# Transformations of elliptic hypergeometric integrals 

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#### Abstract

We prove a pair of transformations relating elliptic hypergeometric integrals of different dimensions, corresponding to the root systems $B C_{n}$ and $A_{n}$; as a special case, we recover some integral identities conjectured by van Diejen and Spiridonov. For $B C_{n}$, we also consider their "Type II" integral. Their proof of that integral, together with our transformation, gives rise to pairs of adjoint integral operators; a different proof gives rise to pairs of adjoint difference operators. These allow us to construct a family of biorthogonal abelian functions generalizing the Koornwinder polynomials, and satisfying the analogues of the Macdonald conjectures. Finally, we discuss some transformations of Type II-style integrals. In particular, we find that adding two parameters to the Type II integral gives an integral invariant under an appropriate action of the Weyl group $E_{7}$.


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## 1 Introduction

In recent work, van Diejen and Spiridonov [5, 6, 32] have produced a number of conjectural elliptic hypergeometric integration formulae, common generalizations of Spiridonov's elliptic beta integral [31] and $q$-hypergeometric integration identities due to Gustafson [11. In particular, for the $B C_{n}$ root system, they gave two conjectures, "Type I" and "Type II" (proved as Corollary 3.2 and Theorem 6.1]below), and showed that the Type I integral would imply the Type II integral. In an appropriate limit, their Type II integral transforms via residue calculus into a sum originally conjectured by Warnaar 36 (and proved by Rosengren 24]). In fact, Warnaar also conjectured a more general formula, a Bailey-type transformation identity, rather than a Jackson-type summation identity. This suggests that there should be transformation formulae on the integral level as well; this is the topic of the present work.

Their Type I integral can be thought of as the ultimate generalization of an integral identity used by Anderson [1] in his proof of the Selberg integral (which the Type II integral generalizes). While Anderson's proof of this identity (based on a clever change of variables) does not appear to generalize any further, some recent investigations of Forrester and the author [8] of a random matrix interpretation of the Anderson integral suggested a different argument, which as we will see does indeed generalize to the elliptic level. While the argument was not powerful enough to directly prove the Type I integral, it was able to prove it for a countably infinite union of submanifolds of parameter space. This suggested that this argument should at least suffice to produce the correct conjecture for a transformation law; in the event, it turned out that it produced not only a conjecture but a proof. We thus obtain an identity relating an $n$-dimensional integral with $2 n+2 m+4$ parameters to an $m$-dimensional integral with transformed parameters; when $m=0$, this gives the van DiejenSpiridonov integral, but the proof requires this degree of freedom. A similar identity for the $A_{n}$ root system follows by a slight modification of the argument; this gives a transformation generalization of a conjecture of Spiridonov [32]. The basic idea for both proofs is that, in an appropriate special case, the transformations can be written as determinants of relatively simple one-dimensional transformations. This "determinantal" case is thus easy to prove; moreover, by taking limits of some of the remaining degrees of freedom, we can transform the $n$-dimensional determinantal identity into a lower-dimensional, but non-determinantal instance of the transformation. Indeed, by repeating this process, starting with a sufficiently large instance of the determinantal case, we can obtain a dense set of special cases of the desired transformation, thus proving the theorem.

As we mentioned, the Type II integral follows as a corollary of the Type I integral. In many ways, this integral is of greater interest, most notably because it generalizes the inner product density for the Koornwinder polynomials [14. Since the inner product density generalizes, it would be natural to suppose that the orthogonal polynomials themselves should generalize. It would be too much to expect them to generalize to orthogonal functions, however; indeed, even in the univariate case, the elliptic analogues of the Askey-Wilson polynomials are merely biorthogonal (these analogues are due to Spiridonov and Zhedanov [33, 34] in the discrete case (generalizing work of Wilson [37]), and Spiridonov [32] in the continuous case (generalizing work of Rahman [17])). With this in mind, we will construct in the sequel a family of functions satisfying biorthogonalilty with respect to the Type II integral.

There are two main ingredients in this construction. The first is a family of difference operators, generalizing some difference operators known to act nicely on the Koornwinder polynomials [21], and satisfying adjointness relations with respect to the elliptic inner product. As a special case, we obtain a difference-operator-based proof of the Type II integral. This suggests that the proof based on the Type I integral should be related to a pair of adjoint integral operators, which form the other main ingredient in our construction. It turns out that the $B C_{n} \leftrightarrow B C_{m}$ transformation plays an important role in understanding these integral operators; indeed, by taking limits of the transformation so that one side becomes a finite sum, we obtain formulas for the images under the integral operator of a spanning set of its domain. The biorthogonal functions are then constructed as the images of suitable sequences of difference and integral operators. (This construction is new even at the level of Koornwinder polynomials.)

As these functions are biorthogonal with respect to a generalization of the Koornwinder density (and indeed contain the Koornwinder polynomials as a special case, although this turns out to be somewhat subtle to prove), one of course expects that they satisfy analogous properties. While the Hecke-algebraic aspects of the Koornwinder theory (see, for instance, [28]) are still quite mysterious at the elliptic level, the main properties, i.e., the "Macdonald conjectures", do indeed carry over. Two of these properties, namely the closed forms for the principal specialization and the nonzero values of the inner product, follow quite easily from the construction and adjointness; the third (evaluation symmetry) will be proved in a follow-up paper [19]. The arguments there are along the same lines as those given in [21] for the Koornwinder case, which were based on Okounkov's $B C_{n}$-symmetric interpolation polynomials [15]. Unlike in the Koornwinder case, however, at the elliptic level the required interpolation functions are actually special cases of the biorthogonal functions.

Just as in [21], these interpolation functions satisfy a number of generalized hypergeometric identities, having Warnaar's multivariate identities and conjectures [36] as special cases. To be precise, they satisfy multivariate analogues of Jackson's summation and Bailey's transformation. The former identity has an integral analogue, extending the Type II integral. In fact, the transformation also has an analogue, stated as Theorem 9.7 below; as a special case, this gives our desired transformation analogue of the Type II integral. The simplest version of this transformation states that an eight-parameter version of the Type II integral is invariant with respect to an action of the Weyl group $E_{7}$; in fact, this action extends formally to an action of $E_{8}$, acting on the parameters in the most natural way.

The plan of the paper is as follows. After defining some notation at the end of this introduction, we proceed in section 2 to discuss Anderson's integral, as motivation for our proof of the $B C_{n}$ and $A_{n}$ integral transformations. These transformations are then stated and proved in sections 3 and 4 , respectively; we also briefly discuss in section 5 some hybrid transformations arising from the fact that the $B C_{1}$ and $A_{1}$ integrals are the same, but the transformations are not. Section 6 then begins our discussion of the biorthogonal functions by describing the three kinds of difference operators, as well as the spaces of functions on which they act, and filtrations of those spaces with respect to which the operators are triangular. Section 7 discusses the corresponding integral operators (all of which are special cases of a single operator defined from the Type I integral), showing that they are triangular with respect to the same filtrations as the difference operators. Then, in section 8 , we combine these ingredients to construct the biorthogonal functions, and describe their main properties. In section 9 , we discuss our Type II transformations. Finally, in an appendix, we give a general result regarding when an integral
of a meromorphic function is meromorphic, and apply it to obtain precise information about the singularities of our integrals.

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## Notation

We will need a number of generalized $q$-symbols in the sequel. First, define the theta function and elliptic Gamma function 27]:

$$
\begin{align*}
\theta(x ; p) & :=\prod_{0 \leq k}\left(1-p^{k} x\right)\left(1-p^{k+1} / x\right)  \tag{1.1}\\
\Gamma(x ; p, q) & :=\prod_{0 \leq j, k}\left(1-p^{j+1} q^{k+1} / x\right)\left(1-p^{j} q^{k} x\right)^{-1} \tag{1.2}
\end{align*}
$$

In each case, the presence of multiple arguments before the semicolon indicates a product; thus, for instance,

$$
\begin{equation*}
\Gamma\left(z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right)=\Gamma\left(z_{i} z_{j} ; p, q\right) \Gamma\left(z_{i} / z_{j} ; p, q\right) \Gamma\left(z_{j} / z_{i} ; p, q\right) \Gamma\left(1 / z_{i} z_{j} ; p, q\right) \tag{1.3}
\end{equation*}
$$

These functions satisfy a number of identities, most notably

$$
\begin{equation*}
\theta(x ; p)=\theta(p / x ; p)=(-x) \theta(1 / x ; p) \tag{1.4}
\end{equation*}
$$

and

$$
\begin{align*}
\Gamma(x ; p, q) & =\Gamma(p q / x ; p, q)^{-1}  \tag{1.5}\\
\Gamma(p x ; p, q) & =\theta(x ; q) \Gamma(x ; p, q)  \tag{1.6}\\
\Gamma(q x ; p, q) & =\theta(x ; p) \Gamma(x ; p, q) \tag{1.7}
\end{align*}
$$

Using the theta function, one can define an elliptic analogue of the $q$-symbol; in fact, just as the elliptic Gamma function is symmetric in $p$ and $q$, we will want our elliptic $q$-symbol to also be symmetric. Thus, we define

$$
\begin{equation*}
\theta(x ; p, q)_{l, m}:=\prod_{0 \leq k<l} \theta\left(p^{k} x ; q\right) \prod_{0 \leq k<m} \theta\left(q^{k} x ; p\right) \tag{1.8}
\end{equation*}
$$

so that

$$
\begin{equation*}
\frac{\Gamma\left((p, q)^{l, m} x ; p, q\right)}{\Gamma(x ; p, q)}=(-x)^{-l m} p^{-l m(l-1) / 2} q^{-l m(m-1) / 2} \theta(x ; p, q)_{l, m} \tag{1.9}
\end{equation*}
$$

where

$$
\begin{equation*}
(p, q)^{l, m}:=p^{l} q^{m} \tag{1.10}
\end{equation*}
$$

We also need some multivariate symbols, indexed (as the biorthogonal functions will be) by pairs of partitions. By convention, we will use bold greek letters to refer to such partition pairs, and extend transformations
and relations of partitions in the obvious way. We then define, following [21],

$$
\begin{align*}
\mathcal{C}_{\boldsymbol{\lambda}}^{0}(x ; t ; p, q) & :=\prod_{1 \leq i} \theta\left(t^{1-i} x ; p, q\right)_{\boldsymbol{\lambda}_{i}}  \tag{1.11}\\
\mathcal{C}_{\boldsymbol{\lambda}}^{-}(x ; t ; p, q) & :=\prod_{1 \leq i \leq j} \frac{\theta\left(t^{j-i} x ; p, q\right)_{\boldsymbol{\lambda}_{i}-\boldsymbol{\lambda}_{j+1}}}{\theta\left(t^{j-i} x ; p, q\right)_{\boldsymbol{\lambda}_{i}-\boldsymbol{\lambda}_{j}}}  \tag{1.12}\\
\mathcal{C}_{\boldsymbol{\lambda}}^{+}(x ; t ; p, q) & :=\prod_{1 \leq i \leq j} \frac{\theta\left(t^{2-i-j} x ; p, q\right)_{\boldsymbol{\lambda}_{i}+\boldsymbol{\lambda}_{j}}}{\theta\left(t^{2-i-j} x ; p, q\right)_{\boldsymbol{\lambda}_{i}+\boldsymbol{\lambda}_{j+1}}} . \tag{1.13}
\end{align*}
$$

We note that each of the above $\mathcal{C}$ symbols extends to a holomorphic function on $x \in \mathbb{C}^{*}$.
Two particular combinations of $\mathcal{C}$ symbols will occur frequently enough to merit their own notation. We define:

$$
\begin{align*}
\Delta_{\boldsymbol{\lambda}}^{0}\left(a \mid \ldots b_{i} \ldots ; t ; p, q\right) & :=\frac{\mathcal{C}_{\boldsymbol{\lambda}}^{0}\left(\ldots b_{i} \ldots ; t ; p, q\right)}{\mathcal{C}_{\boldsymbol{\lambda}}^{0}\left(\ldots p q a / b_{i} \ldots ; t ; p, q\right)}  \tag{1.14}\\
\Delta_{\boldsymbol{\lambda}}\left(a \mid \ldots b_{i} \ldots ; t ; p, q\right) & :=\Delta_{\boldsymbol{\lambda}}^{0}\left(a \mid \ldots b_{i} \ldots ; t ; p, q\right) \frac{\mathcal{C}_{2 \boldsymbol{\lambda}^{2}}^{0}(p q a ; t ; p, q)}{\mathcal{C}_{\boldsymbol{\lambda}}^{-}(p q, t ; t ; p, q) \mathcal{C}_{\boldsymbol{\lambda}}^{+}(a, p q a / t ; t ; p, q)} \tag{1.15}
\end{align*}
$$

We will also need the following notion, where $0<|p|<1$.
Definition 1. A $B C_{n}$-symmetric ( $p$-)theta function of degree $m$ is a holomorphic function $f\left(x_{1}, \ldots x_{n}\right)$ on $\left(\mathbb{C}^{*}\right)^{n}$ such that
$f\left(x_{1}, \ldots x_{n}\right)$ is invariant under permutations of its arguments.
$f\left(x_{1}, \ldots x_{n}\right)$ is invariant under $x_{i} \mapsto 1 / x_{i}$ for each $i$.
$f\left(p x_{1}, x_{2}, \ldots x_{n}\right)=\left(1 / p z_{i}^{2}\right)^{m} f\left(x_{1}, x_{2}, \ldots x_{n}\right)$.
A $B C_{n}$-symmetric ( $p$-)abelian function is a meromorphic function satisfying the above conditions with $m=0$.
In particular, a $B C_{n}$-symmetric theta function of degree $m$ is a $B C_{1}$-symmetric theta function of degree $m$ in each of its arguments. Now, the space of $B C_{1}$-symmetric theta functions of degree $m$ is $m+1$-dimensional, and moreover, any nonzero $B C_{1}$-symmetric theta function vanishes at exactly $2 m$ orbits of points (under multiplication by $p$, and counting multiplicity). Thus we can show that a $B C_{1}$-symmetric theta function vanishes by finding $m+1$ independent points at which it vanishes.

The canonical example of a $B C_{n}$-symmetric theta function of degree 1 is

$$
\begin{equation*}
\prod_{1 \leq i \leq n} \theta\left(u x_{i}^{ \pm 1} ; p\right) \tag{1.16}
\end{equation*}
$$

indeed, the functions for any $n+1$ distinct choices of $u$ form a basis of the space of $B C_{n}$-symmetric theta functions of degree 1 :

$$
\begin{equation*}
f\left(\ldots x_{i} \ldots\right)=\sum_{0 \leq j \leq n} f\left(u_{0}, u_{1}, \ldots u_{j-1}, u_{j+1}, \ldots u_{n}\right) \frac{\prod_{1 \leq i \leq n} \theta\left(u_{j} x_{i}^{ \pm 1} ; p\right)}{\prod_{k \neq j} \theta\left(u_{j} u_{k}^{ \pm 1} ; p\right)} \tag{1.17}
\end{equation*}
$$

for any $B C_{n}$-symmetric theta function $f$ of degree 1 . More generally, the space of $B C_{n}$-symmetric theta functions of degree $m$ is spanned by the set of products of $m$ such functions.

## 2 The Anderson integral

Many of our arguments in the sequel were inspired by considerations of an extremely special (but quite important) case of Corollary 3.2 below, a multivariate integral identity used in Anderson's proof of the Selberg integral.

Theorem 2.1. [1] Let $x_{1}, \ldots x_{n}$ and $s_{1}, \ldots s_{n}$ be sequences of real numbers such that

$$
\begin{equation*}
x_{1}>x_{2}>\cdots>x_{n} \quad \text { and } \quad 0<s_{1}, s_{2}, \ldots s_{n} \tag{2.1}
\end{equation*}
$$

Then

$$
\begin{equation*}
\int_{x_{n} \leq y_{n-1} \leq \cdots \leq x_{2} \leq y_{1} \leq x_{1}} \frac{\prod_{1 \leq i<j \leq n-1}\left|y_{i}-y_{j}\right| \prod_{1 \leq i \leq n-1}^{1 \leq j \leq n}}{}\left|y_{i}-x_{j}\right|^{s_{j}-1} \prod_{1 \leq i \leq n-1} d y_{i}=\frac{\prod_{1 \leq i \leq n} \Gamma\left(s_{i}\right)}{\Gamma(S)} \tag{2.2}
\end{equation*}
$$

where $S=\sum_{1 \leq i \leq n} s_{i}$.
Remark. In fact, although Anderson independently discovered the above integral, it turns out that a more general identity (analogous to Theorem 3.1] below) was discovered in 1905 by Dixon [7] see also the remark above Theorem 2.3 below. However, Anderson was the first to notice the significance of this special case in the theory of the Selberg integral, so we will refer to it as the Anderson integral in the sequel.

Since the integrand is nonnegative, we can normalize to obtain a probability distribution. It turns out that if the $s_{i}$ parameters are all positive integers, then this probability distribution has a natural random matrix interpretation.

Theorem 2.2. [3] [8], §4] Let $A$ be a $S \times S$ complex Hermitian matrix, let $x_{1}>x_{2}>\cdots>x_{n}$ be the list of distinct eigenvalues of $A$, and let $s_{1}, s_{2}, \ldots s_{n}$ be the corresponding multiplicities. Let $\Pi: \mathbb{C}^{S} \rightarrow \mathbb{C}^{S-1}$ be the orthogonal projection onto a hyperplane chosen uniformly at random. Then the $S-1 \times S-1$ Hermitian matrix $B=\Pi A \Pi^{\dagger}$ has eigenvalues $x_{i}$ with multiplicity $s_{i}-1$, together with $n-1$ more eigenvalues $y_{i}$, distributed according to the Anderson distribution.

Remark. A similar statement holds over the reals, except that now the multiplicities correspond to $2 s_{1}$, $2 s_{2}, \ldots$ Similarly, over the quaternions, the multiplicities correspond to $s_{1} / 2, s_{2} / 2, \ldots$

Of particular interest is the generic case, in which the eigenvalues of $A$ are all distinct; that is $s_{1}=s_{2}=$ $\ldots s_{n}=1$ (this is the case considered in [3]). In this case, the Anderson integral is particularly simple to prove. Indeed, the relevant integral is:

$$
\begin{equation*}
\frac{(n-1)!}{\prod_{1 \leq i<j \leq n}\left(x_{i}-x_{j}\right)} \int_{x_{n} \leq y_{n-1} \leq \cdots \leq x_{2} \leq y_{1} \leq x_{1}} \prod_{1 \leq i<j \leq n-1}\left(y_{i}-y_{j}\right) \prod_{1 \leq i \leq n-1} d y_{i} \tag{2.3}
\end{equation*}
$$

In particular, the integrand is simply a Vandermonde determinant,

$$
\begin{equation*}
\prod_{1 \leq i<j \leq n-1}\left(y_{i}-y_{j}\right)=(-1)^{n(n-1) / 2} \operatorname{det}_{1 \leq i, j \leq n-1}\left(y_{i}-x_{n}\right)^{j-1} \tag{2.4}
\end{equation*}
$$

Integrating this row-by-row gives

$$
\begin{align*}
\frac{(-1)^{n(n-1) / 2}(n-1)!}{\prod_{1 \leq i<j \leq n}\left(x_{i}-x_{j}\right)} \operatorname{det}_{1 \leq i, j \leq n-1} \int_{x_{i+1}}^{x_{i}}\left(y-x_{n}\right)^{j-1} d y & =\frac{(-1)^{n(n-1) / 2}(n-1)!}{\prod_{1 \leq i<j \leq n}\left(x_{i}-x_{j}\right)} \operatorname{det}_{1 \leq i, j \leq n-1} \int_{x_{n}}^{x_{i}}\left(y-x_{n}\right)^{j-1} d y  \tag{2.5}\\
& =\frac{(-1)^{n(n-1) / 2}(n-1)!}{\prod_{1 \leq i<j \leq n}\left(x_{i}-x_{j}\right)} \operatorname{det}_{1 \leq i, j \leq n-1} \frac{\left(x_{i}-x_{n}\right)^{j}}{j}  \tag{2.6}\\
& =1 \tag{2.7}
\end{align*}
$$

Now, in general, a Hermitian matrix with multiple eigenvalues can be expressed as a limit of matrices with distinct eigenvalues; this suggests that we should be able to obtain the general integer $s$ Anderson integral as a limit of $s \equiv 1$ Anderson integrals. Indeed, if we integrate over $y_{i}$ and take a limit $x_{i+1} \rightarrow x_{i}$, the result is simply the Anderson distribution with parameters

$$
\begin{equation*}
x_{1}>\ldots x_{i}>x_{i+2}>\ldots x_{n} \quad \text { and } \quad s_{1}, s_{2}, \ldots s_{i}+s_{i+1}, \ldots s_{n-1}, s_{n} \tag{2.8}
\end{equation*}
$$

Combined with the determinantal proof for $s \equiv 1$, we thus obtain by induction a proof of Anderson's integral for arbitrary positive integer $s$. We can then obtain the general case via analytic continuation (for which we omit the argument, as it greatly simplifies in the cases of interest below). The resulting proof is less elegant than Anderson's original proof; however, it has the distinct advantage for our purposes of extending to much more general identities. Indeed, our proofs of Theorems 3.1 and 4.1 below proceed by precisely this sort of induction from large dimensional, but simple, cases.

Remark. Note that the key property of the "determinantal" case is not so much that it is a determinant, but that it is a determinant of univariate instances of the Anderson integral. Indeed, the general Anderson integral can be expressed as a determinant of univariate integrals; in fact, a generalization of the resulting identity was proved by Varchenko 35] even before Anderson's work [1], but without noticing that it could be used to prove the Selberg integral. See also [23] (apparently the first article to observe that Varchenko's identity could be expressed as a multivariate integral). It would be interesting to know if Varchenko's generalized formula can be extended to the elliptic level.

The random matrix interpretation also gives the following result. Given a symmetric function $f$, we define $f(A)$ for a matrix $A$ to be $f$ evaluated at the multiset of eigenvalues of $A$.

Theorem 2.3. Let $A$ be an n-dimensional Hermitian matrix, and let $\Pi$ be a random orthogonal projection as before. Then for any partition $\lambda$,

$$
\begin{equation*}
\mathbf{E}_{\Pi} s_{\lambda}\left(\Pi A \Pi^{\dagger}\right)=\frac{s_{\lambda}\left(1_{n-1}\right) s_{\lambda}(A)}{s_{\lambda}\left(1_{n}\right)} \tag{2.9}
\end{equation*}
$$

Proof. Since $\Pi$ was uniformly distributed, we have

$$
\begin{equation*}
\mathbf{E}_{\Pi} s_{\lambda}\left(\Pi A \Pi^{\dagger}\right)=\mathbf{E}_{\Pi} s_{\lambda}\left(\Pi U A U^{\dagger} \Pi^{\dagger}\right) \tag{2.10}
\end{equation*}
$$

for any unitary matrix $U$. In particular, we can fix $\Pi$ and take expectations over $U$, thus obtaining

$$
\begin{equation*}
\mathbf{E}_{U} s_{\lambda}\left(\Pi U A U^{\dagger} \Pi^{\dagger}\right)=\mathbf{E}_{U} s_{\lambda}\left(U A U^{\dagger} \Pi^{\dagger} \Pi\right)=\frac{s_{\lambda}(A) s_{\lambda}\left(\Pi^{\dagger} \Pi\right)}{s_{\lambda}\left(1_{n}\right)}=\frac{s_{\lambda}(A) s_{\lambda}\left(1_{n-1}\right)}{s_{\lambda}\left(1_{n}\right)} \tag{2.11}
\end{equation*}
$$

Here we have used the fact

$$
\begin{equation*}
\mathbf{E}_{U} s_{\lambda}\left(U A U^{\dagger} B\right)=\frac{s_{\lambda}(A) s_{\lambda}(B)}{s_{\lambda}\left(1_{n}\right)} \tag{2.12}
\end{equation*}
$$

from the theory of zonal polynomials, or equivalently from the fact that Schur functions are irreducible characters of the unitary group.

In other words, the Anderson distribution for $s \equiv 1$ acts as a raising integral operator on Schur polynomials, taking an $n-1$-variable Schur polynomial to the corresponding $n$-variable Schur polynomial. Similarly, the Anderson densities for $s \equiv \frac{1}{2}$ and $s \equiv 2$ act as raising operators on the real and quaternionic zonal polynomials. This suggests that in general, an Anderson distribution with constant $s$ should take polynomials to polynomials (mapping an appropriate Jack polynomial to the corresponding $n$-variable Jack polynomial).

Indeed, we have the following fact, even for nonconstant $s$.
Theorem 2.4. Let $y_{1}, y_{2}, \ldots y_{n-1}$ be distributed according to the Anderson distribution with parameters $s_{1}$, $\ldots s_{n}>0, x_{1}>\cdots>x_{n}>0$. Then as a function of $a_{1}, a_{2}, \ldots a_{m}$,

$$
\begin{equation*}
\mathbf{E}_{y}\left(\prod_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq m}}\left(a_{j}-y_{i}\right)\right)=\frac{\Gamma(S)}{\Gamma(S+m)} \frac{\prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}\left(a_{i}-x_{j}\right)^{1-s_{j}}}{\prod_{1 \leq i<j \leq m}\left(a_{j}-a_{i}\right)} \prod_{1 \leq i \leq m} \frac{\partial}{\partial a_{i}} \prod_{1 \leq j \leq n}\left(a_{i}-x_{j}\right)^{s_{j}} \prod_{1 \leq i<j \leq m}\left(a_{j}-a_{i}\right) \tag{2.13}
\end{equation*}
$$

In particular, the left-hand side is a polynomial in the $x_{j}$.
Proof. If $m=0$, this simply states that $\mathbf{E}_{y}(1)=1$; we may thus proceed by induction on $m$. Suppose the theorem holds for $m=m_{0}$, and consider what becomes of that instance when $s_{n}=1$. In that case, the density is essentially independent of $x_{n}$, in that $x_{n}$ only affects the normalization and the domain of integration. Thus if we multiply both sides by $\prod_{0 \leq i<j \leq n}\left(x_{i}-x_{j}\right)^{1-s_{i}-s_{j}}$, we can differentiate by $x_{n}$ to obtain an $n$-1-dimensional integral. If we then set $x_{n}=a_{m_{0}+1}$ and renormalize, the result is the $n-1$-dimensional case of the theorem with $m=m_{0}+1$.

That the right-hand side is a polynomial in the $x_{i}$ is straightforward, and thus the left-hand side is also polynomial.

Remark 1. Compare the proof of Theorem 7.1 given in Remark 2 following the theorem.
Remark 2. The left-hand side of the above identity was studied by Barsky and Carpentier [2] using Anderson's change of variables; they did not obtain as simple a right-hand side, however.

Corollary 2.5. As an integral operator, the Anderson distribution takes symmetric functions to polynomials; if $s_{1}=s_{2}=\ldots s_{n}=s$, it maps symmetric functions to symmetric functions.

