APPLICATION OF GRANULAR MECHANICS TO THE ANALYSIS OF SOLID PROPELLANTS*

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ABSTRACT

In this paper a theoretical method is developed for analyzing the mechanical behavior of granular solid propellant materials. The granular nature of the material is specifically taken into account and, the analysis allows for a media composed of non-uniformly sized particles with random stacking configuration. The voids between the particles are assumed to be filled with an elastic, homogeneous binder material. Three types of internal forces are assumed to be acting; the normal and tangential contact forces between the granular particles and, the elastic stresses in the binder.

The paper consists of three main parts. First a model is developed to represent a general granular medium. Subsequently, in the second part this model is used to analyze the response of a granular medium to hydrostatic pressure loading. Finally the stress-strain relations are derived for a general loading condition. Because of the presence of the non-conservative frictional forces between the granular particles, the deformation of such a medium depends on the loading history. Consequently the stress-strain relations are in a differential, or incremental form.

INTRODUCTION

The fact that the solid propellants are granular in nature has usually been neglected in their characterization and analysis, instead the propellants have usually been described by a homogeneous continuum model. This approach has been successful mainly for two reasons: first, the homogeneous model was able, very satisfactorily, to describe most of the mechanical behavior of the propellant and secondly, the use of homogeneous analysis, as opposed to granular characterization, leads to relatively simple and standard mathematics. Nevertheless inspite of the above advantages of the homogeneous model the need for granular analysis of the propellant has been recognized for some time. The reason for this is that certain phenomena are direct consequence of the granular nature of the propellants and therefore the theoretical analysis of these can only be performed based on a granular model. One example of such a phenomena is the dewetting effect in which the rupture occurs of the bond between the binder and the oxidizer paritcles. Obviously such a phenomena could never be treated on the basis of homogeneous model.

* This work was sponsored by the Aerojet-General Corporation, Solid Rocket Plant, Sacramento Division, Sacramento, California, under the Contract No. P.O. S-420061-OP.

In the present paper it is intended to develop a method of analysis of granular media which can be directly applied to solid propellant materials. The paper is in three main parts. First a granular model is constructed to represent a material which contains an arbitrary number of groups of different size particles. The space between the particles is assumed to be filled with an elastic homogeneous material, which will be referred to as the binder. The particles are assumed to be stacked in an arbitrary configruation which is made to depend on the relative amounts of the binder and the granular material. In the second part of the paper this model is used to predict the response of a granular material to a hydrostatic compression. Finally the analysis is extended to a granular material under a general loading condition. This is done by obtaining the relationship between the deformation tensor for the whole material and the internal forces. These internal forces are the normal and tangential contact forces between the particles and the stresses in the binder material. The tangential forces are non-conservative and therefore the relationship between them and the deformation has to be written in an incremental form. This in turn leads to an incremental form for the stress-strain laws.

Since little previous analysis has been done in the area of granular media with irregular configuration, see for example the state-of-the-art review in Reference 1, the present analysis has to begin from basic principles. The basis of the analysis is the solution by Hertz (Ref. 2) of the elastic contact problem between two spherical bodies and the extension of this solution by Mindlin (Ref. 3) to include combined normal and tangential forces. These two analyses provide the necessary mathematical tools which will be used in this paper.

DEVELOPMENT OF A GRANULAR MODEL

A granular model which is to represent a solid propellant has to satisfy certain broad requirements. First the model has to take into consideration the irregular nature of the particles in the propellant and be able to predict the number and nature of contact points between these particles. Secondly, since the configuration can change during loading the model has to be such as to allow for this. Finally the model has to possess enough simplicity so that it can be handled mathematically. We shall try to create such a model.

It can be quickly established that a mathematically rigorous description of an irregular granular material is not possible and therefore certain approximations are necessary. In our analysis these approximations will be introduced when we average certain properties of the medium.

We shall assume that the granular medium is composed of x components, each component being a group of different size particles. We let N_1, N_2, $N_3 \ldots N_x$ denote the number of particles of the respective componts. It should be pointed out that assuming x discreet sizes in no way limits the generality of the present analysis since what follows could be developed for a granular model possessing a continuous distribution of particle sizes.

We now define an average radius of the particles in the ith component as follows:

$$R_i = \sqrt[3]{\frac{3}{4\pi} V_i} \tag{1}$$

where; V_i is the volume of the ith component particles.

We now introduce the average diameter of all the particles and define it as follows:

$$R = \sqrt[3]{\frac{\sum_i N_i R_i^3}{\sum N_i}} \tag{2}$$

It should be noted that definitions in equations (1) and (2) both are based on equal volume concept.

We shall now proceed to calculate the number of contact points, or in the granular mechanics terminology the coordination number. An exact evaluation of the coordination number for each component is not possible and therefore we have to proceed in a semi-empirical manner. Figure 1 below contains a two dimensional representation of a granular

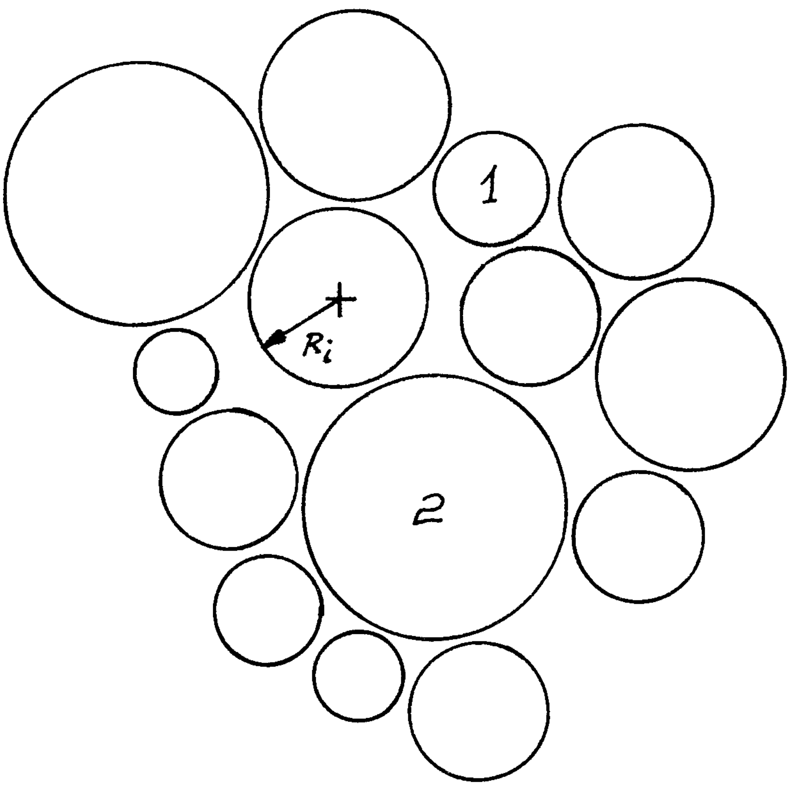

FIGURE 1. TWO DIMENSIONAL REPRESENTATION OF A GRANULAR MEDIUM.

