

# Thermal Fluctuations: Modes versus the Continuum

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**Abstract.** The thermal fluctuation spectrum of the signal received on a patch electrode is examined and it is shown that the spectrum shows both the modes of the plasma and a continuous spectrum related to the independent-particle motions of plasma electrons. Modes whose axial phase velocity are more than 3-4 times the electron thermal speed are lightly Landau-damped and are clearly separated from the continuum. Long wavelength modes are "acoustic" in nature. If the axial phase velocity of a mode becomes less than 1-2 times the electron thermal speed, then the mode becomes strongly Landau-damped and it merges into the continuum. The mode velocities are of the order of  $\omega_p a$ , where  $a$  is the plasma radius, so that the plasma radius must be at least several deBye lengths in order to have lightly damped modes. In general, the spectrum is a mixture of a continuous spectrum together with a finite number of modes which are Landau-damped by varying amounts, depending on their phase velocity relative to the electron thermal speed. Only in the extreme limit,  $\omega_p a \ll v_{th}$  does the continuous spectrum tend to a Gaussian of width  $k v_{th}$ , characteristic of independent particles. The effect of the "load impedance" on the measurements is also discussed.

## INTRODUCTION

Two procedures have been described recently for determining the temperature of pure electron plasmas at or near thermal equilibrium by measuring the spectrum of the fluctuating charge on a patch electrode. The first [1] method employs the narrow resonant peaks associated with the modes of the plasma, and the second [2] makes use of the broad continuous spectrum associated with the independent particle motion. A simple model of the plasma column, using the warm plasma dielectric function, is used to calculate the input admittance  $Y_p(\omega)$  of a patch electrode.  $Y_p(\omega)$  gives the patch current when a voltage is applied, and it reflects the dynamical processes within the plasma. For example, when the frequency of the voltage applied to the electrode is close to one of the mode frequencies of the plasma, the patch current can be very large if the modes are lightly damped.

Since many non-neutral plasmas are at, or close to, thermal equilibrium, a thermodynamic argument can be used to obtain the fluctuation spectrum from  $Y_p(\omega)$ . The plasma can be treated as an electrical circuit and Nyquist's theorem [3] for electric circuits can be used to obtain the fluctuation spectrum. Alternatively, one could use the fluctuation-dissipation theorem [4] to obtain the same result. According to Nyquist's theorem, the fluctuation spectrum of the patch current is related to the dissipative part

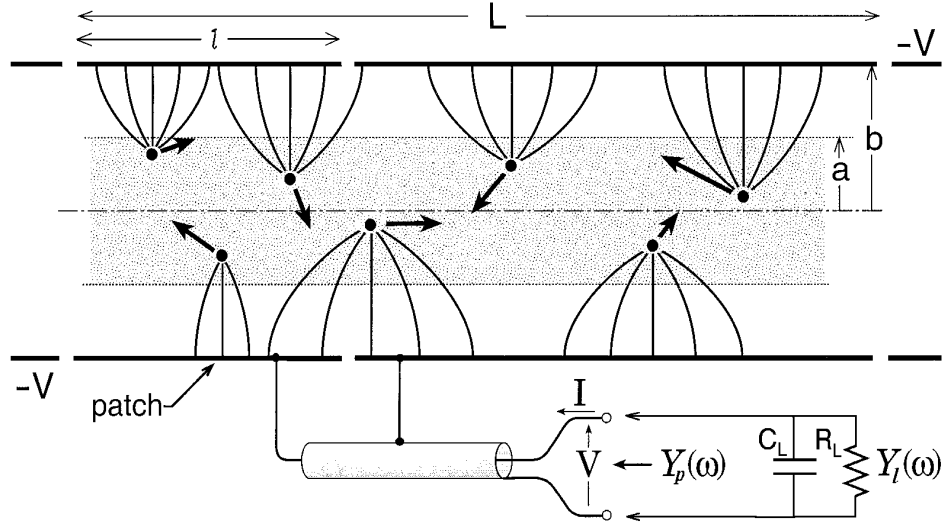


FIGURE 1. Schematic of nonneutral plasma with cylindrical patch electrode and coaxial cable connection to the measurement instrumentation, represented by  $Y_l(\omega)$ .

