda_1$

if $r \geq 3$. We define the following functions:

$$\begin{aligned}
H(\lambda) &= \prod_{i=1}^{t-1} h_i - \prod_{i=1}^{t-2} h_i - \prod_{i=2}^{t-1}(h_i - 2) - \sum_{i=1}^{t-3} h_1 h_2 \cdots h_i (h_{i+2} - 2)(h_{i+3} - 2) \cdots (h_{t-1} - 2), \\
S(\lambda) &= \prod_{i=1}^{t-1}(h_i - 2), \\
\delta((1^2)) &= 1, \\
\delta(\lambda) &= h_1 \delta(\lambda - \widehat{c}) - |\eta_{(\lambda - \widehat{c}) - \widehat{\rho}_2}| \quad \text{where for this we assume } \lambda_1 = \lambda_2 \text{ with } \lambda_3 < \lambda_1 \text{ if } r \geq 3.
\end{aligned}$$

By convention we mean $H(\lambda) = h_1$ when $t = 2$ and $H(\lambda) = h_1 h_2 - h_1 - (h_2 - 2)$ when $t = 3$. Recursively, we have

$$\delta(\lambda) = \prod_{i=1}^{t-1} h_i - \prod_{i=1}^{t-2} h_i - |\eta_{(\lambda - \widehat{c}) - \widehat{\rho}_2}| - \sum_{i=1}^{t-3} h_1 h_2 \cdots h_i |\eta_{\lambda - \widehat{c}_{i+1} - \widehat{\rho}_2}|.$$

The motivation for the above functions will become apparent in Lemma 3.5 and Proposition 3.6.

Lemma 3.4 *Let $\lambda \vdash n$, $\lambda_1 = \lambda_2 \geq 3$ and $\lambda_3 < \lambda_1$ if $r \geq 3$. Then $H(\lambda) > S(\lambda) > 0$.*

Proof. We set $t = \lambda_1 \geq 3$. Note that $h_1 > h_2 > \cdots > h_t = 2$. Clearly, $S(\lambda) > 0$ as $h_i \geq 3$ for all $i \leq t-1$. We proceed by induction on t. For $t = 3$, $h_1 > h_2 \geq 3$, and $H(\lambda) = h_1 h_2 - h_1 - (h_2 - 2) > (h_1 - 2)(h_2 - 2) = S(\lambda) > 0$ since h_1 and h_2 are greater than 2. Let $t > 3$. Then

$$H(\lambda) = h_1 \left(\prod_{i=2}^{t-1} h_i - \prod_{i=2}^{t-2} h_i - \prod_{i=3}^{t-1}(h_i - 2) - \sum_{i=2}^{t-3} h_2 \cdots h_i (h_{i+2} - 2) \cdots (h_{t-1} - 2) \right) - \prod_{i=2}^{t-1}(h_i - 2).$$

By the definition of $H(\lambda - \widehat{c})$ and the inductive hypothesis,

$$\left(\prod_{i=2}^{t-1} h_i - \prod_{i=2}^{t-2} h_i - \prod_{i=3}^{t-1}(h_i - 2) - \sum_{i=2}^{t-3} h_2 \cdots h_i (h_{i+2} - 2) \cdots (h_{t-1} - 2) \right) = H(\lambda - \widehat{c}) > S(\lambda - \widehat{c}).$$

Therefore,

$$\begin{aligned}
H(\lambda) &> h_1 S(\lambda - \widehat{c}) - \prod_{i=2}^{t-1}(h_i - 2) \\
&= h_1 \prod_{i=2}^{t-1}(h_i - 2) - \prod_{i=2}^{t-1}(h_i - 2) \\
&= (h_1 - 1) \prod_{i=2}^{t-1}(h_i - 2) \\
&> \prod_{i=1}^{t-1}(h_i - 2) = S(\lambda).
\end{aligned}$$

■

Lemma 3.5 *Let $\lambda \vdash n$, $\lambda_1 = \lambda_2$ and $\lambda_3 < \lambda_1$ if $r \geq 3$. Then $\delta(\lambda) > 0$.*

Proof. Since $\delta((1^2)) = 1 > 0$, we may assume that $\lambda \neq (1^2)$. If $\lambda_1 = \lambda_2 = 2$ then $\delta(\lambda) = h_1\delta((1^2)) - 1 > 0$. So we may assume that $\lambda_1 = \lambda_2 \geq 3$. By Lemma 3.4, it suffices to show that $\delta(\lambda) \geq H(\lambda)$. Indeed, by Proposition 3.3, we have, for all $i \in \{0, 1, \ldots, t-3\}$,

$$|\eta_{(\lambda - \widehat{c}_{i+1}) - \widehat{\rho}_2}| \leq \prod_{j=i+2}^{t-1} (h_j - 2).$$

So

$$\begin{aligned} \delta(\lambda) &\geq \prod_{i=1}^{t-1} h_i - \prod_{i=1}^{t-2} h_i - \prod_{i=2}^{t-1} (h_i - 2) - \sum_{i=1}^{t-3} h_1 h_2 \cdots h_i (h_{i+2} - 2)(h_{i+3} - 2) \cdots (h_{t-1} - 2) \\ &= H(\lambda). \end{aligned}$$

■

Proposition 3.6 *Let $\lambda \vdash n$, $\lambda_1 = \lambda_2$ and $\lambda_3 < \lambda_1$ if $r \geq 3$. Then*

$$|\eta_\lambda| \geq |\eta_{\lambda - \widehat{\rho}}| + \delta(\lambda) > |\eta_{\lambda - \widehat{\rho}}|.$$

Proof. The second inequality follows immediately from Lemma 3.5. We shall prove the first inequality by induction on $\lambda_1 \geq 1$. If $\lambda_1 = 1$, then $\lambda = (1^2)$, $|\eta_\lambda| = 1$, $|\eta_{\lambda - \widehat{\rho}}| = 0$ and $\delta(\lambda) = 1$, so the inequality holds. Let $\lambda_1 > 1$. By induction, $|\eta_{\lambda - \widehat{c}}| \geq |\eta_{(\lambda - \widehat{c}) - \widehat{\rho}}| + \delta(\lambda - \widehat{c})$. Then using Proposition 3.1,

$$\begin{aligned} |\eta_\lambda| &= h_1 |\eta_{\lambda - \widehat{c}}| - |\eta_{(\lambda - \widehat{c}) - \widehat{\rho}}| \\ &\geq h_1 |\eta_{\lambda - \widehat{c}}| - (|\eta_{\lambda - \widehat{c}}| - \delta(\lambda - \widehat{c})) \quad \text{(by induction)} \\ &= (h_1 - 1)|\eta_{\lambda - \widehat{c}}| + \delta(\lambda - \widehat{c}). \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned} |\eta_{\lambda - \widehat{\rho}}| &\leq (h_1 - 1)|\eta_{(\lambda - \widehat{c}) - \widehat{\rho}}| + |\eta_{(\lambda - \widehat{c}) - \widehat{\rho}_2}| \\ &\leq (h_1 - 1)(|\eta_{\lambda - \widehat{c}}| - \delta(\lambda - \widehat{c})) + |\eta_{(\lambda - \widehat{c}) - \widehat{\rho}_2}| \quad \text{(by induction)} \\ &= (h_1 - 1)|\eta_{\lambda - \widehat{c}}| - (h_1 - 1)\delta(\lambda - \widehat{c}) + |\eta_{(\lambda - \widehat{c}) - \widehat{\rho}_2}|. \end{aligned} \tag{5}$$

As $\delta(\lambda) = h_1 \delta(\lambda - \widehat{c}) - |\eta_{(\lambda - \widehat{c}) - \widehat{\rho}_2}|$, it follows from (4) and (5) that

$$|\eta_\lambda| \geq |\eta_{\lambda - \widehat{\rho}}| + \delta(\lambda).$$

■

Proposition 3.7 *Let $\lambda \vdash n$, $\lambda_1 = \lambda_2$. Suppose* ASP holds for all partitions of size smaller than n. Then

$$|\eta_\lambda| > |\eta_{\lambda - \widehat{\rho}}|.$$

Proof. If $\lambda_3 < \lambda_1$ for $r \geq 3$ or $r = 2$, then the assertion is true by Proposition 3.6, even without the assumptions that ASP holds for smaller partitions. So we may assume that $\lambda_3 = \lambda_1$. We proceed by induction on $\lambda_1 \geq 1$. For $\lambda_1 = 1$, clearly $|\eta_{(1^n)}| = n - 1 > n - 2 = |\eta_{(1^{n-1})}|$. Let $\lambda_1 > 1$. By induction, we assume that $|\eta_{\lambda - \widehat{c}}| > |\eta_{(\lambda-\widehat{c})-\widehat{\rho}}|$. It follows from Proposition 3.1 that

$$\begin{aligned} |\eta_\lambda| &= h|\eta_{\lambda-\widehat{c}}| - |\eta_{(\lambda-\widehat{c})-\widehat{\rho}}| \\ &> h|\eta_{\lambda-\widehat{c}}| - |\eta_{\lambda-\widehat{c}}| \\ &= (h-1)|\eta_{\lambda-\widehat{c}}| \\ &> (h-1)|\eta_{(\lambda-\widehat{c})-\widehat{\rho}}|. \end{aligned} \tag{6}$$