medium composed of different size particles. It can be noted from this figure that, on the average, the smaller particles have a smaller number of contact points than the bigger ones. This can be explained as follows: any sphere of given size will, when it is in contact with another smaller sphere, subtend a larger solid angle at the center of the smaller sphere than when it is in contact with a larger one. Therefore this solid angle can be used as the measure of number of immediate neighbors that any size particle can have. In our present model we shall introduce a number, called the compaction number, which will be defined as the maximum number of average spheres that can be accommodated around the ith component sphere. This we will write as

$$C_i = \frac{4\pi}{\theta_i} \tag{3}$$

where; C_i is the compaction number

θ_i is the solid angle that a sphere of radius R subtends at the center of a sphere radius R_i, when contact exists.

It can easily be shown that

$$\theta_i = \left(1 - \frac{\sqrt{R_i^2 + 2RR_i}}{R_i + R}\right) 2\pi \tag{4}$$

therefore

$$C_i = \frac{2(R + R_i)}{R + R_i - \sqrt{R_i^2 + 2RR_i}} \tag{5}$$

In an actual configuration the number of neighbors that the ith sphere possesses will be less than C_i since this quantity was defined assuming complete composition, however C_i can be used as some measure of the actual number of neighbors.

It is now important to determine an expression for the coordination number of each different size particles. Any given sphere will be in contact with spheres of different size, the number of jth particles in contact with any ith particle will be denoted by α_{ij}. Clearly the total number of contact points γ_i on the ith particle is therefore

$$\gamma_i = \sum_j \alpha_{ij} \tag{6}$$

where the summation sign extends over j = 1 to x. The number α_{ij} will, first of all, be proportional to the number of possible neighbors that the ith particle can possess, that is C_i. Secondly, the smaller neighbors of the ith particle will be less likely to form a contact, this is illustrated in figure 1. It can be seen from this figure that the small sphere, marked 1, is further away from touching the ith sphere than the larger sphere marked 2. Now since the smallness, or the bigness, of any particle can be measured by the compaction number, therefore we can express the effect of the size on a probability of contact by the following qualitative relation,

$$\alpha_{ij} \propto C_i C_j \tag{7}$$

There is also a compatibility relation that has to be satisfied, this is that the total number of contact points, between the ith and jth particles, on all the ith particles and all the jth particles has to be equal. This mathematically is equivalent to

$$N_i \alpha_{ij} = N_j \alpha_{ji} \tag{8}$$

Therefore we finally write a relation for α_{ij} which satisfies the two conditions (7) and (8),

$$\alpha_{ij} = k \frac{c_i c_j}{N_i} \tag{9}$$

The coefficient k will be a function of the particular configuration which the model is suppose to represent and its evaluation will in general be by experimental methods. In order to account for the fact that the number of contact points will change with applied loading, and therefore with the change of compaction, we can write $k = k\,(p)$ where p is fractional porosity of the granular material. We have now sufficient information regarding the configuration of the granular material to enable us to solve certain problems. To illustrate this, the response to an external hydrostatic pressure will now be investigated.

RESPONSE TO HYDROSTATIC COMPRESSION

It is assumed that the granular medium is subject to a hydrostatic pressure the problem is to calculate the change of volume, or the bulk modulus. The space between the granules, which we shall refer to as voids, will be assumed to contain compressible liquid which has zero shear modulus. It is expected that, under the present loading condition, this liquid will closely approximate solid propellant binder.

The equation relating the pressure to volume change will be obtained by using the virtual work theorem. This is a similar approach to that used by Brandt (Ref. 4) in his analysis and therefore we will extend his method for one degree of freedom to our model which has many degrees.

As the external load is applied there will be two elastic deformations. The liquid contained in the voids will be compressed and the granules will be locally deformed at the points of contact. We shall denote the decrease in radius of ith sphere at the point of contact with the jth sphere by ΔR_{ij}. The original total volume of the granular material is

$$V = \frac{1}{1-p} \sum_i N_i \frac{4\pi}{3} R_i^3 \tag{10}$$

where p is the volume fraction of the total volume of the voids. As the hydrostatic pressure is applied there will be a decrease in the total volume, this decrease we denote by ΔV and it is easy to see that

$$\Delta V = \frac{1}{1-p} \sum_i N_i \, 4\pi \, R_i^2 \, \Delta R_i \tag{11}$$

where ΔR_i is the average change of radius of the ith particle and it is defined as

$$\Delta R_i = \sum_j \frac{\Delta R_{ij} \alpha_{ij}}{\gamma_i} \tag{12}$$

The change of volume of the voids will be equal to the total change of volume ΔV less the decrease in volume of all the particles at the point of contact. However the change of volume of the particles due to the elastic deformation can be shown to be of order $(\Delta R_{ij})^2$ and therefore to be consistent with Hertz's theory it has to be neglected with respect to ΔV. It follows, therefore, that the total change of volume is equal to the change of the void volume.

There are three types of energies involved in this problem. These are; the external work done by the pressure q_e, the elastic energy stored in the deformed particles, and finally the energy stored in the compressed void liquid.

Let us consider now a deformed state defined by the quantities ΔR_{ij} and imagine small virtual changes in these quantities $\delta(\Delta R_{ij})$. These virtual changes will produce small changes in the energy state of the system. The virtual change in the external work done is

$$\delta W = q_e \, \delta(\Delta V) \tag{13}$$

and from equation (11)

$$(\Delta V) = \frac{1}{1-p} \sum_i N_i \, 4\pi R_i^2 \, \delta(\Delta R_i)$$

Therefore

$$\delta W = \frac{q_e}{1-p} \sum_i \sum_j N_i \, 4\pi \frac{R_i^2 \alpha_{ij}}{\gamma_i} \, \delta(\Delta R_{ij}) \tag{14}$$

