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ON SHOCK WAVES IN SOLIDS

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Dedicated to Fred Wan on the occasion of his seventieth birthday

Abstract. This paper describes some recent theoretical results pertaining to the experimentally-observed relation between the speed of a shock wave in a solid and the particle velocity immediately behind the shock. The new feature in the present analysis is the assumption that compressive strains are limited by a materially-determined critical value, and that the internal energy density characterizing the material is unbounded as this critical strain is approached. It is shown that, with this assumption in force, the theoretical relation between shock speed and particle velocity is consistent with many experimental observations in the sense that it is asymptotically linear for strong shocks of the kind often arising in the laboratory.

1. **Introduction.** Experiments that produce shock waves in solids have been carried out for many years and form the core of a discipline that has come to be known as *shock-physics*. Sometimes motivated by the need to understand the response of materials to very high pressures such as those found deep in the earth, experiments that produce high-pressure shock waves often involve a rapidly moving projectile, or "flyer plate", impacting a cylindrical specimen on one of its plane faces. Values of the propagation velocity of the resulting plane shock wave and the particle velocity immediately behind it are measured in such experiments. A very striking common feature of these measurements over a wide variety of materials and a wide range of flyer-plate speeds is a very nearly linear relation between these two velocities, a fact long noted, studied and used in the shock-physics literature; see for example, [1, 2, 3]. A large compendium of data illustrating this feature for many materials may be found in [4].

A recent study [5] suggests that, in thermoelastic materials, this linear relation between shock speed and particle velocity may arise from singular behavior of the governing constitutive law at very severe compressive strain. In the present paper, I present the main results derived in [5] without most of the supporting analytical details.

In the next section, we summarize the version of nonlinear thermoelasticity that applies for uniaxial strain, which is the appropriate kinematic framework for the description of the experiments described above. We also specify the special subclass of thermoelastic materials to be employed here, and we make precise the assumption of singular behavior of the governing internal energy density at severe compression; this assumption comprises the new feature of the model. In Section 3, we sketch the

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parts of the theory of shock waves needed here, and we reduce the problem of finding the relation between shock speed and particle velocity to its simplest form. In the last section, we find the asymptotic solution of this problem for severely compressive shocks, and we determine the implications of reconciling this theoretical result with its experimentally observed counterpart. Though the analysis could be carried out for a fully general thermoelastic material, we restrict attention to a special subclass of such materials in order to keep the details as simple as possible.

2. **A class of thermoelastic materials.** We shall be concerned with a space-filling thermoelastic medium whose unstrained configuration is stress-free at the absolute temperature θ_0. In dynamic uniaxial strain, a particle P whose location in the unstrained configuration is x, y, z is carried at time t to the point $x+u(x,t)$, y, z, where u is the axial displacement. The associated strain and particle velocity are given by $\epsilon = u_x$ and $v = u_t$, respectively, where the subscripts indicate partial derivatives. The ratio of the specific volume V (or volume per unit mass) at the particle P at time t to its specific volume V_0 in the undeformed configuration at the temperature θ_0 is given by $V/V_0 = 1+\epsilon$; to keep this positive, one requires that $\epsilon > -1$. The unachievable strain $\epsilon = -1$ may thus be thought of as infinite compression.

Let $\sigma(x,t)$ and $\theta(x,t)$ be the axial normal component of the Piola-Kirchhoff stress tensor and the absolute temperature, respectively, at the particle P at time t. For a thermoelastic material, these quantities are constitutively related to the strain $\epsilon(x,t)$ and the entropy per unit mass $\eta(x,t)$ by

$$\sigma = \rho e_\epsilon(\epsilon,\eta), \quad \theta = e_\eta(\epsilon,\eta); \tag{1}$$

here ρ is the mass per unit undeformed volume, assumed to be a given constant, and the function $e = e(\epsilon,\eta)$ is the internal energy per unit mass at the strain ϵ and the specific entropy η that characterizes the given thermoelastic material under consideration. In (1), the subscripts ϵ and η indicate partial derivatives.