On the other hand, since ASP holds for $\eta_{\lambda-\widehat{\rho}}$, $\eta_{(\lambda-\widehat{c})-\widehat{\rho}}$, and $\eta_{(\lambda-\widehat{c})-\widehat{\rho}_2}$, $\lambda_2 = \lambda_3$, and Proposition3.1

$$\begin{aligned} |\eta_{\lambda-\widehat{\rho}}| &= h'|\eta_{(\lambda-\widehat{c})-\widehat{\rho}}| - |\eta_{(\lambda-\widehat{c})-\widehat{\rho}_2}| \\ &< h'|\eta_{(\lambda-\widehat{c})-\widehat{\rho}}| \\ &= (h-1)|\eta_{(\lambda-\widehat{c})-\widehat{\rho}}|, \end{aligned} \tag{7}$$

where $h' = h - 1$ is the hook of $\lambda - \widehat{\rho}$. It follows immediately from (6) and (7) that $|\eta_\lambda| > |\eta_{\lambda-\widehat{\rho}}|$. ■

Remark 3.1 Notice it is not generally true that $|\eta_\lambda| > |\eta_{\lambda-\widehat{\rho}}|$ as if $\lambda = (2, 1^{n-2})$, $|\eta_\lambda| = 1$ and $|\eta_{\lambda-\widehat{\rho}}| = n - 3$ by Lemma 2.2.

Proof of Theorem 1.2. We assume by induction on the size of λ that ASP holds for partitions of size smaller than n. Moreover, by Proposition 3.1 and Proposition 3.2, we may assume that $\lambda_1 = \lambda_2$. Recall that $h = \lambda_1 + r - 1$, where r is the length of the first column of λ. Then, by Proposition 3.7, $|\eta_{\lambda-\widehat{c}}| > |\eta_{(\lambda-\widehat{c})-\widehat{\rho}}|$, and so $h|\eta_{\lambda-\widehat{c}}| > |\eta_{\lambda-\widehat{c}-\widehat{\rho}}|$. In view of the main recurrence

$$\eta_\lambda = (-1)^h \left((-1)^{\lambda_1} h \eta_{\lambda-\widehat{c}} + \eta_{(\lambda-\widehat{c})-\widehat{\rho}}\right),$$

we deduce that

$$\begin{aligned} \operatorname{sign}(\eta_\lambda) &= \operatorname{sign}\left((-1)^h (-1)^{\lambda_1} \eta_{\lambda-\widehat{c}}\right) \\ &= (-1)^{h+\lambda_1} \cdot (-1)^{(n-r)-(\lambda_1-1)} \quad \text{(by induction)} \\ &= (-1)^{n-\lambda_1}. \end{aligned}$$

■

Corollary 3.8 *Propositions 3.1, 3.2, 3.7 hold without the restriction that* ASP holds for all partitions of smaller size. Furthermore, in the case $\lambda_1 \equiv \lambda_2 \pmod 2$, $|\eta_\lambda| = h|\eta_{\lambda-\widehat{c}}| - |\eta_{\lambda-\widehat{h}}|$.

Proof. The first part follows from Theorem 1.2 which states that ASP holds for all partitions. The second follows as $h|\eta_{\lambda-\widehat{c}}| > |\eta_{\lambda-\widehat{h}}|$ as shown in the proof of Theorem 1.2. ■

4 A lower bound for $|\eta_\lambda|$

In this section we prove the lower bound for η_λ when $\lambda_1 \geq \lfloor \frac{n}{2} \rfloor$. This will be the proof of Theorem 1.3 (i). We begin by giving a lower bound for $|\eta_{(a,a)}|$.

Lemma 4.1 *Let $n = 2a \geq 2$. Then $|\eta_{(a,a)}| \geq 2D_a + D_{a-1}$.*

Proof. The assertion holds for $a = 1, 2, 3$. We may assume $a \geq 4$ and proceed by induction on a. By the main recurrence and induction,

$$\begin{aligned}
|\eta_{(a,a)}| &\geq (a+1)|\eta_{(a-1,a-1)}| - |\eta_{(a-1)}| \\
&\geq (a+1)(2D_{a-1} + D_{a-2}) - D_{a-1} \\
&= 2(a+1)D_{a-1} + (a+1)D_{a-2} - D_{a-1} \\
&= 2aD_{a-1} + (a+1)D_{a-2} + D_{a-1} \\
&\geq 2(D_a - 1) + (a+1)D_{a-2} + D_{a-1} \quad \text{(by Lemma 2.1)} \\
&\geq 2D_a + D_{a-1}
\end{aligned}$$

since $(a+1)D_{a-2} - 2 \geq 0$ for all $a \geq 4$. ■

We are now ready to prove the lower bound of Theorem 1.3 (i).

Proposition 4.2 *Suppose $\lambda \vdash n$ with its first part equal to $\lambda_1 \geq \lfloor \frac{n}{2} \rfloor$. Then $|\eta_\lambda| \geq D_{\lambda_1} + (n - \lambda_1)D_{\lambda_1 - 1} = |\eta_{(\lambda_1, 1^{n-\lambda_1})}|$. This is the lower bound needed for Theorem 1.3 (i).*

Proof. We use induction on $n \geq 1$. Since all the cases for $n \leq 11$ can be done by inspection, we let $n \geq 12$. Furthermore, by Lemma 2.2, the assertion holds with equality when λ is a hook regardless of λ_1. We may therefore assume that λ is not a hook.

We first show that the assertion holds for the following special cases:

Case I. $n = 2b + 1$ and $\lambda = (b, b, 1)$.

By the main recurrence,

$$\begin{aligned}
|\eta_\lambda| &\geq (b+2)|\eta_{(b-1,b-1)}| - |\eta_{(b-1)}| \\
&\geq (b+2)(2D_{b-1} + D_{b-2}) - D_{b-1} \quad \text{(by Lemma 4.1)} \\
&= 2bD_{b-1} + 3D_{b-1} + (b+2)D_{b-2} \\
&\geq 2(D_b - 1) + 3D_{b-1} + (b+2)D_{b-2} \quad \text{(by Lemma 2.1)} \\
&\geq D_b + bD_{b-1} - 3 + 3D_{b-1} + (b+2)D_{b-2} \quad \text{(by Lemma 2.1)} \\
&= D_b + (b+1)D_{b-1} + 2D_{b-1} + (b+2)D_{b-2} - 3 \\
&\geq D_b + (b+1)D_{b-1},
\end{aligned}$$

as for $b \geq 4$, $2D_{b-1} + (b+2)D_{b-2} - 3 \geq 0$.

Case II. $n = 2b + 1$ and $\lambda = (b, c, d)$ with $d \geq 2$.

Since $d \geq 2$, by Theorem 1.5, $|\eta_{\lambda-\widehat{h}}| = |\eta_{(c-1,d-1)}| \leq \frac{D_{b-1}}{b-2} = D_{b-2} + D_{b-3}$. It follows from the main recurrence that

$$
\begin{aligned}
|\eta_{(b,c,d)}| &\geq (b+2)|\eta_{(b-1,c-1,d-1)}| - (D_{b-2} + D_{b-3}) \\
&\geq (b+2)\left(D_{b-1} + ((2b+1) - 3 - (b-1))D_{b-2}\right) - (D_{b-2} + D_{b-3}) \quad \text{(by induction)} \\
&= bD_{b-1} + 2D_{b-1} + (b+2)(b-1)D_{b-2} - (D_{b-2} + D_{b-3}) \\
&\geq D_b - 1 + 2D_{b-1} + (b+2)(D_{b-1} - 1) - (D_{b-2} + D_{b-3}) \quad \text{(by Lemma 2.1)} \\
&= D_b + (b+1)D_{b-1} + 3D_{b-1} - (b+3) - D_{b-2} - D_{b-3} \\
&\geq D_b + (b+1)D_{b-1}, \qquad (8)
\end{aligned}
$$

as for $b \geq 5$, $3D_{b-1} - (b+3) - D_{b-2} - D_{b-3} \geq 0$.

From now on in this section, we may assume that either $\lambda_1 > \frac{n}{2}$ or $\lambda_1 = \frac{n}{2}$ and λ has at least 3 rows or $\lambda_1 = \frac{n-1}{2}$ and λ has at least 4 rows. The case of $\lambda_1 = \frac{n}{2}$ and just two rows is Lemma 4.1.

In particular, $|\lambda - \widehat{h}| \leq \lambda_1 - 2$ and so $|\eta_{\lambda-\widehat{h}}| \leq D_{\lambda_1-2}$.