From Hertz's theory of contact the force between the ith and the jth sphere is given by

$$F_{ij} = \frac{4\sqrt{2}}{3} \frac{E}{1-\nu^2} \left(\frac{R_i R_j}{R_i + R_j} \right)^{\frac{1}{2}} (\Delta R_{ij})^{\frac{3}{2}} \tag{15}$$

where; E and ν are the Young's modulus and the Poisson's ratio of the granular material respectively. Therefore the virtual change of the strain energy E_1 stored in the particles of the medium is

$$\delta E_1 = \sum_i \sum_j N_i \alpha_{ij} F_{ij} \, \delta(\Delta R_{ij}) \tag{16}$$

substituting from equation (15) it follows

$$\delta E_1 = \sum_i \sum_j N_i \alpha_{ij} \frac{4\sqrt{2}}{3} \frac{E}{1-\nu^2} \left(\frac{R_i R_j}{R_i + R_j} \right)^{\frac{1}{2}} (\Delta R_{ij})^{\frac{3}{2}} \, \delta(\Delta R_{ij}) \tag{17}$$

Consider now the strain energy change in the void liquid, if q is the pressure in the liquid then, by definition of the bulk modulus,

$$dq = -\beta \frac{dV}{V} \tag{18}$$

integrating equation (18)

$$q = -\beta \ln \frac{V}{V_o} \tag{19}$$

where; $V = V_o - \Delta V$ and

V_o is the initial volume of the voids and is given by

$$V_o = \frac{p}{1-p} \sum_i N_i \frac{4\pi}{3} R_i^3 \tag{20}$$

It follows from equation (19), by expanding in series of $\Delta V/V_o$ and neglecting second order terms, that

$$q = \beta \frac{\Delta V}{V_o} \tag{21}$$

Therefore the virtual change in the strain energy E_2 of the void liquid is

$$\delta E_2 = q\, \delta(\Delta V)$$

$$= \beta \frac{\Delta V}{V_o} \frac{1}{1-p} \sum_i \sum_j N_i\, 4\pi\, R_i^2 \frac{\alpha_{ij}}{\gamma_i} \delta(\Delta R_{ij}) \tag{22}$$

For equilibrium of the system the virtual change in the external work must be equal to the changes in the two strain energies, and therefore

$$\delta W = \delta E_1 + \delta E_2 \tag{23}$$

and substituting from equations (14), (17) and (22) it follows

$$\sum_i \sum_j \Big[\frac{q_e}{1-p} N_i 4\pi \frac{R_i^2 \alpha_{ij}}{\gamma_i} = N_i \alpha_{ij} \frac{4\sqrt{2}}{3} \frac{E}{1-\nu^2} \left(\frac{R_i R_j}{R_i + R_j}\right)^{\frac{1}{2}} (\Delta R_{ij})^{\frac{3}{2}} + \beta \frac{\Delta V}{V_o} \frac{1}{1-p} N_i 4\pi R_i^2 \frac{\alpha_{ij}}{\gamma_i} \Big] \delta(\Delta R_{ij}) \tag{24}$$

Since the virtual changes $\delta(\Delta R_{ij})$ are arbitrary therefore equation (24) has to be satisfied for each of these quantities separately. However not all $\delta(\Delta R_{ij})$ quantities are independent since from Hertz contact theory it follows that

$$\Delta R_{ij} = \Delta R_{ji} \tag{25}$$

Therefore equation (24) represents a x/2(x+1) degree of freedom system and it has to be satisfied for each degree of freedom. This leads to x/2(x+1) equations

$$q_e = \beta \frac{\Delta V}{V_0} + \frac{\frac{2\sqrt{2}}{3\pi} \frac{E}{1-\nu^2} \left(\frac{R_i R_j}{R_i + R_j}\right)^{\frac{1}{2}} (\Delta R_{ij})^{\frac{3}{2}}}{\left(\frac{R_i^2}{\gamma_i} + \frac{R_j^2}{\gamma_j}\right)} \tag{26}$$

in the derivation of equation (26) the relation (8) was used. The set of x/2(x+1) equations represented by (26) can now be solved for ΔR_{ij} and using the relation

$$\Delta V = \frac{1}{1-p} \sum_i \sum_j N_i 4\pi \frac{R_i^2}{\gamma_i} \alpha_{ij} \Delta R_{ij}$$

and substituting for ΔR_{ij}, obtained from (26), a relationship between q_e and ΔV follows. This is done as follows, from equation (26)

$$\Delta R_{ij} = \left[\frac{\left(\frac{R_i^2}{\gamma_i} + \frac{R_j^2}{\gamma_j}\right)}{\frac{2\sqrt{2}}{3\pi} \frac{E}{1-\nu^2} \left(\frac{R_i R_j}{R_j + R_i}\right)^{\frac{1}{2}}}\right]^{\frac{2}{3}} \left(q_e - \beta \frac{\Delta V}{V_0}\right)^{\frac{2}{3}} \tag{27}$$

Substituting into equation for ΔV from (27)

$$\Delta V = \frac{1}{1-p} \sum_i \sum_j N_i \frac{4\pi R_i^2}{\gamma_i} \alpha_{ij} \left[\frac{\frac{R_i^2}{\gamma_i} + \frac{R_j^2}{\gamma_j}}{\frac{2\sqrt{2}}{3\pi} \frac{E}{1-\nu^2} \left(\frac{R_i R_j}{R_j + R_i}\right)^{\frac{1}{2}}}\right]^{\frac{2}{3}} \left(q_e - \beta \frac{\Delta V}{V_0}\right)^{\frac{2}{3}} \tag{28}$$

Rearranging equation (28) we obtain the final expression for the relation between the external pressure q_e and the volume change ΔV,

$$q_e = \beta \frac{\Delta V}{V_0} + \left[\frac{(1-p)\Delta V}{\sum_i \sum_j N_i \frac{4\pi R_i^2}{\gamma_i} \alpha_{ij} \left[\frac{\frac{R_i^2}{\gamma_i} + \frac{R_j^2}{\gamma_j}}{\frac{2\sqrt{2}}{3\pi} \frac{E}{1-\nu^2} \left(\frac{R_i R_j}{R_j + R_i}\right)^{\frac{1}{2}}}\right]^{\frac{2}{3}}}\right]^{3/2} \tag{29}$$

It should be noted that for very small volume change equation (29) reduces to

$$q_e = \beta \frac{\Delta V}{V_o}$$

and therefore the granular medium will respond with the bulk modulus of the binder.

In equation (26) there are a number of quantities which depend on the porosity p. As the hydrostatic pressure increases the porosity decreases and therefore the quantities which depend on p will also change. If the porosity change is appreciable then it has to be taken into consideration in the solution of (26) and the solution will no longer be as simple as indicated above. In such a case then an approximate solution to (26) would have to be obtained. Such an approximate solution could be carried out by dividing the change of volume into a number of small changes and solving (26) for each small change assuming that the porosity is constant over any particular step.