Remark. A $q$-integral analogue of the Corollary was proved by Okounkov [16], who credits a private communication from Olshanski for the Corollary itself.

We can also obtain integral operators on symmetric functions by fixing one or two of the $x$ parameters and allowing their multiplicities to vary; the result is then a symmetric function in the remaining $x$ parameters. In particular, Anderson's proof of the Selberg integral acquires an interpretation in terms of pairs of adjoint integral operators.

## 3 The $B C_{n} \leftrightarrow B C_{m}$ transformation

For all nonnegative integers $\mathrm{m}, \mathrm{n}$, and parameters $p, q, t_{0} \ldots t_{2 m+2 n+3}$ satisfying

$$
\begin{equation*}
|p|,|q|,\left|t_{0}\right|, \ldots\left|t_{2 m+2 n+3}\right|<1, \quad \prod_{0 \leq r \leq 2 m+2 n+3} t_{r}=(p q)^{m+1}, \tag{3.1}
\end{equation*}
$$

define

$$
\begin{equation*}
I_{B C_{n}}^{(m)}\left(t_{0}, t_{1}, \ldots ; p, q\right):=\frac{(p ; p)^{n}(q ; q)^{n}}{2^{n} n!} \int_{T^{n}} \frac{\prod_{1 \leq i \leq n} \prod_{0 \leq r \leq 2 m+2 n+3} \Gamma\left(t_{r} z_{i}^{ \pm 1} ; p, q\right)}{\prod_{1 \leq i<j \leq n} \Gamma\left(z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right) \prod_{1 \leq i \leq n} \Gamma\left(z_{i}^{ \pm 2} ; p, q\right)} \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}, \tag{3.2}
\end{equation*}
$$

a contour integral over the unit torus. We can extend this to a meromorphic function on the set

$$
\begin{equation*}
\mathcal{P}_{m n}:=\left\{\left(t_{0}, t_{1}, \ldots t_{2 m+2 n+3}, p, q\right)\left|\prod_{0 \leq r \leq 2 m+2 n+3} t_{r}=(p q)^{m+1}, 0<|p|,|q|<1\right\}\right. \tag{3.3}
\end{equation*}
$$

by replacing the unit torus with the $n$-th power of an arbitrary (possibly disconnected) contour that contains the points of the form $p^{i} q^{j} t_{r}, i, j \geq 0$ and excludes their reciprocals. We thus find that the resulting function is analytic away from points where $t_{r} t_{s}=p^{-i} q^{-j}$ for some $0 \leq r, s \leq 2 m+2 n+3,0 \leq i, j$. (In fact, its singularities consist precisely of simple poles along the hypersurfaces $t_{r} t_{s}=p^{-i} q^{-j}$ with $0 \leq r<s \leq 2 m+2 n+3,0 \leq i, j$; see the appendix.)

Note in particular that $I_{B C_{0}}^{(m)}\left(t_{0}, t_{1}, \ldots t_{2 m+3} ; p, q\right)=1$.
Theorem 3.1. The following holds for $m, n \geq 0$ as an identity in meromorphic functions on $\mathcal{P}_{m n}$.

$$
\begin{equation*}
I_{B C_{n}}^{(m)}\left(t_{0}, t_{1}, \ldots t_{2 m+2 n+3} ; p, q\right)=\prod_{0 \leq r<s \leq 2 m+2 n+3} \Gamma\left(t_{r} t_{s} ; p, q\right) I_{B C_{m}}^{(n)}\left(\frac{\sqrt{p q}}{t_{0}}, \frac{\sqrt{p q}}{t_{1}}, \ldots \frac{\sqrt{p q}}{t_{2 m+2 n+3}} ; p, q\right) . \tag{3.4}
\end{equation*}
$$

Remark. If $\sqrt{p q}<\left|t_{r}\right|<1$ for all $r$, both contours may be taken to be the unit torus.
Taking $m=0$ gives the following:

## Corollary 3.2.

$$
\begin{equation*}
I_{B C_{n}}^{(0)}\left(t_{0}, t_{1}, \ldots t_{2 n+3} ; p, q\right)=\prod_{0 \leq r<s \leq 2 n+3} \Gamma\left(t_{r} t_{s} ; p, q\right), \tag{3.5}
\end{equation*}
$$

This is the "Type I" identity conjectured by van Diejen and Spiridonov [6], who showed that it would follow from the fact that the integral vanishes of $t_{0} t_{1}=p q$, but were unable to prove that fact. The case $n=1$ of the $m=0$ integral is, however, known: it is an elliptic beta integral due to Spiridonov 31]; it also happens to agree with the case $n=1$ of Theorem 6.1 below. A direct proof of the corollary has since been given by Spiridonov [30]; see also the remark following the second proof of Theorem 6.1] below.

Our strategy for proving Theorem 3.1]is as follows. We first observe that in a certain extremely special ("determinantal") case, each integrand can be expressed as a product of simple determinants, and thus the integrals themselves can be expressed as determinants. The agreement of corresponding entries of the determinants then follows from the $m=n=1$ instance of the determinantal case (Lemma 3.3 below).

The next crucial observation is that if we take the limit $t_{1} \rightarrow p q / t_{0}$ in an instance of the general identity for given values of $m$ and $n$, the result is an instance of the general identity with $m$ diminished by 1 ; similarly,
the limit $t_{1} \rightarrow 1 / t_{0}$ decreases $n$ by 1 . It turns out, however, that the determinantal case is not preserved by those operations; thus by starting with ever larger determinantal cases and dropping down to the desired $m$ and $n$, we obtain an ever increasing collection of proved special cases of the identity. The full set of special cases obtained is in fact dense, and thus the theorem will follow.

Remark. A similar inductive argument based on a determinantal case was applied in [25] to prove the summation analogue of Corollary 3.2 (see also Remark 3 following Theorem 7.1); it is worth noting, therefore, that the present argument is not in fact a generalization of Rosengren's. This is not to say that the arguments are unrelated; indeed, in a sense, the two arguments are dual. In fact, Rosengren's determinantal case turns out to be precisely Lemma 6.2 below, which is thus related to the difference operators we will be considering in the sequel. These, in turn, are related (by Theorem 7.1] among other things) to the integral operators we will define using Theorem 3.1 The duality is most apparent on the series level; if one interprets the sum as a sum over partitions, the two arguments are precisely related by conjugation of partitions. The main distinction for our purposes is that Rosengren's argument, while superior in the series case (as it does not require analytic continuation), does not appear to extend to the integral case.

The base case for the determinantal identity is the following:
Lemma 3.3. The theorem holds for $m=n=1$, assuming the parameters have the form

$$
\begin{equation*}
\left(t_{0}, t_{1}, t_{2}, \ldots t_{7}\right)=\left(a_{0}, q / a_{0}, a_{1}, q / a_{1}, b_{0}, p / b_{0}, b_{1}, p / b_{1}\right) \tag{3.6}
\end{equation*}
$$

In other words, if we define

$$
\begin{equation*}
F\left(a_{0}, a_{1}: b_{0}, b_{1} ; p, q\right):=\frac{(p ; p)(q ; q)}{2} \int_{C} \frac{\theta\left(z^{2} ; q\right) \theta\left(z^{-2} ; p\right)}{\theta\left(a_{0} z^{ \pm 1}, a_{1} z^{ \pm 1} ; q\right) \theta\left(b_{0} z^{ \pm 1}, b_{1} z^{ \pm 1} ; p\right)} \frac{d z}{2 \pi \sqrt{-1} z} \tag{3.7}
\end{equation*}
$$

with contour as appropriate, then

$$
\begin{equation*}
F\left(a_{0}, a_{1}: b_{0}, b_{1} ; p, q\right)=\frac{\theta\left(a_{0} a_{1} / q, a_{0} / a_{1} ; p\right) \theta\left(b_{0} b_{1} / p, b_{0} / b_{1} ; q\right)}{\theta\left(a_{0} a_{1} / q, a_{0} / a_{1} ; q\right) \theta\left(b_{0} b_{1} / p, b_{0} / b_{1} ; p\right)} F\left(\frac{\sqrt{p q}}{b_{0}}, \frac{\sqrt{p q}}{b_{1}}: \frac{\sqrt{p q}}{a_{0}}, \frac{\sqrt{p q}}{a_{1}} ; p, q\right) \tag{3.8}
\end{equation*}
$$

Proof. We first observe that it suffices to prove the Laurent series expansion

$$
\begin{equation*}
b_{1} \theta\left(b_{0} b_{1}^{ \pm 1} ; p\right) \frac{z^{-1} \theta\left(z^{2} ; p\right)}{\theta\left(b_{0} z^{ \pm 1}, b_{1} z^{ \pm 1} ; p\right)}=(p ; p)^{-2} \sum_{k \neq 0} \frac{b_{0}^{k}+\left(p / b_{0}\right)^{k}-b_{1}^{k}-\left(p / b_{1}\right)^{k}}{1-p^{k}} z^{k} \tag{3.9}
\end{equation*}
$$

valid for $|p|<\left|b_{0}\right|,\left|b_{1}\right|<1$, and $z$ in a neighborhood of the unit circle. Indeed, the desired integral is the constant term of the product of two such expressions, and is thus expressed as an infinite sum, each term of which already satisfies the desired transformation!

Consider the sum

$$
\begin{equation*}
\sum_{k>0} \frac{b_{0}^{k}+\left(p / b_{0}\right)^{k}-b_{1}^{k}-\left(p / b_{1}\right)^{k}}{1-p^{k}} \frac{z^{k}+z^{-k}}{k} \tag{3.10}
\end{equation*}
$$

which clearly differentiates (by $z \frac{d}{d z}$ ) to the stated sum. If we expand $\left(1-p^{k}\right)^{-1}$ in a geometric series, each term can then be summed over $k$ as a linear combination of logarithms. We conclude that

$$
\begin{equation*}
\sum_{k>0} \frac{b_{0}^{k}+\left(p / b_{0}\right)^{k}-b_{1}^{k}-\left(p / b_{1}\right)^{k}}{1-p^{k}} \frac{z^{k}+z^{-k}}{k}=\log \left(\frac{\theta\left(b_{1} z^{ \pm 1} ; p\right)}{\theta\left(b_{0} z^{ \pm 1} ; p\right)}\right) \tag{3.11}
\end{equation*}
$$

Now, the derivative

$$
\begin{equation*}
z \frac{d}{d z} \log \left(\frac{\theta\left(b_{1} z^{ \pm 1} ; p\right)}{\theta\left(b_{0} z^{ \pm 1} ; p\right)}\right) \tag{3.12}
\end{equation*}
$$

is an elliptic function antisymmetric under $z \mapsto z^{-1}$, with only simple poles, and those at points of the form $\left(p^{k} b_{j}\right)^{ \pm 1}$. It follows that

$$
\begin{equation*}
\frac{d}{d \log z} \log \left(\frac{\theta\left(b_{1} z^{ \pm 1} ; p\right)}{\theta\left(b_{0} z^{ \pm 1} ; p\right)}\right)=C\left(b_{0}, b_{1}, p\right) \frac{z^{-1} \theta\left(z^{2} ; p\right)}{\theta\left(b_{0} z^{ \pm 1}, b_{1} z^{ \pm 1} ; p\right)} \tag{3.13}
\end{equation*}
$$

for some factor $C\left(b_{0}, b_{1}, p\right)$ independent of $z$. Comparing asymptotics at $z=b_{0}$ gives the desired result.
Remark. If we take the limit $p \rightarrow 1$ in the above Laurent series expansion, we obtain

$$
\begin{equation*}
\lim _{p \rightarrow 1}(1-p)(p ; p)^{2} b_{1} \theta\left(b_{0} b_{1}^{ \pm 1} ; p\right) \frac{z^{-1} \theta\left(z^{2} ; p\right)}{\theta\left(b_{0} z^{ \pm 1}, b_{1} z^{ \pm 1} ; p\right)}=\sum_{k \neq 0} \frac{b_{0}^{k}+b_{0}^{-k}-b_{1}^{k}-b_{1}^{-k}}{k} z^{k} \tag{3.14}
\end{equation*}
$$

If $\left|b_{0}\right|=\left|b_{1}\right|=1, \Re\left(b_{0}\right)>\Re\left(b_{1}\right)$, then the limit is (up to a factor of $2 \pi \sqrt{-1} \operatorname{sgn}(\Im(z))$ ) the Fourier series expansion of the indicator function for the arcs such that $\Re\left(b_{0}\right) \geq \Re(z) \geq \Re\left(b_{1}\right)$. In particular, this explains how an integral like the Anderson integral, with its relatively complicated domain of integration, can be a limiting case of Corollary 3.2 The corresponding limit applied to Theorem 3.1 gives an identity of Dixon [7].

Lemma 3.4. If $m=n$, then (3.4) holds on the codimension $2 n+2$ subset parametrized by:

$$
\begin{equation*}
t_{2 r}=a_{r}, t_{2 r+1}=q / a_{r}, t_{2 n+2+2 r}=b_{r}, t_{2 n+2+2 r+1}=p / b_{r} \tag{3.15}
\end{equation*}
$$

Proof. By taking a determinant of instances of (3.8), we obtain the identity:

$$
\begin{align*}
\operatorname{det}_{1 \leq i, j \leq n} & \left(\int_{C^{n}} \frac{\theta\left(z^{2} ; p\right) \theta\left(z^{-2} ; q\right)}{\theta\left(a_{0} z^{ \pm 1}, a_{i} z^{ \pm 1} ; q\right) \theta\left(b_{0} z^{ \pm 1}, b_{j} z^{ \pm 1} ; p\right)} \frac{d z}{2 \pi \sqrt{-1} z}\right)=  \tag{3.16}\\
& \operatorname{det}_{1 \leq i, j \leq n}\left(\frac{\theta\left(a_{0} a_{i} / q, a_{0} / a_{i} ; p\right) \theta\left(b_{0} b_{j} / p, b_{0} / b_{j} ; q\right)}{\theta\left(a_{0} a_{i} / q, a_{0} / a_{i} ; q\right) \theta\left(b_{0} b_{j} / p, b_{0} / b_{j} ; p\right)} \int_{C^{\prime n}} \frac{\theta\left(z^{2} ; p\right) \theta\left(z^{-2} ; q\right)}{\theta\left(\frac{\sqrt{p q}}{a_{0}} z^{ \pm 1}, \frac{\sqrt{p q}}{a_{i}} z^{ \pm 1} ; p\right) \theta\left(\frac{\sqrt{p q}}{b_{0}} z^{ \pm 1}, \frac{\sqrt{p q}}{b_{j}} z^{ \pm 1} ; q\right)} \frac{d z}{2 \pi \sqrt{-1} z}\right)
\end{align*}
$$

Consider the determinant on the left. The $p$-theta functions in that integral are independent of $j$, while the $q$-theta functions are independent of $i$. We may thus expand that determinant of integrals as an integral of a product of two determinants:

$$
\begin{align*}
\operatorname{det}_{1 \leq i, j \leq n}\left(\int_{C^{n}}\right. & \left.\frac{\theta\left(z^{2} ; p\right) \theta\left(z^{-2} ; q\right)}{\theta\left(a_{0} z^{ \pm 1}, a_{i} z^{ \pm 1} ; q\right) \theta\left(b_{0} z^{ \pm 1}, b_{j} z^{ \pm 1} ; p\right)} \frac{d z}{2 \pi \sqrt{-1} z}\right) \\
& =\frac{1}{n!} \int_{C^{n}} \operatorname{det}_{1 \leq i, j \leq n}\left(\frac{\theta\left(z_{j}^{-2} ; q\right)}{\theta\left(a_{0} z_{j}^{ \pm 1}, a_{i} z_{j}^{ \pm 1} ; q\right)}\right) \operatorname{det}_{1 \leq j, i \leq n}\left(\frac{\theta\left(z_{j}^{2} ; p\right)}{\theta\left(b_{0} z_{j}^{ \pm 1}, b_{i} z_{j}^{ \pm 1} ; p\right)}\right) \prod_{1 \leq j \leq n} \frac{d z_{j}}{2 \pi \sqrt{-1} z_{j}} \tag{3.17}
\end{align*}
$$

These determinants can in turn be explicitly evaluated, using the following identity:

$$
\begin{equation*}
\operatorname{det}_{1 \leq i, j \leq n}\left(\frac{1}{a_{i}^{-1} \theta\left(a_{i} z_{j}^{ \pm 1} ; q\right)}\right)=(-1)^{n(n-1) / 2} \frac{\prod_{1 \leq i<j \leq n} a_{i}^{-1} \theta\left(a_{i} a_{j}^{ \pm 1} ; q\right) \prod_{1 \leq i<j \leq n} z_{i}^{-1} \theta\left(z_{i} z_{j}^{ \pm 1} ; q\right)}{\prod_{1 \leq i, j \leq n} a_{i}^{-1} \theta\left(a_{i} z_{j}^{ \pm 1} ; q\right)} . \tag{3.18}
\end{equation*}
$$

(This is, for instance, a special case of a determinant identity of Warnaar 36, and can also be obtained as a special case of the Cauchy determinant.) The resulting identity is precisely the desired result.

As we mentioned above, the other key element to the proof is an understanding of the limit of (3.4) as $t_{1} \rightarrow p q / t_{0}$. On the left-hand side, the integral is perfectly well-defined when $t_{1}=p q / t_{0}$, but the right-hand side ends up identifying two poles that should be separated. Thus we need to understand how $I_{B C_{n}}^{(m)}\left(t_{0}, t_{1}, \ldots\right)$ behaves as $t_{1} \rightarrow 1 / t_{0}$.

Lemma 3.5. We have the limit:

$$
\begin{equation*}
\lim _{t_{1} \rightarrow t_{0}^{-1}} \frac{I_{B C_{n}}^{(m)}\left(t_{0}, t_{1}, \ldots ; p, q\right)}{\Gamma\left(t_{0} t_{1} ; p, q\right) \prod_{2 \leq r<2 m+2 n+3} \Gamma\left(t_{0} t_{r}, t_{1} t_{r} ; p, q\right)}=I_{B C_{n}}^{(m-1)}\left(t_{2}, t_{3}, \ldots ; p, q\right) \tag{3.19}
\end{equation*}
$$

Proof. If we deform the contour on the left through the points $t_{1}$ and $1 / t_{1}$, the resulting integral will have a finite limit, and will thus be annihilated by the factor of $\Gamma\left(t_{0} t_{1}\right)$ in the denominator. In other words, the desired limit is precisely the limit of the sum of residues corresponding to the change of contour. By symmetry, each variable contributes equally, as do $t_{1}$ and $1 / t_{1}$; we thus find (using the identity

$$
\begin{equation*}
\lim _{y \rightarrow x} \Gamma(x / y)(1-x / y)=1 /(p ; p)(q ; q) \tag{3.20}
\end{equation*}
$$

which is easily verified):

$$
\begin{align*}
\lim _{t_{1} \rightarrow t_{0}^{-1}} & \frac{I_{B C_{n}}^{(m)}\left(t_{0}, t_{1}, \ldots ; p, q\right)}{\Gamma\left(t_{0} t_{1} ; p, q\right) \prod_{2 \leq r<2 m+2 n+3} \Gamma\left(t_{r} t_{0}, t_{r} t_{1} ; p, q\right)} \\
& =\lim _{t_{1} \rightarrow t_{0}^{-1}} \frac{(p ; p)^{n-1}(q ; q)^{n-1}}{2^{n-1}(n-1)!} \frac{\Gamma\left(t_{0} / t_{1} ; p, q\right) \prod_{2 \leq r<2 m+2 n+3} \Gamma\left(t_{r} / t_{1} ; p, q\right)}{\Gamma\left(1 / t_{1}^{2} ; p, q\right) \prod_{2 \leq r<2 m+2 n+3} \Gamma\left(t_{0} t_{r} ; p, q\right)}  \tag{3.21}\\
& \quad \int_{C^{n}} \frac{\prod_{1 \leq i<n} \Gamma\left(t_{0} z_{i}^{ \pm 1} ; p, q\right)}{\prod_{1 \leq i<n} \Gamma\left(z_{i}^{ \pm 1} / t_{1} ; p, q\right)} \frac{\prod_{1 \leq i<n} \prod_{2 \leq r<2 m+2 n+3} \Gamma\left(t_{r} z_{i}^{ \pm 1} ; p, q\right)}{\prod_{1 \leq i<j<n} \Gamma\left(z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right) \prod_{1 \leq i<n} \Gamma\left(z_{i}^{ \pm 2} ; p, q\right)} \prod_{1 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} \\
& =I_{B C_{n}}^{(m-1)}\left(t_{2}, t_{3}, \ldots t_{2 m+2 n+3} ; p, q\right) \tag{3.22}
\end{align*}
$$

as required.
We can now prove Theorem 3.1
Proof. For $m, n \geq 0$, let $\mathcal{C}_{m n}$ be the set of parameters $c_{0} c_{1} \ldots c_{m+n+1}=(p q)^{m+1}$ such that the theorem holds on the manifold with

$$
\begin{equation*}
t_{2 i} t_{2 i+1}=c_{i}, 0 \leq i \leq m+n+1 \tag{3.23}
\end{equation*}
$$

Thus, for instance, Lemma 3.4 states that the point $(q, q, q, \ldots q, p, p, p \ldots p)$ is in $\mathcal{C}_{n n}$.
The key idea is that if $\left(c_{0}, c_{1}, c_{2}, c_{3}, \ldots, c_{m+n+1}\right) \in \mathcal{C}_{m n}$, then we also have:

$$
\begin{align*}
\left(c_{0} c_{1}, c_{2}, c_{3}, \ldots c_{m+n+1}\right) & \in \mathcal{C}_{m(n-1)}  \tag{3.24}\\
\left(c_{0} c_{1} / p q, c_{2}, c_{3}, \ldots c_{m+n+1}\right) & \in \mathcal{C}_{(m-1) n} \tag{3.25}
\end{align*}
$$

so long as the generic point on the corresponding manifolds gives well-defined integrals; in other words, so long as none of the $c_{i}$ are of the form $p^{i} q^{j}, i, j>0$ or $p^{-i} q^{-j}, i, j \leq 0$. Indeed, if we use Lemma 3.5 to take the limit $t_{2} \rightarrow p q / t_{0}$ in the generic identity corresponding to $\left(c_{0}, c_{1}, \ldots c_{m+n+1}\right)$, we find that in the left-hand side,
the gamma factors corresponding to $t_{2}$ and $t_{0}$ cancel, while on the right-hand side, the residue formula gives an $n$-1-dimensional integral; the result is the generic identity corresponding to ( $c_{0} c_{1} / p q, \ldots c_{m+n+1}$ ). The other combination follows symmetrically.

Thus, starting with the point $(q, q, \ldots q, p, p, \ldots p) \in C_{N N}$ for $N$ sufficiently large, we can combine the $q$ 's with each other to obtain an arbitrary collection of values of the form $q^{j+1} p^{-k}$ with $j, k \geq 0$, and similarly combine the $p$ 's to values of the form $p^{j+1} q^{-k}$, subject only to the global condition that their product is $(p q)^{m+1}$. In other words (taking $N \rightarrow \infty$ ), the theorem holds for a dense set of points, and thus holds in general.

## 4 The $A_{n} \leftrightarrow A_{m}$ transformation

Consider the following family of $A_{n}$-type integrals:

$$
\begin{align*}
& I_{A_{n}}^{(m)}\left(Z \mid t_{0}, \ldots t_{m+n+1} ; u_{0}, \ldots u_{m+n+1} ; p, q\right)  \tag{4.1}\\
& \quad:=\frac{(p ; p)^{n}(q ; q)^{n}}{(n+1)!} \int_{\prod_{0 \leq i \leq n} z_{i}=Z} \frac{\prod_{0 \leq i \leq n} \prod_{0 \leq r<m+n+2} \Gamma\left(t_{r} z_{i}, u_{r} / z_{i} ; p, q\right)}{\prod_{0 \leq i<j \leq n} \Gamma\left(z_{i} / z_{j}, z_{j} / z_{i} ; p, q\right)} \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} .
\end{align*}
$$

If $\left|u_{r}\right|<|Z|^{1 /(n+1)}<1 /\left|t_{r}\right|$, we may take the contour to be the torus of radius $|Z|^{1 /(n+1)}$; outside this range, we must choose the contour to meromorphically continue the integral. Such contour considerations can be greatly simplified by multiplying by a test function $f(Z)$ holomorphic on $\mathbb{C}^{*}$ and integrating over $Z$. In the resulting integral, the correct contour has the form $C^{n+1}$, where $C$ contains all points of the form $p^{j} q^{k} u_{r}, j, k \geq 0$, $0 \leq r \leq m+n+1$ and excludes all points of the form $p^{-j} q^{-k} / t_{r}$.

Note that unlike the $B C_{n}$ case, the $A_{n}$ integral is not equal to 1 for $n=0$; instead, we pick up the value of the integrand at $Z$ :

$$
\begin{equation*}
I_{A_{0}}^{(m)}\left(Z \mid t_{0}, \ldots t_{m+1} ; u_{0}, \ldots u_{m+1} ; p, q\right)=\prod_{0 \leq r<m+2} \Gamma\left(t_{r} Z, u_{r} / Z ; p, q\right) \tag{4.2}
\end{equation*}
$$

We also observe that the $Z$ parameter is not a true degree of freedom; indeed:

$$
\begin{equation*}
I_{A_{n}}^{(m)}\left(c^{n+1} Z \mid \ldots t_{i} \ldots ; \ldots u_{i} \ldots ; p, q\right)=I_{A_{n}}^{(m)}\left(Z \mid \ldots c t_{i} \ldots ; \ldots c^{-1} u_{i} \ldots ; p, q\right) \tag{4.3}
\end{equation*}
$$

In particular, we could in principle always take $Z=1$ (in which case it will be omitted), although this is sometimes notationally inconvenient.

Theorem 4.1. For otherwise generic parameters satisfying $\prod_{0 \leq r<m+n+2} t_{r} u_{r}=(p q)^{m+1}$,

$$
\begin{equation*}
I_{A_{n}}^{(m)}\left(Z \mid \ldots t_{i} \ldots ; \ldots u_{i} \ldots ; p, q\right)=\prod_{0 \leq r, s<m+n+2} \Gamma\left(t_{r} u_{s} ; p, q\right) I_{A_{m}}^{(n)}\left(Z \left\lvert\, \ldots T^{\frac{1}{m+1}} / t_{i} \ldots\right. ; \ldots U^{\frac{1}{m+1}} / u_{i} \ldots ; p, q\right) \tag{4.4}
\end{equation*}
$$

where $T=\prod_{0 \leq r<m+n+2} t_{r}, U=\prod_{0 \leq r<m+n+2} u_{r}$.
Remark. It appears that this can be viewed as an integral analogue of a series transformation of Rosengren [26] and Kajihara and Noumi [12], in that the latter should be derivable via residue calculus from the former.