Let r be the number of rows of λ. Suppose $\lambda - \widehat{c}$ has s rows for some $r \geq s \geq 2$. Note that by the main recurrence

$$
\begin{aligned}
|\eta_\lambda| &\geq h|\eta_{\lambda-\widehat{c}}| - |\eta_{\lambda-\widehat{h}}| \\
&\geq (\lambda_1 + r - 1)|\eta_{\lambda-\widehat{c}}| - D_{\lambda_1-2} \\
&= \lambda_1|\eta_{\lambda-\widehat{c}}| + (r-1)|\eta_{\lambda-\widehat{c}}| - D_{\lambda_1-2}.
\end{aligned}
$$

We proceed by induction on the size of λ to conclude $|\eta_{\lambda-\widehat{c}}| \geq D_{\lambda_1-1} + (n - r - (\lambda_1 - 1))D_{\lambda_1-2}$.

$$
\begin{aligned}
|\eta_\lambda| \geq\ & \lambda_1\big(D_{\lambda_1-1} + (n - r - (\lambda_1-1))D_{\lambda_1-2}\big) \\
&+ (r-1)\big(D_{\lambda_1-1} + (n - r - (\lambda_1-1))D_{\lambda_1-2}\big) - D_{\lambda_1-2} \\
=\ & (D_{\lambda_1} \pm 1) + (n - \lambda_1 - (r-1))(D_{\lambda_1-1} \pm 1) + (n - \lambda_1 - (r-1))D_{\lambda_1-2} \\
&+ (r-1)\big(D_{\lambda_1-1} + (n - r - (\lambda_1-1))D_{\lambda_1-2}\big) - D_{\lambda_1-2} \\
=\ & D_{\lambda_1} + (n-\lambda_1)D_{\lambda_1-1} \pm 1 \pm (n - \lambda_1 - (r-1)) \\
&+ r(n - \lambda_1 - (r-1))D_{\lambda_1-2} - D_{\lambda_1-2}
\end{aligned}
$$

It remains to show that $\pm 1 \pm (n-h) + r(n-h)D_{\lambda_1-2} - D_{\lambda_1-2}$ is positive. Notice

$$
\begin{aligned}
\pm 1 \pm (n-h) + r(n-h)D_{\lambda_1-2} - D_{\lambda_1-2} &\geq -1 - (n-h) + r(n-h)D_{\lambda_1-2} - D_{\lambda_1-2} \\
&= -1 - n + h + \big(r(n-h) - 1\big)D_{\lambda_1-2}.
\end{aligned}
$$

Note that $r \geq 2$, $\lambda_1 \geq 4$ and $n - h \geq 1$. So $-1-n+h+\big(r(n-h)-1\big)D_{\lambda_1-2} \geq -(n-h)-1+D_{\lambda_1-2} \geq -(\lambda_1-1) - 1 + D_{\lambda_1-2}$ which is positive for $\lambda_1 = 6$ and larger.

Hence, the result follows. ■

5 An upper bound for $|\eta_\lambda|$

In this section, we give the upper bound needed for Theorem 1.3 (i) for $|\eta_\lambda|$ except for some cases when $\lambda_1 = \frac{n}{2}$ for n even or $\lambda_1 = \frac{n-1}{2}$ for n odd.

Proposition 5.1 *Let $\lambda \vdash n$, with $\lambda_1 \geq \frac{n-1}{2}$, $n \geq 2$, and suppose that $\lambda \neq (\frac{n}{2}, \frac{n}{2})$ when n is even or $\lambda \neq (\frac{n-1}{2}, c, d)$ for any $c \leq \frac{n-1}{2}$, $d \geq 1$ when n is odd. Then*

$$|\eta_\lambda| \leq (n - \lambda_1 + 1)D_{\lambda_1} + D_{\lambda_1 - 1}. \tag{9}$$

Proof. It is readily checked that the theorem holds for $n = 2, \ldots, 13$. We shall let $n \geq 14$ so that $\lambda_1 \geq 7$. For $r = 1$, $|\eta_\lambda| = D_n \leq D_n + D_{n-1}$, and so the theorem is true. Let $r \geq 2$.

If λ is a hook then $|\eta_\lambda| = D_{\lambda_1} + (n - \lambda_1)D_{\lambda_1 - 1}$ (Lemma 2.2), which is clearly less than the right hand side of (9). So we may assume that λ is not a hook.

If $\lambda_1 > \frac{n}{2}$, then $|\lambda - \widehat{h}| \leq \lambda_1 - 2$. If $\lambda_1 = \frac{n}{2}$ (when n is even), we still have $|\lambda - \widehat{h}| \leq \lambda_1 - 2$ since we are excluding the case $\lambda = (\frac{n}{2}, \frac{n}{2})$ so that $r \geq 3$. If $\lambda_1 = \frac{n-1}{2}$ (when n is odd), we do have $|\lambda - \widehat{h}| \leq \lambda_1 - 2$ since we are excluding the shapes $(\frac{n-1}{2}, c, d)$ with 3 rows so that $r \geq 4$. Therefore, in all cases, $|\eta_{\lambda - \widehat{h}}| \leq D_{\lambda_1 - 2}$.

To validate the inductive hypothesis for $\eta_{\lambda - \widehat{c}}$, $\lambda - \widehat{c} \vdash n - r$, we need to check that $\lambda_1 - 1 \geq \frac{n-r-1}{2}$, $n - r \geq 2$ and that $\lambda - \widehat{c} \neq (\frac{n-r}{2}, \frac{n-r}{2})$ when $n - r$ is even or $\lambda - \widehat{c} \neq (\frac{n-r-1}{2}, c', d')$ for any $c' \leq \frac{n-r-1}{2}$, $d' \geq 1$ when $n - r$ is odd. Indeed, $\lambda_1 - 1 \geq \frac{n-1}{2} - 1 = \frac{n-3}{2} \geq \frac{n-r-1}{2}$ as $r \geq 2$. Also, $n \geq \lambda_1 + r - 1$ and since $\lambda_1 \geq 7$, $n - r \geq 2$. Suppose $\lambda - \widehat{c} = \left(\frac{n-r}{2}, \frac{n-r}{2}\right)$. Then $\frac{n-r}{2} = \lambda_1 - 1 \geq \frac{n-3}{2}$ so that r is 2 or 3. If $r = 2$, the hypothesis gives $\lambda = (a, b)$ with $a > b$ as $(\frac{n}{2}, \frac{n}{2})$ is excluded so $\lambda - \widehat{c} \neq (\frac{n-2}{2}, \frac{n-2}{2})$. This means $r = 3$ and $\lambda = (\frac{n-1}{2}, \frac{n-1}{2}, 1)$ which is excluded by the hypothesis. If $\lambda - \widehat{c} = (\frac{n-r-1}{2}, c', d')$ for some $c' \leq \frac{n-r-1}{2}$ and $d' \geq 1$, then $\frac{n-r-1}{2} = \lambda_1 - 1 \geq \frac{n-3}{2}$ which implies that $r \leq 2$, which is not possible because $\lambda - \widehat{c}$ has 3 rows. Therefore, by induction,

$$|\eta_{\lambda - \widehat{c}}| \leq ((n - r) - (\lambda_1 - 1) + 1)D_{\lambda_1 - 1} + D_{\lambda_1 - 2}.$$

Consequently,

$$\begin{aligned}
|\eta_\lambda| &\leq (\lambda_1 + r - 1)|\eta_{\lambda - \widehat{c}}| + |\eta_{\lambda - \widehat{h}}| \\
&\leq (\lambda_1 + r - 1)|\eta_{\lambda - \widehat{c}}| + D_{\lambda_1 - 2} \\
&\leq (\lambda_1 + r - 1)\left((n - r) - (\lambda_1 - 1) + 1)D_{\lambda_1 - 1} + D_{\lambda_1 - 2}\right) + D_{\lambda_1 - 2} \\
&= (\lambda_1 + r - 1)\left((n - \lambda_1 + 1) - (r - 1)\right)D_{\lambda_1 - 1} + (\lambda_1 + r)D_{\lambda_1 - 2} \\
&= \lambda_1(n - \lambda_1 + 1)D_{\lambda_1 - 1} + (r - 1)(n - 2\lambda_1 - r + 2)D_{\lambda_1 - 1} + (\lambda_1 + r)D_{\lambda_1 - 2} \\
&= (n - \lambda_1 + 1)(D_{\lambda_1} \pm 1) + (r - 1)(n - 2\lambda_1 - r + 2)D_{\lambda_1 - 1} + (\lambda_1 - 1)D_{\lambda_1 - 2} + (r + 1)D_{\lambda_1 - 2} \\
&= (n - \lambda_1 + 1)D_{\lambda_1} \pm (n - \lambda_1 + 1) + (r - 1)(n - 2\lambda_1 - r + 2)D_{\lambda_1 - 1} + D_{\lambda_1 - 1} \pm 1 + (r + 1)D_{\lambda_1 - 2}.
\end{aligned}$$

It sufices to show that

$$(n - \lambda_1 + 2) + (r - 1)(n - 2\lambda_1 - r + 2)D_{\lambda_1 - 1} + (r + 1)D_{\lambda_1 - 2} \leq 0. \tag{10}$$