The porosity is defined by the following relation

$$p = \frac{V_o - \Delta V}{V - \Delta V} \tag{30}$$

and expanding in the powers of $\Delta V/V$ it follows

$$p = p_o - (1-p_o)\frac{\Delta V}{V} \tag{31}$$

where; $p_o = V_o/V$ and it is the original porosity. Therefore the change of porosity Δp is given by

$$\Delta p = p - p_o = -(1-p_o)\frac{\sum_i \sum_j N_i \frac{4\pi R_i^2}{\gamma_i} \alpha_{ij} \, \Delta R_{ij}}{\sum_i N_i \frac{4\pi}{3} R_i^3} \tag{32}$$

It can be seen that the change of porosity will usually be small since it is of order of ΔR_{ij} and therefore for most practical applications it can be neglected. However, depending on the accuracy required from the calculation, its effect can always be evaluated as indicated above.

DEVELOPMENT OF THE "STRESS-STRAIN" LAWS FOR GENERAL LOADING

The relations between the deformations and the various forces acting in the material can be regarded, with the analogy with the continuum material, as the stress-strain laws. This terminology will be used in the present analysis.

The first concept which has to be developed is that of the surface in a granular material. Components of the resultant force on this surface will then be defined. Mathematically a plane surface can be defined in a granular material in the same way as in the continuum, however, if this is done then the resultant force on this surface will be a function of the inter-granular forces, the stress in the filler, and also the internal stresses inside the granular particles. This last dependence arises since the surface will in general pass through various particles cutting them at arbitrary angles. Inclusion in the analysis of the internal stresses would lead to an extremely complex situation since the stress anslysis of the various granular particles would be required. Therefore a new surface is introduced so that its position and direction are approximately defined by the original plane surface but it does not pass through the interior of the granular particles. This new surface, over which the forces will be computed, weaves between the particles and is therefore non-planar. In two dimensions the situation can be represented by the Figure 2 below. This new surface is still not uniquely determined since it can move an arbitrary distance above and below the reference plane. However, it is intended that this surface follow the reference plane as closely as the size of the particles permits. In general this means that the maximum distance from the reference plane should not exceed one-half of the equivalent diameter of the particle of maximum size.

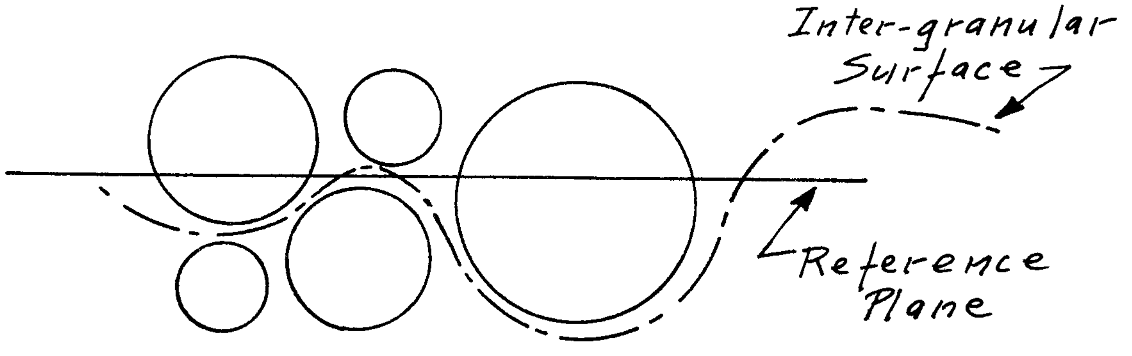

FIGURE 2. INTER-GRANULAR SURFACE.

Particles of maximum size can be denoted by the subscript unity without any loss of generality and therefore the maximum radius will be denoted by R_1 . The inter-granular surface, henceforth designated simply "the surface," will lie anywhere inside a plane region of thickness $2R_1$ having the reference plane as its center.

The next step is to evaluate the number of particles of each size which will be adjacent to the surface. Denoting the number of particles per unit volume by n_i , subscript indicating the ith group, we have from the previous section

$$n_i = \frac{N_i(1-p)}{\sum_j N_j \frac{4\pi}{3} R_j^3} \tag{33}$$

In the region of the surface the number of particles of the ith group, per unit area of reference plane, will be given by

$$R_1 n_i = \frac{R_1 N_i(1-p)}{\sum_j N_j \frac{4\pi}{3} R_j^3} \tag{34}$$

Not all the particles contained in this surface region will have the same probability of being adjacent to the surface. On qualitative grounds it can easily be deduced that the smaller particles will have a lesser chance of coming into contact with the surface that the larger ones. This is illustrated by Figure 3.

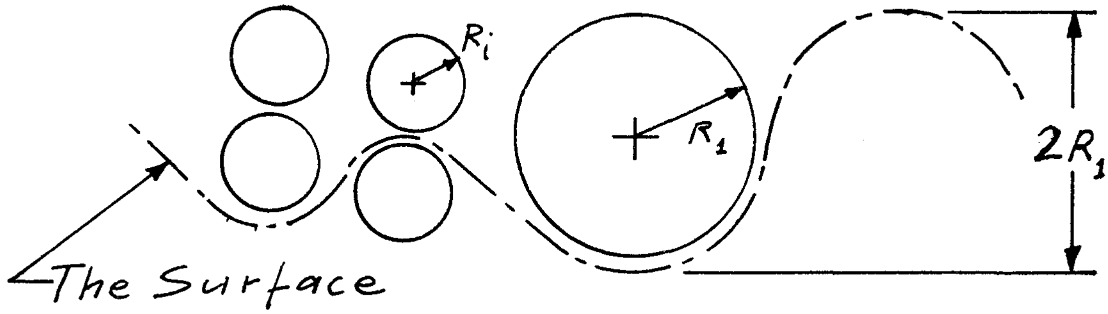

FIGURE 3. AMPLITUDE OF THE INTER-GRANULAR SURFACE.

In fact the probability of any particle to be adjacent to the surface can be taken to be proportional to the radius. The largest particle with radius R_1 will most definitely be adjacent to the surface and therefore its probability can be taken as unity. On this basis the probability for the ith particle is R_i/R_1. Using equation (34) the number of particles β_i of each size adjacent to the surface can be written as

$$\beta_i = R_1 n_i \frac{R_i}{R_1} = \frac{R_i N_i (1-P)}{\sum_j N_j \frac{4\pi}{3} R_j^3} \tag{35}$$

The amount of area of each adjacent particle which will be in contact with the surface will also have to be determined. We can see from the above sketch that the largest particles will have about one-half of their area adjacent to the surface while the smaller ones will have less. This again can be put in terms of probability based on the radius of the particles. We can assume that the area fraction a_i of the ith particle in contact with the surface is given by

$$a_i = \frac{1}{2} \frac{R_i}{R_1} \tag{36}$$

In engineering applications only the average deformation of the granular material will be observed. This deformation can be represented by a strain tensor in the same way as for a continuum. This tensor is denoted here by $\epsilon_{k\ell}$ where k and ℓ refer to a cartesian orthogonal coordinate system. Consider now <u>two</u> particles in contact, the vector joining the centers is denoted by (R_i+R_j). On the average all the distances in the medium will change according to the strain field $\epsilon_{k\ell}$ and therefore the change of the distance between the two particles considered can be written as

$$(R_i + R_j)^2 - (R_i^o + R_j^o)^2 = 2\epsilon_{k\ell}(R_i^o + R_j^o)_k (R_i^o + R_j^o)_\ell \tag{37}$$

where R_i^o denotes the original radius and $(R_i^o + R^o{}_j)_k$ is a component of the vector joining the particle centers. The summation in equation (37) is on k, ℓ = 1, 2, 3.

