Local Quantum Measurement and No-Signaling Imply Quantum Correlations

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We show that, assuming that quantum mechanics holds locally, the finite speed of information is the principle that limits all possible correlations between distant parties to be quantum mechanical as well. Local quantum mechanics means that a Hilbert space is assigned to each party, and then all local positive-operator-valued measurements are (in principle) available; however, the joint system is not necessarily described by a Hilbert space. In particular, we do not assume the tensor product formalism between the joint systems. Our result shows that if any experiment would give nonlocal correlations beyond quantum mechanics, quantum theory would be invalidated even locally.

Quantum correlations between spacelike separated systems are, in the words of Schrödinger, "*the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought"[1]. Indeed, the increasing experimental support [2] for correlations violating Bell inequalities [3] is at odds with local realism. Quantum correlations have been investigated with increasing success [4], but what is the principle that limits them [5]?

Consider two experimenters, Alice and Bob, at two distant locations. They share a preparation of a bipartite physical system, on which they locally perform one of several measurements. This shared preparation may thereby cause the distribution over the possible two outcomes to be correlated. In nature, such nonlocal correlations cannot be arbitrary. For example, it is a consequence of relativity that information cannot propagate faster than light. The existence of a finite upper bound on the speed of information is known as the principle of *no-signaling*. This principle implies that if the events corresponding to Alice's and Bob's measurements are separated by spacelike intervals, then Alice cannot send information to Bob by just choosing a particular measurement setting. Equivalently, the probability distribution over possible outcomes on Bob's side cannot depend on Alice's choice of measurement setting, and vice versa. Quantum mechanics, like all modern physical theories, obeys the principle of no-signaling.

But is no-signaling the only limitation for correlations observed in nature? Bell [3] initiated the study of these limitations based on inequalities, such as the Clauser-Horne-Shimony-Holt (CHSH) expression [6]. It is convenient to describe this inequality in terms of a game played by Alice and Bob. Suppose we choose two bits $x, y \in \{0, 1\}$ uniformly and independently at random, and hand them to Alice and Bob, respectively. We say that the players win if they are able to return answers $a, b \in \{0, 1\}$, respectively, such that $xy = a + b \mod 2$. Alice and Bob can agree on

any strategy beforehand, that is, they can choose to share any preparation possible in a physical theory, and choose any measurements in that theory, but there is no further exchange of information during the game. The probability that the players win is

$$\frac{1}{4} \sum_{x,y \in \{0,1\}} \sum_{\substack{a,b \in \{0,1\} \\ xy = a+b \bmod 2}} p(a, b | M_A^x, M_B^y), \qquad (1)$$

where $p(a, b | M_A^x, M_B^y)$ denotes the probability that Alice and Bob obtain measurement outcomes a and b when performing the measurements M_A^x and M_B^y, respectively (any preprocessing or postprocessing can be taken as part of the measurement operation). Classically, i.e., in any local realistic theory, this probability is bounded by [6]

$$p_{\text{classical}} \leq 3/4. \qquad (2)$$

Such an upper bound is called a Bell inequality.

Crucially, Alice and Bob can violate this inequality using quantum mechanics [3]. The corresponding bound is [7]

$$p_{\text{quantum}} \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}, \qquad (3)$$

and there exists a shared quantum state and measurements that achieve it [6]. Further, there is now compelling experimental evidence that nature violates Bell inequalities and does not admit a local realistic description [2]. Yet, there exist stronger no-signaling correlations (outside quantum mechanics) which achieve success probability $p_{\text{no-signal}} = 1$ [5]. So why, then, is nature not more nonlocal [8]?

Studying limitations on nonlocal correlations thus forms an essential element of understanding nature. On one hand, it provides a systematic method to both theoretically and experimentally compare candidate physical theories [9]. On the other hand, it crucially affects our understanding of information in different settings such as cryptography and communication complexity [10–13]. For example, if na-

ture would admit $p_{\text{no-signal}} = 1$, any two-party communication problem could be solved using only a single bit of communication, independent of its size [11]. Also, for the special case of the CHSH inequality, it is known that the bound (3) is a consequence of information theoretic constraints such as uncertainty relations [12] or the recently proposed principle of information causality [13]. However, characterizing general correlations remains a difficult challenge [14], and it is interesting to consider what other constraints may impose limits on quantum correlations.

Result.—We forge a fundamental link between local quantum theory and nonlocal quantum correlations. In particular, we show that if Alice and Bob are *locally quantum*, then relativity theory implies that their nonlocal correlations admit a quantum description. The assumption of being locally quantum may thus provide another "reason" why the correlations we observe in nature are restricted by more than the principle of no-signaling itself. Figure 1 states our result.