of the plasma admittance,  $\text{Re}\{Y_p(\omega)\}$ . Fig.1. shows schematically a typical experimental geometry. The patch electrode is connected to the measuring apparatus with a coaxial cable. The input to the measuring apparatus represents a "load"  $Y_l(\omega)$  on the plasma and is assumed to consist of a resistance  $R_l$  and a capacitance  $C_l$  in parallel.  $C_l$  also includes the capacitance of the coaxial cable. The fluctuation spectrum can be obtained by measuring either the voltage  $V(\omega)$  at the coax terminals, or the current  $I(\omega)$  which flows to the load. Nyquist's theorem gives either the fluctuating open-circuit voltage which appears across the terminals, or the fluctuating short-circuit current which flows. The latter is

$$I_{sc}^2(\omega) = 4\kappa T_p \text{Re}\{Y_p(\omega)\}, \quad (1)$$

where  $\kappa$  is Boltzmann's constant and  $T_p$  is the plasma temperature. In general, the load is neither a short-circuit, nor a open-circuit, so the effect of the load admittance must be taken into account. At the end of the paper various load effects are discussed. However, to simplify the present discussion, it is assumed that the load is simply the capacitance  $C_L$  and that  $|Y_p| \ll |Y_l|$ . Then the spectral density of the potential which appears across  $C_L$  is  $V^2(\omega) = I_{sc}^2(\omega)/\omega^2 C_L^2 = q_{sc}^2/C_L^2$ , with  $q_{sc}$  being the fluctuating charge on the sector electrode. First the calculation of  $Y_p(\omega)$  is outlined, and then some illustrative results are given.

## ADMITTANCE CALCULATION

In this section the calculation of the input admittance  $Y_p(\omega)$  is outlined. To simplify the calculation, a number of reasonable assumptions and approximations are made: a) the patch electrode extends over a length,  $l$ , at one end of the plasma, b) the plasma has constant density up to radius  $a$ , and the density is zero outside this radius, c) the plasma length is  $L$ , and periodic boundary conditions are used which are appropriate to

specular reflection of electrons at each end, d) the plasma is described by its dielectric tensor,  $\mathbf{K}$ .  $\nabla \cdot \mathbf{K} \cdot \nabla \phi = 0$  is solved for the potential,  $\phi$ .  $K_1$ ,  $K_2$ , and  $K_3$  are the perpendicular, Hall, and parallel components of  $\mathbf{K}$ , respectively. Assumption (c) allows us to write the potential and radial electric field, with  $k_n = n\pi/L$ , as

$$\phi = \sum_{mn} \phi_{mn} \cos(k_n z) e^{im\theta - i\omega t} \quad (2a)$$

$$E_r = \sum_{mn} E_{rmn} \cos(k_n z) e^{im\theta - i\omega t} \quad (2b)$$

Solving the potential equation leads to  $\phi_{mn}(r) \sim AI_m(k_n r) + BK_m(k_n r)$  for  $a \leq r \leq b$ , and  $\phi_{mn}(r) \sim J_m(Tr)$  for  $r < a$  with  $T^2 = -k_n^2 K_3/K_1$ .  $T$  is the radial wave number. Assuming that the potential  $\phi_{mn}(b)$ , the potential of one Fourier component at the wall, is known these solutions can be fitted together so as to give  $E_{rmn}(b)$ , the radial electric field at the wall. We define the quantity  $\chi_{mn}(\omega) = -bE_{rmn}(b)/\phi_{mn}$ , which is the logarithmic derivative of the potential at the wall.  $\chi_{mn}$  gives the response of the plasma to an applied potential on the wall.  $\chi_{mn}$  is a function of the frequency,  $\omega$ , and is useful in calculating the current flowing to the sector probe,  $I(\omega)$ . This current is a displacement current at the surface of the patch electrode, whose area is  $S_p$ ,

$$I(\omega) = -i\omega\epsilon_o \int_{S_p} (-E_r) dS, \quad (3)$$

which can be evaluated in terms of  $E_{rmn}(b)$  using Eq. (2b). If the patch electrode is at potential  $V(\omega)e^{-i\omega t}$ , the coefficients  $\phi_{mn}(b)$  can be evaluated and the input admittance of the patch electrode can be shown to be

$$Y_p(\omega) = \frac{I(\omega)}{V(\omega)} = \frac{2i\omega\epsilon_o S_p^2}{S_{tot} b} \sum_{mn} |M_{mn}|^2 \chi_{mn}(\omega) \quad (4)$$

where  $M_{mn} = \frac{1}{S_p} \int_{S_p} \cos(k_n z) e^{im\theta} dS \leq 1$ , is a sector factor,  $S_{tot} = 2\pi bL$ ,