The hypothesis on λ_1 gives $n - 2\lambda_1 \leq 1$. Suppose $n - 2\lambda_1 = 1$ which because of the partitions excluded implies $r \geq 4$. Then (10) becomes

$$(\lambda_1 + 3) - (r - 1)(r - 3)D_{\lambda_1 - 1} + (r + 1)D_{\lambda_1 - 2} \leq 0$$

By using the estimation $(\lambda_1 - 1)D_{\lambda_1 - 2} - 1 \leq D_{\lambda_1 - 1}$, it suffices to show that

$$(\lambda_1 + 3) + (r + 1)D_{\lambda_1 - 2} \leq (r - 1)(r - 3)\left((\lambda_1 - 1)D_{\lambda_1 - 2} - 1\right),$$

that is

$$\lambda_1 + 3 + (r-1)(r-3) \leq ((r-1)(r-3)(\lambda_1 - 1) - (r+1))\, D_{\lambda_1 - 2}. \tag{11}$$

Since $\lambda_1 \geq 7$, $D_{\lambda_1-2} \geq D_5 = 44$. Therefore, it is enough to show that

$$\lambda_1 + 3 + (r-1)(r-3) \leq 44\,((r-1)(r-3)(\lambda_1 - 1) - (r+1))\,.$$

For some of the computations below we used the computer algebra package Maple. Now we have

$$45r^2 - 136r + 182 \leq (44r^2 - 176r + 131)\lambda_1.$$

As λ is not a hook, we have $r \leq \lambda_1$, and so we are done if

$$45r^2 - 136r + 182 \leq (44r^2 - 176r + 131)r,$$

$$0 \leq 44r^3 - 221r^2 + 267r - 182,$$

which indeed holds for all $r \geq 4$ again using Maple as for example $44r^3 - 221r^2$ is positive for $r \geq 6$ and smaller values can be checked.

Suppose first that $r = 2$. As we are excluding the shape $\lambda = (\frac{n}{2}, \frac{n}{2})$, we must have $\lambda = (a, b)$ with $a > b$ and so $n - 2\lambda_1 \leq -1$. From (10) it is enough to show that

$$\begin{aligned} (n - \lambda_1 + 2) - D_{\lambda_1-1} + 3D_{\lambda_1-2} &\leq 0 \quad \text{or} \\ (n - \lambda_1 + 2) + 3D_{\lambda_1-2} &\leq D_{\lambda_1-1} = (\lambda_1 - 1)D_{\lambda_1-2} \pm 1, \end{aligned}$$

which is true if

$$(n - \lambda_1 + 2) \leq (\lambda_1 - 4)D_{\lambda_1-2} - 1.$$

As $n - \lambda_1 + 2 \leq \lambda_1 + 1$, it suffices to check that

$$1 + \frac{6}{\lambda_1 - 4} = \frac{\lambda_1 + 2}{\lambda_1 - 4} \leq D_{\lambda_1-2},$$

which clearly holds for all $\lambda_1 \geq 7$. This shows we may assume $r \geq 3$.

From now on, we may assume that $n - 2\lambda_1 \leq 0$. Again, from (10), it is enough to show that

$$(n - \lambda_1 + 2) + (r-1)(-r+2)(\lambda_1 - 2)(D_{\lambda_1-2} + D_{\lambda_1-3}) + (r+1)D_{\lambda_1-2} \leq 0 \tag{12}$$

where we used $D_{\lambda_1-1} = (\lambda_1 - 2)(D_{\lambda_1-2} + D_{\lambda_1-3})$

As $n - 2\lambda_1 \leq 0$ and $r \geq 3$, rearranging (12) we need to show that

$$(n - \lambda_1 + 2) + ((r+1) - (r-1)(r-2)(\lambda_1 - 2))\, D_{\lambda_1-2} \leq (r-1)(r-2)(\lambda_1 - 2)D_{\lambda_1-3}.$$

Since $\lambda_1 - 2 \geq \frac{n}{2} - 2$, the coefficient of D_{λ_1-2} is less than or equal to

$$(r+1) - (r-1)(r-2)(\frac{n-4}{2})$$

which is nonpositive for $r \geq 3$, $n > 10$. It now suffices to show that

$$(n - \lambda_1 + 2) \leq (r-1)(r-2)(\lambda_1 - 2)D_{\lambda_1-3},$$

which is true for $\lambda_1 \geq 7$, $r \geq 3$. ■

Proposition 5.2 *Suppose $\lambda \vdash n$ with $\lambda_1 \geq \frac{n-1}{2}$, $n \geq 3$, and $\lambda \neq (\frac{n}{2}, \frac{n}{2})$ when n is even or $\lambda \neq (\frac{n-1}{2}, c, d)$ for any $c \leq \frac{n-1}{2}$, $d \geq 1$ when n is odd. Then $|\eta_\lambda| < |\eta_{(\lambda_1+1, 1^{n-\lambda_1-1})}| = D_{\lambda_1+1} + (n - \lambda_1 - 1)D_{\lambda_1}$. In particular the upper bound in Theorem 1.3 (i) holds except for the partitions excluded in the statement.*

Proof. In view of Proposition 5.1, we only need to show that

$$(n - \lambda_1 + 1)D_{\lambda_1} + D_{\lambda_1 - 1} < D_{\lambda_1+1} + (n - \lambda_1 - 1)D_{\lambda_1}.$$

Subtracting $(n - \lambda_1 - 1)D_{\lambda_1}$ leaves

$$\begin{aligned} 2D_{\lambda_1} + D_{\lambda_1-1} &< D_{\lambda_1+1} \\ &= (\lambda_1 + 1)D_{\lambda_1} \pm 1. \end{aligned}$$

Next, subtracting $2D_{\lambda_1}$ yields

$$D_{\lambda_1-1} < (\lambda_1 - 1)D_{\lambda_1} \pm 1$$

It is sufficient to show that

$$\begin{aligned} D_{\lambda_1-1} &< (\lambda_1 - 1)D_{\lambda_1} - 1 \\ &= (\lambda_1 - 1)^2(D_{\lambda_1-1} + D_{\lambda_1-2}) - 1 \end{aligned}$$

This is true for $\lambda_1 \geq 3$ (when $n \geq 7$) while the result can be verified separately for small n. ■

6 Partitions with few parts

In this section we give the upper bounds needed for Theorem 1.3 (i) not covered by Proposition 5.2.

6.1 The cases (a, b), (b, b) and $(b, b, 1)$

Lemma 6.1 *For $a \geq b > 1$, the following formulae hold:*

$$\begin{aligned} \eta_{(a,b)} &= (-1)^{a+1}D_{b-1} - (a+1)\eta_{(a-1,b-1)} \\ \eta_{(a,b-1,1)} &= (-1)^{a+2}D_{b-2} + (a+2)n_{(a-1,b-2)} \end{aligned}$$

Proof. This is just an application of the main recurrence. Notice $(-1)^{a+1}(-1)^a = -1$ as one of a and $a+1$ is even with the other being odd. Similarly $(-1)^{a+2}(-1)^a = 1$ as they are either both even or both odd. ■.

We give an explicit formula for $\eta_{(a,b)}$ which is not specifically needed in the remainder of the paper but gives some indication of how the values could be computed.

Lemma 6.2 *Suppose $(a, b) \vdash n$ with $b > 0$. Then*

$$\begin{aligned} \eta_{(a,b)} = {} & (-1)^{a+1}\big(D_{b-1} + (a+1)D_{b-2} + (a+1)aD_{b-3} + \cdots + (a+1)a(a-1)\ldots(a-b+3)D_0\big) \\ & + (-1)^b(a+1)a(a-1)\ldots(a-b+3)(a-b+2)D_{a-b}. \end{aligned}$$

As this in not needed in the sequel, we omit the proof which is a straightforward iteration of Lemma 6.1

Next, we find the upper bounds needed for Theorem 1.3 (i) for $|\eta_{(b,b)}|$ and $|\eta_{(b,b,1)}|$ respectively.

Lemma 6.3 *For all b, $|\eta_{(b,b)}| \leq D_{b+1} + (b-1)D_b$. If $b \geq 4$, then $|\eta_{(b,b)}| \leq D_{b+1} + (b-3)D_b$ with equality only for $b = 4$.*

Proof. The fist inequality is implied by the second for $b \geq 4$ and the small cases can be done by inspection.