For $m=0$, we obtain the following integral conjectured by Spiridonov 32]:

Corollary 4.2. For otherwise generic parameters satisfying $\prod_{0 \leq r<n+2} t_{r} u_{r}=p q$,

$$
\begin{equation*}
I_{A_{n}}^{(0)}\left(\ldots t_{i} \ldots ; \ldots u_{i} \ldots ; p, q\right)=\prod_{0 \leq r, s<n+2} \Gamma\left(t_{r} u_{s} ; p, q\right) \prod_{0 \leq r<n+2} \Gamma\left(U / u_{r}, T / t_{r} ; p, q\right) \tag{4.5}
\end{equation*}
$$

The main difficulty with applying the $B C_{n}$ approach in this case is the fact that the variables are coupled by the condition $\prod_{i} z_{i}=Z$; in general the integral over this domain of the usual sort of product of determinants will not be expressible as a determinant of univariate integrals. Another difficulty is that, in any event, even in the "right" specialization, the integrand is not quite expressible as a product of determinants. As we shall see, it turns out that these problems effectively cancel each other.

In particular, we note the extra factor in the following determinant identity.
Lemma 4.3. [10]

$$
\begin{equation*}
\operatorname{det}_{0 \leq i, j<n}\left(\frac{\theta\left(t x_{i} y_{j} ; p\right)}{\theta\left(t, x_{i} y_{j} ; p\right)}\right)=\frac{\theta\left(t \prod_{0 \leq i<n} x_{i} y_{i} ; p\right)}{\theta(t ; p)} \prod_{0 \leq i<j<n} x_{j} y_{j} \theta\left(x_{i} / x_{j}, y_{i} / y_{j} ; p\right) \prod_{0 \leq i, j<n} \theta\left(x_{i} y_{j} ; p\right)^{-1} \tag{4.6}
\end{equation*}
$$

Proof. Consider the function

$$
\begin{equation*}
F\left(t ; \ldots x_{i} \ldots ; \ldots y_{i} \ldots\right):=\prod_{0 \leq i<j<n}\left(x_{j} y_{j}\right)^{-1} \theta\left(x_{i} / x_{j}, y_{i} / y_{j} ; p\right)^{-1} \prod_{0 \leq i, j<n} \theta\left(x_{i} y_{j} ; p\right) \operatorname{det}_{0 \leq i, j<n}\left(\frac{\theta\left(t x_{i} y_{j} ; p\right)}{\theta\left(t, x_{i} y_{j} ; p\right)}\right) \tag{4.7}
\end{equation*}
$$

This is clearly holomorphic on $\left(\mathbb{C}^{*}\right)^{2 n}$ for $t$ fixed; moreover, since

$$
\begin{equation*}
F\left(t ; p x_{0}, x_{1}, \ldots x_{n-1} ; \ldots y_{i} \ldots\right)=-\left(t \prod_{0 \leq j<n}\left(x_{j} y_{j}\right)\right)^{-1} F\left(t ; x_{0}, x_{1}, \ldots x_{n-1} ; \ldots y_{i} \ldots\right) \tag{4.8}
\end{equation*}
$$

we conclude that $F$ vanishes if $t \prod_{0 \leq j<n}\left(x_{j} y_{j}\right)=1$; indeed, $F\left(x_{0}\right)$ is a degree one theta function, and thus uniquely determined by its multiplier. Thus the function

$$
\begin{equation*}
\theta\left(t \prod_{0 \leq i<n} x_{i} y_{i} ; p\right)^{-1} F\left(t ; x_{0}, x_{1}, \ldots x_{n-1} ; \ldots y_{i} \ldots\right) \tag{4.9}
\end{equation*}
$$

is still holomorphic, and indeed we verify that it is an abelian function of all variables except $t$, so is in fact a function of $t$ alone. The remaining factors can thus be recovered from the limiting case:

$$
\begin{equation*}
\lim _{\substack{y_{i} \rightarrow x_{i}^{-1} \\ i=0 \ldots n-1}} F\left(t ; x_{0}, x_{1}, \ldots x_{n-1} ; \ldots y_{i} \ldots\right)=1 \tag{4.10}
\end{equation*}
$$

Remark. A presumably related application of this determinant to hypergeometric series identities can be found in [12].

Lemma 4.4. The theorem holds for the special case

$$
\begin{equation*}
I_{A_{n-1}}^{(n-1)}\left(Z \mid \ldots x_{i} \ldots, \ldots \frac{q}{y_{i}} \ldots ; \ldots \frac{p}{x_{i}} \ldots, \ldots y_{i} \ldots ; p, q\right) \tag{4.11}
\end{equation*}
$$

Proof. We first observe that the integral:

$$
\begin{equation*}
\int \frac{\theta(s x z ; p)}{\theta(s, x z ; p)} \frac{\theta(t y / z ; q)}{\theta(t, y / z ; q)} \frac{d z}{2 \pi \sqrt{-1} z} \tag{4.12}
\end{equation*}
$$

is symmetric in $x$ and $y$, as follows from the change of variable $z \mapsto y z / x$. It thus follows that the determinant

$$
\begin{equation*}
\operatorname{det}_{0 \leq i, j<n}\left(\int \frac{\theta\left(s x_{i} z ; p\right)}{\theta\left(s, x_{i} z ; p\right)} \frac{\theta\left(t y_{j} / z ; q\right)}{\theta\left(t, y_{j} / z ; q\right)} \frac{d z}{2 \pi \sqrt{-1} z}\right) \tag{4.13}
\end{equation*}
$$

is invariant under exchanging the roles of the $x$ and $y$ variables. As before, we can write this as a multiple integral of a product of two determinants:

$$
\begin{align*}
n!\operatorname{det}_{0 \leq i, j<n}\left(\int \frac{\theta\left(s x_{i} z ; p\right)}{\theta\left(s, x_{i} z ; p\right)}\right. & \left.\frac{\theta\left(t y_{j} / z ; q\right)}{\theta\left(t, y_{j} / z ; q\right)} \frac{d z}{2 \pi \sqrt{-1} z}\right) \\
= & \int \operatorname{det}_{0 \leq i, j<n}\left(\frac{\theta\left(s x_{i} z_{j} ; p\right)}{\theta\left(s, x_{i} z_{j} ; p\right)}\right) \operatorname{det}_{0 \leq i, j<n}\left(\frac{\theta\left(t y_{i} / z_{j} ; q\right)}{\theta\left(t, y_{i} / z_{j} ; q\right)}\right) \prod_{0 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}  \tag{4.14}\\
= & \prod_{0 \leq i<j<n} x_{j} y_{j} \theta\left(x_{i} / x_{j} ; p\right) \theta\left(y_{i} / y_{j} ; q\right) \\
& \int \frac{\theta(s X Z ; p) \theta(t Y / Z ; q)}{\theta(s ; p) \theta(t ; q)} \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(x_{i} z_{j} ; p\right) \theta\left(y_{i} / z_{j} ; q\right)} \prod_{0 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} \tag{4.15}
\end{align*}
$$

where $X=\prod_{i} x_{i}, Y=\prod_{i} y_{i}, Z=\prod_{i} z_{i}$. We thus conclude:

$$
\begin{align*}
& \int \frac{\theta(s X Z ; p) \theta(t Y / Z ; q)}{\theta(s ; p) \theta(t ; q)} \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(x_{i} z_{j} ; p\right) \theta\left(y_{i} / z_{j} ; q\right)} \prod_{0 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}  \tag{4.16}\\
& \quad=\prod_{0 \leq i<j<n} \frac{\theta\left(x_{i} / x_{j} ; q\right) \theta\left(y_{i} / y_{j} ; p\right)}{\theta\left(x_{i} / x_{j} ; p\right) \theta\left(y_{i} / y_{j} ; q\right)} \int \frac{\theta(s Y Z ; p) \theta(t X / Z ; q)}{\theta(s ; p) \theta(t ; q)} \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(y_{i} z_{j} ; p\right) \theta\left(x_{i} / z_{j} ; q\right)} \prod_{0 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}
\end{align*}
$$

Now, if we replace $s$ in this identity by $p^{k} s$, we find:

$$
\begin{align*}
& \int \frac{\theta(s X Z ; p) \theta(t Y / Z ; q)}{(X Z)^{k} \theta(s ; p) \theta(t ; q)} \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(x_{i} z_{j} ; p\right) \theta\left(y_{i} / z_{j} ; q\right)} \prod_{0 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}  \tag{4.17}\\
& \quad=\prod_{0 \leq i<j<n} \frac{\theta\left(x_{i} / x_{j} ; q\right) \theta\left(y_{i} / y_{j} ; p\right)}{\theta\left(x_{i} / x_{j} ; p\right) \theta\left(y_{i} / y_{j} ; q\right)} \int \frac{\theta(s Y Z ; p) \theta(t X / Z ; q)}{(Y Z)^{k} \theta(s ; p) \theta(t ; q)} \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(y_{i} z_{j} ; p\right) \theta\left(x_{i} / z_{j} ; q\right)} \prod_{0 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}
\end{align*}
$$

As this is true for all integers $k$, we find that

$$
\begin{align*}
& \int f(X Z) \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(x_{i} z_{j} ; p\right) \theta\left(y_{i} / z_{j} ; q\right)} \prod_{0 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}  \tag{4.18}\\
& \quad=\prod_{0 \leq i<j<n} \frac{\theta\left(x_{i} / x_{j} ; q\right) \theta\left(y_{i} / y_{j} ; p\right)}{\theta\left(x_{i} / x_{j} ; p\right) \theta\left(y_{i} / y_{j} ; q\right)} \int f(Y Z) \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(y_{i} z_{j} ; p\right) \theta\left(x_{i} / z_{j} ; q\right)} \prod_{0 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}
\end{align*}
$$

for any function $f$ holomorphic in a neighborhood of the contour (the dependence on $s$ and $t$ having been absorbed in $f$ ). But this implies

$$
\begin{align*}
\int_{\prod_{0 \leq i<n} z_{i}=Z} & \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(x_{i} z_{j} ; p\right) \theta\left(y_{i} / z_{j} ; q\right)} \prod_{1 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}  \tag{4.19}\\
& =\prod_{0 \leq i<j<n} \frac{\theta\left(x_{i} / x_{j} ; q\right) \theta\left(y_{i} / y_{j} ; p\right)}{\theta\left(x_{i} / x_{j} ; p\right) \theta\left(y_{i} / y_{j} ; q\right)} \int_{\prod_{0 \leq i<n} z_{i}=Z X / Y} \frac{\prod_{0 \leq i<j<n} \theta\left(z_{i} / z_{j} ; p\right) \theta\left(z_{j} / z_{i} ; q\right)}{\prod_{0 \leq i, j<n} \theta\left(y_{i} z_{j} ; p\right) \theta\left(x_{i} / z_{j} ; q\right)^{-1}} \prod_{1 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} .
\end{align*}
$$

Applying the change of variables $z_{i} \rightarrow(X / Y)^{1 / n} z_{i}$ on the right gives the desired result.
We also have the following analogue of Lemma 3.5 with essentially the same proof.
Lemma 4.5. We have the limit:

$$
\begin{equation*}
\lim _{u_{0} \rightarrow t_{0}^{-1}} \frac{I_{A_{n}}^{(m)}\left(Z \mid t_{0} \ldots t_{m+n+1} ; u_{0} \ldots u_{m+n+1} ; p, q\right)}{\Gamma\left(t_{0} u_{0} ; p, q\right) \prod_{0<r<m+n+2} \Gamma\left(t_{0} u_{r}, t_{r} u_{0} ; p, q\right)}=I_{A_{n-1}}^{(m)}\left(t_{0} Z \mid t_{1} \ldots t_{m+n+1} ; u_{1} \ldots u_{m+n+1} ; p, q\right) . \tag{4.20}
\end{equation*}
$$

Theorem 4.1 follows as in the proof of Theorem 3.1 except that in the definition of $\mathcal{C}_{m n}$, we take $t_{i} u_{i}=c_{i}$; we have

$$
\begin{equation*}
(q, q, \ldots q, p, p, \ldots p) \in \mathcal{C}_{n n} \tag{4.21}
\end{equation*}
$$

and if $\left(c_{0} \ldots c_{m+n+1}\right) \in \mathcal{C}_{m n}$, then

$$
\begin{align*}
\left(c_{0} c_{1}, c_{2}, \ldots c_{m+n+1}\right) & \in \mathcal{C}_{m(n-1)}  \tag{4.22}\\
\left(c_{0} c_{1} / p q, c_{2}, \ldots c_{m+n+1}\right) & \in \mathcal{C}_{m(n-1)} \tag{4.23}
\end{align*}
$$

as long as both sides of the corresponding identities are generically well-defined. As before, this shows that $\mathcal{C}_{m n}$ is dense, and thus the Theorem 4.1 holds in general.

## 5 Mixed transformations

Consider the integral associated to $A_{1}$. If we eliminate $z_{2}$ from the integral using the relation $z_{1} z_{2}=1$, we find that the result is invariant under $z_{1} \mapsto z_{1}^{-1}$, and is thus an instance of the $B C_{1}$ integral. Indeed, if

$$
\begin{equation*}
\prod_{i} t_{i} u_{i}=(p q)^{m+1} \tag{5.1}
\end{equation*}
$$

then

$$
\begin{align*}
I_{A_{1}}^{(m)}\left(t_{0} \ldots t_{m+2} ; u_{0} \ldots u_{m+2} ; p, q\right) & =\frac{(p ; p)(q ; q)}{2} \int \frac{\Gamma\left(t_{r} z^{ \pm 1}, u_{r} z^{ \pm 1} ; p, q\right)}{\Gamma\left(z^{ \pm 2} ; p, q\right)} \frac{d z}{2 \pi \sqrt{-1} z}  \tag{5.2}\\
& =I_{B C_{1}}^{(m)}\left(t_{0} \ldots t_{m+2}, u_{0} \ldots u_{m+2} ; p, q\right) \tag{5.3}
\end{align*}
$$

As a consequence, we obtain an identity between the $m=1$ integrals of types $A_{n}$ and $B C_{n}$.
Corollary 5.1. If $\prod_{0 \leq i \leq n+2} t_{i} u_{i}=(p q)^{2}$, then
$I_{A_{n}}^{(1)}\left(\ldots t_{i} \ldots ; \ldots u_{i} \ldots ; p, q\right)=\prod_{0 \leq i<j \leq n+2} \Gamma\left(T / t_{i} t_{j}, U / u_{i} u_{j} ; p, q\right) I_{B C_{n}}^{(1)}\left(\ldots(U / T)^{1 / 4} t_{i} \ldots, \ldots(T / U)^{1 / 4} u_{i} \ldots ; p, q\right)$,
where $T=\prod_{0 \leq i \leq n+2} t_{i}, U=\prod_{0 \leq i \leq n+2} u_{i}$.
In particular, since the $B C_{n}$ integral is symmetric in its $2 n+6$ parameters, we obtain an $S_{2 n+6}$ symmetry of $I_{A_{n}}^{(1)}$. We thus obtain a total of $n+4$ essentially different transformations of the $A_{n}$ integral, corresponding to the $n+4$ double cosets of $S_{n+3} \times S_{n+3}$ in $S_{2 n+6}$.

Corollary 5.2. Let $k$ be an integer $0 \leq k \leq n+3$. Then

$$
\begin{align*}
& I_{A_{n}}^{(1)}\left(t_{0}, \ldots, t_{n+2} ; u_{0}, \ldots, u_{n+2} ; p, q\right) \\
&=\prod_{\substack{0 \leq r<k \\
k \leq s \leq n+2}} \Gamma\left(t_{r} u_{s}, t_{s} u_{r}, T / t_{r} t_{s}, U / u_{r} u_{s} ; p, q\right) I_{A_{n}}^{(1)}\left(t_{0}^{\prime}, \ldots, t_{n+2}^{\prime} ; u_{0}^{\prime}, \ldots, u_{n+2}^{\prime} ; p, q\right) \tag{5.5}
\end{align*}
$$

where

$$
\begin{align*}
& t_{r}^{\prime}= \begin{cases}(T / U)^{(n+1-k) / 2(n+1)}\left(T_{k} / U_{k}\right)^{1 /(n+1)} u_{r}, & 0 \leq r<k \\
(U / T)^{k / 2(n+1)}\left(T_{k} / U_{k}\right)^{1 /(n+1)} t_{r}, & k \leq r \leq n+2\end{cases}  \tag{5.6}\\
& u_{r}^{\prime}= \begin{cases}(U / T)^{(n+1-k) / 2(n+1)}\left(U_{k} / T_{k}\right)^{1 /(n+1)} t_{r}, & 0 \leq r<k \\
(T / U)^{k / 2(n+1)}\left(U_{k} / T_{k}\right)^{1 /(n+1)} u_{r}, & k \leq r \leq n+2\end{cases}  \tag{5.7}\\
& T
\end{aligned}=\prod_{0 \leq r \leq n+2} t_{r}, ~ t_{r} \quad \begin{aligned}
& 0 \leq r \leq n+2 \tag{5.8}
\end{align*}
$$

For $k=0$, we obtain the identity transformation, while for $k=n+3$, we simply switch the $t_{i}$ and $u_{i}$ parameters (corresponding to taking $z \mapsto 1 / z$ in the integral). The case $k=1$ was stated as equation (6.11) of [32] (conditional on Corollary [4.2). Again, apparently related series identities are known; see [26] and 12].

## 6 Difference operators

The following identity was originally conjectured by van Diejen and Spiridonov [6] (their "Type II" integral):
Theorem 6.1. For otherwise generic parameters satisfying $|p|,|q|,|t|<1$ and $t^{2 n-2} \prod_{0 \leq r \leq 5} t_{r}=p q$,

$$
\begin{align*}
\frac{(p ; p)^{n}(q ; q)^{n} \Gamma(t ; p, q)^{n}}{2^{n} n!} \int_{C^{n}} \prod_{1 \leq i<j \leq n} \frac{\Gamma\left(t z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right)}{\Gamma\left(z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right)} \prod_{1 \leq i \leq n} & \frac{\prod_{0 \leq r \leq 5} \Gamma\left(t_{r} z_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(z_{i}^{ \pm 2} ; p, q\right)} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}  \tag{6.1}\\
& =\prod_{0 \leq j<n} \Gamma\left(t^{j+1} ; p, q\right) \prod_{0 \leq r<s \leq 5} \Gamma\left(t^{j} t_{r} t_{s} ; p, q\right),
\end{align*}
$$

where the contour $C=C^{-1}$ contains all points of the form $p^{i} q^{j} t_{r}$ for $i, j \geq 0$, excludes their reciprocals, and contains the contours $p^{i} q^{j} t C$ for $i, j \geq 0$. (In particular, if $\left|t_{r}\right|<1$ for $0 \leq r \leq 5, C$ may be taken to be the unit circle.)

Proof. 66 Suppose $t^{2 n} \prod_{0 \leq r \leq 5} t_{r}=p q$, and consider the double integral

$$
\begin{align*}
\int_{C^{n+1}} \int_{C^{\prime n}} & \frac{\prod_{\substack{0 \leq i \leq n \\
1 \leq n \leq n}} \Gamma\left(\sqrt{t} x_{i}^{ \pm 1} y_{j}^{ \pm 1} ; p, q\right)}{\prod_{0 \leq i<j \leq n} \Gamma\left(x_{i}^{ \pm 1} x_{j}^{ \pm 1} ; p, q\right) \prod_{1 \leq i<j \leq n} \Gamma\left(y_{i}^{ \pm 1} y_{j}^{ \pm 1} ; p, q\right)}  \tag{6.2}\\
& \prod_{0 \leq i \leq n} \frac{\Gamma\left(t^{n} t_{0} x_{i}^{ \pm 1} ; p, q\right) \prod_{1 \leq r \leq 5} \Gamma\left(t_{r} x_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(x_{i}^{ \pm 2} ; p, q\right)} \prod_{1 \leq i \leq n} \frac{\Gamma\left(p q t^{-n-1 / 2} y_{i}^{ \pm 1} / t_{0}, t^{-1 / 2} t_{0} y_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(y_{i}^{ \pm 2} ; p, q\right)} \\
& \prod_{1 \leq i \leq n} \frac{d y_{i}}{2 \pi \sqrt{-1} y_{i}} \prod_{0 \leq i \leq n} \frac{d x_{i}}{2 \pi \sqrt{-1} x_{i}}
\end{align*}
$$

Both the $x$ and $y$ integrals can be evaluated via Corollary 3.2 comparing both sides gives a recurrence for the left-hand side of (6.1), the unique solution of which is the right-hand side, as required.

We will discuss this proof (of which Anderson's proof of the Selberg integral is a limiting case) in greater detail in the sequel; for the moment, however, it will be instructive to consider a different proof. The main ingredient in the alternate proof is the following identity:

Lemma 6.2. Let $n$ be a nonnegative integer, and let $u_{0}, u_{1}, u_{2}, u_{3}, t$ satisfy $t^{n-1} u_{0} u_{1} u_{2} u_{3}=p$. Then

$$
\begin{align*}
\sum_{\sigma \in\{ \pm 1\}^{n}} \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 3} \theta\left(u_{r} z_{i}^{\sigma_{i}} ; p\right)}{\theta\left(z_{i}^{2 \sigma_{i}} ; p\right)} \prod_{1 \leq i<j \leq n} \frac{\theta\left(t z_{i}^{\sigma_{i}} z_{j}^{\sigma_{j}} ; p\right)}{\theta\left(z_{i}^{\sigma_{i}} z_{j}^{\sigma_{j}} ; p\right)} & =\prod_{0 \leq i<n} \theta\left(t^{i} u_{0} u_{1}, t^{i} u_{0} u_{2}, t^{i} u_{0} u_{3} ; p\right)  \tag{6.3}\\
& =\prod_{0 \leq i<n} \theta\left(t^{i} u_{0} u_{1}, t^{i} u_{0} u_{2}, t^{i} u_{1} u_{2} ; p\right) \tag{6.4}
\end{align*}
$$

Proof. We first observe that the condition on the $u_{r}$ ensures that every term in the above sum is invariant under all translations $z_{i} \rightarrow p z_{i}$, and thus the same is true of their sum. Moreover, the sum is manifestly invariant under permutations of the $z_{i}$ as well as reflections $z_{i} \rightarrow 1 / z_{i}$. Thus if we multiply the sum by

$$
\begin{equation*}
\prod_{1 \leq i \leq n} z_{i}^{-1} \theta\left(z_{i}^{2} ; p\right) \prod_{1 \leq i<j \leq n} z_{i}^{-1} \theta\left(z_{i} z_{j}, z_{i} z_{j}^{-1} ; p\right), \tag{6.5}
\end{equation*}
$$

the result is a (holomorphic) theta function anti-invariant under the same group. But any such theta function is a multiple of the above product; it thus follows that the desired sum has no singularities in $z_{i}$, and must therefore be independent of $z_{i}$.

To evaluate the sum, we may therefore specialize $z_{i}=u_{0} t^{n-i}$, in which case all but one of the terms in the sum vanish, so the sum is given by the remaining term (that with $\sigma_{i}=1$ for all $i$ ):

$$
\begin{equation*}
\frac{\prod_{0 \leq r \leq 3} \theta\left(u_{0} u_{r} n^{n-i} ; p\right)}{\theta\left(u_{0}^{2} t^{2 n-2 i} ; p\right)} \prod_{1 \leq i<j \leq n} \frac{\theta\left(u_{0}^{2} t^{2 n+1-i-j} ; p\right)}{\theta\left(u_{0}^{2} t^{2 n-i-j} ; p\right)} \tag{6.6}
\end{equation*}
$$

The factors involving $u_{0}^{2}$ cancel, and we are thus left with the evaluation claimed above.
Proof. (of Theorem 6.1) Divide the integral by the claimed evaluation, and consider the result as a meromorphic function on the set $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} t_{4} t_{5}=p q$. We claim that this function is invariant under the translations

$$
\begin{align*}
& \left(t_{0}, t_{1}, t_{2}, t_{3}, t_{4}, t_{5}\right) \rightarrow\left(p^{1 / 2} t_{0}, p^{1 / 2} t_{1}, p^{1 / 2} t_{2}, p^{-1 / 2} t_{3}, p^{-1 / 2} t_{4}, p^{-1 / 2} t_{5}\right)  \tag{6.7}\\
& \left(t_{0}, t_{1}, t_{2}, t_{3}, t_{4}, t_{5}\right) \rightarrow\left(q^{1 / 2} t_{0}, q^{1 / 2} t_{1}, q^{1 / 2} t_{2}, q^{-1 / 2} t_{3}, q^{-1 / 2} t_{4}, q^{-1 / 2} t_{5}\right), \tag{6.8}
\end{align*}
$$

and all permutations thereof. It will then follow that the ratio is a constant; to evalute the constant, we may then consider the limit $t_{1} \rightarrow t^{1-n} t_{0}^{-1}$ as in Lemma 3.5 above. (In other words, we apply the special case of the residue formula of van Diejen and Spiridonov in which the resulting sum consists of precisely one term.)