Using the results of the Hertz theory (Ref. 2) of contact the normal force between the two particles can be written as

$$F_{ij} = \begin{cases} \frac{2}{3} \frac{E}{1-\nu^2} \left(\frac{R_i^o R_j^o}{R_i^o + R_j^o} \right)^{\frac{1}{2}} \left[2\epsilon_{kl} (R_i^o + R_j^o)_k (R_i^o + R_j^o)_l \right] & \text{when } R_i + R_j < R_i^o + R_j^o \\ 0 & \text{when } R_i + R_j > R_i^o + R_j^o \end{cases} \tag{38}$$

where E is the Young's modulus and ν is the Poisson's ratio of the granules. The components of this force along the three cartesian axes can be written as

$$(F_{ij})_m = F_{ij} \frac{(R_i^o + R_j^o)_m}{R_i^o + R_j^o} \tag{39}$$

The contribution from the normal contact forces to the components of the total force acting on the inter-granular surface can now be obtained by combining equations (35), (36), (38) and (39). Denoting these components by τ_k^1 it follows that

$$\begin{aligned} \tau_k^1 &= \sum_i \sum_j \alpha_{ij} a_i \beta_i (F_{ij})_k \\ &= \sum_i \sum_j \frac{R_i N_i (1-P)}{\sum_i N_i \frac{4\pi}{3} R_i^3} \alpha_{ij} \frac{R_i}{2R_1} \frac{2}{3} \frac{E}{1-\nu^2} \left(\frac{R_i^o R_j^o}{R_i^o + R_j^o} \right)^{\frac{1}{2}} \times \\ &\quad \left[2\epsilon_{lm} (R_i^o + R_j^o)_l (R_i^o + R_j^o)_m \right] \end{aligned} \tag{40}$$

where it is understood that $\tau_k^1 = 0$ when $(R_i + R_j) > (R_i^o + R_j^o)$ and the summations on l and m is still implied. The above expression still depends on the distribution of contact points since this affects the components $(R_i^o + R_j^o)_k$. The discussion of this distribution will be left until later in the analysis.

Next we turn our attention to the tangential forces between the granules. In order to evaluate these forces it is necessary to obtain the expressions for the slip at each contact point. The slip can occur in two ways, it can be due to the angular motion of one particle around another while both remain with the same orientation, also it can be produced if the centers of the two particles remain in fixed position and the particles rotate about the centers. These two situations are illustrated in two dimensions in Figure 4.

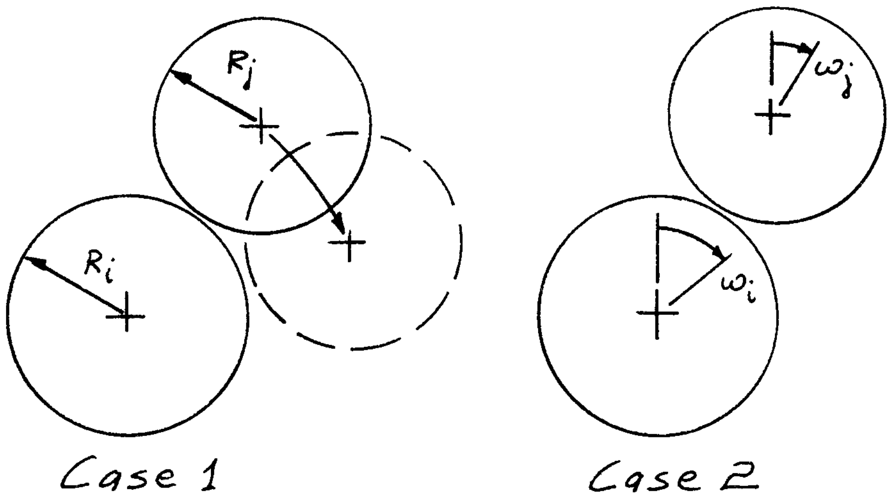

FIGURE 4. TANGENTIAL MOTION OF PARTICLES.

The amount of slip in the Case 1 can be related to the shear deformation of the medium. For clarity we shall first develop this for the above two dimensional situation. Consider the two particles imbedded in a deformable medium as shown below in Figure 5.

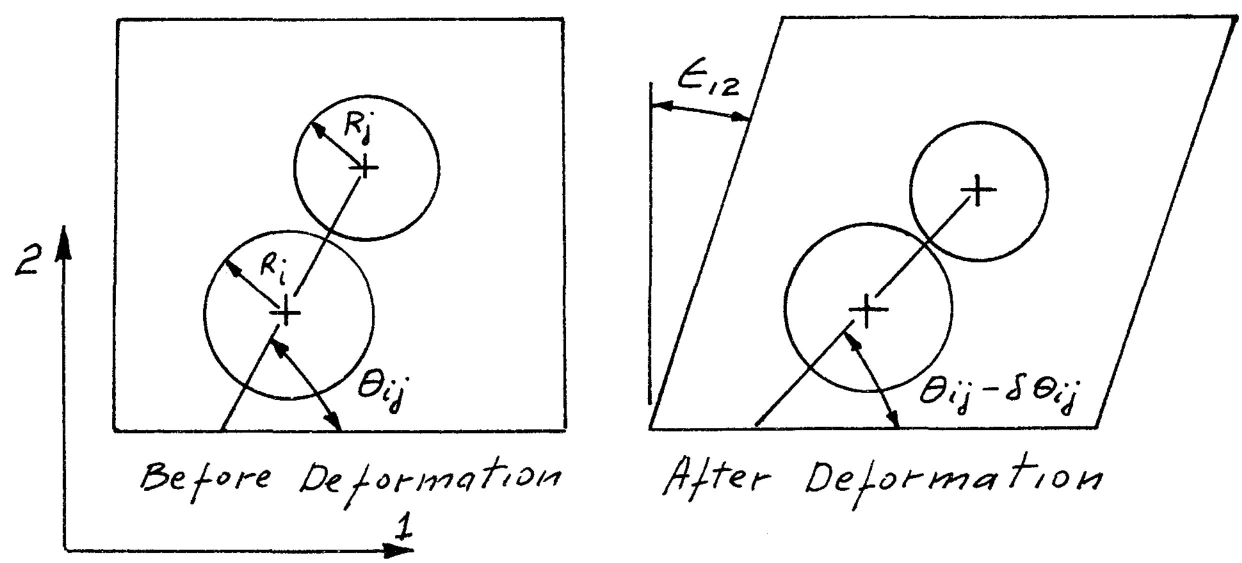

FIGURE 5. RELATION BETWEEN SLIP AND SHEAR STRAIN.