For the second inequality we use induction on b and note it is equality for $b = 4$. Suppose $b > 4$.

$$\begin{aligned}
|\eta_{(b,b)}| &= h|\eta_{(b-1,b-1)}| - |\eta_{(b-1)}| \qquad \text{(by Corollary 3.8)}\\
&= (b+1)|\eta_{(b-1,b-1)}| - D_{b-1}\\
&\leq (b+1)\big(D_b + (b-4)D_{b-1}\big) - D_{b-1}\\
&= D_{b+1} \pm 1 + (b-4)bD_{b-1} + (b-4)D_{b-1} - D_{b-1}\\
&= D_{b+1} \pm 1 + (b-4)D_b \pm (b-4) + (b-5)D_{b-1}
\end{aligned}$$

Subtracting $D_{b+1} + (b-3)D_b$ we need

$$\pm 1 - D_b \pm (b-4) + (b-5)D_{b-1} \leq 0.$$

This is

$$\pm 1 \pm (b-4) + (b-5)D_{b-1} \leq D_b = bD_{b-1} \pm 1$$

which is

$$\pm 1 \pm (b-4) \leq 5D_{b-1} \pm 1.$$

Taking the worst case of signs $+$ on the left and $-$ on the right this is true if $b-2 \leq 5D_{b-1}$ which is true for all $b \geq 5$. Notice this is a strict inequality for $b \geq 3$. We have already noted the lemma holds for $b = 4$. However, the induction step does not apply with $b = 4$. ■

We now consider the case $\eta_{(b,b,1)}$.

Lemma 6.4 *Suppose $b > 1$. Then $|\eta_{(b,b,1)}| \leq D_{b+1} + bD_b$ with equality only for $b = 2$.*

Proof. This can be checked by hand for $b \leq 4$ and so we assume $b \geq 5$.

$$\begin{aligned}
|\eta_{(b,b,1)}| &= (b+2)|\eta_{(b-1,b-1)}| - D_{(b-1)} \qquad \text{(by Corollary 3.8)}\\
&\leq (b+1+1)\big(D_b + (b-4)D_{b-1}\big) - D_{b-1} \qquad \text{(by Lemma 6.3)}\\
&= D_{b+1} \pm 1 + D_b + (b+2)(b-4)D_{b-1} - D_{b-1}\\
&= D_{b+1} \pm 1 + D_b + (b-2)bD_{b-1} - 9D_{b-1}\\
&= D_{b+1} \pm 1 + D_b + (b-2)D_b \pm (b-2) - 9D_{b-1}\\
&= D_{b+1} + (b-1)D_b \pm 1 - 9D_{b-1} \pm (b-2)
\end{aligned}$$

Subtracting $D_{b+1} + bD_b$ gives $-D_b \pm 1 - 9D_{b-1} \pm (b-2)$, which we must show is less that 0. Taking the worst case with both signs $+$ we need $-9D_{b-1} + b - 1 < D_b$. This is true if $b - 1 < D_b$ which is certainly true for $b \geq 5$. ■

6.2 The case $\lambda = (b, c, d)$ with $b = \frac{n-1}{2}$ and $d \geq 2$

Lemma 6.5 *Let $n = 2b+1 \geq 3$. If $\lambda = (b, c, d) \vdash n$ with $b = \frac{n-1}{2}$, $c < b$, $d \geq 2$. Then*

$$|\eta_\lambda| \leq D_{b+1} + bD_b.$$

Proof. The smallest case is $n = 7$ where it holds. So let $n \geq 9$ so that $b \geq 4$. Note that $|\lambda - \widehat{c}| = n - 3 \geq 3$, $b - 1 = \frac{n-3}{2}$. Moreover, $\lambda - \widehat{c} \neq (\frac{n-3}{2}, \frac{n-3}{2})$ since $c < b$ and also $\lambda - \widehat{c} \neq (\frac{n-4}{2}, c', d')$ since $n - 3$ is even. By Proposition 5.1,

$$|\eta_{\lambda-\widehat{c}}| \leq bD_{b-1} + D_{b-2}.$$

On the other hand, since $|\lambda - \widehat{h}| = n - (b+2) = b - 1$, we have

$$|\eta_{\lambda-\widehat{h}}| \leq D_{b-1}.$$

Therefore,

$$\begin{aligned}
|\eta_\lambda| &\leq h|\eta_{\lambda-\widehat{c}}| + |\eta_{\lambda-\widehat{h}}| \\
&\leq (b+2)(bD_{b-1} + D_{b-2}) + D_{b-1} \\
&= ((b+2)b+1)D_{b-1} + (b+2)D_{b-2} \\
&= (b+1)^2 D_{b-1} + (b-1)D_{b-2} + 3D_{b-2} \\
&= (b+1)bD_{b-1} + (b+1)D_{b-1} + D_{b-1} \pm 1 + 3D_{b-2} \\
&\leq (b+1)(D_b + 1) + bD_{b-1} + 2D_{b-1} + 3D_{b-2} + 1 \\
&\leq D_{b+1} + 1 + (b+1) + D_b + 1 + 2D_{b-1} + 3D_{b-2} + 1 \\
&= D_{b+1} + D_b + 2D_{b-1} + 3D_{b-2} + (b+4) \\
&\leq D_{b+1} + bD_b,
\end{aligned}$$

the last inequality holds since for all $b \geq 4$,

$$(b-1)D_b = (b-1)^2 D_{b-1} + (b-1)^2 D_{b-2} \geq 9D_{b-1} + 9D_{b-2} \geq 2D_{b-1} + 3D_{b-2} + (b+4).$$

■

Note the exceptions to Proposition 5.2 with $\lambda_1 = \frac{n-1}{2}$ other than $\eta_{(b,b,1)}$ have $d \geq 2$.

Corollary 6.6 *Theorem 1.3 (i) has been proved.*

Proof. This follows from Proposition 5.2 and Lemmas 6.3, 6.4, and 6.5. ■

7 A second upper bound for $|\eta_\lambda|$

In this section we provide another upper bound for η_λ when λ_1 is small compared to n. This is the upper bound needed for Theorem 1.3 (ii). In particular we prove the following theorem.

Proposition 7.1 *Let $b = \lfloor \frac{n}{2} \rfloor$ and suppose $\lambda \vdash n$. If $\lambda_1 < b$, then $|\eta_\lambda| \leq D_{b+1} + (n-b-1)D_b$ with strict inequality for $n \geq 6$.*

Remark 7.1 One implication of the proposition and Corollay 6.6 is that $|\eta_\lambda| \le D_{c+1} + (n-c-1)D_c$ for any c greater than or equal the maximum of $b = \lfloor \frac{n}{2} \rfloor$ and λ_1 by the use of Lemma 2.4.

Proof. Assume $\lambda_1 < b$ and in particular $\lambda_1 \le b-1$.

We will use induction on $|\lambda|$. As $\lambda_1 < \frac{n}{2}$ there must be at least three rows and we can assume $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ with $r \ge 3$.

If λ were a hook the result would follow from Lemma 2.4.

We know from the main recurrence that

$$|\eta_\lambda| \le h|\eta_{\lambda-\widehat{c}}| + |\eta_{\lambda-\widehat{h}}|.$$

Notice $\lambda_1 - 1 \le b-2$ and $\lambda - \widehat{c} = (\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_s - 1) \vdash n-r$ with $s \le r$. Also $\lambda_2 - 1 \le b-2$ as $\lambda_2 \le \lambda_1$ and $\lambda - \widehat{h} = (\lambda_2 - 1, \cdots, \lambda_s - 1) \vdash n-h$.

We will apply the induction assumption to $\lambda - \widehat{c}$ and $\lambda - \widehat{h}$. Notice $\lfloor \frac{n-h}{2} \rfloor \le b-2$ as $h \ge 4$. In particular $|\eta_{\lambda-\widehat{h}}| \le D_{b-1} + (n-h-b+1)D_{b-2}$ by Remark 7.1. Notice $n-h-b+1 \le b-1$ as $n-h \le 2b-2$ as $\lfloor \frac{n-h}{2} \rfloor \le b-2$. This means $|\eta_{\lambda-\widehat{h}}| \le D_{b-1} + (b-1)D_{b-2} \le D_{b-1} + D_{b-1} + 1 = 2D_{b-1} + 1$.

We now exclude the case in which n is odd and $r = 3$ which we will do later. With this assumption, $\lfloor \frac{n-r}{2} \rfloor \le b-2$. Now by Remark 7.1 we see $|\eta_{\lambda-\widehat{c}}| \le D_{b-1} + (n-r-b+1)D_{b-2}$.

For these cases we have

$$\begin{aligned} |\eta_\lambda| &\le h|\eta_{\lambda-\widehat{c}}| + |\eta_{\lambda-\widehat{h}}| \\ &\le h\big(D_{b-1} + (n-r-b+1)D_{b-2}\big) + |\eta_{\lambda-\widehat{h}}|. \end{aligned}$$

Notice $b = \lfloor \frac{n}{2} \rfloor$ and so $2b \ge n-1$. Also notice that $h \le n-1$ as λ is not a hook and in particular $hD_{b-1} \le 2bD_{b-1} \le 2D_b + 2$. This means $hD_{b-1} \le 2D_b + 2$.

Also $n-r-b+1 \le n-b-2 \le b-1$ as $b = \lfloor \frac{n}{2} \rfloor$. In particular $(n-r-b+1)D_{b-2} \le (b-1)D_{b-2} \le D_{b-1} + 1$ and so $h(n-r-b+1)D_{b-2} \le 2b(D_{b-1} + 1) \le 2D_b + 2 + 2b$.