Let

$$c = \frac{e_\eta}{e_{\eta\eta}}, \quad g = -\frac{e_{\epsilon\eta}}{e_\eta}\ ; \tag{2}$$

c is the *specific heat at constant strain* of the material, and g is a measure of thermomechanical coupling, often called the *modified Grüneisen parameter*, that is related at small deformations to the coefficient of thermal expansion of the material. Note that, in general, both c and g depend on the state (ϵ,η) at the particle P. Although it is not essential to our argument, we now make two assumptions in order to keep the presentation as simple as possible: we assume that c and g are absolute positive constants. It can then be shown that the internal energy $e(\epsilon,\eta)$ takes the form

$$e(\epsilon,\eta) = W(\epsilon) + gc\theta_0\,\epsilon + c\theta_0 \exp\left(\frac{\eta-\eta_0-gc\,\epsilon}{c}\right), \tag{3}$$

where $W(\epsilon)$ is arbitrary except that

$$W(0) = W'(0) = 0. \tag{4}$$

The second of these normalizations on $W(\epsilon)$ at $\epsilon = 0$ arises from the requirement that the stress should vanish at the reference state $\epsilon = 0, \eta = \eta_0$; the condition

$W(0) = 0$ is convenient and may always be arranged, since adding an arbitrary constant to the internal energy density does not effect the stress or the temperature as given by the derivatives of $e(\epsilon, \eta)$ in (1).

When the internal energy density has the form (3), the stress σ and the temperature θ follow from (1) as

$$\sigma = \rho W'(\epsilon) + \rho c g \theta_0 [1 - E(\epsilon, \eta)], \quad \theta = \theta_0 E(\epsilon, \theta), \tag{5}$$

where we have abbreviated by writing

$$E(\epsilon, \eta) = \exp\left(\frac{\eta - \eta_0}{c} - g\epsilon\right). \tag{6}$$

Elimination of $E(\epsilon, \eta)$ between σ and θ in (5) yields the expression for stress in terms of strain and temperature:

$$\sigma = W'(\epsilon) - \rho c g(\theta - \theta_0). \tag{7}$$

The special choice $W(\epsilon) = m\epsilon^2/2$, where m is the isothermal elastic modulus for uniaxial strain, reduces (7) to the stress-strain-temperature relation of classical linear thermoelasticity [6], which is applicable to problems in which the strain ϵ and the departure of the temperature θ from the referential temperature θ_0 are both small in magnitude. The problem to be discussed here does not fall into this category.

In addition to the normalization conditions (4), we make the following assumptions about $W(\epsilon)$:

$$W''(\epsilon) > 0, \;\; W'''(\epsilon) < 0. \tag{8}$$

The first of these requirements assures that stress (7) is an increasing function of strain at fixed temperature; the second requires that this function be concave. These assumptions are not unconventional.

On the other hand, our final assumption about $W(\epsilon)$ *is* unconventional and is the critical new ingredient in the theory of thermoelastic shock waves as presented here. We assume there is a compressive strain $-\epsilon_\star$, short of infinite compression so that $0 < \epsilon_\star < 1$, at which the internal energy blows up: $W(\epsilon) \to +\infty$ as $\epsilon \to -\epsilon_\star+$, so that only strains ϵ greater than $\epsilon_\star$ are permitted. More precisely, it is now assumed that

$$W(\epsilon) \sim \frac{K}{k}(\epsilon + \epsilon_\star)^{-k} \quad \text{as } \epsilon \to -\epsilon_\star+, \tag{9}$$

where $\epsilon_\star$, K and k are given material constants. Since the ratio V/V_0 of current to undeformed specific volume is given by $1 + \epsilon$, it follows that $V/V_0 > 1 - \epsilon_\star$.

An internal energy density of the form (3) was used by Clifton [11] in his model for the so-called “failure waves” arising in impact experiments on glass. Because his model involves a solid-to-solid phase transition, it necessarily violates the assumptions (8).

Thermoelastic materials governed by an internal energy density of the form (3) are members of a class of materials known as those of *Mie-Grüneisen* type. For a fuller discussion of such materials, the reader may consult the extensive study of waves in thermoelastic materials due to Dunn and Fosdick [12].

While the requirement that the compressive strain must exceed the value $-\epsilon_\star$ and the associated asymptotic assumption (9) do not usually enter models used in shock physics as far as I know, there are analogous restrictions in other areas that may be viewed as precedents. The most closely related one arises in the model known as the van der Waals fluid, which is intended to correct the ideal gas model so as to account for the transition to the liquid phase; see §76 of [7] or §3-5 of [8]. One of the two departures from the ideal gas arising in the van der Waals model imposes a lower limit on the volume occupied by the gas. This limit is viewed as arising from the aggregated volume of the gas molecules thought of as billiard balls, and it enters the van der Waals constitutive law as a material constant. A second analog to our proposed limit on attainable strain arises in Gent's model [9] for rubber, which was suggested by experiments and which imposes a finite limit on the strain achievable *in tension*. More generally, the need to postulate the nature of material response at severe deformations also arises in the study of nonlinear behavior near singularities in solids; for example, specifying the asymptotic behavior at severe deformations of the governing energy density is essential in the analysis of nonlinear effects at the tip of a crack [10].