Since both sides are symmetric in $p$ and $q$, it suffices to consider the $q$ translation. If we factor the integrand as

$$
\begin{equation*}
\Delta^{(n)}\left(z_{1}, z_{2}, \ldots z_{n}\right) \Delta^{(n)}\left(z_{1}^{-1}, z_{2}^{-1}, \ldots z_{n}^{-1}\right) \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} \tag{6.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta^{(n)}\left(z_{1}, z_{2}, \ldots z_{n}\right)=\prod_{1 \leq i \leq n} \frac{\Gamma\left(t_{0} z_{i}, t_{1} z_{i}, t_{2} z_{i}, t_{3} z_{i}, t_{4} z_{i}, t_{5} z_{i}, p z_{i} /\left(t^{n-1} t_{0} t_{1} t_{2}\right) ; p, q\right)}{\Gamma\left(z_{i}^{2}, p /\left(z_{i} t^{n-1} t_{0} t_{1} t_{2}\right) ; p, q\right)} \prod_{1 \leq i<j \leq n} \frac{\Gamma\left(t z_{i} z_{j}^{ \pm 1} ; p, q\right)}{\Gamma\left(z_{i} z_{j}^{ \pm 1} ; p, q\right)} \tag{6.10}
\end{equation*}
$$

and similarly let $\tilde{\Delta}^{(n)}$ be the corresponding product with parameters

$$
\begin{equation*}
\left(q^{-1 / 2} t_{3}, q^{-1 / 2} t_{4}, q^{-1 / 2} t_{5}, q^{1 / 2} t_{0}, q^{1 / 2} t_{1}, q^{1 / 2} t_{2}\right) \tag{6.11}
\end{equation*}
$$

(permuting the parameters to make the transformation an involution), then we find that

$$
\begin{equation*}
\frac{\tilde{\Delta}^{(n)}\left(\ldots q^{1 / 2} z_{i} \ldots\right)}{\Delta^{(n)}\left(\ldots z_{i} \ldots\right)}=\prod_{1 \leq i \leq n} \frac{\theta\left(t_{0} z_{i}, t_{1} z_{i}, t_{2} z_{i}, p z_{i} / t^{n-1} t_{0} t_{1} t_{2} ; p\right)}{\theta\left(z_{i}^{2} ; p, q\right)} \prod_{1 \leq i<j \leq n} \frac{\theta\left(t z_{i} z_{j} ; p\right)}{\theta\left(z_{i} z_{j} ; p\right)} \tag{6.12}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\sum_{\sigma_{i} \in\{ \pm 1\}^{n}} \frac{\tilde{\Delta}^{(n)}\left(\ldots q^{1 / 2} z_{i}^{\sigma_{i}} \ldots\right)}{\Delta^{(n)}\left(\ldots z_{i}^{\sigma_{i}} \ldots\right)}=\prod_{0 \leq i<n} \theta\left(t^{i} t_{0} t_{1}, t^{i} t_{0} t_{2}, t^{i} t_{1} t_{2} ; p\right) \tag{6.13}
\end{equation*}
$$

by Lemma 6.2 Similarly,

$$
\begin{equation*}
\sum_{\sigma_{i} \in\{ \pm 1\}^{n}} \frac{\Delta^{(n)}\left(\ldots q^{1 / 2} z_{i}^{\sigma_{i}} \ldots\right)}{\tilde{\Delta}^{(n)}\left(\ldots z_{i}^{\sigma_{i}} \ldots\right)}=\prod_{0 \leq i<n} \theta\left(t^{i} t_{3} t_{4} / q, t^{i} t_{3} t_{5} / q, t^{i} t_{4} t_{5} / q ; p\right) . \tag{6.14}
\end{equation*}
$$

Now, consider the integral:

$$
\begin{equation*}
\int_{C} \tilde{\Delta}^{(n)}\left(\ldots q^{1 / 2} z_{i} \ldots\right) \Delta^{(n)}\left(\ldots z_{i}^{-1} \ldots\right) \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} \tag{6.15}
\end{equation*}
$$

where the contour is chosen to contain the points $p^{i} q^{j} t_{r}$ for $i, j \geq 0$, exclude their reciprocals, and contain the contours $t C$ and $t C^{-1}$; here we note that the poles of $\tilde{\Delta}^{(n)}\left(\ldots q^{1 / 2} z_{i} \ldots\right)$ are a subset of the poles of $\Delta^{(n)}\left(\ldots z_{i} \ldots\right)$, so this constraint on the contour is still reasonable. If we then perform the change of variable $z_{i} \mapsto q^{-1 / 2} / z_{i}$, we find that the new contour is legal for the transformed parameters. In other words, we have

$$
\begin{align*}
\int_{C} \tilde{\Delta}^{(n)}\left(\ldots q^{1 / 2} z_{i} \ldots\right) \Delta^{(n)}\left(\ldots z_{i}^{-1} \ldots\right) & \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}  \tag{6.16}\\
& =\int_{C^{\prime}} \Delta^{(n)}\left(\ldots q^{1 / 2} z_{i} \ldots\right) \tilde{\Delta}^{(n)}\left(\ldots z_{i}^{-1} \ldots\right) \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}
\end{align*}
$$

Since the constraints on the contours are symmetrical under $z_{i} \mapsto 1 / z_{i}$, we may symmetrize the integrands, losing the same factor of $2^{n}$ on both sides. The theorem follows upon applying equations (6.13) and (6.14) to simplify the symmetrized integrands.

Remark. One can also prove Corollary 3.2 by a similar argument, based on the straightforward identity

$$
\begin{equation*}
\sum_{\sigma \in\{ \pm 1\}^{n}} \frac{\theta\left(\prod_{0 \leq r<n+2} t_{r} / \prod_{1 \leq i \leq n} z_{i}^{\sigma_{i}} ; p\right) \prod_{1 \leq i \leq n} \prod_{0 \leq r<n+2} \theta\left(t_{r} z_{i}^{\sigma_{i}} ; p\right)}{\prod_{1 \leq i \leq j \leq n} \theta\left(z_{i}^{\sigma_{i}} z_{j}^{\sigma_{j}} ; p\right)}=\prod_{0 \leq r<s<n+2} \theta\left(t_{r} t_{s} ; p\right) \tag{6.17}
\end{equation*}
$$

Define a $q$-difference operator $D_{q}^{(n)}\left(u_{0}, u_{1}, u_{2}, u_{3} ; t, p\right)$ by setting

$$
\begin{equation*}
\left(D_{q}^{(n)}\left(u_{0}, u_{1}, u_{2}, u_{3} ; t, p\right) f\right)\left(\ldots z_{i} \ldots\right):=\sum_{\sigma \in\{ \pm 1\}^{n}} \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 3} \theta\left(u_{r} z_{i}^{\sigma_{i}} ; p\right)}{\theta\left(z_{i}^{2 \sigma_{i}} ; p\right)} \prod_{1 \leq i<j \leq n} \frac{\theta\left(t z_{i}^{\sigma_{i}} z_{j}^{\sigma_{j}} ; p\right)}{\theta\left(z_{i}^{\sigma_{i}} z_{j}^{\sigma_{j}} ; p\right)} f\left(\ldots q^{\sigma_{i} / 2} z_{i} \ldots\right) \tag{6.18}
\end{equation*}
$$

Thus Lemma 6.2 gives a formula for the image of 1 under $D_{q}^{(n)}\left(u_{0}, u_{1}, u_{2}, u_{3} ; t, p\right)$ when $t^{n-1} u_{0} u_{1} u_{2} u_{3}=p$. Moreover, the resulting proof of Theorem 6.1 would appear to be based on an adjointness relation between two such difference operators, as we will confirm below.

To make this precise, we need some suitable spaces of functions on which to act. Let $A^{(n)}\left(u_{0} ; p, q\right)$ be the space of $B C_{n}$-symmetric $p$-abelian functions $f$ such that

$$
\begin{equation*}
\prod_{1 \leq i \leq n} \theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{0, m} f\left(\ldots z_{i} \ldots\right) \tag{6.19}
\end{equation*}
$$

is holomorphic for sufficiently large $m$; that is, $f$ is smooth except at the points $p^{k} u_{0} / q^{l}, p^{k} q^{l} / u_{0}$ for $k \in \mathbb{Z}$, $1 \leq l \leq m$, where it has at most simple poles. The canonical (multiplication) map from the tensor product of $A^{(n)}\left(u_{0} ; p, q\right)$ and $A^{(n)}\left(u_{0} ; q, p\right)$ to the space of meromorphic functions on $\left(\mathbb{C}^{*}\right)^{n}$ is generically injective; denote the image by $\mathcal{A}^{(n)}\left(u_{0} ; p, q\right)$. In particular, we observe that if $f \in \mathcal{A}^{(n)}\left(u_{0} ; p, q\right)$, then

$$
\begin{equation*}
\prod_{1 \leq i \leq n} \theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m} f\left(\ldots z_{i} \ldots\right) \propto \prod_{1 \leq i \leq n} \frac{\Gamma\left(u_{0} z_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(p^{-l} q^{-m} u_{0} z_{i}^{ \pm 1} ; p, q\right)} f\left(\ldots z_{i} \ldots\right) \tag{6.20}
\end{equation*}
$$

is holomorphic for sufficiently large $l, m$.
Remark. Our main motivation for considering the large space $\mathcal{A}^{(n)}\left(u_{0} ; p, q\right)$, rather than the smaller spaces in which the functions are actually abelian, is that such product functions already appear in the family of univariate biorthogonal functions considered by Spiridonov [32 Appendix A].

We now define

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right) f:=\frac{D_{q}^{(n)}\left(u_{0}, u_{1}, u_{2}, t^{1-n} p / u_{0} u_{1} u_{2} ; t, p\right) f}{\prod_{1 \leq i \leq n} \theta\left(t^{n-i} u_{0} u_{1}, t^{n-i} u_{0} u_{2}, t^{n-i} u_{1} u_{2} ; p\right)} \tag{6.21}
\end{equation*}
$$

We will also need a shift operator $T_{\omega, q}^{(n)}$ :

$$
\begin{equation*}
\left(T_{\omega, q}^{(n)} f\right)\left(\ldots z_{i} \ldots\right)=f\left(\ldots q^{1 / 2} z_{i} \ldots\right) \tag{6.22}
\end{equation*}
$$

Note that this maps $B C_{n}$-symmetric $q$-abelian functions to $B C_{n}$-symmetric $q$-abelian functions.
Lemma 6.3. The operator $\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right)$ induces a linear transformation

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right): \mathcal{A}^{(n)}\left(\sqrt{q} u_{0} ; p, q\right) \rightarrow \mathcal{A}^{(n)}\left(u_{0} ; p, q\right) \tag{6.23}
\end{equation*}
$$

Moreover, the corresponding map

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right): A^{(n)}\left(\sqrt{q} u_{0} ; p, q\right) \otimes A^{(n)}\left(\sqrt{q} u_{0} ; q, p\right) \rightarrow A^{(n)}\left(u_{0} ; p, q\right) \otimes A^{(n)}\left(u_{0} ; q, p\right) \tag{6.24}
\end{equation*}
$$

can be decomposed as

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right)=\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right) \otimes T_{\omega, q}^{(n)} \tag{6.25}
\end{equation*}
$$

Proof. Let

$$
\begin{aligned}
& g \in A^{(n)}\left(\sqrt{q} u_{0} ; p, q\right) \\
& h \in A^{(n)}\left(\sqrt{q} u_{0} ; q, p\right) .
\end{aligned}
$$

A straightforward computation, using the fact that $h$ is $q$-abelian, gives:

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right)(g h)=\left(\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right) g\right)\left(T_{\omega, q}^{(n)} h\right) \tag{6.26}
\end{equation*}
$$

as required. That $T_{\omega, q}^{(n)} h \in A^{(n)}\left(u_{0} ; q, p\right)$ is straightforward; that

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t, p\right) g \tag{6.27}
\end{equation*}
$$

is $p$-abelian follows as in the proof of Lemma 6.2. Finally, we observe that this function is holomorphic at $z_{i}=u_{0}$, as required.

The desired adjointness relation can then be stated as follows. For parameters satisfying $t^{2 n-2} u_{0} u_{1} t_{0} t_{1} t_{2} t_{3}=$ $p q$, define a scalar product between $\mathcal{A}^{(n)}\left(u_{0} ; p, q\right)$ and $\mathcal{A}^{(n)}\left(u_{1} ; p, q\right)$ as follows:

$$
\begin{array}{r}
\langle f, g\rangle_{t_{0}, t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t, p, q}:=\frac{1}{Z} \int_{C^{n}} f\left(\ldots z_{i} \ldots\right) g\left(\ldots z_{i} \ldots\right) \prod_{1 \leq i<j \leq n} \frac{\Gamma\left(t z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right)}{\Gamma\left(z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right)}  \tag{6.28}\\
\\
\prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 5} \Gamma\left(t_{r} z_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(z_{i}^{ \pm 2} ; p, q\right)} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}}
\end{array}
$$

where

$$
\begin{align*}
Z & =\frac{2^{n} n!}{(p ; p)^{n}(q ; q)^{n} \Gamma(t ; p, q)^{n}} \prod_{1 \leq i \leq n} \Gamma\left(t^{i} ; p, q\right) \prod_{0 \leq r<s \leq 5} \Gamma\left(t^{n-i} t_{r} t_{s} ; p, q\right)  \tag{6.29}\\
t_{4} & =u_{0}  \tag{6.30}\\
t_{5} & =u_{1} \tag{6.31}
\end{align*}
$$

and the contour is chosen as in Theorem 6.1 except that we first absorb the singularities of $f$ and $g$ into the factors $\Gamma\left(u_{r} z_{i}^{ \pm 1} ; p, q\right)$ of the integrand. In particular, we have

$$
\begin{equation*}
\langle 1,1\rangle_{t_{0}, t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t, p, q}=1 \tag{6.32}
\end{equation*}
$$

Theorem 6.4. If $f \in \mathcal{A}^{(n)}\left(q^{1 / 2} u_{0} ; p, q\right), g \in \mathcal{A}^{(n)}\left(u_{1} ; p, q\right)$ and $t^{2 n-2} u_{0} u_{1} t_{0} t_{1} t_{2} t_{3}=p q$, then

$$
\begin{equation*}
\left\langle\mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right) f, g\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t, p, q}=\left\langle f, \mathcal{D}_{q}^{(n)}\left(u_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime} ; t, p\right) g\right\rangle_{t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime} ; u_{0}^{\prime}, u_{1}^{\prime} ; t, p, q}, \tag{6.33}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, u_{0}^{\prime}, u_{1}^{\prime}\right)=\left(q^{1 / 2} t_{0}, q^{1 / 2} t_{1}, q^{-1 / 2} t_{2}, q^{-1 / 2} t_{3}, q^{1 / 2} u_{0}, q^{-1 / 2} u_{1}\right) \tag{6.34}
\end{equation*}
$$

Proof. The second proof of Theorem 6.1 applies, essentially without change.
To understand the significance of this result, we need to introduce a filtration of the space $\mathcal{A}^{(n)}\left(u_{0} ; p, q\right)$. Let $\Lambda_{n}$ be the set of partitions of at most $n$ parts, and let $\subset$ denote the inclusion partial order; we also let $\subset$ denote the product partial order on $\Lambda_{n} \times \Lambda_{n}$. Then for any pair of partitions $\lambda, \mu \in \Lambda_{n}$, we define

$$
\begin{equation*}
\mathcal{A}_{\lambda \mu}^{(n)}\left(u_{0} ; t ; p, q\right) \tag{6.35}
\end{equation*}
$$

to be the subspace of $\mathcal{A}^{(n)}\left(u_{0} ; p, q\right)$ consisting of functions $f$ such that whenever $(\kappa, \nu) \not \subset(\lambda, \mu)$, we have the limit

$$
\lim _{\substack{z_{i} \rightarrow p^{-\kappa_{i} q^{-\nu}} \begin{array}{c}
-t_{i} \\
i=1 \ldots n \tag{6.36}
\end{array}}} \prod_{u_{0}} \theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m} f\left(\ldots z_{i} \ldots\right)=0
$$

whenever

$$
\begin{equation*}
\prod_{1 \leq i \leq n} \theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m} f\left(\ldots z_{i} \ldots\right) \tag{6.37}
\end{equation*}
$$

is holomorphic. Note that enlarging $l$ or $m$ multiplies the equation by a (possibly zero) scalar, so we really have only one equation for each pair $(\kappa, \nu)$.

Remark. In the univariate case ( $37,17,18,33,34,32]$ ), this filtration simply corresponds to a sequence of allowed poles. Given the role played by vanishing conditions in the theory of Koornwinder polynomials [15, 21, it would seem to be natural to generalize the forbiddance of a pole to the vanishing (after clearing the denominator) at an appropriate point, thus obtaining our filtration.

Lemma 6.5. For generic $u_{0}, p, q$, $t$, the filtration $\mathcal{A}_{\lambda}^{(n)}\left(u_{0} ; t ; p, q\right)$ is tight in the sense that

$$
\begin{equation*}
\operatorname{dim} \mathcal{A}_{\boldsymbol{\lambda}}^{(n)}\left(u_{0} ; t ; p, q\right)=1+\operatorname{dim} \sum_{\boldsymbol{\kappa} \subsetneq \boldsymbol{\lambda}} \mathcal{A}_{\boldsymbol{\kappa}}^{(n)}\left(u_{0} ; t ; p, q\right) \tag{6.38}
\end{equation*}
$$

for any partition pair $\boldsymbol{\lambda} \in \Lambda_{n}^{2}$. In particular, each space in the filtration is finite-dimensional.
Proof. Let $\boldsymbol{\lambda}=(\lambda, \mu), \boldsymbol{\kappa}=(\kappa, \nu)$. Since the spaces

$$
\begin{equation*}
\mathcal{A}_{\lambda \mu}^{(n)}\left(u_{0} ; t ; p, q\right) \tag{6.39}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{(\kappa, \nu) \subsetneq(\lambda, \mu)} \mathcal{A}_{\kappa \nu}^{(n)}\left(u_{0} ; t ; p, q\right) \tag{6.40}
\end{equation*}
$$

differ by a single equation, their dimensions differ by at most 1 ; it thus suffices to construct a function in the former but not in the latter.

Define a function $F_{\lambda \mu}^{(n)}\left(u_{0}: \ldots z_{i} \ldots\right)$ by the following product:

$$
\begin{equation*}
F_{\lambda \mu}^{(n)}\left(u_{0}: \ldots z_{i} \ldots ; t ; p, q\right)=\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \lambda_{1}}} \frac{\theta\left(p^{j} q t^{-\lambda_{j}^{\prime}} z_{i}^{ \pm 1} / u_{0} ; q\right)}{\theta\left(p^{j} q z_{i}^{ \pm 1} / u_{0} ; q\right)} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \mu_{1}}} \frac{\theta\left(p q^{j} t^{-\mu_{j}^{\prime}} z_{i}^{ \pm 1} / u_{0} ; p\right)}{\theta\left(p q^{j} z_{i}^{ \pm 1} / u_{0} ; p\right)} \tag{6.41}
\end{equation*}
$$

It follows as in the proof of Lemma 6.3 of [15] that

$$
\begin{equation*}
F_{\lambda \mu}^{(n)}\left(u_{0}: \ldots z_{i} \ldots\right) \in \mathcal{A}_{\lambda \mu}^{(n)}\left(u_{0} ; t ; p, q\right) \tag{6.42}
\end{equation*}
$$

on the other hand, we find that

$$
\begin{equation*}
\lim _{\substack{z_{i} \rightarrow p^{-\lambda_{i}} q^{-\mu_{i}} t^{i-1} u_{0} \\ i=1 \ldots n}} \prod_{1 \leq i \leq n} \theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m} F_{\lambda \mu}^{(n)}\left(u_{0}: \ldots z_{i} \ldots ; t ; p, q\right) \tag{6.43}
\end{equation*}
$$

is generically nonzero.
Remark. The function $F_{\lambda \mu}^{(n)}$ is a special case of the interpolation functions introduced below (Definition 5). Indeed, one can show that

$$
\begin{equation*}
F_{\boldsymbol{\lambda}}^{(n)}\left(u_{0} ; ; t, p, q\right)=\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; p q t^{-n} / u_{0}, u_{0} ; t ; p, q\right) \tag{6.44}
\end{equation*}
$$

The existence of such a factorizable special case of the interpolation functions will turn out to be crucial to the arguments of 19.

The reason we have introduced this filtration is the following fact:
Lemma 6.6. The difference operator $\mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right)$ is triangular with respect to the above filtration; that is, for all $\boldsymbol{\lambda} \in \Lambda_{n}^{2}$,

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right) \mathcal{A}_{\boldsymbol{\lambda}}^{(n)}\left(\sqrt{q} u_{0} ; t ; p, q\right) \subset \mathcal{A}_{\boldsymbol{\lambda}}^{(n)}\left(u_{0} ; t ; p, q\right), \tag{6.45}
\end{equation*}
$$

with equality for generic values of the parameters.
Proof. Let $\boldsymbol{\lambda}=(\lambda, \mu)$. Choose $l \geq \lambda_{1}, m \geq \mu_{1}$, and consider a function

$$
\begin{equation*}
f \in \mathcal{A}_{\lambda \mu}^{(n)}\left(\sqrt{q} u_{0} ; t ; p, q\right) \tag{6.46}
\end{equation*}
$$

For $\kappa \subset l^{n}, \nu \subset m^{n}$, define

$$
\begin{align*}
& C_{\kappa \nu}(f)=\lim _{z_{i} \rightarrow p^{-\kappa_{i}} q_{i=1 / 2-\nu_{i}}^{i} t^{i-1}} \prod_{u_{0}} \theta\left(p q^{1 / 2} z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m} f\left(\ldots z_{i} \ldots\right) \tag{6.47}
\end{align*}
$$

We claim that we can write

$$
\begin{equation*}
C_{\kappa \nu}^{\prime}=\sum_{\nu \subset \rho} c_{\kappa \nu \rho} C_{\kappa \rho} \tag{6.49}
\end{equation*}
$$

where the coefficients $c_{\kappa \nu \rho}$ are meromorphic and independent of the choice of $f$. Indeed, this follows readily from the definition of $\mathcal{D}$; compare the proof of Theorem 3.2 of [21]. More precisely, we see that a given term of the corresponding sum involves the specialization

$$
\begin{equation*}
\lim _{\substack{z_{i} \rightarrow p^{-\kappa_{i}} q_{i} / 2-\nu_{i} \\ i=1 \ldots n}} \prod_{t^{i-1} u_{0}}\left(p q^{1 / 2} z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m} f\left(\ldots z_{i} \ldots\right) \tag{6.50}
\end{equation*}
$$

if the sequence $\frac{1-\sigma_{i}}{2}+\nu_{i}$ does not induce a partition, then the remaining factors vanish, while if it does give a partition, that partition necessarily contains $\nu$. We also find that the diagonal coefficient $c_{\kappa \nu \nu}$ is generically nonzero; the result follows.

Now, given a pair of spaces with corresponding tight filtrations, equipped with a (sufficiently general) scalar product, there is a unique (up to scalar multiples) orthogonal pair of bases compatible with the filtration. In the case of the above scalar product, this suggests the following definition.
"Definition". For all partition pairs $\boldsymbol{\lambda} \in \Lambda_{n}^{2}$, the function

$$
\begin{equation*}
R_{\lambda}^{(n)}\left(\ldots z_{i} \ldots ; t_{0}, t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right) \tag{6.51}
\end{equation*}
$$

is defined to be the unique (up to scalar multiples) element of $\mathcal{A}_{\boldsymbol{\lambda}}^{(n)}\left(u_{0} ; t ; p, q\right)$ such that

$$
\begin{equation*}
\left\langle R_{\lambda}^{(n)}\left(\ldots z_{i} \ldots ; t_{0}, t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right), g\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3} ; u_{0} ; u_{1} ; t ; p, q}=0 \tag{6.52}
\end{equation*}
$$

whenever $g \in \mathcal{A}_{\kappa}^{(n)}\left(u_{1} ; t ; p, q\right)$ for some $\boldsymbol{\kappa} \subsetneq \boldsymbol{\lambda}$.
Since our adjoint difference operators preserve the filtrations, they would necessarily be diagonal in the corresponding bases, if they were well-defined. Unfortunately, we have as yet no reason to believe that the scalar product is nondegenerate relative to the filtration; that is, that its restriction to $\mathcal{A}_{\lambda}^{(n)}\left(u_{0} ; t ; p, q\right)$ and $\mathcal{A}_{\boldsymbol{\lambda}}^{(n)}\left(u_{1} ; t ; p, q\right)$ is nondegenerate for all partition pairs. If this condition were to fail for a given pair $\boldsymbol{\kappa}$, then the function $R_{\boldsymbol{\lambda}}^{(n)}$ would not be uniquely determined for $\boldsymbol{\lambda} \supsetneq \boldsymbol{\kappa}$, and the argument breaks down.

There is one special case in which we can prove the scalar product generically nondegenerate.
Proposition 6.7. For generic parameters satisfying $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} u_{0} u_{1}=p q$, and any partition $\lambda \in \Lambda_{n}$, the scalar product $\left\rangle_{t_{0}, t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q}\right.$ is nondegenerate between $\mathcal{A}_{0 \lambda}^{(n)}\left(u_{0} ; t ; p, q\right)$ and $\mathcal{A}_{0 \lambda}^{(n)}\left(u_{1} ; t ; p, q\right)$.

Proof. To show a scalar product generically nondegenerate, it suffices to exhibit a nondegenerate specialization. Choose $l$ such that the spaces

$$
\begin{equation*}
\theta\left(p q / u_{0} ; p, q\right)_{0, l} \mathcal{A}_{0 \lambda}^{(n)}\left(u_{0} ; t ; p, q\right) \quad \text { and } \quad \theta\left(p q / u_{1} ; p, q\right)_{0, l} \mathcal{A}_{0 \lambda}^{(n)}\left(u_{1} ; t ; p, q\right) \tag{6.53}
\end{equation*}
$$

consist of holomorphic functions, and specialize the parameters so that every parameter except $u_{0}, u_{1}$ is real, between 0 and 1 , while $u_{0}$ and $u_{1}$ are complex conjugates satisfying $0<\left|u_{0}\right|=\left|u_{1}\right|<q^{l}$. (This is possible as long as $p<q^{2 l-1} t^{2 n-2}$.) Then the contour in the scalar product can be taken to be the unit torus, on which the weight function is clearly strictly positive. Moreover, the filtrations with respect to $u_{0}$ and $u_{1}$ are conjugate to each other. The scalar product thus becomes a positive definite Hermitian inner product, and is therefore nondegenerate.

This in particular proves the existence and uniqueness of the above biorthogonal functions, as long as one of the partitions is trivial. In general, however, it is unclear how to construct a manifestly nondegenerate instance of the scalar product. We will therefore give a more direct construction of these functions, and by computing their scalar products show that this problem generically does not arise. (In addition, the above construction gives functions that are only guaranteed to be orthogonal when the corresponding pairs of partitions are distinct but comparable; it will follow below (as one would expect) that comparability is not necessary.)

To do this, we need a different adjoint pair of difference operators.

First, define

$$
\begin{equation*}
\mathcal{D}_{q}^{-(n)}\left(u_{0} ; t, p\right):=D_{q}^{(n)}\left(u_{0}, q u_{0}, p / u_{0}, \frac{1}{t^{n-1} u_{0} q} ; t, p\right) \tag{6.54}
\end{equation*}
$$

Next, define

$$
\begin{align*}
&\left(\mathcal{D}_{q}^{+(n)}\left(u_{0}: u_{1}: u_{2}, u_{3}, u_{4} ; t, p\right) f\right)\left(\ldots z_{i} \ldots\right) \\
&:= \prod_{1 \leq i \leq n} \frac{\theta\left(p q t^{n-i} u_{1} / u_{0} ; p\right)}{\prod_{2 \leq r \leq 5} \theta\left(u_{r} t^{n-i} u_{1} ; p\right)}  \tag{6.55}\\
& \sum_{\sigma \in\{ \pm 1\}^{n}} \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq r \leq 5} \theta\left(u_{r} z_{i}^{\sigma_{i}} ; p\right)}{\theta\left(p q z_{i}^{\sigma_{i}} / u_{0} ; p\right) \theta\left(z_{i}^{2 \sigma_{i}} ; p\right)} \prod_{1 \leq i<j \leq n} \frac{\theta\left(t z_{i}^{\sigma_{i}} z_{j}^{\sigma_{j}} ; p\right)}{\theta\left(z_{i}^{\sigma_{i}} z_{j}^{\sigma_{j}} ; p\right)} f\left(\ldots q^{\sigma_{i} / 2} z_{i} \ldots\right),
\end{align*}
$$

where $u_{5}=p^{2} q / t^{n-1} u_{0} u_{1} u_{2} u_{3} u_{4}$. Note that aside from the normalization factor, $\mathcal{D}_{q}^{+(n)}$ is symmetric in $u_{1}$ through $u_{5}$.