In particular,

$$|\eta_\lambda| \le 4D_b + 2D_{b-1} + 2b + 5.$$

But $4D_b + 2D_{b-1} + 2b + 5 < D_{b+1} + (n-b-1)D_b$ using $D_{b+1} \ge (b+1)D_b - 1$ as long as $b \ge 5$ and the smaller cases can be done by hand.

We have one more case in which n is odd, $r = 3$ and $b' = \lfloor \frac{n-3}{2} \rfloor = b-1$. Here $h = \lambda_1 + 2 \le b+1$. By the induction assumption and Remark 7.1, $|\eta_{\lambda-\widehat{c}}| \le D_b + (b-2)D_{b-1}$ as $n-r-b = n-3-\frac{n-1}{2} = b-2$ and $\lambda_1 - 1 < b'$. As shown above $|\eta_{\lambda-\widehat{h}}| \le 2D_{b-1} + 1$.

$$\begin{aligned} |\eta_\lambda| &\le (\lambda_1 + 2)|\eta_{\lambda-\widehat{c}}| + |\eta_{\lambda-\widehat{h}}| \\ &\le (b+1)\big(D_b + (b-2)D_{b-1}\big) + 2D_{b-1} + 1 \\ &= D_{b+1} \pm 1 + (b^2 - b - 2)D_{b-1} + 2D_{b-1} + 1 \\ &\le D_{b+1} + b^2 D_{b-1} - bD_{b-1} - 2D_{b-1} + 2D_{b-1} + 2 \\ &= D_{b+1} + bD_b \pm b - D_b \pm 1 + 2. \\ &\le D_{b+1} + (b-1)D_b + b + 3. \end{aligned}$$

But this is less than $D_{b+1} + (n-b-1)D_b$ as long as $b \geq 4$ and as usual the smaller cases follow by hand. The needed condition is $b + 3 < D_b$ as here $n - b - 1 = b$. ■

Corollary 7.2 *Theorem 1.3 has now been proven.*

Proof. This follows from Corollary 6.6 and Proposition 7.1. ■

8 The case $\lambda_1 = n - 2$

In this section we prove the part of Theorem 1.4 when $\lambda_1 = n - 2$. Before we begin, we need some preliminary calculations.

Lemma 8.1 *Let $n \geq 4$.*

(1) $\eta_{(n-2,2)} = \frac{(n-1)}{(n-3)} D_{n-2}$.

(2) $\eta_{(n-2,1^2)} = nD_{n-3} + (-1)^n$.

Proof. The second is a direct application of Lemma 2.2. The first follows from Lemma 6.1 and Lemma 2.2. ■

Lemma 8.2 *Let $n \geq 6$. Then $|\eta_{(n-2,2)}| > |\eta_{(n-2,1^2)}|$.*

Proof. We must show that $\frac{n-1}{n-3} D_{n-2} = \frac{n-1}{n-3}((n-3)(D_{n-3} + D_{n-4})) > nD_{n-3} + (-1)^n$, i.e. $(n-1)(D_{n-3} + D_{n-4}) > nD_{n-3} + (-1)^n$. Subtracting $(n-1)D_{n-3}$ from both sides, we need to show that $(n-1)D_{n-4} > D_{n-3} + (-1)^n$. Since $D_{n-3} = (n-3)D_{n-4} + (-1)^{n-3}$, it suffices to show that $(n-1)D_{n-4} > (n-3)D_{n-4}$. But this is true as $2D_{n-4} > 0$ for all $n \geq 6$. ■

Corollary 8.3 *Theorem 1.4 has been proven when $\lambda_1 = n - 2$.*

Proof. This follows by Lemma 8.2. ■

9 The case $\lambda_1 = n - 3$

In this section we prove Theorem 1.4 in the case $\lambda_1 = n - 3$.

Lemma 9.1 *Let $n \geq 6$.*

(1) $\eta_{(n-3,3)} = (-1)^{n-2} - \frac{(n-2)(n-3)}{(n-5)} D_{n-4}$.

(2) $\eta_{(n-3,2,1)} = -\frac{n-1}{n-4} D_{n-3} = -(n-1)(D_{n-4} + D_{n-5})$.

(3) $\eta_{(n-3,1^3)} = (-1)^n (1 + (-1)^{n-3} nD_{n-4}) = -nD_{n-4} + (-1)^n$.

Proof. The third is Lemma 2.2. The second follows from Lemma 2.3 and Lemma 2.1. For the first use the main recurrence and substitute the value obtained in Lemma 8.1. ■

Lemma 9.2 *Let $n \geq 6$. Then $|\eta_{(n-3,3)}| > |\eta_{(n-3,2,1)}|$.*

Proof. We assume $n \geq 9$ and check the smaller values by hand. We shall prove that $\frac{n-1}{n-4}D_{n-3} < \frac{(n-2)(n-3)}{n-5}D_{n-4} - 1$. As $D_{n-3} - 1 \leq (n-3)D_{n-4}$, it is enough to show that $\frac{n-1}{n-4}D_{n-3} < \frac{n-2}{n-5}D_{n-3} - \frac{n-2}{n-5} - 1$, which is $\frac{n-2}{n-5} + 1 < \left(\frac{n-2}{n-5} - \frac{n-1}{n-4}\right) D_{n-3} = \frac{3}{(n-4)(n-5)}D_{n-3}$. As $\frac{D_{n-3}}{n-4} = D_{n-4} + D_{n-5}$, it suffices to prove that $\frac{2n-7}{3} < n-5 < D_{n-4} + D_{n-5}$. This is true since $D_{n-4} = (n-5)(D_{n-5} + D_{n-6}) > n-5$. Recall $n \geq 9$ here. ■

Lemma 9.3 *Let $n \geq 6$. Then $|\eta_{(n-3,2,1)}| \geq |\eta_{(n-3,1^3)}|$ with equality only for $n = 6$.*

Proof. Check this is equality if $n = 6$ and so assume $n \geq 7$.

We want to show that $nD_{n-4} + 1 < \frac{n-1}{n-4}D_{n-3}$.

$$\begin{aligned} nD_{n-4} + 1 &= (n-3)D_{n-4} + 3D_{n-4} + 1 \\ &\leq D_{n-3} + 3D_{n-4} + 2 \\ &< \frac{n-1}{n-4}D_{n-3}. \end{aligned}$$

This will hold provided the following equivalent inequalities hold.

$$\begin{aligned} (n-4)D_{n-3} + 2(n-4) + 3(n-4)D_{n-4} &< (n-1)D_{n-3} \\ 2(n-4) + 3(n-4)D_{n-4} &< 3D_{n-3} \\ (n-4)(D_{n-4} + 2/3) &< D_{n-3}, \end{aligned}$$

which is true since $D_{n-3} = (n-4)(D_{n-4} + D_{n-5}) \geq (n-4)(D_{n-4} + 1) > (n-4)(D_{n-4} + 2/3)$ for $n \geq 7$. ■

Corollary 9.4 *Theorem 1.4 is proven in the case $\lambda_1 = n-3$.*

Proof. This follow from Lemmas 9.2 and 9.3. ■

10 The case $\lambda_1 = n - 4$

In this section we prove Theorem 1.4 in the case $\lambda_1 = n-4$. Notice $n \geq 8$ here.

This will be proved by a series of lemmas.

Lemma 10.1 *For $a \geq 4$ the following values hold.*

$$\begin{aligned}
\eta_{(a,4)} &= 2(-1)^{a+1} - (a+1)\eta_{(a-1,3)} \\
\eta_{(a,3,1)} &= (-1)^{a+2} + (a+2)\big(D_{a-1} + 2(D_{a-2} + D_{a-3})\big) \\
\eta_{(a,2^2)} &= (-1)^{a+1}(a+3) + (a+2)(a+1)D_{a-2} \\
\eta_{(a,2,1^2)} &= (a+3)\big((-1)^{a+3} + aD_{a-2}\big) \\
\eta_{(a,1^4)} &= (-1)^a + (a+4)D_{a-1} = D_a + 4D_{a-1}.
\end{aligned}$$

Proof. These are straightforward applications of the main recurrence, Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 8.1 . ■

Lemma 10.2 *If $a \geq 4$, then $\eta_{(a,4)} > \eta_{(a,3,1)}$.*

Proof. By Lemma 9.1 and Lemma 10.1,

$$\begin{aligned}
\eta_{(a,4)} &= 2(-1)^{a+1} - (a+1)\eta_{(a-1,3)} \\
&= 2(-1)^{a+1} - (a+1)\left((-1)^a - \frac{a(a-1)}{(a-3)}D_{a-2}\right) \\
&\geq \frac{(a+1)a(a-1)}{(a-3)}D_{a-2} - (a+3) \\
&\geq \frac{(a+1)a}{a-3}\left(D_{a-1} - 1\right) - (a+3). \\
&= \frac{(a+1)a}{a-3}D_{a-1} - \left(\frac{(a+1)a}{a-3} + a + 3\right) \\
&= \frac{(a+1)a}{a-3}D_{a-1} - \left(2a + 7 + \frac{12}{a-3}\right). \qquad (13)
\end{aligned}$$