$$\chi_{mn}(\omega) = G_3 \frac{F(\omega) + G_1}{F(\omega) + G_2}, \quad (5)$$

$$F(\omega) = K_1 T a \left[ \frac{J'_m(Ta)}{J_m(Ta)} \right] + m K_2, \quad G_1 = k a \frac{I'_m(kb) K'_m(ka) - K'_m(kb) I'_m(ka)}{K'_m(kb) I_m(ka) - I'_m(kb) K_m(ka)},$$

$$G_2 = k a \frac{I_m(kb) K'_m(ka) - K_m(kb) I'_m(ka)}{K_m(kb) I_m(ka) - I_m(kb) K_m(ka)}, \quad G_3 = k b \frac{K'_m(kb) I_m(ka) - I'_m(kb) K_m(ka)}{K_m(kb) I_m(ka) - I_m(kb) K_m(ka)},$$

and  $T^2 = -k^2(K_3/K_1)$ . Thus  $Y_p(\omega)$  is a weighted sum over the various  $\chi_{mn}$ . Note that  $G_1$ ,  $G_2$ , and  $G_3$  are just functions of the geometry and the axial wave number  $k = k_n$ . For a frequency corresponding to a mode  $\chi_{mn}$ , the plasma response becomes very large. The mode frequencies are thus determined by the vanishing of the denominator of Eq. 5.

## RESULTS

In the remainder of this paper axisymmetry ( $m = 0$ ), and a strong magnetic field so that  $K_1 \approx 1$ , and  $K_2 \approx 0$ , will be assumed. In addition, it will be assumed that electron parallel motion obeys the kinetic equation so that  $K_3 = 1 - (\omega_p^2/k^2 v_{th}^2) Z'(\omega/k v_{th})$ , where  $Z'$  is the derivative of the plasma dispersion function and  $v_{th}^2 = 2\kappa T_p/m$ . In the cold plasma limit  $K_3 = 1 - \omega_p^2/\omega^2$  and for  $b/a = 2$ , this leads to the undamped cold plasma modes shown in Fig. 2. For low frequencies and long wavelengths, the dispersion is acoustic in nature,  $\omega \sim k$ , with the lowest radial mode having the highest axial phase velocity,  $v_{ph} = \omega/k$ , shown by the dashed line. Higher order radial modes have lower axial phase velocities and in Fig. 2. lie below the lowest mode. If the phase velocity of a mode is less than 2-3 times  $v_{th}$  that mode will be strongly Landau-damped.

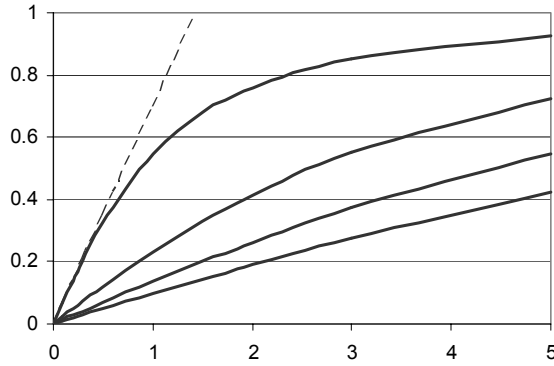


FIGURE 2. Radial mode frequencies,  $\omega/\omega_p$  versus  $ka$ , for  $m = 0$ ,  $b/a = 2$ , and  $T_p = 0$ . The dashed line indicates low frequency asymptote for the lowest ( $l = 1$ ) radial mode.

Furthermore, for  $ka, kb \ll 1$  (long wavelengths),  $G_1 \rightarrow 0$ ,  $G_2 \approx G_3 \approx 1/\ln(b/a)$ . In this limit,

$$\chi_{0n} \approx \frac{1}{-\frac{J_0(Ta)}{Ta J_1(Ta)} + \ln(b/a)} \quad (6)$$

with  $T^2 = -K_3 k_n^2$ . The plasma response,  $\chi_{mn}$ , becomes very large when the excitation frequency corresponds to a mode so the denominator of Eq. 6 must vanish. Thus, for a given mode,  $b/a$  determines  $Ta$ , irrespective of  $\omega$  and  $k$  for long wavelengths. For a cold plasma, and for frequencies well below the plasma frequency,  $K_3 \approx -\omega_p^2/\omega^2$ , and the low frequency phase velocity is  $v_{ph} = \omega/k = \omega_p a / Ta$ .  $Ta$  is less than 2.405 for the lowest mode, with  $Ta \approx 1.45$  for  $b/a = 2$ . Thus the ratio of velocity of the lowest mode to thermal velocity is  $\sim 0.7 \omega_p a / v_{th}$ , i.e. of the order of the plasma radius divided by the deBye length.