The deformation of the matrix medium is given by the shear strain, ϵ_{12} say, and therefore the change of angle $\delta\theta_{ij}$ is given by

$$\delta\theta_{ij} = \epsilon_{12}\theta_{ij} \tag{41}$$

Therefore if the particles remained with the original orientation in space then the amount of slip $S_{ij}{}^{1}$ due to the relative position change would be given by

$$S^{1}_{ij} = \delta\theta_{ij}(R_i + R_j) = \epsilon_{12}(R_i + R_j)\theta_{ij} \tag{42}$$

The additional slip $S_{ij}{}^{2}$ due to the rotation of the particles about their own centers is obviously

$$S^{2}_{ij} = -(R_i\omega_i + R_j\omega_j) \tag{43}$$

Therefore the total slip S_{ij} is

$$S_{ij} = S^{1}_{ij} + S^{2}_{ij} = \epsilon_{12}(R_i + R_j)\theta_{ij} - (R_i\omega_i + R_j\omega_j) \tag{44}$$

These ideas will now be extended to the three dimensional case. The components of rotation, about the three cartesian axes, of each particle are denoted by $(\omega_i)_k$. Also, there will be three angles $(\theta_{ij})_k$ one associated with each axis, these angles are shown in the Figure 6 below.

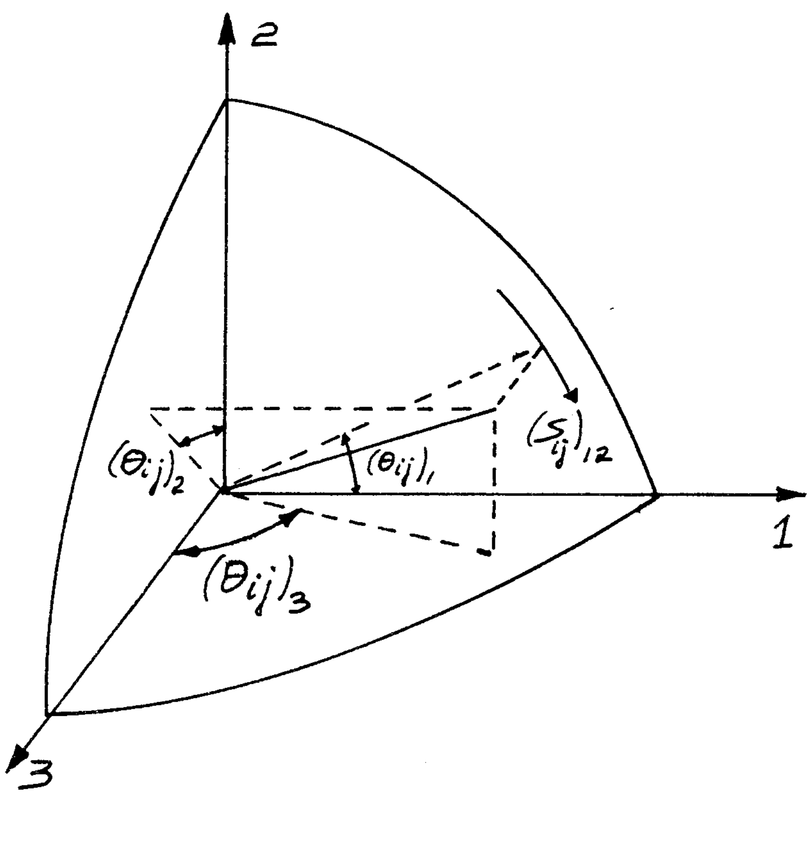

FIGURE 6. THREE-DIMENSIONAL SLIP

Using the same arguments as for the two dimensional case the projection of the slip line in each of the cartesian planes can be written as

$$(S_{ij})_{12} = \epsilon_{12}\left\{\sqrt{(R_i+R_j)_1^2 + (R_i+R_j)_2^2}\right\}(\theta_{ij})_1 - \left\{(\omega_i)_3\sqrt{(R_i)_1^2+(R_i)_2^2} + (\omega_j)_3\sqrt{(R_j)_1^2+(R_j)_2^2}\right\} \tag{45}$$

with similar expressions for $(S_{ij})_{23}$ and $(S_{ij})_{31}$. Components of these projections along the three axes are

$$(S_{ij})_1 = (S_{ij})_{12}\sin(\theta_{ij})_1 - (S_{ij})_{31}\sin(\theta_{ij})_3 \tag{46}$$

with similar expressions for $(S_{ij})_2$ and $(S_{ij})_3$. These three components define the magnitude and the direction of the slip between the ith and jth particle.

As mentioned before the tangential force acting between the particles of the material is a frictional force and therefore the relation between the amount of slip and the load is dependent on deformation history. In such a case the relation between the load and slip has to be written in an incremental form. This idea was introduced in reference 4 where vibration of a regular granular material was analyzed. Writing equation (46) in an incremental form we obtain

$$\delta(S_{ij})_1 = \delta(S_{ij})_{12}\sin(\theta_{ij})_1 - \delta(S_{ij})_{31}\sin(\theta_{ij})_3 \tag{47}$$

where, from equation (45),

$$\delta(S_{ij})_{12} = \delta(\epsilon_{12})\left\{\sqrt{(R_i+R_j)^2+(R_i+R_j)_2^2}\right\}(\theta_{ij})_1 - \delta(\omega_i)_3\sqrt{(R_i)_1^2+(R_i)_2^2} - \delta(\omega_j)_3\sqrt{(R_j)_1^2+(R_j)_2^2} \tag{48}$$

with similar expression for $\delta(S_{ij})_{31}$. Denoting the tangential contact force by T_{ij} we can write the incremental relation between this force and the slip S_{ij} in the form

$$\delta(S_{ij}) = C_T\,\delta(T_{ij}) \tag{49}$$

where C_T is defined as the tangential compliance. More will be said about this quantity later. Equation (17) can be written in the component form

$$\delta(T_{ij})_k = \frac{1}{C_T}\,\delta(S_{ij})_k \tag{50}$$

where $\delta(S_{ij})_k$ can be expressed in terms of the incremental changes in the strains $\delta(\epsilon_{k\ell})$ and the incremental changes in rotation $(\omega_i)_k$ from equations (47) and (48).