On the other hand, by Lemma 10.1,

$$\begin{aligned}
\eta_{(a,3,1)} &= (-1)^{a+2} + (a+2)\big(D_{a-1} + 2(D_{a-2} + D_{a-3})\big) \\
&\leq 1 + (a+2)\left(D_{a-1} + \frac{2}{a-2}D_{a-1}\right) \\
&= 1 + (a+2)\left(\frac{a}{a-2}D_{a-1}\right). \qquad (14)
\end{aligned}$$

Therefore, it is enough to show that the right hand side of (13) is more than the right hand side of (14), i.e.

$$\begin{aligned}
a\left(\frac{(a+1)(a-2)-(a+2)(a-3)}{(a-2)(a-3)}\right)D_{a-1} &> 2a + 8 + \frac{12}{a-3} \\
\frac{4aD_{a-1}}{(a-2)(a-3)} &> 2a + 8 + \frac{12}{a-3} \\
\frac{4a}{a-3}(D_{a-2} + D_{a-3}) &> 2a + 8 + \frac{12}{a-3} \\
4a\left(D_{a-3} + D_{a-4} + \frac{D_{a-3}}{a-3}\right) &> 2a + 8 + \frac{12}{a-3} \\
D_{a-3} + D_{a-4} + \frac{D_{a-3}}{a-3} &> \frac{1}{2} + \frac{2}{a} + \frac{3}{(a-3)a} \qquad (15)
\end{aligned}$$

Note that the right hand side of (15) is less than 2 for $a \geq 4$. Therefore, the inequality holds whenever $a \geq 6$. The lemma also holds for $a = 4, 5$ by inspection. ■

Lemma 10.3 *If $a \geq 4$, then $\eta_{(a,3,1)} > \eta_{(a,2^2)}$.*

Proof.

$$\begin{aligned}
\eta_{(a,3,1)} - \eta_{(a,2^2)} &= (-1)^{a+2} - (-1)^{a+1}(a+3) + (a+2)\big(D_{a-1} + 2(D_{a-2} + D_{a-3}) - (a+1)D_{a-2}\big) \\
&= (-1)^{a+2}(a+4) + (a+2)\big(D_{a-1} + 2(D_{a-2} + D_{a-3})\big) - (a-1)D_{a-2} - 2D_{a-2}\big) \\
&= (-1)^{a+2}(a+4) + (a+2)\big((a-1)D_{a-2} + (-1)^{a-1} + 2(D_{a-2} + D_{a-3}) \\
&\qquad -(a-1)D_{a-2} - 2D_{a-2}\big) \\
&= (-1)^{a+2}(a+4) + (a+2)\big((-1)^{a-1} + 2D_{a-3}\big) \\
&= (-1)^{a+2}(2) + (a+2)(2)D_{a-3}
\end{aligned}$$

This is positive for $a = 4$ and for $a \geq 5$ it is positive as $D_{a-3} \geq 1$. ■

Lemma 10.4 *If $a \geq 4$, then $\eta_{(a,2,2)} > \eta_{(a,2,1^2)}$.*

Proof.

$$\begin{aligned}
\eta_{(a,2,2)} - \eta_{(a,2,1^2)} &= (-1)^{a+1}(a+3) + (a+2)(a+1)D_{a-2} - (a+3)\big((-1)^{a+3} + aD_{a-2}\big) \\
&= \big((a+2)(a+1) - (a+3)a\big)D_{a-2} \\
&= (a^2 + 3a + 2 - a^2 - 3a)D_{a-2} \\
&= 2D_{a-2}.
\end{aligned}$$

This is positive as $D_{a-2} \geq 1$. ■

Lemma 10.5 *If $a \geq 4$, then $\eta_{(a,2,1^2)} > \eta_{(a,1^3)}$.*

Proof.

$$\begin{aligned}
\eta_{(a,2,1^2)} - \eta_{(a,1^4)} &= (a+3)\big((-1)^{a+3} + aD_{a-2}\big) - \big((-1)^a + (a+4)D_{a-1}\big) \\
&= (a+4)(-1)^{a+3} + (a+3)aD_{a-2} - (a+4)\big((a-1)D_{a-2} + (-1)^{a-1}\big) \\
&= (a+4)\big((-1)^{a+3} - (-1)^{a-1}\big) + \big((a+3)a - (a+4)(a-1)\big)D_{a-2} \\
&= \big(a^2 + 3a - (a^2 + 3a - 4)\big)D_{a-2} \\
&= 4D_{a-2}.
\end{aligned}$$

This is positive as $D_{a-2} \geq 1$. ■

Corollary 10.6 *Theorem 1.4 has been proven for the case $\lambda_1 = n - 4$.*

Proof. This is Lemmas 10.2, 10.3, 10.4, and 10.5. ■

This completes the proof of Theorem 1.4 using Theorem 1.5, Corollary 8.3, Corollary 9.4 and Corollary 10.6.

11 Some Values of η_λ

In this section we tabulate values of η_λ for some small values of n.

$n = 2$

λ	η_λ
2	1
1^2	-1

$n = 3$

λ	η_λ
3	2
$2,1$	-1
1^3	2

$n = 4$

λ	η_λ		λ	η_λ
4	9		$2,1^2$	1
$3,1$	-3		1^4	-3
$2,2$	3			

$n = 5$

λ	η_λ		λ	η_λ
5	44		$2^2,1$	-4
$4,1$	-11		$2,1^3$	-1
$3,2$	4		1^5	4
$3,1^2$	4			

$n = 6$

λ	η_λ		λ	η_λ
6	265		$3,1^3$	-5
$5,1$	-53		2^3	7
$4,2$	15		$2^2,1^2$	5
$4,1^2$	13		$2,1^4$	1
3^2	-11		1^6	-5
$3,2,1$	-5			

$n = 7$

λ	η_λ		λ	η_λ
7	1854		$3,2^2$	6
$6,1$	-309		$3,2,1^2$	6
$5,2$	66		$3,1^4$	6
$5,1,1$	62		$2^3,1$	-9
$4,3$	-21		$2^2,1^3$	-6
$4,2,1$	-18		$2,1^5$	-1
$4,1^3$	-15		1^7	6
$3^2,1$	14			

$n = 8$

λ	η_λ		λ	η_λ
8	14833		$4,1^4$	17
$7,1$	-2119		$3^2,2$	-19
$6,2$	371		$3^2,1^2$	-17
$6,1^2$	353		$3,2^2,1$	-7
$5,3$	-89		$3,2,1^3$	-7
$5,2,1$	-77		$3,1^5$	-7
$5,1^3$	-71		2^4	13
4^2	53		$2^3,1^2$	11
$4,3,1$	25		$2^2,1^4$	7
$4,2^2$	23		$2,1^6$	1
$4,2,1^2$	21		1^8	-7

$n = 9$

λ	η_λ		λ	η_λ
9	133496		$4,2^2,1$	-27
$8,1$	-16687		$4,2,1^3$	-24
$7,2$	2472		$4,1^5$	-19
$7,1^2$	2384		3^3	32
$6,3$	-463		$3^2,2,1$	23
$6,2,1$	-424		$3^2,1^3$	20
$6,1^3$	-397		$3,2^3$	8
$5,4$	128		$3,2^2,1^2$	8
$5,3,1$	104		$3,2,1^4$	8
$5,2^2$	92		$3,1^6$	8
$5,2,1^2$	88		$2^4,1$	-16
$5,1^4$	80		$2^3,1^3$	-13
$4^2,1$	-64		$2^2,1^5$	-8
$4,3,2$	-31		$2,1^7$	-1
$4,3,1^2$	-29		1^9	8

$n = 10$

λ	η_λ	λ	η_λ	λ	η_λ	λ	η_λ
10	1334961	$6,1^4$	441	$4,3,2,1$	36	$3,2^2,1^3$	-9
$9,1$	-148329	$5,5$	-309	$4,3,1^3$	33	$3,2,1^5$	-9
$8,2$	19071	$5,4,1$	-149	$4,2^3$	33	$3,1^7$	-9
$8,1^2$	18541	$5,3,2$	-125	$4,2^2,1^2$	31	2^5	21
$7,3$	-2967	$5.3.1^2$	-119	$4,2,1^4$	27	$2^4,1^2$	19
$7,2,1$	-2781	$5,2^2,1$	-105	$4,1^6$	21	$2^3,1^4$	15
$7,1^3$	-2649	$5,2,1^3$	-99	$3^3,1$	-39	$2^2,1^6$	9
$6,4$	621	$5,1^5$	-89	$3^2,2^2$	-29	$2,1^8$	1
$6,3,1$	529	$4^2,2$	81	$3^2,2,1^2$	-27	1^{10}	-9
$6,2^2$	495	$4^2,1^2$	75	$3^2,1^4$	-23		
$6,2,1^2$	477	$4,3^2$	39	$3,2^3,1$	-9		