3. **Shock waves.** Suppose a plane moving through our thermolastic medium in the positive x-direction with velocity $\dot{s} > 0$ carries discontinuities in particle velocity, strain, specific entropy, stress and temperature. Viewing the thermomechanical process as adiabatic and requiring continuity of displacement, balance of linear momentum, balance of total energy and the satisfaction of the second law of thermodynamics across such a shock wave imposes the following respective jump conditions:

$$[\![\epsilon]\!]\dot{s} + [\![v]\!] = 0, \quad [\![\sigma]\!] + \rho\dot{s}[\![v]\!] = 0, \quad \rho[\![e]\!] - <\sigma> [\![\epsilon]\!] = 0, \quad [\![\eta]\!] \leq 0. \tag{10}$$

Here, for example, $[\![\epsilon]\!] = \epsilon^+ - \epsilon^-$, where ϵ^+, ϵ^- are the values of strain on the right and left sides of the shock, respectively, and $<\sigma>= (1/2)(\sigma^+ + \sigma^-)$. For background surrounding these equations, see §2 of [12].

Suppose further that the material ahead of the shock wave is at rest in the unstrained, stress-free state at the temperature θ_0, so that $v^+ = 0, \epsilon^+ = \sigma^+ = 0, \eta^+ = \eta_0$, and $\theta^+ = \theta_0$. Dropping the superscripted minuses from the back-state quantities such as $\epsilon^-, \eta^-, ...$, using the constitutive law (1) and specializing (10) to these circumstances shows that the back-state quantities must satisfy

$$\dot{s}\epsilon + v = 0, \quad e_\epsilon(\epsilon, \eta) + \dot{s}v = 0, \quad e(\epsilon, \eta) - \frac{1}{2}e_\epsilon(\epsilon, \eta)\epsilon = c\theta_0, \quad \eta_0 \leq \eta. \tag{11}$$

The first three assertions in (11) comprise three equations involving the four unknowns $\epsilon, \dot{s}, \eta$ and v. Pretending that the back-state particle velocity v is given and positive, we wish to solve these three equations for the remaining three unknowns in terms of v. In particular, once we know $\dot{s} = \hat{\dot{s}}(v)$, we shall have determined the relation between shock speed and particle velocity whose near-linearity has been observed experimentally for many materials, as discussed in the preceding section.

Eliminating $\dot{s}$ between the first two equations in (11) immediately yields two equations for the back-state strain ϵ and the back-state entropy η in terms of the back-state particle velocity v:

$$\epsilon e_\epsilon(\epsilon, \eta) = v^2, \quad e(\epsilon, \eta) = c\theta_0 + \frac{1}{2}v^2. \tag{12}$$

Specializing (12) to the Mie-Grüneisen material described by (3) gives

$$\epsilon W'(\epsilon) - c\theta_0 g\,\epsilon(E-1) = v^2, \quad W(\epsilon) + c\theta_0 g\,\epsilon + c\theta_0(E-1) = \frac{1}{2}v^2, \tag{13}$$

where E is given in (6). Eliminating E reduces (13) to

$$F(\epsilon) = v^2, \tag{14}$$

where the function F is defined by

$$F(\epsilon) = \frac{\epsilon W'(\epsilon) + g\epsilon W(\epsilon) + c\theta_0 g^2\epsilon^2}{1 + g\epsilon/2}. \tag{15}$$

If (14) can be solved to determine $\epsilon = \hat{\epsilon}(v)$, then the relation between shock speed and back-state particle velocity that we ultimately seek is obtained from $(11)_1$: $\dot{s} = -v/\hat{\epsilon}(v)$.

Because the strains of interest to us are compressive, negative values of ϵ are the relevant ones. The most notable property of $F(\epsilon)$ is that it has two singular points where it blows up: at $\epsilon = -2/g$, because of the denominator in (15), and at $\epsilon = -\epsilon_\star$, because of the assumption (9). Let

$$\epsilon_{\min} = \min(\epsilon_\star, 2/g). \tag{16}$$

In using $F(\epsilon)$, we must work to the right of $\epsilon = -\epsilon_{\min}$. From (4), (15), it follows that $F(0) = 0$. Using (8), it is possible to show that $F(\epsilon)$ decreases monotonically from $+\infty$ at $\epsilon = -\epsilon_{\min}$ to zero at $\epsilon = 0$. It then follows that (14) has a unique root $\epsilon = \hat{\epsilon}(v) < 0$ for every $v > 0$, and that $\hat{\epsilon}(v)$ decreases from 0 to $-\epsilon_{\min}$ as v increases from 0 to ∞. It further follows that $\dot{s}$ is uniquely determined as

$$\dot{s} = \hat{\dot{s}}(v) = -v/\hat{\epsilon}(v) > 0 \ \text{ for every } v > 0. \tag{17}$$