These act as lowering and raising operators with respect to the filtration:
Lemma 6.8. For all $\boldsymbol{\lambda} \in \Lambda_{n}^{2}$ with $(0,1)^{n} \subset \boldsymbol{\lambda}$,

$$
\begin{equation*}
\mathcal{D}_{q}^{-(n)}\left(u_{0} ; t, p\right) \mathcal{A}_{\lambda}^{(n)}\left(q^{3 / 2} u_{0} ; t ; p, q\right) \subset \mathcal{A}_{\lambda-(0,1)^{n}}^{(n)}\left(u_{0} ; t ; p, q\right) \tag{6.56}
\end{equation*}
$$

Similarly, for all $\boldsymbol{\lambda} \in \Lambda_{n}^{2}$,

$$
\begin{equation*}
\mathcal{D}_{q}^{+(n)}\left(u_{0}: u_{1}: u_{2}, u_{3}, u_{4} ; t, p\right) \mathcal{A}_{\lambda}^{(n)}\left(q^{-1 / 2} u_{0} ; t ; p, q\right) \subset \mathcal{A}_{\lambda+(0,1)^{n}}^{(n)}\left(u_{0} ; t ; p, q\right) \tag{6.57}
\end{equation*}
$$

Moreover, the restriction of $\mathcal{D}^{-(n)}$ is generically surjective, while the restriction of $\mathcal{D}^{+(n)}$ is generically injective. Proof. As above.

Theorem 6.9. If $f \in \mathcal{A}^{(n)}\left(q^{-1 / 2} u_{0} ; p, q\right), g \in \mathcal{A}^{(n)}\left(u_{1} ; p, q\right)$ and $t^{2 n-2} u_{0} u_{1} t_{0} t_{1} t_{2} t_{3}=p q$, then

$$
\begin{equation*}
\left\langle\mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{0}: t_{1}, t_{2}, t_{3} ; t, p\right) f, g\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t, p, q}=C\left\langle f, \mathcal{D}_{q}^{-(n)}\left(u_{1}^{\prime} ; t, p\right) g\right\rangle_{t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime} ; u_{0}^{\prime}, u_{1}^{\prime} ; t, p, q}, \tag{6.58}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, u_{0}^{\prime}, u_{1}^{\prime}\right)=\left(q^{1 / 2} t_{0}, q^{1 / 2} t_{1}, q^{1 / 2} t_{2}, q^{1 / 2} t_{3}, q^{-1 / 2} u_{0}, q^{-3 / 2} u_{1}\right) \tag{6.59}
\end{equation*}
$$

and

$$
\begin{equation*}
C=\prod_{1 \leq i \leq n} \frac{\theta\left(t^{n-i} t_{1} t_{2}, t^{n-i} t_{1} t_{3}, t^{n-i} t_{2} t_{3}, p q t^{n-i} t_{0} / u_{0} ; p\right)}{\theta\left(t^{n-i} t_{0} u_{1} / q, t^{n-i} t_{1} u_{1} / q, t^{n-i} t_{2} u_{1} / q, t^{n-i} t_{3} u_{1} / q, t^{n-i} u_{0} u_{1} / q, t^{n-i} u_{0} u_{1} / q^{2}, p t_{0} u_{1} t^{2 n-1-i} ; p\right)} \tag{6.60}
\end{equation*}
$$

## 7 Integral operators

Just as our second proof of Theorem 6.1 is related to an adjoint pair of difference operators, the argument of van Diejen and Spiridonov is related to an adjoint pair of integral operators. To understand these operators, we first need to understand what happens to the $I_{B C_{n}}^{(0)}$ integral when the integrand is multiplied by an element of $\mathcal{A}\left(t_{0} ; p, q\right)$. We define a corresponding integral operator as follows.

Definition 2. If $f \in \mathcal{A}\left(u_{0} ; p, q\right)$, then $I^{(n)}\left(u_{0} ; p, q\right) f$ is the function on the set $\prod_{0 \leq r \leq 2 n+3} u_{r}=p q$ defined by

$$
\begin{align*}
\left(\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right) f\right) & \left(u_{1}, \ldots u_{2 n+3}\right) \\
= & \frac{(p ; p)^{n}(q ; q)^{n}}{2^{n} n!\prod_{0 \leq r<s \leq 2 n+3} \Gamma\left(u_{r} u_{s} ; p, q\right)} \int_{C^{n}} f\left(\ldots z_{i} \ldots\right) \prod_{1 \leq i<j \leq n} \frac{1}{\Gamma\left(z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right)}  \tag{7.1}\\
& \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 2 n+3} \Gamma\left(u_{r} z_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(z_{i}^{ \pm 2} ; p, q\right)} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}},
\end{align*}
$$

with the usual conventions about the choice of contour.
In particular, by Corollary 3.2 it follows that

$$
\begin{equation*}
\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right) 1=1 . \tag{7.2}
\end{equation*}
$$

To determine the action of this integral operator in general, it suffices to consider $f$ in a spanning set. We may thus restrict our attention to functions of the form

$$
\begin{equation*}
f\left(\ldots z_{i} \ldots\right)=\prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq l} x_{j}^{-1} \theta\left(x_{j} z_{i}^{ \pm 1} ; q\right) \prod_{1 \leq j \leq m} y_{j}^{-1} \theta\left(y_{j} z_{i}^{ \pm 1} ; p\right)}{\theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m}} \tag{7.3}
\end{equation*}
$$

If we write the theta functions in the numerator as a ratio of elliptic $\Gamma$ functions, and similarly absorb the denominator factors into a ratio of elliptic $\Gamma$ functions, we find that the resulting integral is proportional to an integral of type $I_{B C_{n}}^{(l+m)}$ in which the extra $2 l+2 m$ parameters have pairwise products $p^{2} q$ and $p q^{2}$. If we then apply Theorem 3.1 we find that the right-hand side becomes a sum via residue calculus. We thus obtain the following result.

Theorem 7.1. If

$$
\begin{equation*}
f\left(\ldots z_{i} \ldots\right)=\prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq l} x_{j}^{-1} \theta\left(x_{j} z_{i}^{ \pm 1} ; q\right) \prod_{1 \leq j \leq m} y_{j}^{-1} \theta\left(y_{j} z_{i}^{ \pm 1} ; p\right)}{\theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m}} \tag{7.4}
\end{equation*}
$$

and $\prod_{0 \leq r \leq 2 n+3} u_{r}=p q$, then

$$
\begin{align*}
\left(\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right) f\right) & \left(u_{1} \ldots u_{2 n+3}\right) \\
= & \prod_{1 \leq r \leq 2 n+3} \frac{1}{\theta\left(p q / u_{0} u_{r} ; p, q\right)_{l, m}}  \tag{7.5}\\
& \left(\prod_{1 \leq i \leq l}\left(1+R\left(x_{i}\right)\right) \frac{\theta\left(p^{-l} u_{0} x_{i} ; q\right) \prod_{1 \leq r \leq 2 n+3} \theta\left(u_{r} x_{i} ; q\right)}{x_{i}^{n} \theta\left(x_{i}^{2} ; q\right)} \prod_{1 \leq i<j \leq l} \frac{\theta\left(p x_{i} x_{j} ; q\right)}{\theta\left(x_{i} x_{j} ; q\right)}\right) \\
& \left(\prod_{1 \leq i \leq m}\left(1+R\left(y_{i}\right)\right) \frac{\theta\left(q^{-m} u_{0} y_{i} ; p\right) \prod_{1 \leq r \leq 2 n+3} \theta\left(u_{r} y_{i} ; p\right)}{y_{i}^{n} \theta\left(y_{i}^{2} ; p\right)} \prod_{1 \leq i<j \leq m} \frac{\theta\left(q y_{i} y_{j} ; p\right)}{\theta\left(y_{i} y_{j} ; p\right)}\right)
\end{align*}
$$

where $R\left(x_{k}\right)$ is an operator acting on $g\left(\ldots x_{i} \ldots\right)$ via the substitution $x_{k} \mapsto x_{k}^{-1}$; thus the factors in parentheses are sums of $2^{l}$ and $2^{m}$ terms respectively.

Since the factors in parentheses are clearly holomorphic in $u_{1} \ldots u_{2 n+3}$, and the given functions span $\mathcal{A}\left(u_{0} ; p, q\right)$, we obtain the following as an immediate consequence:

Corollary 7.2. If $f \in \mathcal{A}\left(u_{0} ; p, q\right)$ is such that

$$
\begin{equation*}
\theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m} f\left(\ldots z_{i} \ldots\right) \tag{7.6}
\end{equation*}
$$

is holomorphic, then

$$
\begin{equation*}
\prod_{1 \leq r \leq 2 n+3} \theta\left(p q / u_{0} u_{r} ; p, q\right)_{l, m}\left(\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right) f\right)\left(u_{1} \ldots u_{2 n+3}\right) \tag{7.7}
\end{equation*}
$$

is holomorphic on the set $\prod_{0 \leq r \leq 2 n+3} u_{r}=p q$.
Remark 1. Similarly, the left-hand side of (7.5) is manifestly a holomorphic $q$-theta function in the $x$ 's, and a holomorphic $p$-theta function in the $y$ 's; that this is true of the right-hand side follows from a symmetrization argument analogous to those we have just encountered in studying difference operators. And, indeed, the two sums are really just minor variants of the difference operators we have already seen.

Remark 2. As the above argument is based on Theorem 3.1 it cannot be directly applied in the limit $p \rightarrow 0$. In fact, one can also derive this result from Corollary 3.2 for which direct, non-elliptic, proofs are known in the $p \rightarrow 0$ limit [11]. The basic observation is that if two of the parameters have product $q$, i.e., if two of the $\Gamma$ factors combine to produce a factor of the form

$$
\begin{equation*}
\prod_{1 \leq i \leq n} \frac{1}{\theta\left(a z_{i}^{ \pm 1} ; q\right)} \tag{7.8}
\end{equation*}
$$

then the integrand is essentially invariant under $a \mapsto 1 / a$ (aside from an overall constant). However, the integral does not share this invariance, because inverting $a$ changes the constraint on the contour. The two contours differ only in whether they contain the points $z=a^{ \pm 1}$; as a result, the difference in the two integrals is (proportional to) the $n$-1-dimensional integral of the residue at that point. This $n$ - 1 -dimensional integral simplifies to the above form, with $l=1, m=0$; the difference of the original $n$-dimensional integrals simplifies to the desired right-hand side. This argument can then be repeated as necessary to prove the theorem for arbitrary values of $l, m \geq 0$.

Remark 3. The fact that we obtain an $l+m$-tuple sum is, of course, directly related to the fact that we needed $2 l+2 m \Gamma$ factors to represent the numerator of $f$. In general, if we took

$$
\begin{equation*}
f\left(\ldots z_{i} \ldots\right)=\prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq m} \theta\left(x_{j} z_{i}^{ \pm 1} ; p, q\right)_{a_{j}, b_{j}}}{\theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{\sum a_{j}, \sum b_{j}}} \tag{7.9}
\end{equation*}
$$

residue calculus would again give a sum, this time a $2 m$-tuple sum (i.e., the product of an $m$-tuple sum for $p$ and an $m$-tuple sum for $q$ ). On the other hand, we could also compute $\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right) f$ by specialization of Theorem 7.1. which would give a sum with $2^{\sum a_{j}+b_{j}}$ terms. The fact that this sum simplifies underlies Rosengren's arguments in section 7 of [25].

Remark 4. It is particularly striking that the right-hand side factors as a product of two sums, one involving only $q$-theta functions, and one involving only $p$-theta functions. This factorization phenomenon appears to
hold quite generally in the theory of elliptic hypergeometric integrals, but only when the relevant balancing condition holds.

Since $\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right)$ takes $B C_{n}$-symmetric functions to $A_{2 n+2}$-symmetric functions, it is not quite suitable for our purposes. However, we can readily obtain $B C$-symmetric functions by suitable specialization.
Definition 3. Define operators $\mathcal{I}_{t}^{+(n)}\left(u_{0} ; p, q\right), \mathcal{I}_{t}^{(n)}\left(u_{0}: u_{1}, u_{2} ; p, q\right)$, and $\mathcal{I}_{t}^{-(n)}\left(u_{0}: u_{1}, u_{2}, u_{3}, u_{4} ; p, q\right)$ by:

$$
\begin{align*}
\left(\mathcal{I}_{t}^{+(n)}\left(u_{0} ; p, q\right) f\right)\left(z_{1} \ldots z_{n+1}\right) & =\left(\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right) f\right)\left(\frac{t^{-n-1} p q}{u_{0}}, \ldots \sqrt{t} z_{i}^{ \pm 1} \ldots\right)  \tag{7.10}\\
\left(\mathcal{I}_{t}^{(n)}\left(u_{0}: u_{1}, u_{2} ; p, q\right) f\right)\left(z_{1} \ldots z_{n}\right) & =\left(\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right) f\right)\left(u_{1}, u_{2}, \frac{t^{-n} p q}{u_{0} u_{1} u_{2}}, \ldots \sqrt{t} z_{i}^{ \pm 1} \ldots\right)  \tag{7.11}\\
\left(\mathcal{I}_{t}^{-(n)}\left(u_{0}: u_{1}, u_{2}, u_{3}, u_{4} ; p, q\right) f\right)\left(z_{1} \ldots z_{n-1}\right) & =\left(\mathcal{I}^{*(n)}\left(u_{0} ; p, q\right) f\right)\left(u_{1}, u_{2}, u_{3}, u_{4}, \frac{t^{1-n} p q}{u_{0} u_{1} u_{2} u_{3} u_{4}}, \ldots \sqrt{t} z_{i}^{ \pm 1} \ldots\right) \tag{7.12}
\end{align*}
$$

Theorem 7.3. The above operators are triangular with repsect to the filtration of $\mathcal{A}^{(n)}\left(u_{0} ; p, q\right)$; to be precise,

$$
\begin{align*}
\mathcal{I}_{t}^{+(n)}\left(u_{0} ; p, q\right) \mathcal{A}_{\boldsymbol{\lambda}}^{(n)}\left(u_{0} ; t ; p, q\right) & \subseteq \mathcal{A}_{\boldsymbol{\lambda}}^{(n+1)}\left(t^{1 / 2} u_{0} ; t ; p, q\right)  \tag{7.13}\\
\mathcal{I}_{t}^{(n)}\left(u_{0}: u_{1}, u_{2} ; p, q\right) \mathcal{A}_{\boldsymbol{\lambda}}^{(n)}\left(u_{0} ; t ; p, q\right) & \subseteq \mathcal{A}_{\boldsymbol{\lambda}}^{(n)}\left(t^{1 / 2} u_{0} ; t ; p, q\right) \tag{7.14}
\end{align*}
$$

and, if $\boldsymbol{\lambda}_{n}=(0,0)$,

$$
\begin{equation*}
\mathcal{I}_{t}^{-(n)}\left(u_{0}: u_{1}, u_{2}, u_{3}, u_{4} ; p, q\right) \mathcal{A}_{\lambda}^{(n)}\left(u_{0} ; t ; p, q\right) \subseteq \mathcal{A}_{\boldsymbol{\lambda}}^{(n-1)}\left(t^{1 / 2} u_{0} ; t ; p, q\right) \tag{7.15}
\end{equation*}
$$

Moreover, $\mathcal{I}_{t}^{+(n)}\left(u_{0} ; p, q\right)$ is generically injective, and $\mathcal{I}_{t}^{-(n)}\left(u_{0}: u_{1}, u_{2}, u_{3}, u_{4} ; p, q\right)$ is generically surjective.
Proof. It suffices to consider the action of the operators on the functions

$$
\begin{equation*}
F_{\lambda \mu}^{(n)}\left(u_{0}: \ldots z_{i} \ldots ; t ; p, q\right)=\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \lambda_{1}}} \frac{\theta\left(p^{j} q t^{-\lambda_{j}^{\prime}} z_{i}^{ \pm 1} / u_{0} ; q\right)}{\theta\left(p^{j} q z_{i}^{ \pm 1} / u_{0} ; q\right)} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \mu_{1}}} \frac{\theta\left(p q^{j} t^{-\mu_{j}^{\prime}} z_{i}^{ \pm 1} / u_{0} ; p\right)}{\theta\left(p q^{j} z_{i}^{ \pm 1} / u_{0} ; p\right)} \tag{7.16}
\end{equation*}
$$

considered above. Applying Theorem[7.1] we find that each term of the resulting sum is also of this form, with appropriately constrained partitions. The one exception is $\mathcal{I}_{t}^{-(n)}$ in the case when $\lambda_{n}$ or $\mu_{n}>0$, which we will consider below.

As promised, the integral operators indeed satisfy appropriate adjointness relations.
Theorem 7.4. If $f \in \mathcal{A}^{(n)}\left(u_{0} ; p, q\right), g \in \mathcal{A}^{(n)}\left(t^{-1 / 2} u_{1} ; p, q\right)$ and $t^{2 n-2} u_{0} u_{1} t_{0} t_{1} t_{2} t_{3}=p q$, then

$$
\begin{equation*}
\left\langle\mathcal{I}_{t}^{(n)}\left(u_{0}: t_{0}, t_{1} ; p, q\right) f, g\right\rangle_{t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime} ; u_{0}^{\prime}, u_{1}^{\prime} ; t ; p, q}=\left\langle f, \mathcal{I}_{t}^{(n)}\left(u_{1}^{\prime}: t_{2}^{\prime}, t_{3}^{\prime} ; p, q\right) g\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q} \tag{7.17}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, u_{0}^{\prime}, u_{1}^{\prime}\right)=\left(t^{1 / 2} t_{0}, t^{1 / 2} t_{1}, t^{-1 / 2} t_{2}, t^{-1 / 2} t_{3}, t^{1 / 2} u_{0}, t^{-1 / 2} u_{1}\right) \tag{7.18}
\end{equation*}
$$

Similarly, if $f \in \mathcal{A}^{(n)}\left(u_{0} ; p, q\right), g \in \mathcal{A}^{(n-1)}\left(u_{1}^{\prime} ; p, q\right)$ and $t^{2 n-2} u_{0} u_{1} t_{0} t_{1} t_{2} t_{3}=p q$, then

$$
\begin{equation*}
\left\langle\mathcal{I}_{t}^{-(n)}\left(u_{0}: t_{0}, t_{1}, t_{2}, t_{3} ; p, q\right) f, g\right\rangle_{t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, u_{0}^{\prime}, u_{1}^{\prime} ; t ; p, q}=\left\langle f, \mathcal{I}_{t}^{+(n-1)}\left(u_{1}^{\prime} ; p, q\right) g\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q} \tag{7.19}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, u_{0}^{\prime}, u_{1}^{\prime}\right)=\left(t^{1 / 2} t_{0}, t^{1 / 2} t_{1}, t^{1 / 2} t_{2}, t^{1 / 2} t_{3}, t^{1 / 2} u_{0}, t^{-1 / 2} u_{1}\right) \tag{7.20}
\end{equation*}
$$

Proof. In each case, the definition of the integral operators allows us to express the inner products as double integrals; the stated identities correspond to changing the order of integration.

Recall that for the operators $\mathcal{D}^{-}$and $\mathcal{I}^{-}$, we were only able to show triangularity with respect to a portion of the filtration; for some functions, the methods we used were insufficient to understand the images. The key observation for dealing with those cases is that the difficult case for $\mathcal{D}^{-}$is precisely the (generic) image of $\mathcal{I}^{+}$, and similarly the difficult case for $\mathcal{I}^{-}$is the image of $\mathcal{D}^{+}$. Thus to complete our understanding of the action of these operators on the filtration, it will suffice to prove the following result.

Theorem 7.5. For any function $f \in \mathcal{A}\left(u_{0} ; p, q\right)$,

$$
\begin{equation*}
\mathcal{D}_{q}^{-(n)}\left(q^{-3 / 2} t^{1 / 2} u_{0} ; t, p\right) \mathcal{I}_{t}^{+(n-1)}\left(u_{0} ; p, q\right) f=0 \tag{7.21}
\end{equation*}
$$

Similarly, for any function $f \in \mathcal{A}^{(n)}\left(q^{-1 / 2} u_{0} ; p, q\right)$,

$$
\begin{equation*}
\mathcal{I}_{t}^{-(n)}\left(u_{0}: t_{0}, t_{1}, t_{2}, t_{3} ; p, q\right) \mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{0}, t_{1}, t_{2}, t_{3} ; t, p\right) f=0 \tag{7.22}
\end{equation*}
$$

Proof. For the first identity, take

$$
\begin{equation*}
f\left(\ldots z_{i} \ldots\right)=\prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq l} x_{j}^{-1} \theta\left(x_{j} z_{i}^{ \pm 1} ; q\right) \prod_{1 \leq j \leq m} y_{j}^{-1} \theta\left(y_{j} z_{i}^{ \pm 1} ; p\right)}{\theta\left(p q z_{i}^{ \pm 1} / u_{0} ; p, q\right)_{l, m}} \tag{7.23}
\end{equation*}
$$

we can thus compute its image via Theorem 7.1 and the definition of $\mathcal{D}^{-(n)}$. The vanishing of the resulting sum follows as a special case of Lemma 7.6 below.

For the second identity, we can argue as in the proof of adjointness of the difference operators to express the image as the integral of $f\left(\ldots z_{i} \ldots\right)$ with respect to an appropriate $B C_{n}$-symmetric density. That this density vanishes identically follows from Lemma 7.8 below.

Lemma 7.6. For arbitrary parameters satisfying $v w \prod_{1 \leq i \leq n} q_{i}^{2}=1$, and generic $z_{1}, \ldots z_{n}$,

$$
\begin{equation*}
\prod_{1 \leq i \leq n}\left(1+R\left(z_{i}\right)\right) \frac{\theta\left(v q_{i} z_{i}, w q_{i} z_{i} ; p\right)}{\theta\left(z_{i}^{2} ; p\right)} \prod_{1 \leq i<j \leq n} \frac{q_{j}^{-1} \theta\left(q_{i} q_{j} z_{i} z_{j}, q_{j} z_{i} / q_{i} z_{j} ; p\right)}{\theta\left(z_{i} z_{j}, z_{i} / z_{j} ; p\right)}=0 \tag{7.24}
\end{equation*}
$$

Proof. For $n=1$, the summand is manifestly antisymmetric under $R\left(z_{i}\right)$, and thus the lemma follows in that case. Thus assume $n>1$, set $v=u / Q, w=(u Q)^{-1}$ with $Q:=\prod_{1 \leq i \leq n} q_{i}$, and consider the sum as a function of $u$. We readily verify that it is a $B C_{1}$-symmetric theta function in $u$ of degree $n$; we thus need only show that it vanishes at more than $n$ independent points. If $u=Q z_{n} / q_{n}$, the terms involving $R\left(z_{n}\right)$ vanish; moreover, if we pull out $B C_{n-1}$-symmetric factors, we obtain a special case of the $n-1$-dimensional sum. By symmetry, the identity holds for any point of the form $u=Q z_{i}^{ \pm 1} / q_{i}$; since $n>1$, these $2 n$ points are generically independent, and the result follows.

We note the following related result in passing:

Corollary 7.7. For arbitrary parameters satisfying tuvw $\prod_{1 \leq i \leq n} q_{i}^{2}=1$, and generic $z_{1}, \ldots z_{n}$,

$$
\begin{equation*}
t \theta(u v, u w, v w ; p) \prod_{1 \leq i \leq n}\left(1+R\left(z_{i}\right)\right) \frac{\theta\left(t z_{i} / q_{i}, u q_{i} z_{i}, v q_{i} z_{i}, w q_{i} z_{i} ; p\right)}{z_{i} \theta\left(z_{i}^{2} ; p\right)} \prod_{1 \leq i<j \leq n} \frac{q_{j}^{-1} \theta\left(q_{i} q_{j} z_{i} z_{j}, q_{j} z_{i} / q_{i} z_{j} ; p\right)}{\theta\left(z_{i} z_{j}, z_{i} / z_{j} ; p\right)} \tag{7.25}
\end{equation*}
$$

is symmetric under permutations of $t, u, v, w$.
Proof. The sum is manifestly symmetric in $u, v, w$, so it suffices to show that it is invariant under the exchange of $t$ and $u$. Thus take the difference of the given sum and its image upon exchanging $t$ and $u$. If we then set $t=q_{n+1} y_{n+1}, u=q_{n+1} / y_{n+1}$, we obtain the $n+1$-dimensional instance of the lemma.

Lemma 7.8. For generic values of $y_{1}, \ldots y_{n}$,

$$
\begin{equation*}
\prod_{1 \leq i \leq n}\left(1+R\left(y_{i}\right)\right) \prod_{1 \leq i<j \leq n} \frac{\theta\left(u y_{i} y_{j} ; p\right)}{\theta\left(y_{i} y_{j} ; p\right)} \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq r \leq n-1} \theta\left(u^{-1 / 2} z_{r}^{ \pm 1} y_{i} ; p\right)}{y_{i}^{n-2} \theta\left(y_{i}^{2} ; p\right)}=0 \tag{7.26}
\end{equation*}
$$

Proof. When $n=1$, the summand is antisymmetric under $R\left(y_{1}\right)$, and the sum therefore vanishes. Now, consider the sum for general $n$ as a function of $z_{n-1}$. This is manifestly a $B C_{1}$-symmetric theta function of degree $n$; it thus suffices to show that it vanishes at more than $n$ independent points. If $z_{n-1}=u^{1 / 2} y_{n}$, the terms coming from $R\left(y_{n}\right)$ vanish; we thus obtain an instance of the $n-1$-dimensional sum, which vanishes by induction. By symmetry, the sum vanishes at any point of the form $z_{n-1}=u^{1 / 2} y_{i}^{ \pm 1}$; this gives $2 n$ independent values at which the sum vanishes, proving the lemma.

A similar argument applies to the following result, which can also be obtained from Theorem [3.1]via residue calculus.