Let us explain more formally what we mean by being locally quantum (see also Fig. 1). We say that Alice is locally quantum if her physical system can be described by means of a Hilbert space $\mathcal{H}_A$ of some fixed finite dimension d, on which she can perform *any* local quantum measurement [positive-operator-valued measurement (POVM)] $M_A = \{Q_a\}_a$ given by bounded operators $Q_a \in \mathcal{B}(\mathcal{H}_A)$. The probability $p(a|M_A)$ that she obtains an outcome a for measurement $M_A = \{Q_a\}_a$ is given by a function $\mathcal{B}(\mathcal{H}_A) \to [0, 1]$ applied to the POVM elements. Gleason's theorem for POVM elements implies that the state of Alice's system is then described by a state $\rho_A \in \mathcal{B}(\mathcal{H}_A)$ [15], and similarly for Bob, where we use $\mathcal{H}_B$ and $M_B = \{R_b\}_b$ to denote his Hilbert space and measurements, respectively. Conceptually, this means that quantum

mechanics describes Alice and Bob's local physical systems.

However, we make no *a priori* assumption about the nature of the joint system held by Alice and Bob. In particular, we do not assume that it is described by a tensor product of their local Hilbert spaces, or that their joint system is quantum mechanical. This means that Alice and Bob can share any possible preparation which assigns probabilities to local POVM measurements. That is, their preparation is simply a function ω such that the probabilities of observing outcomes a and b for measurements $M_A = \{Q_a\}_a$ and $M_B = \{R_b\}_b$ are given by $p(a, b|M_A, M_B) = \omega(Q_a, R_b)$. In particular, the state of their joint system may not be described by any density matrix.

Nevertheless, we are able to show that just from the assumptions that Alice and Bob are locally quantum and that the no-signaling principle is obeyed, it follows that there exists a Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, a state $\rho_{AB} \in \mathcal{B}(\mathcal{H}_{AB})$, and measurements $\tilde{M}_A = \{\tilde{Q}_a\}_a$ and $\tilde{M}_B = \{\tilde{R}_b\}_b$ for Alice and Bob, such that

$$p(a, b|M_A, M_B) = \omega(Q_a, R_b) = \text{tr}((\tilde{Q}_a \otimes \tilde{R}_b)\rho_{AB}). \quad (4)$$

That is, all correlations can be reproduced quantum mechanically.

Implications.—Our result solves an important piece of the puzzle of understanding nonlocal correlations, and their relation to the rich local phenomena we encounter in quantum theory such as Bohr's complementarity principle, Heisenberg uncertainty, and Kochen-Specker noncontextuality. In particular, it implies that if we obey local quantum statistics we can never hope to surpass a Tsirelson-type bound on p_{quantum} like that of (3), ruling out the possibility of such striking differences with respect
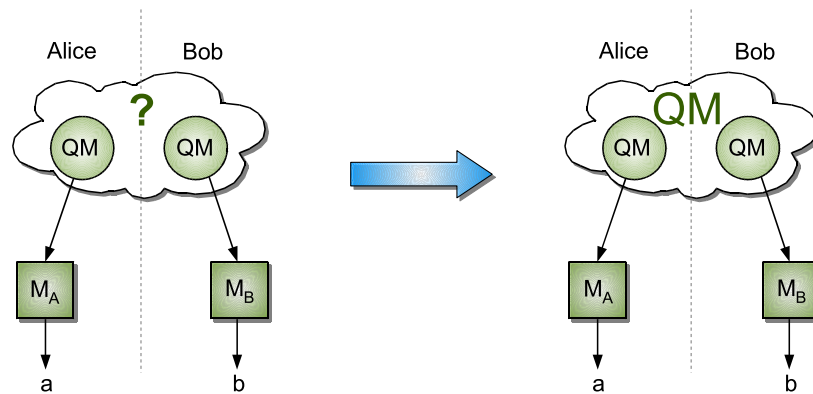

FIG. 1 (color online).   If the principle of no-signaling is obeyed and Alice and Bob are locally quantum, their nonlocal correlations can be obtained in quantum mechanics. Alice and Bob are *locally quantum* if their local systems can be described by a Hilbert space and they can choose to measure *any* local POVM $M_A = \{Q_a\}_a$ and $M_B = \{R_b\}_b$. A shared preparation between Alice and Bob corresponds to a function ω on the pair of POVM elements such that $p(a, b|M_A, M_B) = \omega(Q_a, R_b)$. This, with no-signaling, implies that the marginal distributions are given by Born's rule, $p(a|M_A) = \text{tr}(Q_a \rho_A)$ and $p(b|M_B) = \text{tr}(R_b \rho_B)$, where ρ_A and ρ_B are quantum states (but the state of their joint system may not be quantum). We show that, in this setting, for any preparation ω there exist a joint quantum state σ_{AB} and a relabeling of POVM measurements $\{\tilde{M}_A\}$ and $\{\tilde{M}_B\}$, such that $\omega(Q_a, R_b) = p(a, b|M_A, M_B) = \text{tr}((\tilde{Q}_a \otimes \tilde{R}_b)\sigma_{AB})$, where $\tilde{M}_A = \{\tilde{Q}_a\}_a$ and $\tilde{M}_B = \{\tilde{R}_b\}_b$.