As in the case of the normal contact forces the tangential forces acting on the inter-granular surface will have force components. Denoting these components, per unit area of the reference plane, by $\tau_m^{\ 2}$ we can write, by using similar arguments as in equation (40),

$$\delta(\tau_m^{\ 2}) = \sum_i \sum_j \alpha_{ij}\, a_i \beta_i\, \delta(T_{ij})_m$$

$$= \sum_i \sum_j \alpha_{ij}\, a_i \beta_i\, \frac{1}{C_T}\, \delta(S_{ij})_m \tag{51}$$

It still remains to consider the contribution to the forces on the surface due to the stresses in the filler material. In the present analysis it is assumed that the material in the voids is much softer than the granular material. In view of this the change in bulk deformation in the total material will be taken up by the filler. This follows from the fact that the volume change of the granules at the points of contact is of second order as was shown in the previous section. Therefore the normal strains in the filler denoted by e_{kk} will actually be larger than the average strains ϵ_{kk} of the whole medium, we write

$$e_{kk} = A\epsilon_{kk} \tag{52}$$

where A is some magnification factor. Since the apparent volume dilation of the total granular material is equal to the change of void volume, therefore

$$\sum_k (e_{kk})\, pV = \sum_k (A\,\epsilon_{kk})\, pV = \left(\sum_k \epsilon_{kk}\right) V \tag{53}$$

and $$A = \frac{1}{p}$$

The shear strains in the filler material will on the average be the same as for the overall material. Assuming classical elastic behavior for the filler the stresses $\sigma_{k\ell}$ in this material are

$$\sigma_{k\ell} = \lambda\, \frac{1}{p} \left[\sum_k \epsilon_{kk}\right] \delta_{k\ell} + 2\mu\, \epsilon_{k\ell} \tag{54}$$

where λ and μ are the Lame's constants, and $\delta_{k\ell}$ is the Kronecker delta. The contribution to the components of force per unit area of this surface from the stresses in the filler is denoted by $\tau_k^{\ 3}$ and it can be written

$$\tau_k^{\ 3} = \frac{B_\ell}{p}\left[\lambda\left(\sum_k \epsilon_{kk}\right) \delta_{k\ell} + 2\mu p\, \epsilon_{k\ell}\right] \tag{55}$$

(summation on ℓ)

where B_ℓ is some multiplying constant which depends on the orientation of the surface with respect to the cartesian coordinate system and also on the amount of filler in contact with the surface. If the inter-granular surface is chosen perpendicular to the one of the cartesian axis, ℓ say, then B_ℓ will only be function of the latter effect. In agreement with the results for the hydrostatic compression case previously analyzed we choose $B_{\ell} = p$. Therefore

$$\tau_k^3 = \lambda\left(\sum_k \epsilon_{kk}\right)\delta_{k\ell} + 2\mu p \epsilon_{k\ell} \tag{56}$$

Finally the kth component of the total force acting on an intergranular surface can be obtained by combining equations (40), (51) and (56). Because of equation (51) this final expression has to be written in the incremental form

$$\begin{aligned} \delta(\tau_m) &= \delta(\tau_m^1) + \delta(\tau_m^2) + \delta(\tau_m^3) \\ &= \sum_i \sum_j \frac{R_i N_i (1-p)}{\sum N_i \frac{4\pi}{3} R_i^3} \alpha_{ij} \frac{R_i}{2R_1} \frac{2}{3} \frac{E}{1-\nu^2} \left(\frac{R_i^0 R_j^0}{R_i^0 + R_j^0}\right)^{\frac{1}{2}} \times \\ &\quad \left[2\delta(\epsilon_{k\ell})(R_i^0 + R_j^0)_k (R_i^0 + R_j^0)_\ell\right] \\ &\quad + \sum_i \sum_j \alpha_{ij} a_i \beta_i \frac{1}{C_T} \delta(S_{ij})_m \\ &\quad + \lambda \sum_k \left[\delta(\epsilon_{kk})\right] \delta_{m\ell} + 2\mu p\, \delta(\epsilon_{m\ell}) \end{aligned} \tag{57}$$

The above equation is the required "stress-strain" law for the granular material. It relates to the incremental changes in the components of the resultant force, acting on an interior surface of a granular material perpendicular to the ℓ-axis, to the incremental changes of the strains $\delta(\epsilon_{k\ell})$ and the incremental rotation components $\delta(\omega_i)$. Before equation (57) can be applied it is necessary to develop further ideas regarding the tangential compliance C_T, the components of rotation $(\omega_j)_k$ and, the distribution of contact points on the granular particle.

We shall first deal with the tangential compliance C_T. As discussed before the form of this function depends on the history of loading. Based on Mindlin's original work on the tangential force (Ref. 3) Mindlin and Deresiewicz (Ref. 6) have examined a number of different loading histories in detail. We shall use the results of this analysis in our investigation. We shall reproduce the results from reference 6 for only certain loading conditions which are the most likely to be of practical interest. For any other more complex histories the necessary expression for C_T can always be developed using the approach of reference 6. For the sake of simplicity we shall denote the normal force by F and the corresponding tangential force by T.

Case 1. F and T are increased at any arbitrary rates. For this case

$$C_T = \frac{2-\nu}{8\mu a}\left[f\frac{dF}{dT} + \left(1 - f\frac{dF}{dT}\right)\left(1 - \frac{T}{fF}\right)^{-\frac{1}{3}}\right] \tag{58}$$

when $0 < \frac{dF}{dT} < \frac{1}{f}$

and $$C_T = \frac{2-\nu}{8\mu a} \qquad \text{when } \frac{dF}{dT} \geqslant \frac{1}{f} \tag{59}$$

In the above expressions ν is the Poisson's ratio, μ is the shear modulus, a is the instantaneous radius of contact and, f is the static coefficient of friction between the granular particles.