Theorem 7.9. Choose integers $m \geq l \geq 0$, and suppose $q^{m-l} t_{0} t_{1} t_{2} t_{3}=q$. Then we have the following identity.

$$
\begin{align*}
\prod_{1 \leq i \leq m}(1 & \left.+R\left(x_{i}\right)\right) \frac{\theta\left(t_{0} x_{i}, t_{1} x_{i}, t_{2} x_{i}, t_{3} x_{i} ; p\right) \prod_{1 \leq r \leq l} \theta\left(q^{-1 / 2} x_{i} y_{r}^{ \pm 1} ; p\right)}{x_{i}^{l+1} \theta\left(x_{i}^{2} ; p\right)} \prod_{1 \leq i<j \leq m} \frac{\theta\left(q x_{i} x_{j} ; p\right)}{\theta\left(x_{i} x_{j} ; p\right)} \\
= & \prod_{0 \leq i<m-l} \frac{\theta\left(q^{i} t_{0} t_{1}, q^{i} t_{0} t_{2}, q^{i} t_{0} t_{3} ; p\right)}{\left(-q^{1 / 2}\right)^{m-1} t_{0}}  \tag{7.27}\\
& \prod_{1 \leq i \leq l}\left(1+R\left(y_{i}\right)\right) \frac{\theta\left(\frac{q^{1 / 2}}{t_{0}} y_{i}, \frac{q^{1 / 2}}{t_{1}} y_{i}, \frac{q^{1 / 2}}{t_{2}} y_{i}, \frac{q^{1 / 2}}{t_{3}} y_{i} ; p\right) \prod_{1 \leq r \leq m} \theta\left(q^{-1 / 2} y_{i} x_{r}^{ \pm 1} ; p\right)}{y_{i}^{m+1} \theta\left(y_{i}^{2} ; p\right)} \prod_{1 \leq i<j \leq l} \frac{\theta\left(q y_{i} y_{j} ; p\right)}{\theta\left(y_{i} y_{j} ; p\right)}
\end{align*}
$$

Proof. By the usual symmetry argument, we find that both sides are $B C_{m}$-symmetric theta functions of degree $l$ in $x$. By induction, both sides agree if $x_{m}$ is of the form $t_{r}$ or $q^{-1 / 2} y_{i}^{ \pm 1}$; this gives $2 l+4$ independent points at which the functions agree, which shows that they agree everywhere.

This gives rise to some commutation relations between our difference and integral operators.
Corollary 7.10. For any function $f \in \mathcal{A}^{(n)}\left(q^{1 / 2} u_{0} ; p, q\right)$,

$$
\begin{align*}
& \mathcal{I}_{t}^{(n)}\left(u_{0}: t_{0}, t_{1} ; p, q\right) \mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right) f=\mathcal{D}_{q}^{(n)}\left(t^{1 / 2} u_{0}, t^{1 / 2} t_{0}, t^{1 / 2} t_{1} ; t, p\right) \mathcal{I}_{t}^{(n)}\left(q^{1 / 2} u_{0}: q^{1 / 2} t_{0}, q^{1 / 2} t_{1} ; p, q\right) f  \tag{7.28}\\
& \mathcal{I}_{t}^{(n)}\left(u_{0}: t_{0}, t_{1} ; p, q\right) \mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{2} ; t, p\right) f=\mathcal{D}_{q}^{(n)}\left(t^{1 / 2} u_{0}, t^{1 / 2} t_{0}, t^{-1 / 2} t_{2} ; t, p\right) \mathcal{I}_{t}^{(n)}\left(q^{1 / 2} u_{0}: q^{1 / 2} t_{0}, q^{-1 / 2} t_{1} ; p, q\right) f  \tag{7.29}\\
& \quad \mathcal{I}_{t}^{+(n)}\left(u_{0} ; p, q\right) \mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right) f=\mathcal{D}_{q}^{(n+1)}\left(t^{1 / 2} u_{0}, t^{-1 / 2} t_{0}, t^{-1 / 2} t_{1} ; t, p\right) \mathcal{I}_{t}^{+(n)}\left(q^{1 / 2} u_{0} ; p, q\right) f \tag{7.30}
\end{align*}
$$

while for any function $f \in \mathcal{A}^{(n)}\left(q^{-1 / 2} u_{0} ; p, q\right)$,

$$
\begin{align*}
\mathcal{I}_{t}^{(n)}\left(u_{0}: t_{0}, t_{1} ; p, q\right) & \mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{0}: t_{1}, t_{2}, t_{3} ; t, p\right) f  \tag{7.31}\\
& =\mathcal{D}_{q}^{+(n)}\left(t^{1 / 2} u_{0}: t^{1 / 2} t_{0}: t^{1 / 2} t_{1}, t^{-1 / 2} t_{2}, t^{-1 / 2} t_{3} ; t, p\right) \mathcal{I}_{t}^{(n)}\left(q^{-1 / 2} u_{0}: q^{1 / 2} t_{0}, q^{1 / 2} t_{1} ; p, q\right) f
\end{align*}
$$

Proof. In each case, arguing as in the proof of adjointness of difference operators transforms the left-hand side into an integral of $f$ against a $B C_{n}$-symmetric density which itself can be transformed via the theorem to give the right-hand side.

## 8 Biorthogonal functions

Now that we have suitable difference and integral operators, we are now in a position to construct the desired biorthogonal functions.

Definition 4. For each integer $n \geq 0$, we define a family of functions

$$
\begin{equation*}
\tilde{\mathcal{R}}_{\lambda}^{(n)}\left(z_{1} \ldots z_{n} ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right) \in \mathcal{A}_{\lambda}^{(n)}\left(u_{0} ; t ; p, q\right) \tag{8.1}
\end{equation*}
$$

indexed by a partition pair $\boldsymbol{\lambda}$ of length at most $n$ and with parameters satisfying $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} u_{0} u_{1}=p q$, as follows. For $n=0$, we take

$$
\begin{equation*}
\tilde{\mathcal{R}}^{(0)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right):=1 \tag{8.2}
\end{equation*}
$$

Otherwise, if $\boldsymbol{\lambda}_{n}=(0,0)$, we set

$$
\begin{align*}
& \tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)  \tag{8.3}\\
& \quad:=\mathcal{I}_{t}^{+(n-1)}\left(t^{-1 / 2} u_{0} ; p, q\right) \tilde{\mathcal{R}}_{\lambda}^{(n-1)}\left(; t^{1 / 2} t_{0}: t^{1 / 2} t_{1}, t^{1 / 2} t_{2}, t^{1 / 2} t_{3} ; t^{-1 / 2} u_{0}, t^{1 / 2} u_{1} ; t ; p, q\right)
\end{align*}
$$

If $(0,1)^{n} \subset \boldsymbol{\lambda}$, set

$$
\begin{align*}
& \tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)  \tag{8.4}\\
& \quad:=\mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{0}: t_{1}, t_{2}, t_{3} ; t, p\right) \tilde{\mathcal{R}}_{\lambda-(0,1)^{n}}^{(n)}\left(; q^{1 / 2} t_{0}: q^{1 / 2} t_{1}, q^{1 / 2} t_{2}, q^{1 / 2} t_{3} ; q^{-1 / 2} u_{0}, q^{-3 / 2} u_{1} ; t ; p, q\right)
\end{align*}
$$

Finally, if $(1,0)^{n} \subset \boldsymbol{\lambda}$, but $(0,1)^{n} \not \subset \boldsymbol{\lambda}$, set

$$
\begin{align*}
& \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)  \tag{8.5}\\
& \quad:=\mathcal{D}_{p}^{+(n)}\left(u_{0}: t_{0}: t_{1}, t_{2}, t_{3} ; t, q\right) \tilde{\mathcal{R}}_{\lambda-(1,0)^{n}}^{(n)}\left(; p^{1 / 2} t_{0}: p^{1 / 2} t_{1}, p^{1 / 2} t_{2}, p^{1 / 2} t_{3} ; p^{-1 / 2} u_{0}, p^{-3 / 2} u_{1} ; t ; p, q\right)
\end{align*}
$$

Remark. The above definition closely resembles, and indeed was inspired by, Okounkov's integral representation for interpolation polynomials [15]; in fact, in an appropriate limit, our $\mathcal{I}^{+(n)}$ becomes Okounkov's integral operator (which can thus be expressed as a contour integral, rather than a $q$-integral).

It is clear that this inductively defines a family of functions as described; note also that the last relation still holds if $(1,1)^{n} \subset \boldsymbol{\lambda}$, since the corresponding $p$ - and $q$-difference operators "commute". In addition, it is clear that these functions should agree with the functions $R$ we attempted to define above, aside from the fact that the scalar multiplication freedom has been eliminated:

Proposition 8.1. The functions $\tilde{\mathcal{R}}$ satisfy the normalization condition

$$
\begin{equation*}
\tilde{\mathcal{R}}_{\lambda}^{(n)}\left(\ldots t^{n-i} t_{0} \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)=1 \tag{8.6}
\end{equation*}
$$

Since the "diagonal" coefficients of the + operators with respect to the filtration are generically nonzero, we find that they form a section of the filtration; that is:

Proposition 8.2. For any partition pair $\boldsymbol{\lambda}$, and for generic values of the parameters, the functions

$$
\begin{equation*}
\tilde{\mathcal{R}}_{\kappa}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right) \tag{8.7}
\end{equation*}
$$

for $\boldsymbol{\kappa} \subset \boldsymbol{\lambda}$ form a basis of $\mathcal{A}_{\boldsymbol{\lambda}}^{(n)}(t ; p, q)$.
Also, since each of the + operators used above factors as a tensor product, we find that the same holds for our family of functions.

Lemma 8.3. Each function $\tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}$ is a product of a q-abelian and a p-abelian function; more precisely, we have

$$
\begin{align*}
\tilde{\mathcal{R}}_{\lambda \mu}^{(n)}\left(\ldots z_{i} \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)= & \tilde{\mathcal{R}}_{\lambda 0}^{(n)}\left(\ldots z_{i} \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)  \tag{8.8}\\
& \tilde{\mathcal{R}}_{0 \mu}^{(n)}\left(\ldots z_{i} \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)
\end{align*}
$$

Similarly, from adjointness and Theorem 7.5, we can conclude:
Theorem 8.4. The functions $\tilde{\mathcal{R}}_{\lambda}^{(n)}$ satisfy the biorthogonality relation

$$
\begin{equation*}
\left\langle\tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right), \tilde{\mathcal{R}}_{\kappa}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{1}, u_{0} ; t ; p, q\right)\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t, p, q}=0 \tag{8.9}
\end{equation*}
$$

whenever $\boldsymbol{\kappa} \neq \boldsymbol{\lambda}$. In particular, $\tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}$ is orthogonal to the space $\mathcal{A}_{\boldsymbol{\kappa}}^{(n)}(t ; p, q)$ whenever $\boldsymbol{\lambda} \not \subset \boldsymbol{\kappa}$.
Remark. In particular, it follows that our functions agree with the univariate biorthogonal functions considered in [32, Appendix A]. Note that in the univariate case, the definition involves only the raising difference operators; the integral operators are unnecessary. This gives rise to a generalized Rodriguez-type formula; compare [18].

Theorem 8.5. The functions $\tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}$ satisfy the difference equations:

$$
\begin{align*}
& \mathcal{D}_{p}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, q\right) \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; p^{1 / 2} t_{0}: p^{1 / 2} t_{1}, p^{-1 / 2} t_{2}, p^{-1 / 2} t_{3} ; p^{1 / 2} u_{0}, p^{-1 / 2} u_{1} ; t ; p, q\right)  \tag{8.10}\\
& \\
& =\tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)  \tag{8.11}\\
& \begin{aligned}
\mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right) \tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; q^{1 / 2} t_{0}: q^{1 / 2} t_{1}, q^{-1 / 2} t_{2}, q^{-1 / 2} t_{3} ; q^{1 / 2} u_{0}, q^{-1 / 2} u_{1} ;\right. & t ; p, q) \\
& =\tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)
\end{aligned}
\end{align*}
$$

and the integral equation

$$
\begin{equation*}
\mathcal{I}_{t}^{(n)}\left(u_{0}: t_{0}, t_{1} ; p, q\right) \tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)=\tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; t^{1 / 2} t_{0}: t^{1 / 2} t_{1}, t^{-1 / 2} t_{2}, t^{-1 / 2} t_{3} ; t^{1 / 2} u_{0}, t^{-1 / 2} u_{1} ; t ; p, q\right) \tag{8.12}
\end{equation*}
$$

Proof. Since each of the operators respects the factorization of $\tilde{\mathcal{R}}_{\lambda \mu}^{(n)}$, it suffices to consider the cases $\lambda=0$ or $\mu=0$, which are clearly equivalent. In particular, the inner product is now generically nondegenerate, and thus $\tilde{\mathcal{R}}_{\lambda 0}^{(n)}$ and $\tilde{\mathcal{R}}_{0 \mu}^{(n)}$ are uniquely determined by biorthogonality and the normalization condition. Since each of the three operators we are considering has triangular adjoint, the left-hand sides satisfy biorthogonality; on the other hand, we readily compute that each operator preserves the normalization condition.

Remark. This gives rise to an alternate proof of the commutation relations of Corollary [7.10 by comparing the actions of the two sides on the appropriate basis of biorthogonal functions. Similarly, one obtains the commutation relations:

$$
\begin{align*}
& \mathcal{I}_{t}^{(n+1)}\left(u_{0}: t_{0}, t_{1} ; p, q\right) \mathcal{I}_{t}^{+(n)}\left(t^{-1 / 2} u_{0} ; p, q\right)=\mathcal{I}_{t}^{+(n)}\left(u_{0} ; p, q\right) \mathcal{I}_{t}^{(n)}\left(t^{-1 / 2} u_{0}: t^{1 / 2} t_{0}, t^{1 / 2} t_{1} ; p, q\right)  \tag{8.13}\\
& \mathcal{I}_{t}^{(n)}\left(u_{0}: t_{0}, t_{1} ; p, q\right) \mathcal{I}_{t}^{(n)}\left(t^{-1 / 2} u_{0}: t^{-1 / 2} t_{0}, t^{1 / 2} t_{2} ; p, q\right)=\mathcal{I}_{t}^{(n)}\left(u_{0}: t_{0}, t_{2} ; p, q\right) \mathcal{I}_{t}^{(n)}\left(t^{-1 / 2} u_{0}: t^{-1 / 2} t_{0}, t^{1 / 2} t_{1} ; p, q\right)  \tag{8.14}\\
& \mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right) \mathcal{D}_{q}^{(n)}\left(q^{1 / 2} u_{0}, q^{1 / 2} t_{0}, q^{-1 / 2} t_{2} ; t, p\right)=\mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{2} ; t, p\right) \mathcal{D}_{q}^{(n)}\left(q^{1 / 2} u_{0}, q^{1 / 2} t_{0}, q^{-1 / 2} t_{1} ; t, p\right)  \tag{8.15}\\
& \mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{0}: t_{1}, t_{2}, t_{3} ; t, p\right) \mathcal{D}_{q}^{(n)}\left(q^{-1 / 2} u_{0}, q^{1 / 2} t_{0}, q^{1 / 2} t_{1} ; t, p\right) \\
&=\mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right) \mathcal{D}_{q}^{+(n)}\left(q^{1 / 2} u_{0}: q^{1 / 2} t_{0}: q^{1 / 2} t_{1}, q^{-1 / 2} t_{2}, q^{-1 / 2} t_{3} ; t, p\right) \tag{8.16}
\end{align*}
$$

In contrast to Corollary 7.10, it is unclear how to prove these commutation relations directly.
Corollary 8.6. For any partition $\lambda$,

$$
\begin{equation*}
T_{\omega, p}^{(n)} \tilde{\mathcal{R}}_{0 \lambda}^{(n)}\left(; p^{1 / 2} t_{0}: p^{1 / 2} t_{1}, p^{-1 / 2} t_{2}, p^{-1 / 2} t_{3} ; p^{1 / 2} u_{0}, p^{-1 / 2} u_{1} ; t ; p, q\right)=\tilde{\mathcal{R}}_{0 \lambda}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right) \tag{8.17}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\tilde{\mathcal{R}}_{0 \lambda}^{(n)}\left(; p^{k_{0}} t_{0}: p^{k_{1}} t_{1}, p^{k_{2}} t_{2}, p^{k_{3}} t_{3} ; p^{l_{0}} u_{0}, p^{l_{1}} u_{1} ; t ; p, q\right)=\tilde{\mathcal{R}}_{0 \lambda}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right) \tag{8.18}
\end{equation*}
$$

for all choices of integers $k_{\text {? }}, l_{\text {? }}$ such that $k_{0}+k_{1}+k_{2}+k_{3}+l_{0}+l_{1}=0$.
Proof. The first claim follows from the fact that

$$
\begin{equation*}
\mathcal{D}_{p}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, q\right) f=T_{\omega, p}^{(n)} f \tag{8.19}
\end{equation*}
$$

for any $p$-abelian function $f$. Now, when $l_{0}=0$, the second claim follows from the definition of $\mathcal{R}_{0 \lambda}^{(n)}$ and the fact that $\mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{0}: t_{1}, t_{2}, t_{3} ; t, p\right)$ is a $p$-abelian function of the $u_{\text {? }}$ and $t_{\text {? }}$ parameters. Iterating the first claim and using the fact that $\tilde{\mathcal{R}}_{0 \lambda}^{(n)}$ is $p$-abelian gives an instance of the second claim with $l_{0}=1$, and thus the claim holds in general.

To see how the operators act when $t_{0}$ is not among the parameters of the operator, we need to determine how $\tilde{\mathcal{R}}^{(n)}$ changes when we switch $t_{0}$ and $t_{1}$. This leaves the biorthogonality relation unchanged, so multiplies the function by a constant; to determine that constant, it suffices to compute the following evaluation.

Proposition 8.7. For generic values of the parameters,

$$
\begin{equation*}
\tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(\ldots t^{n-i} t_{1} \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)=\frac{\mathcal{C}_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} t_{1} t_{2}, t^{n-1} t_{1} t_{3}, p q t^{n-1} t_{0} / u_{0}, t^{1-n} / t_{1} u_{1} ; t ; p, q\right)}{\mathcal{C}_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} t_{0} t_{2}, t^{n-1} t_{0} t_{3}, p q t^{n-1} t_{1} / u_{0}, t^{1-n} / t_{0} u_{1} ; t ; p, q\right)} \tag{8.20}
\end{equation*}
$$

Proof. This follows by comparing the actions of $\mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{0}: t_{1}, t_{2}, t_{3} ; t, p\right)$ and $\mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{1}: t_{0}, t_{2}, t_{3} ; t, p\right)$. This gives a recurrence for the desired specialization, having the right-hand side as unique solution.

It will be convenient at this point to introduce "hatted" parameters. These are defined as follows. First, we have:

$$
\begin{equation*}
\hat{t}_{0}=\sqrt{t_{0} t_{1} t_{2} t_{3} / p q}=\frac{t^{1-n}}{\sqrt{u_{0} u_{1}}} . \tag{8.21}
\end{equation*}
$$

The remaining parameters are then defined by giving invariants of the transformation. To be precise, we define $\hat{t}_{1}, \hat{t}_{2}, \hat{t}_{3}, \hat{u}_{0}$, and $\hat{u}_{1}$ by insisting that

$$
\begin{equation*}
\hat{t}_{0} \hat{t}_{1}=t_{0} t_{1} \quad \hat{t}_{0} \hat{t}_{2}=t_{0} t_{2} \quad \hat{t}_{0} \hat{t}_{3}=t_{0} t_{3} \quad \frac{\hat{u}_{0}}{\hat{t}_{0}}=\frac{u_{0}}{t_{0}} \quad \frac{\hat{u}_{1}}{\hat{t}_{0}}=\frac{u_{1}}{t_{0}} \tag{8.22}
\end{equation*}
$$

Note in particular that

$$
\begin{equation*}
t^{2 n-2} \hat{t}_{0} \hat{t}_{1} \hat{t}_{2} \hat{t}_{3} \hat{u}_{0} \hat{u}_{1}=p q \tag{8.23}
\end{equation*}
$$

The action of the hat transformation on the $t$ parameters is, of course, quite familiar from the theory of Koornwinder polynomials [14, 28] (aside from the factor of $p$ required to preserve symmetry); the action on the $u$ parameters is then essentially forced by the balancing condition. We furthermore define $z_{i}\left(\boldsymbol{\lambda} ; \hat{t}_{0}\right):=$ $(p, q)^{\boldsymbol{\lambda}_{i}} t^{n-i} \hat{t}_{0}$.

In the following formulas, the ratios of $\Gamma$ functions that appear are sometimes ill-defined, in that some of the factors vanish. These should be interpreted by multiplying the argument of each $\Gamma$ function by the same scale factor, then taking the limit as that scale factor approaches 1. Alternatively, it turns out in each case that the ratio can be formally expressed in terms of theta functions alone, and that upon doing so, the resulting formula is well-defined. Similar comments apply to ratios of $\theta$ functions. In particular, we note that

$$
\begin{equation*}
\prod_{1 \leq i \leq n} \frac{\theta\left(v z_{i}(\boldsymbol{\lambda} ; w)^{ \pm 1} ; p\right)}{\theta\left(v z_{i}(0,0 ; w)^{ \pm 1} ; p\right)}=\prod_{1 \leq i \leq n} \frac{\Gamma\left(q v z_{i}(\boldsymbol{\lambda} ; w)^{ \pm 1}, v z_{i}(0,0 ; w)^{ \pm 1} ; p, q\right)}{\Gamma\left(v z_{i}(\boldsymbol{\lambda} ; w)^{ \pm 1}, q v z_{i}(0,0 ; w)^{ \pm 1} ; p, q\right)} \propto \frac{\mathcal{C}_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} q v w, t^{n-1} p q w / v ; p, q\right)}{\mathcal{C}_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} v w, t^{n-1} p w / v ; p, q\right)} \tag{8.24}
\end{equation*}
$$

where the constant of proportionality is independent of $v$.
Corollary 8.8. We have the difference equations

$$
\begin{align*}
& \mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t, p\right) \mathcal{D}_{q}^{(n)}\left(q^{1 / 2} u_{0}, q^{-1 / 2} t_{2}, q^{-1 / 2} t_{3} ; t, p\right) \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; q u_{0}, q^{-1} u_{1} ; t ; p, q\right)  \tag{8.25}\\
&=\frac{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{D}}\left(\hat{t}_{1}: \hat{t}_{0} ; t ; p: q\right)}{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{D}}\left(\hat{u}_{0}: \hat{t}_{0} ; t ; p: q\right)} \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right), \\
& \mathcal{D}_{q}^{(n)}\left(u_{0}, t_{2}, t_{3} ; t, p\right) \mathcal{D}_{q}^{(n)}\left(q^{1 / 2} u_{0}, q^{-1 / 2} t_{0}, q^{-1 / 2} t_{1} ; t, p\right) \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; q u_{0}, q^{-1} u_{1} ; t ; p, q\right)  \tag{8.26}\\
&=\frac{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{D}}\left(\hat{t}_{1} / q: \hat{t}_{0} ; t ; p: q\right)}{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{D}}\left(\hat{u}_{0}: \hat{t}_{0} ; t ; p: q\right)} \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right),
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{D}}(v: w ; t ; p: q):=\prod_{1 \leq i \leq n} \frac{\theta\left(v z_{i}(\boldsymbol{\lambda} ; w)^{ \pm 1} ; p\right)}{\theta\left(v z_{i}(0,0 ; w)^{ \pm 1} ; p\right)} \tag{8.27}
\end{equation*}
$$

Similarly,

$$
\begin{align*}
& \mathcal{I}_{t}^{(n)}\left(t^{1 / 2} u_{0}: t^{-1 / 2} t_{2}, t^{-1 / 2} t_{3} ; p, q\right) \mathcal{I}_{t}^{(n)}\left(u_{0}: t_{0}, t_{1} ; p, q\right) \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; p, q\right)  \tag{8.28}\\
&=\frac{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{I}}\left(\hat{t}_{1}: \hat{t}_{0} ; t ; p, q\right)}{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{I}}\left(\hat{u}_{0}: \hat{t}_{0} ; t ; p, q\right)} \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; t u_{0}, u_{1} / t ; p, q\right) \\
& \begin{aligned}
\mathcal{I}_{t}^{(n)}\left(t^{1 / 2} u_{0}: t^{-1 / 2} t_{0}, t^{-1 / 2} t_{1} ; p, q\right) \mathcal{I}_{t}^{(n)}\left(u_{0}: t_{2}, t_{3} ; p, q\right) & \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; p, q\right) \\
& =\frac{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{I}}\left(\hat{t}_{1} / t: \hat{t}_{0} ; t ; p, q\right)}{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{I}}\left(\hat{u}_{0}: \hat{t}_{0} ; t ; p, q\right)} \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; t u_{0}, u_{1} / t ; p, q\right)
\end{aligned} \tag{8.29}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{I}}(v: w ; t ; p, q):=\prod_{1 \leq i \leq n} \frac{\Gamma\left(v z_{i}(\boldsymbol{\lambda} ; w)^{ \pm 1}, t v z_{i}(0,0 ; w)^{ \pm 1} ; p, q\right)}{\Gamma\left(t v z_{i}(\boldsymbol{\lambda} ; w)^{ \pm 1}, v z_{i}(0,0 ; w)^{ \pm 1} ; p, q\right)} \tag{8.30}
\end{equation*}
$$

The - and + operators give similar equations:
Theorem 8.9.

$$
\begin{align*}
\mathcal{D}_{q}^{-(n)}\left(u_{0} ; t, p\right) \mathcal{D}_{q}^{+(n)}\left(q^{3 / 2} u_{0}: \frac{t_{0}}{q^{1 / 2}}: \frac{t_{1}}{q^{1 / 2}}, \frac{t_{2}}{q^{1 / 2}}, \frac{t_{3}}{q^{1 / 2}} ; t, p\right) & \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; q u_{0}, q^{-1} u_{1} ; t ; p, q\right)  \tag{8.31}\\
& =C \frac{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{D}}\left(\hat{t}_{0} / q: \hat{t}_{0} ; p\right)}{\mathcal{E}_{\lambda}^{\mathcal{D}}\left(\hat{u}_{0}: \hat{t}_{0} ; p\right)} \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)
\end{align*}
$$

where

$$
\begin{equation*}
C=\prod_{1 \leq i \leq n} \frac{\theta\left(t^{n-i} u_{0} t_{0}, t^{n-i} u_{0} t_{1}, t^{n-i} u_{0} t_{2}, t^{n-i} u_{0} t_{3}, p t^{n-i} t_{0} / q u_{0}, p q t^{n-i}, t^{n-i} t_{0} t_{1} t_{2} t_{3} / p q^{2} ; p\right)}{\theta\left(p t^{2 n-i-1} t_{0} u_{1}, t^{n-i} t_{0} t_{1} / q, t^{n-i} t_{0} t_{2} / q, t^{n-i} t_{0} t_{3} / q ; p\right)} \tag{8.32}
\end{equation*}
$$

Similarly,

$$
\begin{align*}
\mathcal{I}_{t}^{-(n+1)}\left(t^{1 / 2} u_{0}: \frac{t_{0}}{t^{1 / 2}}, \frac{t_{1}}{t^{1 / 2}}, \frac{t_{2}}{t^{1 / 2}}, \frac{t_{3}}{t^{1 / 2}} ; p, q\right) \mathcal{I}_{t}^{+(n)}\left(u_{0} ; p, q\right) & \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; p, q\right)  \tag{8.33}\\
& =\frac{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{I}}\left(t / \hat{t}_{0}: \hat{t}_{0} ; t ; p, q\right)}{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{I}}\left(\hat{u}_{0}: \hat{t}_{0} ; t ; p, q\right)} \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; t u_{0}, u_{1} / t ; p, q\right)
\end{align*}
$$

Proof. In each case, by adjointness, both sides satisfy biorthogonality, and must therefore be proportional. To determine the constant of proportionality, we can compare to one of the corresponding equations from Corollary 8.8. Indeed, the fact of proportionality shows that the relevant products of difference (or integral) operators differ in their action only by a diagonal transformation; as a result, we can compute the ratio of their constants of proportionality using any section of the filtration. In particular, it is straightforward to compute diagonal coefficients using the sections with which we proved triangularity in the first place, thus giving the desired result.