to information processing as those pointed out in [11]. Indeed, if we were able to surpass such bounds, then the local systems of Alice and Bob could not be quantum.

Other recent works also attempt to explain the limitations of quantum correlations. For example, the principle of information causality [13] starts with the assumption that nature demands that certain communication tasks should be hard to solve. Together with the assumption of the no-signaling principle, this allows one to obtain Tsirelson's bound for the special case of the CHSH inequality. In our work, we also assume the no-signaling principle, but combine it with a different assumption, namely, that the world is locally quantum; that is, quantum mechanics correctly describes the laws of nature of local physical systems. Making this assumption we recover the quantum limit on *all* possible nonlocal correlations (not only the Tsirelson's bound for the CHSH inequality).

Proof.—To prove our result, we now proceed in two steps. First, we explain a known characterization of all no-signaling probability assignments to local quantum measurements [16–18]. Second, we use this characterization to show that the resulting correlations can be obtained in quantum mechanics.

From local quantum measurements to positive on pure tensors (POPT) states.—Fix two finite dimensional Hilbert spaces on Alice and Bob's sides. A local quantum measurement (or POVM) consists of a pair of measurements M_A and M_B with outcome labels $\{a\}$ and $\{b\}$, respectively, on Alice's and Bob's Hilbert spaces. Such POVMs are described by complex Hermitian matrices $M_A = \{Q_a\}_a$, $M_B = \{R_b\}_b$, $Q_a, R_b \geq 0$, which sum to the identity, i.e., $\sum_a Q_a = \sum_b R_b = \mathbb{1}$ (see Fig. 1). A preparation shared between Alice and Bob assigns outcome probabilities $p(a, b|M_A, M_B)$ to any choice of measurements M_A and M_b. More precisely, it corresponds to a function ω on the pair of POVM elements such that $p(a, b|M_A, M_B) = \omega(Q_a, R_b)$.

Kläy *et al.* [17] have shown (see supplementary material [19]) that, assuming no-signaling, the shared preparations (or equivalently the functions ω) are in one-to-one correspondence with matrices W_{AB} such that $\mathrm{tr}(W_{AB}) = 1$ and [20]

$$p(a, b|M_A, M_B) = \mathrm{tr}((Q_a \otimes R_b)W_{AB}) \geq 0. \quad (5)$$

The matrices W_{AB} are called positive on pure tensors (POPT) states. All quantum states are POPT states, but there are POPT states that do not correspond to quantum states [21].

Note that POPT states cannot be combined arbitrarily [18]. For example, not all entangled measurements (measurements which are not a convex combination of tensor products $Q_a \otimes R_b$) of POPT states are well defined because they would result in negative "probabilities" for nonquantum POPTs. Specifically, if Alice and Bob share a POPT, and Charlie and Bob share another one, then if Alice and

Charlie come together, entangled measurements between their POPTs are not necessarily defined. This does not affect our result, since we are only interested in the case where we consider parties (here Alice and Charlie together) which are locally quantum.

From POPT states to quantum correlations.—We now show that there exist a quantum state σ_{AB} and a map on POVM measurements

$$f: \{M_A = \{Q_a\}_a\} \mapsto \{\tilde{M}_A = \{\tilde{Q}_a\}_a\}, \quad (6)$$

such that

$$p(a, b|M_A, M_B) = \mathrm{tr}((\tilde{Q}_a \otimes R_b)\sigma_{AB}). \quad (7)$$