Once $\hat{\epsilon}(v)$ and $\hat{\dot{s}}(v)$ are known, one can find the specific entropy $\eta = \hat{\eta}(v)$ from either of the equations in (13), making use of (6). It is then possible to verify that the entropy inequality $(11)_4$ is satisfied.

The results represented by (14), (15) and (17) comprise a special case of a result of Dunn [13]. The assumptions made here - that the Grüneisen parameter and the specific heat are constants and that the state in front of the shock is undisturbed - are not made by Dunn. As recognized by Dunn, two cases are implicit in the definition (16) of $\epsilon_{\min}$:

$$\text{Case 1} \ \ \epsilon_\star > 2/g, \ \text{ so that } \epsilon_{\min} = 2/g; \quad \text{Case 2} \ \ \epsilon_\star < 2/g, \ \text{ so that } \epsilon_{\min} = \epsilon_\star. \tag{18}$$

Dunn identified these two cases; he pursued Case 1, but not Case 2.

We turn next to the asymptotic analysis of (14) for large v.

4. **Asymptotics for severe impact.** For much of the data documented by Marsh [4], values of particle velocity and shock speed that are each two or three times the speed of small-amplitude waves near the unstrained state are common. This suggests that the asymptotic behavior of $\dot{s}$ for large values of v may be relevant. We thus first seek the behavior of the solution $\hat{\epsilon}(v)$ of (14) for large v, and the associated results for the shock-speed - particle-velocity relation (17) in Cases 1 and 2. We then assess the relation between these results and the corresponding experimental findings summarized in [4]. We shall show that consistency between the asymptotic analytical results and the preponderance of observations reported in [4] not only

rules out Case 1, but also determines the order k of the singularity in the assumed large-compression behavior (9) of $W(\epsilon)$, and therefore of the internal energy density $e(\epsilon, \eta)$ itself.

4.1. **Asymptotic solutions of (14) in Cases 1 and 2. Case 1.** $\epsilon_\star > 2/g$ ($\epsilon_{\text{min}} = 2/g$). If v in (14) is large, then in this case ϵ must be near $-2/g$. Indeed, one readily shows that

$$\epsilon = \hat{\epsilon}(v) = -2/g + O(1/v^2) \text{ as } v \to \infty \quad \text{(Case 1)}; \tag{19}$$

it then follows from (17) that

$$\dot{s} = \hat{\dot{s}}(v) = \frac{g}{2}v + O(1/v) \text{ as } v \to \infty \quad \text{(Case 1)}. \tag{20}$$

Case 2. $\epsilon_\star < 2/g$ ($\epsilon_{\text{min}} = \epsilon_\star$). In this case, large v means ϵ must be near $-\epsilon_\star$. Making use of (9) in (14) leads directly to

$$\epsilon = \hat{\epsilon}(v) = -\epsilon_\star + bv^{-\frac{2}{k+1}} + o(v^{-\frac{2}{k+1}}) \text{ as } v \to \infty \quad \text{(Case 2)}, \tag{21}$$

and then to

$$\dot{s} = \hat{\dot{s}}(v) = v/\epsilon_\star + (b/{\epsilon_\star}^2)v^{\frac{k-1}{k+1}} + o(v^{\frac{k-1}{k+1}}) \text{ as } v \to \infty \quad \text{(Case 2)}, \tag{22}$$

where

$$b = \left\{ \frac{\epsilon_\star K}{1 - g\epsilon_\star/2} \right\}^{\frac{1}{k+1}}. \tag{23}$$

(Recall that K was introduced in (9).)