Remark. We thus find that for $v \in\left\{\hat{t}_{1}, \hat{t}_{2}, \hat{t}_{3}, \hat{t}_{0} / q, \hat{t}_{1} / q, \hat{t}_{2} / q, \hat{t}_{3} / q\right\}$, we have a difference operator $\mathcal{D}(v)$ (of "order" 2) such that

$$
\begin{equation*}
\mathcal{D}(v) \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; q u_{0}, q^{-1} u_{1} ; t ; p, q\right)=\frac{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{D}}\left(v: \hat{t}_{0} ; t ; p: q\right)}{\mathcal{E}_{\boldsymbol{\lambda}}^{\mathcal{D}}\left(\hat{u}_{0}: \hat{t}_{0} ; t ; p: q\right)} \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right) \tag{8.34}
\end{equation*}
$$

moreover

$$
\begin{equation*}
\mathcal{D}_{q}^{+(n)}\left(u_{0}: t_{0}: t_{1}, t_{2}, t_{3} ; t, p\right) \mathcal{D}^{-(n)}\left(q^{-1 / 2} u_{0} ; t, p\right) \tag{8.35}
\end{equation*}
$$

essentially gives us such an operator for $v=\hat{t}_{0}$. We conjecture that such an operator exists for all $v$; since the "eigenvalue" is effectively just a $B C_{1}$-symmetric theta function of degree n in $v$, this conjecture certainly holds for $n \leq 7$. Such a collection of difference operators, together with the various spaces of higher-degree difference operators obtained by composing them, would seem to give the analogue of the center of the affine Hecke algebra applicable to our biorthogonal functions. Indeed, in the Koornwinder limit, the conjecture certainly holds, and the resulting space of operators is precisely the subspace of the center of the affine Hecke algebra having degree at most 1 .

In particular, this gives us a recurrence for the nonzero values of the inner product. Define

$$
\begin{equation*}
\Delta^{(n)}\left(z ; t_{0}, t_{1}, t_{2}, t_{3}, t_{4}, t_{5} ; t ; p, q\right)=\prod_{1 \leq i<j \leq n} \frac{\Gamma\left(t z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right)}{\Gamma\left(z_{i}^{ \pm 1} z_{j}^{ \pm 1} ; p, q\right)} \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 5} \Gamma\left(t_{r} z_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(z_{i}^{ \pm 2} ; p, q\right)} \tag{8.36}
\end{equation*}
$$

in other words, this is simply the density with respect to which our functions are biorthogonal.
Theorem 8.10. For any partition pair $\boldsymbol{\lambda}$ of length at most $n$, and for generic values of the parameters,

$$
\begin{align*}
\left\langle\tilde { \mathcal { R } } _ { \boldsymbol { \lambda } } ^ { ( n ) } \left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ;\right.\right. & \left.; p, q), \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{1}, u_{0} ; t ; p, q\right)\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t, p, q}  \tag{8.37}\\
& =\frac{\Delta^{(n)}\left(\ldots z_{i}\left(0,0 ; \hat{t}_{0}\right) \ldots ; \hat{t}_{0}, \hat{t}_{1}, \hat{t}_{2}, \hat{t}_{3}, \hat{u}_{0}, \hat{u}_{1} ; t ; p, q\right)}{\Delta^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\lambda} ; \hat{t}_{0}\right) \ldots ; \hat{t}_{0}, \hat{t}_{1}, \hat{t}_{2}, \hat{t}_{3}, \hat{u}_{0}, \hat{u}_{1} ; t ; p, q\right)} \\
& =\Delta_{\boldsymbol{\lambda}}\left(t^{2 n-2} \hat{t}_{0}^{2} \mid t^{n}, t^{n-1} \hat{t}_{0} \hat{t}_{1}, t^{n-1} \hat{t}_{0} \hat{t}_{2}, t^{n-1} \hat{t}_{0} \hat{t}_{3}, t^{n-1} \hat{t}_{0} \hat{u}_{0}, t^{n-1} \hat{t}_{0} \hat{u}_{1} ; t ; p, q\right)^{-1}  \tag{8.38}\\
& =\Delta_{\boldsymbol{\lambda}}\left(\left.\frac{1}{u_{0} u_{1}} \right\rvert\, t^{n}, t^{n-1} t_{0} t_{1}, t^{n-1} t_{0} t_{2}, t^{n-1} t_{0} t_{3}, \frac{t^{1-n}}{t_{0} u_{0}}, \frac{t^{1-n}}{t_{0} u_{1}} ; t ; p, q\right)^{-1} \tag{8.39}
\end{align*}
$$

This of course, is the direct analogue of the formula for the inner products of Koornwinder polynomials.
If $t_{0} t_{1}=p^{-l} q^{-m} t^{1-n}$, then the integral converts via residue calculus to a sum, and we thus obtain the following discrete biorthogonality property.

Theorem 8.11. For any partition pairs $\boldsymbol{\lambda}, \boldsymbol{\kappa} \subset(l, m)^{n}$, and for otherwise generic parameters satisfying $t_{0} t_{1}=$ $p^{-l} q^{-m} t^{1-n}$,

$$
\begin{array}{r}
\sum_{\boldsymbol{\mu} \subset(l, m)^{n}} \tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\mu} ; t_{0}\right) \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right) \tilde{\mathcal{R}}_{\kappa}^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\mu} ; t_{0}\right) \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{1}, u_{0} ; t ; p, q\right)  \tag{8.40}\\
\frac{\Delta^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\mu} ; t_{0}\right) \ldots ; t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q\right)}{\Delta^{(n)}\left(\ldots z_{i}\left(0,0 ; t_{0}\right) \ldots ; t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q\right)}=0
\end{array}
$$

unless $\boldsymbol{\lambda}=\boldsymbol{\kappa}$.
Remark. Note that when $t_{0} t_{1}=p^{-l} q^{-m} t^{1-n}$, we have

$$
\begin{equation*}
z_{i}\left(\boldsymbol{\mu} ; t_{0}\right)=z_{n+1-i}\left((l, m)^{n}-\boldsymbol{\mu} ; t_{1}\right)^{-1} \tag{8.41}
\end{equation*}
$$

and thus summing over $z_{i}\left(\boldsymbol{\mu} ; t_{1}\right)$ gives the same result.

This result leads to a very important special case of the $\tilde{\mathcal{R}}$ functions.
Corollary 8.12. If $t_{1} u_{1}=t^{1-n}$, then

$$
\begin{equation*}
\tilde{\mathcal{R}}_{\lambda}^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\kappa} ; t_{1}\right) \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)=0 \tag{8.42}
\end{equation*}
$$

unless $\boldsymbol{\lambda} \subseteq \boldsymbol{\kappa}$. Moreover, in this case $\tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}$ is independent of $t_{2}$ and $t_{3}$, and up to scalar multiplication, is independent of $t_{0}$.

Proof. First suppose that we have $t_{0} t_{1}=p^{-l} q^{-m} t^{1-n}$ for $l, m$ such that $\boldsymbol{\lambda}, \boldsymbol{\kappa} \subset(l, m)^{n}$, and consider the discrete biorthogonality relation. We observe that for $f \in \mathcal{A}^{(n)}\left(u_{1} ; p, q\right)$ such that

$$
\begin{equation*}
\prod_{1 \leq i \leq n} \theta\left(p q z_{i}^{ \pm 1} / u_{1} ; p, q\right)_{l, m} f\left(\ldots z_{i} \ldots\right) \tag{8.43}
\end{equation*}
$$

is holomorphic, and for partitions $\boldsymbol{\lambda} \subset(l, m)^{n}$,

$$
\begin{array}{rl}
\lim _{v \rightarrow t^{1-n} / u_{1}} \prod_{1 \leq i \leq n} \theta\left(p q z_{i}(\boldsymbol{\lambda} ; v)^{ \pm 1} / u_{1} ; p, q\right)_{l, m} & f\left(\ldots z_{i}(\boldsymbol{\lambda} ; v) \ldots\right)  \tag{8.44}\\
& =\lim _{\substack{z_{i} \rightarrow(p, q)^{\kappa} \kappa_{i} \\
i=1 . \ldots n}} \prod_{\substack{i-1}} \prod_{1 \leq i \leq n} \theta\left(p q z_{i}^{ \pm 1} / u_{1} ; p, q\right)_{l, m} f\left(\ldots z_{i} \ldots\right)
\end{array}
$$

In other words, if $f \in \mathcal{A}_{\kappa}^{(n)}\left(u_{1} ; t ; p, q\right)$, then the inner product of our function with $f$ can be expressed as a sum over partition pairs contained in $\boldsymbol{\kappa}$, by the very definition of the filtration. The desired vanishing property follows immediately. Moreover, this orthogonality is independent of the specific values for $t_{2}$, $t_{3}$, and thus changing $t_{2}$ or $t_{3}$ can at most multiply our function by a scalar; this scalar must then be 1 by the normalization formula.

We thus find that the result holds whenever $t_{0}$ is of the above form. Since the given quantity is a product of abelian functions of $t_{0}$ for any choice of $\boldsymbol{\lambda}, \boldsymbol{\kappa}$, the fact that it holds for $t_{0}$ of the form $p^{-l} q^{-m} t^{1-n} / t_{1}$ implies that it holds in general. Symmetry in $t_{0}, t_{2}, t_{3}$ then shows that the dependence on $t_{0}$ is only via the normalization.

With this in mind, we consider the following alternate normalization in this case.
Definition 5. The interpolation functions $\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right)$ are defined by

$$
\begin{equation*}
\mathcal{R}_{\lambda}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right)=\Delta_{\lambda}^{0}\left(t^{n-1} t_{0} / u_{0} \mid t^{n-1} t_{0} t_{1}, t_{0} / t_{1} ; t ; p, q\right) \tilde{\mathcal{R}}_{\lambda}^{(n)}\left(; t_{1}: t_{0}, t_{2}, t_{3} ; u_{0}, t^{1-n} / t_{0} ; t ; p, q\right) \tag{8.45}
\end{equation*}
$$

where the right-hand side is independent of the choice of $t_{1}, t_{2}, t_{3}$, as long as $t^{n-1} t_{1} t_{2} t_{3} u_{0}=p q$. The multivariate elliptic binomial coefficient $\left[\begin{array}{c}\boldsymbol{\lambda} \\ \boldsymbol{\kappa}\end{array}\right]_{[a, b] ; ; ; p, q}$ is defined by

$$
\left[\begin{array}{l}
\boldsymbol{\lambda}  \tag{8.46}\\
\boldsymbol{\kappa}
\end{array}\right]_{[a, b] ; t ; p, q}:=\Delta_{\boldsymbol{\kappa}}\left(\left.\frac{a}{b} \right\rvert\, t^{n}, 1 / b ; t ; p, q\right) \mathcal{R}_{\boldsymbol{\kappa}}^{*(n)}\left(\ldots z_{i}\left(\boldsymbol{\lambda} ; t^{1-n} a^{1 / 2}\right) \ldots ; t^{1-n} a^{1 / 2}, b / a^{1 / 2} ; t ; p, q\right)
$$

for $n \geq \ell(\boldsymbol{\lambda}), \ell(\boldsymbol{\kappa})$.

Remark. An alternate definition uses the fact that

$$
\begin{equation*}
\mathcal{R}_{l^{n}+\lambda, m^{n}+\mu}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right)=\frac{\left(p q / t_{0} u_{0}\right)^{-2 l|\mu|-2 m|\lambda|} \prod_{1 \leq i \leq n} \theta\left(t_{0} x_{i}^{ \pm 1} ; p, q\right)_{l, m}}{\prod_{1 \leq i \leq n} \theta\left(\left(p q / u_{0}\right) x_{i}^{ \pm 1} ; p, q\right)_{l, m}} \mathcal{R}_{\lambda, \mu}^{*(n)}\left(; p^{l} q^{m} t_{0}, u_{0} / p^{l} q^{m} ; t ; p, q\right) \tag{8.47}
\end{equation*}
$$

(which follows by two applications of equation (9.12) below) together with the action of $\mathcal{I}^{+(n)}$ to obtain an integral representation generalizing that of (15].

We note in particular that

$$
\begin{equation*}
\mathcal{R}_{\kappa}^{*(n)}\left(\ldots z_{i} \ldots, a, t a, \ldots t^{m-1} a ; a, b ; t ; p, q\right),=\frac{\Delta_{\kappa}\left(\left.\frac{t^{n-1} a}{b} \right\rvert\, t^{n-m}, 1 / b ; t ; p, q\right)}{\Delta_{\kappa}\left(\left.\frac{t^{n-1} a}{b} \right\rvert\, t^{n}, 1 / b ; t ; p, q\right)} \mathcal{R}_{\kappa}^{*(n-m)}\left(\ldots z_{i} \ldots ; t^{m} a, b ; t ; p, q\right) \tag{8.48}
\end{equation*}
$$

and thus the multivariate elliptic binomial coefficient is independent of $n$ (as long as $\ell(\boldsymbol{\lambda}), \ell(\boldsymbol{\kappa}) \leq n$, that is).
The significance of these interpolation functions is that one can express connection coefficients for the biorthogonal functions in terms of multivariate elliptic binomial coefficients. (The proof requires a more thorough study of these binomial coefficients, and will thus be deferred to 19.)

Theorem 8.13. [19] If we define connection coefficients $c_{\boldsymbol{\lambda} \mu}$ by

$$
\begin{equation*}
\tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)=\sum_{\boldsymbol{\mu} \subset \boldsymbol{\lambda}} c_{\boldsymbol{\lambda} \boldsymbol{\mu}} \tilde{\mathcal{R}}_{\boldsymbol{\mu}}^{(n)}\left(; t_{0}: t_{1} v, t_{2}, t_{3} ; u_{0}, u_{1} / v ; t ; p, q\right) \tag{8.49}
\end{equation*}
$$

then

$$
c_{\boldsymbol{\lambda} \boldsymbol{\mu}}=\frac{\Delta_{\boldsymbol{\lambda}}^{0}\left(1 / u_{0} u_{1} \mid 1 / v, t^{n-1} t_{2} t_{3}, p q t^{n-1} t_{0} / u_{0}, t_{1} v / u_{1} ; t ; p, q\right)}{\Delta_{\mu}^{0}\left(v / u_{0} u_{1} \mid v, t^{n-1} t_{2} t_{3}, p q t^{n-1} t_{0} / u_{0}, t_{1} v / u_{1} ; t ; p, q\right)}\left[\begin{array}{l}
\boldsymbol{\lambda}  \tag{8.50}\\
\boldsymbol{\mu}
\end{array}\right]_{\left[1 / u_{0} u_{1}, 1 / v\right] ; t ; p, q}
$$

If the biorthogonal function on the right is specialized to an interpolation function, we obtain the following generalization of Okounkov's binomial formula for Koornwinder polynomials [15]:

## Corollary 8.14.

$$
\begin{equation*}
\tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)=\sum_{\kappa \subseteq \boldsymbol{\lambda}} c_{\kappa} \mathcal{R}_{\kappa}^{*(n)}\left(\ldots z_{i}\left(\boldsymbol{\lambda} ; \hat{t}_{0}\right) \ldots ; \hat{t}_{0}, \hat{u}_{0} ; t ; p, q\right) \mathcal{R}_{\kappa}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right) \tag{8.51}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{\boldsymbol{\kappa}}=\Delta_{\boldsymbol{\kappa}}\left(t^{n-1} t_{0} / u_{0} \mid t^{n}, p q / u_{0} t_{1}, p q / u_{0} t_{2}, p q / u_{0} t_{3} ; t ; p, q\right) \tag{8.52}
\end{equation*}
$$

Since $c_{\boldsymbol{\kappa}}$ above remains the same when the parameters are replaced by their hatted analogues, we obtain the following corollary, the analogue of the "evaluation symmetry" property of Koornwinder polynomials.

Corollary 8.15. For any partition pairs $\boldsymbol{\lambda}, \boldsymbol{\kappa}$ of length at most $n$, and for generic values of the parameters,

$$
\begin{equation*}
\tilde{\mathcal{R}}_{\boldsymbol{\lambda}}^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\kappa} ; t_{0}\right) \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, u_{1} ; t ; p, q\right)=\tilde{\mathcal{R}}_{\boldsymbol{\kappa}}^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\lambda} ; \hat{t}_{0}\right) \ldots ; \hat{t}_{0}: \hat{t}_{1}, \hat{t}_{2}, \hat{t}_{3} ; \hat{u}_{0}, \hat{u}_{1} ; t ; p, q\right) \tag{8.53}
\end{equation*}
$$

Before leaving the topic of biorthogonal functions, it remains to justify our assertions that these are a generalization of Koornwinder polynomials. The inner product clearly can be degenerated into the Koornwinder inner product; the difficulty is the filtration. Indeed, in order to degenerate the inner product, we must take $p \rightarrow 0, u_{0} \rightarrow\{0, \infty\}$, at which point the definition of the filtration breaks. It turns out that the filtration actually does have a well-defined limit; however, we have been unable to find an argument for this other than as a corollary of the following result.

Theorem 8.16. Fix otherwise generic parameters $t_{0}, t_{1}, t_{2}, t_{3}$. Then the limits

$$
\begin{align*}
& \lim _{u_{0} \rightarrow 0} \lim _{p \rightarrow 0} \tilde{\mathcal{R}}_{0 \lambda}^{(n)}\left(\ldots z_{i} \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, \frac{p q}{t^{2 n-2} u_{0} t_{0} t_{1} t_{2} t_{3}} ; t ; p, q\right)  \tag{8.54}\\
& \lim _{u_{0} \rightarrow \infty} \lim _{p \rightarrow 0} \tilde{\mathcal{R}}_{0 \lambda}^{(n)}\left(\ldots z_{i} \ldots ; t_{0}: t_{1}, t_{2}, t_{3} ; u_{0}, \frac{p q}{t^{2 n-2} u_{0} t_{0} t_{1} t_{2} t_{3}} ; t ; p, q\right) \tag{8.55}
\end{align*}
$$

agree, and give a family of $B C_{n}$-symmetric Laurent polynomials orthogonal with respect to the Koornwinder inner product. Moreover, these polynomials are diagonal with respect to dominance of monomials, and thus are precisely the Koornwinder polynomials (normalized to have principal specialization 1).

Proof. The key observation is that, although the definition of the filtration blows up in the limit, the raising difference and integral operators have perfectly fine limits. Consequently, the above limits are indeed welldefined; as the choice $u_{0} \rightarrow 0$ or $u_{0} \rightarrow \infty$ has no effect on the limiting operators, it can have no effect on the limiting functions. Since the space of $B C_{n}$-symmetric $p$-theta functions of degree $m$ tends in the limit $p \rightarrow 0$ to the space of $B C_{n}$-symmetric Laurent polynomials of degree at most $m$ in each variable, our functions become rational functions in that limit. Taking the limit $u_{0} \rightarrow 0, \infty$ causes the poles of the rational functions to move to 0 and $\infty$, thus giving Laurent polynomials. Finally, we observe that because the above limits agree, biorthogonality becomes orthogonality in the limit. (Recall that $\mathcal{R}_{0 \lambda}^{(n)}$ is $p$-abelian in its parameters, so the factor of $p$ in $u_{1}$ can be moved around arbitrarily before taking the limit.) We have thus proved the first claim.

To see that these agree with Koornwinder polynomials, we observe that the operator $\mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t\right)$ also has a well-defined limit; standard arguments ( 21 , Theorem 3.2]) show that the limit is triangular with respect to dominance of monomials. It thus follows from Theorem 8.5 that the limiting polynomials are eigenfunctions of a pair of triangular difference operators, and thus must themselves be triangular. The normalization then follows from Proposition 8.1

Remark. In order to determine the constant of proportionality, i.e., determine the leading coefficient of the limiting polynomial, we need simply determine how the raising operators affect the leading coefficient. For the difference operator, this is straightforward; for the integral operator, we can appeal to 7.1 and, by using the fact

$$
\begin{equation*}
\sum_{\lambda \subset m^{n}}(-1)^{m n-|\lambda|} m_{\lambda}\left(y_{1}, y_{2}, \ldots y_{n}\right) e_{m^{n}-\lambda}\left(z_{1}, z_{2}, \ldots z_{m}, 1 / z_{1}, 1 / z_{2}, \ldots 1 / z_{m}\right)=\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}\left(y_{i}+1 / y_{i}-z_{j}-1 / z_{j}\right) \tag{8.56}
\end{equation*}
$$

(where $m_{\lambda}$ is a $B C_{n}$-symmetric monomial), reduce to the difference operator case. The result, of course, is simply Macdonald's "evaluation" conjecture; Theorem 8.10 then gives the nonzero values of the inner product. (For more details, see [22].) The remaining ("symmetry") conjecture does not follow from the methods given above, however (although there are at least two different arguments for deducing it from evaluation: [4], 15]). The argument we will give in [19] does descend to the Koornwinder case; indeed, the result is precisely the proof given in [21].

It follows from [21, Theorem 7.25] that the filtration has the following limit.
Corollary 8.17. Choose an integer $n \geq 0$, and a partition $\lambda$ of at most $n$ parts. Then the limits $u_{0} \rightarrow 0$ or $u_{0} \rightarrow \infty$ of the space $\mathcal{A}_{0 \lambda}^{(n)}\left(u_{0} ; t ; 0, q\right)$ agree, and are given by the span

$$
\begin{equation*}
\left\langle P_{\mu}\left(x_{1}^{ \pm 1}, x_{2}^{ \pm 1}, \ldots x_{n}^{ \pm 1} ; q, t\right)\right\rangle_{\mu \subset \lambda} \tag{8.57}
\end{equation*}
$$

where $P_{\mu}$ is an ordinary Macdonald polynomial.
Remark. It would be nice to have a direct proof of this Corollary, or the corresponding result for a refined partial order; in particular, for the dominance partial order, the limiting filtration should correspond to dominance of monomials.

## 9 Type II transformations

The connection coefficient formula for our biorthogonal functions, Theorem 8.13, has a number of nice consequences for the multivariate elliptic binomial coefficients. For instance, by taking $v=1$, we obtain the limiting case

$$
\lim _{b \rightarrow 1} \frac{\Delta_{\boldsymbol{\lambda}}^{0}(a \mid b ; t ; p, q)}{\Delta_{\kappa}^{0}(a / b \mid 1 / b ; t ; p, q)}\left[\begin{array}{l}
\boldsymbol{\lambda}  \tag{9.1}\\
\boldsymbol{\kappa}
\end{array}\right]_{[a, b] ; t ; p, q}=\delta_{\boldsymbol{\lambda} \boldsymbol{\kappa}}
$$

Also, if we perform the change of basis corresponding to $t_{1} \rightarrow t_{1} v$, then the change of basis corresponding to $t_{1} v \rightarrow t_{1} v w$, the result should be the same as if we directly changed $t_{1} \rightarrow t_{1} v w$. We thus obtain the following sum:

Theorem 9.1. [19] For otherwise generic parameters satisfying bcde $=p q a$,

$$
\left[\begin{array}{l}
\boldsymbol{\lambda}  \tag{9.2}\\
\boldsymbol{\kappa}
\end{array}\right]_{[a, c] ; t ; p, q}=\frac{\Delta_{\boldsymbol{\kappa}}^{0}(a / c \mid 1 / c, b d, b e, p q a / b ; t ; p, q)}{\Delta_{\boldsymbol{\lambda}}^{0}(a \mid c, b d, b e, p q a / b ; t ; p, q)} \sum_{\boldsymbol{\kappa} \subset \boldsymbol{\mu} \subset \boldsymbol{\lambda}} \Delta_{\boldsymbol{\mu}}^{0}(a / b \mid c / b, p q a, d, e ; t ; p, q)\left[\begin{array}{l}
\boldsymbol{\lambda} \\
\boldsymbol{\mu}
\end{array}\right]_{[a, b] ; t ; p, q}\left[\begin{array}{l}
\boldsymbol{\mu} \\
\boldsymbol{\kappa}
\end{array}\right]_{[a / b, c / b] ; t ; p, q}
$$

In particular,

$$
\sum_{\boldsymbol{\kappa} \subset \boldsymbol{\mu} \subset \boldsymbol{\lambda}}\left[\begin{array}{l}
\boldsymbol{\lambda}  \tag{9.3}\\
\boldsymbol{\mu}
\end{array}\right]_{[a, b] ; t ; p, q}\left[\begin{array}{l}
\boldsymbol{\mu} \\
\boldsymbol{\kappa}
\end{array}\right]_{[a / b, 1 / b] ; t ; p, q}=\delta_{\boldsymbol{\lambda} \boldsymbol{\kappa}}
$$

Remark. Although this identity, along with the other sums mentioned in this section, does indeed follow from Theorem 8.13 we should mention that the argument in 19 proceeds in the opposite direction, using these identities (and others) to prove the binomial formula, and from this Theorem 8.13 On the other hand, the above argument provides a more straightforward interpretation of the identity than that given in [19].

If we take $\boldsymbol{\lambda}=(l, m)^{n}, \boldsymbol{\kappa}=0$ above, the above identity turns out to be a product of two general instances of Warnaar's Jackson-type summation (conjectured in [36], and proved by Rosengren [24]). Warnaar's Schlossertype summation is also a special case; see 19 .

Our reason for discussing this here is that there is an integral analogue of the above sum, generalizing Theorem 6.1

Theorem 9.2. For otherwise generic parameters satisfying $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} u_{0} u_{1}=p q$,

$$
\begin{align*}
&\left\langle\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right), \mathcal{R}_{\boldsymbol{\kappa}}^{*(n)}\left(; t_{1}, u_{1} ; t ; p, q\right)\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q} \\
&= \Delta_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} t_{0} / u_{0} \mid t^{n-1} t_{0} t_{2}, t^{n-1} t_{0} t_{3} ; t ; p, q\right) \Delta_{\boldsymbol{\kappa}}^{0}\left(t^{n-1} t_{1} / u_{1} \mid t^{n-1} t_{1} t_{2}, t^{n-1} t_{1} t_{3}, t^{n-1} t_{1} t_{0}, t^{n-1} t_{1} u_{0} ; t ; p, q\right) \\
& \quad \mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(\ldots z_{i}\left(\boldsymbol{\kappa} ; t_{1} / \sqrt{t^{n-1} t_{1} u_{1}}\right) \ldots ; t_{0} \sqrt{t^{n-1} t_{1} u_{1}}, u_{0} \sqrt{t^{n-1} t_{1} u_{1}} ; t ; p, q\right) \tag{9.4}
\end{align*}
$$

Proof. Using the connection coefficient identity, we may express both interpolation functions as linear combinations of biorthogonal functions. Substituting in the known values for the inner products of the biorthogonal functions, we thus obtain a sum over partition pairs $\boldsymbol{\mu} \subset \boldsymbol{\lambda}, \boldsymbol{\kappa}$. That this sum gives the desired right-hand side is itself a special case of the connection coefficient identity.