If the plasma radius is less than a deBye length, all modes will be strongly Landau-damped. As the deBye length becomes smaller, for example if the plasma temperature is reduced, the modes become less damped. In Fig. 3 we illustrate this by focusing on the contribution to  $q_{sc}^2$ , the charge fluctuation spectrum, from a single wave number,  $k_n$ .

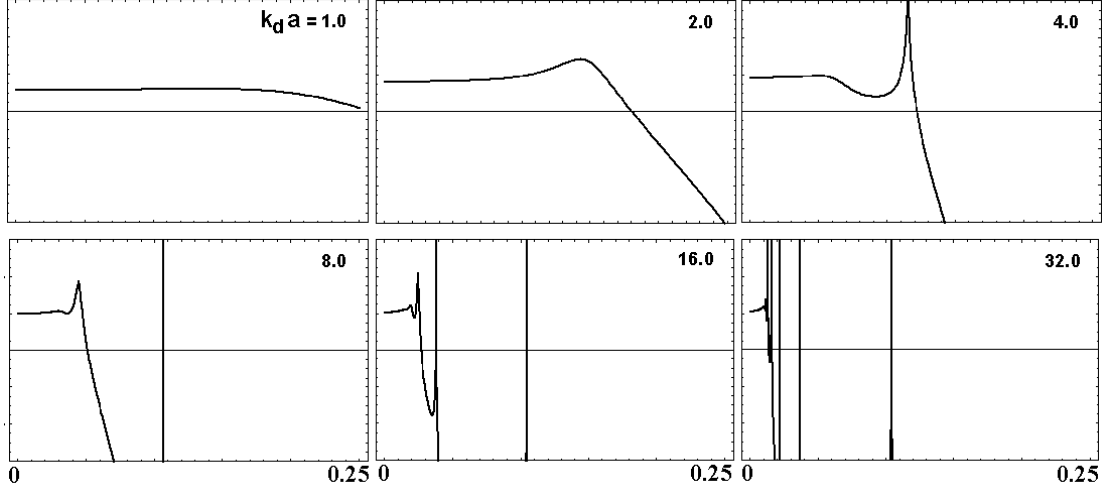


FIGURE 3. Semilog plot of the charge fluctuation spectrum on a sector probe for values of  $k_d a = 1, 2, 4, 8, 16$ , and  $32$ . The ordinate is the normalized frequency  $f = \omega/\omega_p$ . Parameters for this example are  $b/a = 2.5$ ,  $L/b = 11.4$ . When a mode becomes too narrow to resolve, it is shown as a vertical line.

We plot, on a semilogarithmic scale, the charge fluctuation spectrum on a patch electrode for values of  $k_d a = \omega_p a / v_{th} = 1, 2, 4, 8, 16$ , and  $32$ . The ordinate is the normalized frequency  $f = \omega/\omega_p$ . The dimensional factors in front of Eq. 4. have been omitted for this illustration and we have plotted  $Re\{M_{mn}^2 \chi_{mn}(f)/f\}$ , for  $b/a = 2.5$  and  $L/b = 11.4$ . As  $k_d a$  increases due to decreasing plasma temperature, modes become less damped and emerge, one at a time, from the continuous spectrum. The continuous spectrum due to non-resonant particles has a width  $\sim k v_{th}$ , which decreases with decreasing temperature.

When  $k_d a < 1$ , the single wave-number spectrum is nearly Gaussian (a parabola on the semi-log plot) with width  $k_n v_{th}$  (upper left panel of Fig 3.). This is the single wave number charge fluctuation spectrum for an uncorrelated Maxwellian velocity distribution of non-resonant particles. As  $k_d a$  increases due to decreasing temperature, the lowest radial mode begins to emerge from the Gaussian continuum, at first strongly damped, because the mode velocity is the same order as the thermal velocity. With further increase of  $k_d a$ , the continuous spectrum narrows and the lowest mode becomes less damped (peak is narrower and higher). One can also notice a slight downward shift of the mode frequency, due to a smaller "thermal shift". Still further increase in  $k_d a$  causes a further narrowing of the continuum, and higher radial modes (of lower phase velocity) emerge from the continuum. Fig. 3 is for a single wave number,  $k_1$ , but the results for the higher order axial modes,  $k = k_2, k_3, k_4 \dots$  are very similar except for a change in the frequency scale.