$n = 11,\ \lambda_1 \geq 5$

λ	η_λ	λ	η_λ	λ	η_λ	λ	η_λ
11	14684570	$7,3,1$	3338	$6,2^2,1$	-557	$5,3,1^3$	134
$10,1$	-1468457	$7,2^2$	3178	$6,2,1^3$	-530	$5,2^3$	122
$9,2$	166870	$7,2,1^2$	3090	$6,1^5$	-485	$5,2^2,1^2$	118
$9,1^2$	163162	$7,1^4$	2914	$5^2,1$	362	$5,2,1^4$	110
$8,3$	-22249	$6,5$	-905	$5,4,2$	178	$5,1^6$	98
$8,2,1$	-21190	$6,4,1$	-710	$5,4,1^2$	170		
$8,1^3$	-20395	$6,3,2$	-617	$5,3^2$	158		
$7,4$	3706	$6,3,1^2$	-595	$5,3,2,1$	143		

$n = 12,\ \lambda_1 \geq 6$

λ	η_λ	λ	η_λ	λ	η_λ	λ	η_λ
12	176214841	$8,3,1$	24721	$7,2^2,1$	-3531	$6,3,2,1$	694
$11,1$	-16019531	$8,2^2$	23839	$7,2,1^3$	-3399	$6,3,1^3$	661
$10,2$	1631619	$8,2,1^2$	23309	$7,1^5$	-3179	$6,2^3$	637
$10,1^2$	1601953	$8,1^4$	22249	6^2	2119	$6,2^2,1^2$	619
$9,3$	-190709	$7,5$	-4959	$6,5,1$	1033	$6,2,1^4$	583
$9,2,1$	-183557	$7,4,1$	-4169	$6,4,2$	829	$6,1^6$	529
$9,1^3$	-177995	$7,3,2$	-3815	$6,4,1^2$	799		
$8,4$	26701	$7,3,1^2$	-3709	$6,3^2$	739		

$n = 13,\ \ \lambda_1 \geq 6$

λ	η_λ	λ	η_λ	λ	η_λ	λ	η_λ
13	2290792932	$9,1^4$	192828	$7,4,1^2$	4632	$6,4,3$	-996
$12,1$	-190899411	$8,5$	-33363	$7,3^2$	4452	$6,4,2,1$	-933
$11,2$	17621484	$8,4,1$	-29668	$7,3,2,1$	4239	$6,4,1^3$	-888
$11,1^2$	17354492	$8,3,2$	-27811	$7,3,1^3$	4080	$6,3^2,1$	-831
$10,3$	-1835571	$8,3,1^2$	-27193	$7,2^3$	3972	$6,3,2^2$	-793
$10,2,1$	-1779948	$8,2^2,1$	-26223	$7,2^2,1^2$	3884	$6,3,2,1^2$	-771
$10,1^3$	-1735449	$8,2,1^3$	-25428	$7,2,1^4$	3708	$6,3,1^4$	-727
$9,4$	222492	$8,1^5$	-24103	$7,1^6$	3444	$6,2^3,1$	-708
$9,3,1$	209780	$7,6$	7284	$6^2,1$	-2428	$6,2^2,1^3$	-681
$9,2^2$	203952	$7,5,1$	5580	$6,5,2$	-1203	$6,2,1^5$	-636
$9,2,1^2$	200244	$7,4,2$	4764	$6,5,1^2$	-1161	$6,1^7$	-573

$n = 15$

λ	η_λ	λ	η_λ	λ	η_λ	λ	η_λ
15	481066515734	$7^2,1$	18806	$6,2,1^7$	-742	$4,3^2,2^2,1$	-77
$14,1$	-34361893981	$7,6,2$	9350	$6,1^9$	-661	$4,3^2,2,1^3$	-74
$13,2$	2672591754	$7,6,1^2$	9094	5^3	1214	$4,3^2,1^5$	-69
$13,1^2$	2643222614	$7,5,3$	7446	$5^2,4,1$	859	$4,3,2^4$	-73
$12,3$	-229079293	$7,5,2,1$	7089	$5^2,3,2$	742	$4,3,2^3,1^2$	-71
$12,2,1$	-224273434	$7,5,1^3$	6822	$5^2,3,1^2$	714	$4,3,2^3,1^4$	-67
$12,1^3$	-220268551	$7,4^2$	6662	$5^2,2^2,1$	662	$4,3,2,1^6$	-61
$11,4$	22026854	$7,4,3,1$	6174	$5^2,2,1^3$	629	$4,3,1^8$	-53
$11,3,1$	21211046	$7,4,2^2$	5954	$5^2,1^5$	574	$4,2^5,1$	-66
$11,2^2$	20825390	$7,4,2,1^2$	5822	$5,4^2,2$	374	$4,2^4,1^3$	-63
$11,2,1^2$	20558398	$7,4,1^4$	5558	$5,4^2,1^2$	362	$4,2^3,1^5$	-58
$11,1^4$	20024414	$7,3^2,2$	5566	$5,4,3^2$	350	$4,2^2,1^7$	-51
$10,5$	-2447421	$7,3^2,1^2$	5442	$5,4,3,2,1$	329	$4,2,1^9$	-42
$10,4,1$	-2288506	$7,3,2^2,1$	5246	$5,4,3,1^3$	314	$4,1^{11}$	-31
$10,3,2$	-2202685	$7,3,2,1^3$	5087	$5,4,2^3$	302	3^5	134
$10,3,1^2$	-2169311	$7,3,1^5$	4822	$5,4,2^2,1^2$	294	$3^4,2,1$	119
$10,2^2,1$	-2121105	$7,2^4$	4854	$5,4,2,1^4$	278	$3^4,1^3$	110
$10,2,1^3$	-2076606	$7,2^3,1^2$	4766	$5,4,1^6$	254	$3^3,2^3$	98
$10,1^5$	-2002441	$7,2^2,1^4$	4590	$5,3^3,1$	290	$3^3,2^2,1^2$	94
$9,6$	333674	$7,2,1^6$	4326	$5,3^2,2^2$	274	$3^3,2,1^4$	86
$9,5,1$	293702	$7,1^8$	3974	$5,3^2,2,1^2$	266	$3^3,1^6$	74
$9,4,2$	271934	$6^2,3$	-3430	$5,3^2,1^4$	250	$3^2,2^4,1$	62
$9,4,1^2$	266990	$6^2,2,1$	-3205	$5,3,2^3,1$	239	$3^2,2^3,1^3$	59
$9,3^2$	262226	$6^2,1^3$	-3046	$5,3,2^2,1^3$	230	$3^2,2^2,1^5$	54
$9,3,2,1$	254279	$6,5,4$	-1789	$5,3,2,1^5$	215	$3^2,2,1^7$	47
$9,3,1^3$	247922	$6,5,3,1$	-1617	$5,3,1^7$	194	$3^2,1^9$	38
$9,2^3$	244742	$6,5,2^2$	-1543	$5,2^5$	194	$3,2^6$	14
$9,2^2,1^2$	241034	$6,5,2,1^2$	-1501	$5,2^4,1^2$	190	$3,2^5,1^2$	14
$9,2,1^4$	233618	$6,5,1^4$	-1417	$5,2^3,1^4$	182	$3,2^4,1^4$	14
$9,1^6$	222494	$6,4^2,1$	-1411	$5,2^2,1^6$	170	$3,2^3,1^6$	14
$8,7$	-65821	$6,4,3,2$	-1282	$5,2,1^8$	154	$3,2^2,1^8$	14
$8,6,1$	-49546	$6,4,3,1^2$	-1246	$5,1^{10}$	134	$3,2,1^{10}$	14
$8,5,2$	-41701	$6,4,2^2,1$	-1181	$4^3,3$	-331	$3,1^{12}$	14
$8,5,1^2$	-40775	$6,4,2,1^3$	-1141	$4^3,2,1$	-298	$2^7,1$	-49
$8,4,3$	-38146	$6,4,1^5$	-1066	$4^3,1,1,1$	-277	$2^6,1^3$	-46
$8,4,2,1$	-36715	$6,3^3$	-1105	$4^2,3^2,1$	-226	$2^5,1^5$	-41
$8,4,1^3$	-35602	$6,3^2,2,1$	-1054	$4^2,3,2^2$	-210	$2^4,1^7$	-34
$8,3^2,1$	-34961	$6,3^2,1^3$	-1015	$4^2,3,2,1^2$	-202	$2^3,1^9$	-25
$8,3,2^2$	-33991	$6,3,2^3$	-991	$4^2,3,1^4$	-186	$2^2,1^{11}$	-14
$8,3,2,1^2$	-33373	$6,3,2^2,1^2$	-969	$4^2,2^3,1$	-175	$2,1^{13}$	-1
$8,3,1^4$	-32137	$6,3,2,1^4$	-925	$4^2,2^2,1^3$	-166	1^{15}	14
$8,2^3,1$	-31786	$6,3,1^6$	-859	$4^2,2,1^5$	-151		
$8,2^2,1^3$	-30991	$6,2^4,1$	-877	$4^2,1^7$	-130		
$8,2,1^5$	-29666	$6,2^3,1^3$	-850	$4,3^3,2$	-81		
$8,1^7$	-27811	$6,2^2,1^5$	-805	$4,3^3,1^2$	-79		

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