4.2. **Reconciliation of the asymptotic results with observations.** For most of the materials for which data is given in [4], observed values of shock speed $\dot{s}$ and particle velocity v conform closely to an empirical relation of the form

$$\dot{s} = p + qv, \tag{24}$$

where p and q are constants. For most of the materials that support a relation of this form, values of p and q as determined from curve-fitting are reported by Marsh [4]. In the great majority of cases, though not all, for which values of p and q are given, one has

$$1 < p < 8, \text{ and } 1 < q < 2. \tag{25}$$

Case 1 of the asymptotic results above would suggest that

$$p = 0, \quad q = g/2 \quad \text{(Case 1)}. \tag{26}$$

According to (25), the value $p = 0$ is inappropriate for virtually all of the materials in [4]. Moreover, to have $q > 1$ as in (25)would impose the requirement $g > 2$ on the modified Grüneisen parameter. A table of values of this parameter for a large collection of materials may be found on p. 133 of [14]; three-quarters of the materials listed in the table have values of g no larger than 2. Thus Case 1 is unlikely to apply to most materials, a conclusion also reached by Dunn [13].

On the other hand, if Case 2 were to be appropriate, then (22) and (24) would have to be consistent. This would require

$$k = 1 \quad \text{(Case 2)}, \tag{27}$$

thus determining the order of the postulated singularity (9) in the internal energy, and specializing the asymptotic strain-particle velocity relation (21) to

$$\epsilon = \hat{\epsilon}(v) = -\epsilon_\star + \frac{b}{v} + o(1/v) \ \text{ as } v \ \to \ \infty \quad \text{(Case 2).)} \tag{28}$$

This in turn would mean that the coefficients p and q in (24) should be taken to be

$$p = b/{\epsilon_\star}^2, \quad q = 1/\epsilon_\star \quad \text{(Case 2)}. \tag{29}$$

By (9) and (27), the energy singularity would then specialize to

$$W(\epsilon) \sim \frac{K}{\epsilon + \epsilon_\star} \ \text{ as } \ \epsilon \to -\,\epsilon_\star\,+ \quad \text{(Case 2)}, \tag{30}$$

and by (23), b would be given by

$$b = \sqrt{\frac{\epsilon_\star K}{1 - g\epsilon_\star/2}} \quad \text{(Case 2)}. \tag{31}$$

According to (29), restricting $\epsilon_\star$ to the interval $(1/2, 1)$ would assure that q lies in the range given in (25). The defining property $g < 2/\epsilon_\star$ of Case 2 would then imply that $g < 4$; the modified Grüneisen parameter of each material listed in Table 5.1 of [14] meets this requirement. Finally, a suitable choice of the amplitude K of the singularity (30) will assure through (29) and (31) that p lies in the range specified in (25).

Thus the large-v asymptotics in Case 2 would appear to be consistent with the preponderance of observations in [4] if the parameters in the singularity proposed in (9) for the internal energy density are suitably chosen. This has the appealing feature that the source of the quasi-universal observed linear relationship between shock speed and particle velocity lies not in the detailed behavior of $e(\epsilon, \eta)$, but rather in the nature of the singular response at severe compression shared by a large class of materials.

4.3. **Back-state stress and temperature.** Using (13) in (5) furnishes the following expressions for the back-state stress and the temperature-change across the shock in terms of the back-state strain and particle velocity:

$$\frac{\sigma}{\rho} = \frac{v^2}{\epsilon}, \qquad \theta - \theta_0 = \frac{1}{cg}\left[W'(\epsilon) - \frac{v^2}{\epsilon}\right]; \tag{32}$$

for the class of Mie-Grüneisen materials characterized by the internal energy (3), the relations (32) are exact.

After inverting the relation (28) to give v asymptotically in terms of ϵ, we may extract the asymptotic first approximations to back-state stress and temperature-change from (32) as

$$\frac{\sigma}{\rho} = -\,\frac{K}{1 - g\epsilon_\star/2}\,\frac{1}{(\epsilon+\epsilon_\star)^2}\,[1{+}o(1)], \qquad \theta{-}\theta_0 = \frac{K}{c}\,\frac{\epsilon_\star}{2 - g\epsilon_\star}\,\frac{1}{(\epsilon+\epsilon_\star)^2}\,[1{+}o(1)]. \tag{33}$$

The asymptotic result $(33)_1$ for the compressive stress suggests that one might expect to see in experiments a steep rise in the magnitude of back-state stress as the magnitude of the compressive strain gets large. In addition to experimental plots of shock speed versus particle velocity, Marsh [4] includes graphs of the pressure $-\sigma$ against specific volume V; for most of the materials in [4], this steep rise is indeed observed.

It may be remarked that while the absolute error committed by the approximation (28) for the back-state strain is small for large v, the corresponding error committed by either of the approximations in (33), while small compared to the leading term, may itself also be unbounded as ϵ approaches $-\epsilon_\star$. To improve on the first approximations in (33) would involve specifying higher-order terms in the asymptotics (30) at the postulated singularity in $W(\epsilon)$.

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