In order to do so, we associate to each POPT state W_{AB} a map $\mathcal{W}$ from matrices to matrices using the Choi-Jamiołkowski isomorphism. Explicitly, W_{AB} is obtained from $\mathcal{W}$ by acting on Bob's side of the (projection on the) maximally entangled state $|\Phi\rangle$

$$W_{AB} = \mathbb{1} \otimes \mathcal{W}(|\Phi\rangle\langle\Phi|). \quad (8)$$

Because W_{AB} is a POPT, the associated map $\mathcal{W}$ is positive; i.e., it sends positive matrices to positive ones, but it may not be an admissible quantum operation. Nevertheless, if $\mathcal{W}$ still maps POVMs to POVMs we can obtain the POPT correlations by moving the action of $\mathcal{W}$ from the maximally entangled state to the measurement elements. In particular, if $\mathcal{W}$ is unital $[\mathcal{W}(\mathbb{1}) = \mathbb{1}]$, the map

$$f: Q_a \mapsto \tilde{Q}_a = \mathcal{W}(Q_a^T)^T \quad (9)$$

maps POVM measurements to POVM measurements. We then show that (7) holds with $\sigma_{AB} = |\Phi\rangle\langle\Phi|$. Let d be the local dimension of Alice and Bob. If $\mathcal{W}$ is unital, we have

$$\begin{aligned}
\mathrm{tr}((Q_a \otimes R_b)W_{AB}) &= \mathrm{tr}((Q_a \otimes R_b)\mathbb{1} \otimes \mathcal{W}(|\Phi\rangle\langle\Phi|)) \\
&= \mathrm{tr}(|\Phi\rangle\langle\Phi|(\mathbb{1} \otimes \mathcal{W}^*)(Q_a \otimes R_b)) \\
&= \mathrm{tr}(|\Phi\rangle\langle\Phi|(Q_a \otimes \mathcal{W}^*(R_b))) \\
&= \frac{1}{d}\mathrm{tr}(Q_a^T \mathcal{W}^*(R_b)) \\
&= \frac{1}{d}\mathrm{tr}(\mathcal{W}(Q_a^T)R_b) \\
&= \mathrm{tr}((\tilde{Q}_a \otimes R_b)|\Phi\rangle\langle\Phi|), \quad (10)
\end{aligned}$$

where $\mathcal{W}^*$ denotes the adjoint of $\mathcal{W}$. This establishes (7) in the unital case.

In general, $\mathcal{W}$ can be decomposed into a unital map and another map. This other map gives a quantum state σ_{AB} by acting on $|\Phi\rangle$. Then f is defined in terms of the unital map as before. We finish the proof by showing that σ_{AB} is well normalized and (7) is satisfied. For a general positive map, let M be the image of the identity, i.e., $\mathcal{W}(\mathbb{1}) = M$. The matrix M is normalized, $\mathrm{tr}(M)/d = \mathrm{tr}(W_{AB}) = 1$. We assume initially that M is invertible, and define

$$\tilde{\mathcal{W}}(\cdot) = M^{-1/2}\mathcal{W}(\cdot)M^{-1/2}. \qquad (11)$$

The map $\tilde{\mathcal{W}}$ is unital. Further, the quantum state $\sigma_{AB} = |\psi\rangle\langle\psi|$ given by

$$|\psi\rangle = (M^{1/2})^T \otimes \mathbb{1}\Phi \qquad (12)$$

is well normalized; that is, $\mathrm{tr}(\sigma_{AB}) = \mathrm{tr}(M^T)/d = 1$. Thus by defining f as in (9) but in terms of $\tilde{\mathcal{W}}$, we conclude

$$\begin{aligned}
\mathrm{tr}\left((Q_a \otimes R_b)W_{AB}\right) &= \frac{1}{d}\mathrm{tr}\left(\mathcal{W}(Q_a^T)R_b\right) \\
&= \frac{1}{d}\mathrm{tr}((M^{1/2}\tilde{\mathcal{W}}(Q_a^T)M^{1/2})R_b) \\
&= \mathrm{tr}((\tilde{Q}_a \otimes R_b)\sigma_{AB}). \qquad (13)
\end{aligned}$$

If M is not invertible, in order to define $\tilde{\mathcal{W}}$, one can start with the map $(1-\epsilon)\mathcal{W}(\cdot) + \epsilon\mathbb{1}\,\mathrm{tr}(\cdot)$, and then take the limit $\epsilon \to 0$.

Conclusion.—We have shown that being locally quantum is sufficient to ensure that all nonlocal correlations between distant parties can be reproduced quantum mechanically, if the principle of no-signaling is obeyed. This gives us a natural explanation of why quantum correlations are weaker than is required by the no-signaling principle alone; i.e., given that one can describe local physics according to quantum measurements and states, then no-signaling already implies quantum correlations.

It would be interesting to know whether our work can be used to derive more efficient tests for nonlocal quantum correlations than those proposed in [14]. Finally, it is an intriguing question whether one can find new limits on our ability to perform information processing locally based on the limits of nonlocal correlations, which we now know to demand local quantum behavior.

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[21] Note that our result does not rule out the existence of POPT states, but shows that they do not generate stronger than quantum correlations.