Case 2. F decreasing and T increasing at any arbitrary rate. For this case

$$C_T = \frac{2-\nu}{8\mu a}\left[-f\frac{dF}{dT} + \left(1+f\frac{dF}{dT}\right)\left(1-\frac{T}{fF}\right)^{-\frac{1}{3}}\right] \tag{60}$$

Case 3. F increasing and T decreasing at same arbitrary rates. For this case

$$C_T = \frac{2-\nu}{8\mu a}\left[f\frac{dF}{dT} + \left(1-f\frac{dF}{dT}\right)\left(1-\frac{T^*-T}{2fF}\right)^{-\frac{1}{3}}\right] \tag{61}$$

when $0 > \frac{dF}{dT} \geqslant -\frac{1}{f}$

and

$$C_T = \frac{2-\nu}{8\mu a} \tag{62}$$

when $\frac{dF}{dT} \leq -\frac{1}{f}$

In equation (61) T* represents the initial value of the tangential force T.

Case 4. F decreasing and T decreasing at some arbitrary rate. For this case

$$C_T = \frac{2-\nu}{8\mu a}\left[-f\frac{dF}{dT} + \left(1+f\frac{dF}{dT}\right)\left(1-\frac{T^*-T}{2fF}\right)^{-\frac{1}{3}}\right] \tag{63}$$

The above four cases cover the most common possible situation. It should be noted that the frictional force T can never physically exceed the value of fF, this always has to be kept in mind as this analysis is applied. This of course imposes a ceiling on the tangential force and introduces a discontinuity into the analysis. However no additional difficulties arise since the solution of any problem has to be obtained by incremental procedure and this type of discontinuity is easily taken into account.

We shall now deal with the components of rotation $(\omega_j)_k$ of the granules. These quantities are additional deformation parameters which cannot be evaluated on the basis of the analysis so far. To evaluate these parameters it is necessary to consider the rotational equilibrium of the granular particles. There are two types of rotational forces acting on the particles, the tangential forces at the contact points and the tangential forces between the granules and filler material in the voids. The rotational incremental moments due to contact forces can be obtained from equation (50) and these are in the component form, for the ith particle

$$\delta(M_k^1) = \sum_j \delta(T_{ij})_\ell \, \chi_m \, \delta_{\ell m} \tag{64}$$

where $\ell \neq k$, $m \neq k$ and, χ_m are the coordinates of the contact point. The rotational moments due to the filler are little less easy to estimate analytically. However, since the filler is elastic the restoring moment, on the particle, due to its rotation, has to be proportional to the rotation. Therefore the incremental moment due to this effect can be written as

$$\delta(M_k^2) = G \, \delta(\omega_k) \tag{65}$$

where G is some constant which depends on the shear modulus of the filler and the granular geometric configuration. Due to geometric complexities it would be very difficult to estimate G analytically and therefore experimental methods would have to be used. The total restoring incremental moment is therefore

$$\begin{aligned} \delta(M_k) &= \delta(M_k^1) + \delta(M_k^2) \\ &= \sum_j \delta(T_{ij})_\ell \, \chi_m \, \delta_{\ell m} + G \, \delta(\omega_k) = 0 \end{aligned} \tag{66}$$

The three equations (66) provide the additional relations from which the components of rotation can be evaluated.

It remains now to discuss the distribution of the contact points with respect to the cartesian coordinate system. If on any particle there are many points of contact then any arbitrary distribution should yield essentially the same result because of the averaging process. However for cases with few such contact points the problem becomes more involved. In such cases it is necessary to introduce the idea of probability. Consider a spherical particle in the cartesian coordinate system and imagine that its surface is divided into equally small area segments. Each small area is completely defined by the orientation with respect to a cartesian coordinate system. Suppose that this particle possesses α contact points, then the porbability that any contact point falls in some area A_M is $1/M$; where M is the number of area segments into which the surface has been divided. Since there are α contact points than the number of contact points in any area is α/M. This defines how the contact points should be oriented with respect to the cartesian coordinate system.

CONCLUDING REMARKS

Some specific comments regarding this granular analysis are now in order. It is quite probable that the model chosen to represent the medium will only by applicable to certain propellant materials. Such propellants would have reasonably dense packing in order for the averaging processes, assumed in the analysis, to hold. Furthermore, the particle shapes would have to have approximately spherical shapes. For certain propellants, for example such that contain dendritic type particles, this analysis would be wholly inapplicable.

In the course of the development of the analysis a number of empirical assumptions and a number of averaging processes was used. Consequently the analysis is far from rigorous, however it is felt that most important features of the granular material have been taken into consideration. The final proof of such analysis can only be through experimental verification and, it would be a useful extension of the present investigation if such experimental work was carried out and compared with the results of the present analysis. We, therefore, propose that certain simple geometries of highly filled propellant, subject to simple loading conditions, be analyzed by the present method and tested experimentally. This should provide a useful check for the theoretical method.

In spite of the semi-empirical nature of this analysis we feel that it can be useful in analyzing actual propellant materials. What is more important it is hoped that the present investigation will stimulate more work in this area which, up to the present time, has not received much attention.

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LIST OF SYMBOLS

A	constant defined in equation (52)
B_ℓ	constant defined in equation (55)
C_i	compaction number for the ith particle
C_T	tangential compliance
E	Young's modulus of granular particles
E_1, E_2	strain energy of granules and binder, respectively
F	normal contact force
F_{ij}	normal contact force between ith and jth particles
G	constant defined in equation (65)
M_k^1, M_k^2, M_k	moments acting on the granular particles

N_i , N	number of particles of ith size and total number, respectively
R	average radius of all particles
R_i^o, R_i	the initial and instantaneous average radius of ith particle
ΔR_{ij}	changes in radius at the contact point between ith and jth particles
ΔR_i	average change of radius of ith particle
S^1_{ij}, S^2_{ij}, S_{ij}	relative tangential slips between ith and jth particle
T	tangential force
T_{ij}	tangential force between ith and jth particle
V_o , V	initial and instantaneous volume of all particles
V_i	volume of ith particle
ΔV	change in volume
W	external work
a	contact radius
a_i	defined in equation (36)
$e_{k\ell}$	strain in binder
f	coefficient of friction
k	defined in equation (9)
n_i	number of ith particles per unit volume
p_o, p	initial and instantaneous porosity
q	hydrostatic pressure in the binder
q_e	external hydrostatic pressure
x	number of different size particles
x_m	cartesian coordinates
α_{ij}	number of contact points between ith and jth particle
β	bulk modulus of the binder
β_i	defined in equation (35)
γ_i	number of contact points on the ith particle
δ	denotes an infinitesimal change
$\epsilon_{k\ell}$	strain
θ_i	solid angle
θ_{ij}	andle defined in figure 5.
λ, μ	Lame constants
ν	Poisson's ratio
$\sigma_{k\ell}$	stress in the binder
τ^1_k, τ^2_k, τ_k	resultant forces on the inter-granular surface
i	ith particle rotation

Subscripts: i, j - refer to the particles

1, 2, 3, k, ℓ, m - refer to the cartesian orthogonal reference coordinate system.