In Fig. 4. the various wave-number contributions ( $k_1 k_2 k_3 k_4$ ) have been added together as required in Eq. 4, for  $k_d a = 2.3$ . Also shown in Fig. 4. by the dashed curve is an approximate fit to the mode peaks using 4 simple poles. At frequencies below the lowest mode, one can still see the non-resonant particle contribution (labeled NRP in Fig. 4).

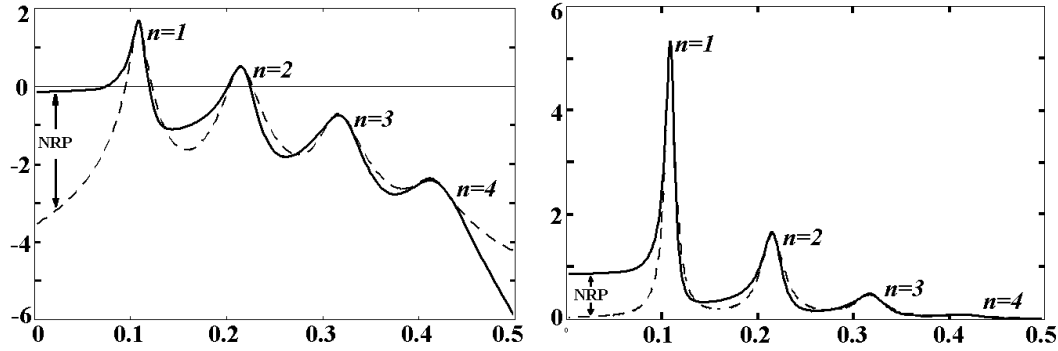


FIGURE 4. Semilog Plot (left) and linear plot (right) of the charge fluctuation spectrum versus  $f = \omega/\omega_p$  for  $k_d a = 2.3$ . Solid curve: calculation using plasma dispersion function, Dashed curve: 4 pole approximation to calculated result.

## DISCUSSION

It has been shown that the thermal fluctuation spectrum of a nonneutral plasma has both a broad continuous spectrum at low frequencies, due to non-resonant particles, together with peaks at the resonant modes of the plasma. A paper at this workshop [6] presents experimental results which shows this.

Modes whose axial phase velocity are more than 3-4 times the electron thermal speed are lightly damped and are clearly separated from the continuum. If the axial phase velocity of a mode becomes less than 1-2 times the electron thermal speed, then the mode becomes strongly Landau-damped and it merges into the continuum. Since mode velocities are of the order of  $\omega_p a$ , where  $a$  is the plasma radius, the plasma radius must be at least several debye lengths in order to have lightly damped modes. In general, the spectrum is a mixture of a continuous spectrum together with a finite number of modes which are Landau-damped by varying amounts, depending on their phase velocity relative to the electron thermal speed. Only in the extreme limit,  $\omega_p a \ll v_{th}$ , does the continuous spectrum tend to a Gaussian of width  $k v_{th}$ , characteristic of independent particles.

In order for the resonant modes to emerge from the non-resonant continuous spectrum, the modes must not be too strongly damped by Landau damping and this generally requires that the plasma be at least a few debye lengths across. Only if  $k_d a \ll 1$ , can one neglect collective effects (screening).

In order to simplify the presentation, it has been assumed that  $|Y_p| \ll |Y_l|$  in the results above. This is often the case, except when the mode damping becomes very small, and  $|Y_p|$  can become relatively large, and comparable with  $|Y_l|$  near a mode resonance. Then the following "load effects" must be accounted for when interpreting experimental data [1]: a) the load affects the magnitude of the signal observed, b) the capacitance of the load can shift the mode frequency slightly, c) the resistance of the load can increase the damping of the mode, d) the load resistance can be an additional source of thermal noise.

While the analysis here has been for cylindrical plasmas, similar results apply for spheroidal plasmas. Although the analysis of mixed cylindrical electrode geometry and

spheroidal plasma geometry can complicate an analysis, nevertheless the patch fluctuation spectra is still related to the input admittance of the plasma.

In the **RESULTS** section of this paper examples were given only for the axisymmetric  $m = 0$  modes. Some non-axisymmetric modes are negative energy modes and can be resistively destabilized. However, when they are not destabilized, the fluctuation spectrum is still related to the plasma admittance, but in a more subtle way [5].

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