Alternatively, we can mimic the proof of Theorem 6.1] using the fact that $\mathcal{D}_{q}^{(n)}\left(u_{0}, t_{0}, t_{3} ; t, p\right)$ acts nicely on $\mathcal{R}_{\lambda}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right)$. If we define

$$
\begin{align*}
& F_{\boldsymbol{\lambda} \boldsymbol{\kappa}}^{(n)}\left(t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q\right)  \tag{9.5}\\
& \quad:=\frac{\left\langle\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right), \mathcal{R}_{\kappa}^{*(n)}\left(; t_{1}, u_{1} ; t ; p, q\right)\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q}}{\Delta_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} t_{0} / u_{0} \mid t^{n-1} t_{0} t_{2}, t^{n-1} t_{0} t_{3} ; t ; p, q\right) \Delta_{\kappa}^{0}\left(t^{n-1} t_{1} / u_{1} \mid t^{n-1} t_{1} t_{2}, t^{n-1} t_{1} t_{3} ; t ; p, q\right)}
\end{align*}
$$

then adjointness gives

$$
\begin{align*}
F_{\boldsymbol{\lambda} \kappa}^{(n)}\left(t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q\right) & =F_{\boldsymbol{\lambda} \kappa}^{(n)}\left(q^{1 / 2} t_{0}, q^{-1 / 2} t_{1}, q^{1 / 2} t_{2}, q^{-1 / 2} t_{3}, q^{1 / 2} u_{0}, q^{-1 / 2} u_{1} ; t ; p, q\right)  \tag{9.6}\\
& =F_{\boldsymbol{\lambda} \kappa}^{(n)}\left(p^{1 / 2} t_{0}, p^{-1 / 2} t_{1}, p^{1 / 2} t_{2}, p^{-1 / 2} t_{3}, p^{1 / 2} u_{0}, p^{-1 / 2} u_{1} ; t ; p, q\right) \tag{9.7}
\end{align*}
$$

and thus

$$
\begin{equation*}
F_{\lambda \kappa}^{(n)}\left(t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q\right)=F_{\lambda \kappa}^{(n)}\left(w t_{0}, t_{1} / w, t_{2} w, t_{3} / w, u_{0} w, u_{1} / w ; t ; p, q\right) \tag{9.8}
\end{equation*}
$$

for any $w \in \mathbb{C}^{*}$. Taking the limit $w \rightarrow \sqrt{t^{n-1} t_{1} u_{1}}$ and expanding via residue calculus, we obtain a sum over partition pairs contained in $\boldsymbol{\kappa}_{1}^{n}$, in which only the term associated to $\boldsymbol{\kappa}$ survives. (Recall that the contour must be deformed around the poles of $\mathcal{R}_{\kappa}^{*(n)}$.) We thus find

$$
\begin{equation*}
F_{\boldsymbol{\lambda} \boldsymbol{\kappa}}^{(n)}\left(t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q\right) \propto \mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(\ldots z_{i}\left(\boldsymbol{\kappa} ; t_{1} / \sqrt{t^{n-1} t_{1} u_{1}}\right) \ldots ; t_{0} \sqrt{t^{n-1} t_{1} u_{1}}, u_{0} \sqrt{t^{n-1} t_{1} u_{1}} ; t ; p, q\right) \tag{9.9}
\end{equation*}
$$

where the constant of proportionality is independent of $\boldsymbol{\lambda}$. This constant can be resolved by taking the limit $w \rightarrow \sqrt{t^{n-1} t_{0} u_{0}}$ in the case $\boldsymbol{\lambda}=0$.

Remark 1. The left-hand side above is invariant under exchanging ( $\left.\boldsymbol{\lambda}, t_{0}, u_{0}\right)$ and $\left(\boldsymbol{\kappa}, t_{1}, u_{1}\right)$. That the righthand side is invariant is a special case of evaluation symmetry (Corollary 8.15). We can also use that same special case of evaluation symmetry to see that this generalizes Theorem 9.1 Indeed, if we specialize so that $t_{0} t_{1}=p^{-l} q^{-m} t^{1-n}$ with $\boldsymbol{\lambda}, \boldsymbol{\kappa} \subset(l, m)^{n}$, then the above left-hand side becomes a sum over $\boldsymbol{\mu} \subset(l, m)^{n}$. Using evaluation symmetry, the factor

$$
\begin{equation*}
\mathcal{R}_{\kappa}^{*(n)}\left(\ldots z_{i}\left(\boldsymbol{\mu} ; t_{0}\right) \ldots ; t_{1}, u_{1}\right)=\mathcal{R}_{\kappa}^{*(n)}\left(\ldots z_{i}\left((l, m)^{n}-\boldsymbol{\mu} ; t_{1}\right) \ldots ; t_{1}, u_{1}\right) \tag{9.10}
\end{equation*}
$$

can be rewritten in terms of

$$
\begin{equation*}
\mathcal{R}_{\boldsymbol{\mu}}^{*(n)}\left(\ldots z_{i}\left((l, m)^{n}-\boldsymbol{\kappa} ; x\right) \ldots ; x, y\right) \tag{9.11}
\end{equation*}
$$

for suitable $x$ and $y$. Replacing $\boldsymbol{\kappa}$ by $(l, m)^{n}-\boldsymbol{\kappa}$ gives Theorem 9.1
Similarly, replacing $\boldsymbol{\kappa}$ by $(l, m)^{n}-\boldsymbol{\kappa}$ in the general version and comparing the results, we find

$$
\begin{equation*}
\mathcal{R}_{l^{n}-\lambda, m^{n}-\mu}^{*(n)}\left(; t_{1}, u_{1} ; t ; p, q\right)=\frac{\left(p q / t_{1} u_{1}\right)^{2 l|\mu|+2 m|\lambda|} \prod_{1 \leq i \leq n} \theta\left(t_{1} x_{i}^{ \pm 1} ; p, q\right)_{l, m}}{\prod_{1 \leq i \leq n} \theta\left(\left(p q / u_{1}\right) x_{i}^{ \pm 1} ; p, q\right)_{l, m}} \mathcal{R}_{\lambda, \mu}^{*(n)}\left(; u_{1} / p^{l} q^{m}, p^{l} q^{m} t_{1} ; t ; p, q\right) \tag{9.12}
\end{equation*}
$$

as both sides have the same inner product with $\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right)$.

Remark 2. Note that the second proof of the theorem did not use the connection coefficient identity, and is thus independent of 19.

If we take $\boldsymbol{\kappa}=0$ above, we obtain the following identity, generalizing Kadell's lemma (see, for instance Corollary 5.14 of [21]).

Corollary 9.3. For otherwise generic parameters satisfying $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} t_{4} t_{5}=p q$,

$$
\begin{equation*}
\left\langle\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; t_{0}, t_{1} ; t ; p, q\right)\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3}, t_{4}, t_{5} ; t ; p, q}=\Delta_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} t_{0} / t_{1} \mid t^{n-1} t_{0} t_{2}, t^{n-1} t_{0} t_{3}, t^{n-1} t_{0} t_{4}, t^{n-1} t_{0} t_{5} ; t ; p, q\right) \tag{9.13}
\end{equation*}
$$

The connection coefficient argument gives the following identity as well.
Theorem 9.4. For otherwise generic parameters satisfying $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} u_{0} u_{1}=p q$,

$$
\begin{align*}
&\left\langle\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; t_{0}, u_{0} ; t ; p, q\right), \mathcal{R}_{\boldsymbol{\kappa}}^{*(n)}\left(; t_{0}, u_{1} ; t ; p, q\right)\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q} \\
&= \Delta_{\boldsymbol{\lambda}}^{0}\left(\left.\frac{t^{n-1} t_{0}}{u_{0}} \right\rvert\, t^{n-1} t_{0} t_{1}, t^{n-1} t_{0} t_{2}, t^{n-1} t_{0} t_{3}, t^{n-1} t_{0} u_{1} ; t ; p, q\right) \\
& \Delta_{\boldsymbol{\kappa}}^{0}\left(\left.\frac{t^{n-1} t_{0}}{u_{1}} \right\rvert\, t^{n-1} t_{0} t_{1}, t^{n-1} t_{0} t_{2}, t^{n-1} t_{0} t_{3}, t^{n-1} t_{0} u_{0} ; t ; p, q\right) \\
& \tilde{R}_{\boldsymbol{\lambda}}^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\kappa}, t_{0}^{\prime}\right) \ldots ; t_{0}^{\prime}: t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime} ; u_{0}^{\prime}, u_{1}^{\prime} ; t ; p, q\right), \tag{9.14}
\end{align*}
$$

where the primed parameters are determined by

$$
\begin{array}{llr}
t^{n-1} t_{0}^{\prime} t_{1}^{\prime}=t^{n-1} t_{0} t_{1} & t^{n-1} t_{0}^{\prime} t_{2}^{\prime}=t^{n-1} t_{0} t_{2} & t^{n-1} t_{0}^{\prime} t_{3}^{\prime}=t^{n-1} t_{0} t_{3} \\
t^{n-1} t_{0}^{\prime} u_{0}^{\prime}=t^{n-1} t_{0} u_{0} & t^{n-1} t_{0}^{\prime} u_{1}^{\prime}=\frac{1}{t^{n-1} t_{0} u_{1}} & t_{0}^{\prime 2}=\frac{t_{0}}{t^{n-1} u_{1}}
\end{array}
$$

Remark. The above transformation of the parameters is involutive, and conjugates the exchange $u_{0} \leftrightarrow u_{1}$ to the "hat" transformation.

A further application of connection coefficients gives the following result, containing both Theorems 9.2 and 9.4 as special cases.

Theorem 9.5. For otherwise generic parameters satisfying $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} u_{0} u_{1}=p q$,

$$
\begin{align*}
&\left\langle\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; t_{0} v, u_{0} ; t ; p, q\right), \mathcal{R}_{\boldsymbol{\kappa}}^{*(n)}\left(; t_{0}, u_{1} ; t ; p, q\right)\right\rangle_{t_{0}, t_{1}, t_{2}, t_{3}, u_{0}, u_{1} ; t ; p, q} \\
&= \Delta_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} t_{0} v / u_{0} \mid t^{n-1} v t_{0} t_{1}, t^{n-1} v t_{0} t_{2}, t^{n-1} v t_{0} t_{3}, t^{n-1} t_{0} u_{1} ; t ; p, q\right) \\
& \Delta_{\boldsymbol{\kappa}}^{0}\left(t^{n-1} t_{0} / u_{1} \mid t^{n-1} t_{0} t_{1}, t^{n-1} t_{0} t_{2}, t^{n-1} t_{0} t_{3}, t^{n-1} t_{0} u_{0} ; t ; p, q\right) \\
& \tilde{R}_{\boldsymbol{\lambda}}^{(n)}\left(\ldots z_{i}\left(\boldsymbol{\kappa}, t_{0}^{\prime}\right) \ldots ; t_{0}^{\prime} v: t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime} ; u_{0}^{\prime}, u_{1}^{\prime} / v ; t ; p, q\right), \tag{9.15}
\end{align*}
$$

with primed parameters as above.
Now, Theorem 9.1 is sufficiently general that the univariate argument for deriving Bailey-type transformations from Jackson-type summations applies, giving the following identity.

Theorem 9.6. [19] The sum

$$
\frac{\Delta_{\boldsymbol{\lambda}}^{0}(a \mid b, a p q / b f ; t ; p, q)}{\Delta_{\boldsymbol{\kappa}}^{0}(a / c \mid b / c, a p q / b d ; t ; p, q)} \sum_{\boldsymbol{\kappa} \subset \boldsymbol{\mu} \subset \boldsymbol{\lambda}} \frac{\Delta_{\boldsymbol{\mu}}^{0}(a / b \mid c / b, f, g ; t ; p, q)}{\Delta_{\boldsymbol{\mu}}^{0}(a / b \mid 1 / b, d, e ; t ; p, q)}\left[\begin{array}{l}
\boldsymbol{\lambda}  \tag{9.16}\\
\boldsymbol{\mu}
\end{array}\right]_{[a, b] ; t ; p, q}\left[\begin{array}{l}
\boldsymbol{\mu} \\
\boldsymbol{\kappa}
\end{array}\right]_{[a / b, c / b] ; t ; p, q}
$$

is symmetric in $b$ and $b^{\prime}$, where $b b^{\prime} d e=c a p q, b b^{\prime} f g=a p q$.
Remark 1. Again, taking $\boldsymbol{\lambda}=(l, m)^{n}, \boldsymbol{\kappa}=0$ gives an identity conjectured by Warnaar, in this case his conjectured multivariate Frenkel-Turaev transformation 36.

Remark 2. This identity can also be obtained by comparing various ways of computing connection coefficients for biorthogonal functions in which $t_{0}, t_{3}, u_{0}$ are left fixed, but $t_{1}, t_{2}, u_{1}$ change.

It turn out that this identity also has an integral analogue. For each integer $n \geq 0$, and partition pairs $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ of length at most $n$, we define a (meromorphic) function

$$
\begin{align*}
& I I_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)  \tag{9.17}\\
& \qquad=\frac{(p ; p)^{n}(q ; q)^{n} \Gamma(t ; p, q)^{n}}{2^{n} n!} \int_{C^{n}} \mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(\ldots x_{i} \ldots ; t_{0}, t_{1} ; t ; p, q\right) \mathcal{R}_{\boldsymbol{\mu}}^{*(n)}\left(\ldots x_{i} \ldots ; t_{2}, t_{3} ; t ; p, q\right) \\
& \\
& \quad \prod_{1 \leq i<j \leq n} \frac{\Gamma\left(t x_{i}^{ \pm 1} x_{j}^{ \pm 1} ; p, q\right)}{\Gamma\left(x_{i}^{ \pm 1} x_{j}^{ \pm 1} ; p, q\right)} \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 7} \Gamma\left(t_{r} x_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(x_{i}^{ \pm 2} ; p, q\right)} \frac{d x_{i}}{2 \pi \sqrt{-1} x_{i}},
\end{align*}
$$

on the domain $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}=p^{2} q^{2}$, where the contour $C^{n}$ is constrained in the usual way by the poles of the integrand.

Theorem 9.7. If $t^{2 n-2} t_{0} t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}=p^{2} q^{2}$ for some nonnegative integer $n$, then

$$
\begin{align*}
& I I_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)= \Delta_{\boldsymbol{\lambda}}^{0}\left(t^{n-1} t_{0} / t_{1} \mid t^{n-1} t_{0} t_{4}, t^{n-1} t_{0} t_{5}\right) \\
& \prod_{\boldsymbol{\mu}}^{0}\left(t^{n-1} t_{2} / t_{3} \mid t^{n-1} t_{2} t_{6}, t^{n-1} t_{2} t_{7}\right) \prod_{1 \leq j \leq n, s \in\{0,1,4,5\}} \\
& \prod_{1 \leq j \leq n} \prod_{\substack{r, s \in s \in\{, 3,6,7\} \\
r<s}} \Gamma\left(t^{n-j} t_{r} t_{s} ; p, q\right)  \tag{9.18}\\
&\left.I t^{n-j} t_{r} t_{s} ; p, q\right) \\
&(n)\left(t_{0} / u, t_{1} / u: u t_{2}, u t_{3}: t_{4} / u, t_{5} / u, u t_{6}, u t_{7} ; t ; p, q\right),
\end{align*}
$$

where $u$ is chosen so that

$$
\begin{equation*}
u^{2}=\sqrt{\frac{t_{0} t_{1} t_{4} t_{5}}{t_{2} t_{3} t_{6} t_{7}}}=\frac{p q t^{1-n}}{t_{2} t_{3} t_{6} t_{7}}=\frac{t_{0} t_{1} t_{4} t_{5}}{p q t^{1-n}} . \tag{9.19}
\end{equation*}
$$

Proof. If, following the second proof of Theorem 9.2 we attempt to mimic the difference operator proof of Theorem 6.1 we immediately encounter the difficulty that we no longer have adjointness between two instances of $\mathcal{D}_{q}^{(n)}$, but rather between an instance of $\mathcal{D}_{q}^{(n)}$ and an instance of $\mathcal{D}_{q}^{+(n)}$. The one exception is when $t^{n-1} t_{0} t_{1} t_{4} t_{5}=p$, in which case

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(t_{0}, t_{1}, t_{4} ; t, p\right) \tag{9.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{D}_{q}^{(n)}\left(\sqrt{q} t_{2}, \sqrt{q} t_{3}, \sqrt{q} t_{6} ; t, p\right) \tag{9.21}
\end{equation*}
$$

are adjoint; the resulting transformation is precisely the case $u^{2}=q$ of the theorem.
To extend this argument, we will thus need to extend the difference operators. With this in mind, we define a difference operator $D_{l, m}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right)$ for $l, m \geq 0$ as follows.

$$
\begin{align*}
D_{0,1}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right) & =D_{q}^{(n)}\left(u_{0}, u_{1}, u_{2}, p / t^{n-1} u_{0} u_{1} u_{2} ; t, p\right)  \tag{9.22}\\
D_{1,0}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right) & =D_{p}^{(n)}\left(u_{0}, u_{1}, u_{2}, q / t^{n-1} u_{0} u_{1} u_{2} ; t, q\right)  \tag{9.23}\\
D_{l+l^{\prime}, m+m^{\prime}}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right) & =D_{l, m}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right) D_{l^{\prime}, m^{\prime}}^{(n)}\left(S_{l, m}^{1 / 2} u_{0}, S_{l, m}^{1 / 2} u_{1}, S_{l, m}^{1 / 2} u_{2} ; t ; p, q\right), \tag{9.24}
\end{align*}
$$

where $S_{l, m}=(p, q)^{l, m}$. Since $D_{l, m}^{(n)}$ is a composition of $p$ - and $q$-difference operators, it itself is a difference operator; that it is well-defined follows by verifying that the two ways of computing $D_{1,1}^{(n)}$ agree.

Lemma 9.8. Let $u_{0}, u_{1}, u_{2}, u_{3}$ be such that $t^{n-1} u_{0} u_{1} u_{2} u_{3}=p q / S_{l, m}$. Then

$$
\begin{align*}
D_{l, m}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right) & \mathcal{R}_{\lambda}^{*(n)}\left(; S_{l, m}^{1 / 2} u_{0}, S_{l, m}^{1 / 2} u_{1} ; t ; p, q\right)  \tag{9.25}\\
& =\frac{\Delta_{\lambda}^{0}\left(t^{n-1} u_{0} / u_{1} \mid p q / u_{1} u_{2}, p q / u_{1} u_{3} ; t ; p, q\right)}{\prod_{1 \leq i \leq n} \prod_{0 \leq r<s \leq 3} \Gamma\left(t^{n-i} u_{r} u_{s} ; p, q\right)} \mathcal{R}_{\lambda}^{*(n)}\left(; u_{0}, u_{1} ; t ; p, q\right)
\end{align*}
$$

In particular,

$$
\begin{equation*}
D_{l, m}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right)=D_{l, m}^{(n)}\left(u_{0}, u_{1}, u_{3} ; t ; p, q\right) . \tag{9.26}
\end{equation*}
$$

Proof. The first claim holds when $(l, m) \in\{(0,1),(1,0)\}$; an easy induction gives it in general.
Since $D_{l, m}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right)$ is a difference operator, it is uniquely determined by this action; since the given formula is symmetric between $u_{2}$ and $u_{3}$, the operator itself is symmetric.

Lemma 9.9. The different instances of $D_{l, m}^{(n)}$ are related by

$$
\begin{equation*}
D_{l, m}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right)=\prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \frac{\Gamma\left(u_{r}^{\prime} x_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(u_{r} x_{i}^{ \pm 1} ; p, q\right)} D_{l, m}^{(n)}\left(u_{0}^{\prime}, u_{1}^{\prime}, u_{2}^{\prime} ; t ; p, q\right) \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \frac{\Gamma\left(S_{l, m}^{1 / 2} u_{r} x_{i}^{ \pm 1} ; p, q\right)}{\Gamma\left(S_{l, m}^{1 / 2} u_{r}^{\prime} x_{i}^{ \pm 1} ; p, q\right)} \tag{9.27}
\end{equation*}
$$

Proof. If $u_{3}^{\prime}=u_{3}$, the result follows by a simple induction; the general case then follows by combining that case with the symmetry between $u_{0}, u_{1}, u_{2}$ and $u_{3}$.

In particular, we can define the operator

$$
\begin{equation*}
D_{l, m}^{(n)}(t ; p, q):=\prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \Gamma\left(u_{r} x_{i}^{ \pm 1} ; p, q\right) D_{l, m}^{(n)}\left(u_{0}, u_{1}, u_{2} ; t ; p, q\right) \prod_{\substack{1 \leq \leq \leq n \\ 0 \leq r \leq 3}} \frac{1}{\Gamma\left(S_{l, m}^{1 / 2} u_{r} x_{i}^{ \pm 1} ; p, q\right)}, \tag{9.28}
\end{equation*}
$$

which is independent of $u_{0}, u_{1}, u_{2}$.
This operator is self-adjoint with respect to the cross-terms in the II density; that is, with respect to

$$
\begin{equation*}
\prod_{1 \leq i<j \leq n} \frac{\Gamma\left(t x_{i}^{ \pm 1} x_{j}^{ \pm 1} ; p, q\right)}{\Gamma\left(x_{i}^{ \pm 1} x_{j}^{ \pm 1} ; p, q\right)} \prod_{1 \leq i \leq n} \frac{d x_{i}}{2 \pi \sqrt{-1} x_{i}} . \tag{9.29}
\end{equation*}
$$

This follows from the fact that

$$
\begin{equation*}
\int f D_{l, m}^{(n)}\left(t_{0}, t_{1}, t_{2} ; t ; p, q\right) g \Delta^{(n)}\left(t_{0}, \ldots, t_{5} ; t ; p, q\right)=\int g D_{l, m}^{(n)}\left(t_{3}^{\prime}, t_{4}^{\prime}, t_{5}^{\prime} ; t ; p, q\right) f \Delta^{(n)}\left(t_{0}^{\prime}, \ldots, t_{5}^{\prime} ; t ; p, q\right) \tag{9.30}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, t_{4}^{\prime}, t_{5}^{\prime}\right)=\left(S_{l, m}^{1 / 2} t_{0}, S_{l, m}^{1 / 2} t_{1}, S_{l, m}^{1 / 2} t_{2}, S_{l, m}^{-1 / 2} t_{3}, S_{l, m}^{-1 / 2} t_{4}, S_{l, m}^{-1 / 2} t_{5}\right) \tag{9.31}
\end{equation*}
$$

which in turn follows by induction from the cases $(l, m) \in\{(0,1),(1,0)\}$.
We also have a sort of commutation relation satisfied by $D_{l, m}^{(n)}(t ; p, q)$.
Lemma 9.10. If $l, m, l^{\prime}, m^{\prime}$ are nonnegative integers and $u_{0} u_{1} u_{2} u_{3}=S_{l, m} S_{l^{\prime}, m^{\prime}} p^{2} q^{2}$, then we have the following identity of difference operators.

$$
\begin{align*}
D_{l, m}^{(n)}(t ; p, q) & \prod_{\substack{1 \leq i \leq n \\
0 \leq r \leq 3}} \Gamma\left(u_{r} x_{i}^{ \pm 1} ; p, q\right) D_{l^{\prime}, m^{\prime}}^{(n)}(t ; p, q) \prod_{\substack{1 \leq i \leq n \\
0 \leq r \leq 3}} \frac{1}{\Gamma\left(S_{l^{\prime}, m^{\prime}}^{-1 / 2} u_{r} x_{i}^{ \pm 1} ; p, q\right)}  \tag{9.32}\\
& =\prod_{\substack{1 \leq i \leq n \\
0 \leq r \leq 3}} \Gamma\left(S_{l, m}^{-1 / 2} u_{r} x_{i}^{ \pm 1} ; p, q\right) D_{l^{\prime}, m^{\prime}}^{(n)}(t ; p, q) \prod_{\substack{1 \leq i \leq n \\
0 \leq r \leq 3}} \frac{1}{\Gamma\left(\left(S_{l, m} S_{l^{\prime}, m^{\prime}}\right)^{-1 / 2} u_{r} x_{i}^{ \pm 1} ; p, q\right)} D_{l, m}^{(n)}(t ; p, q)
\end{align*}
$$

Proof. By comparing actions on $\mathcal{R}_{\lambda}^{*(n)}\left(\left(S_{l, m} S_{l^{\prime}, m^{\prime}}\right)^{1 / 2} u_{0},\left(S_{l, m} S_{l^{\prime}, m^{\prime}}\right)^{1 / 2} t_{0} ; t ; p, q\right)$, we find that

$$
\begin{align*}
D_{l, m}^{(n)}\left(u_{0}, t_{0}, t_{1} ; t ; p, q\right) & D_{l^{\prime}, m^{\prime}}^{(n)}\left(S_{l, m}^{1 / 2} u_{0}, S_{l, m}^{1 / 2} t_{0}, S_{l, m}^{-1 / 2} t_{2} ; t ; p, q\right)  \tag{9.33}\\
& =D_{l^{\prime}, m^{\prime}}^{(n)}\left(u_{0}, t_{0}, t_{2} ; t ; p, q\right) D_{l, m}^{(n)}\left(S_{l^{\prime}, m^{\prime}}^{1 / 2} u_{0}, S_{l^{\prime}, m^{\prime}}^{1 / 2} t_{0}, S_{l^{\prime}, m^{\prime}}^{-1 / 2} t_{1} ; t ; p, q\right)
\end{align*}
$$

Expressing this in terms of $D_{l, m}^{(n)}(t ; p, q)$ and simplifying gives the desired result.
Now, consider an integral of the form

$$
\begin{equation*}
\int\left[D_{l, m}^{(n)}\left(t_{0}, t_{1}, t_{4} ; t ; p, q\right) f\right]\left[D_{l^{\prime}, m^{\prime}}^{(n)}\left(t_{2}, t_{3}, t_{6} ; t ; p, q\right) g\right] \Delta^{(n)}\left(t_{0}, t_{1}, t_{2}, t_{3}, t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right) \tag{9.34}
\end{equation*}
$$

where $f \in \mathcal{A}^{(n)}\left(S_{l, m}^{1 / 2} t_{1} ; p, q\right), g \in \mathcal{A}^{(n)}\left(S_{l^{\prime}, m^{\prime}}^{1 / 2} t_{3} ; p, q\right)$, and the parameters satisfy the relations

$$
\begin{equation*}
t^{n-1} t_{0} t_{1} t_{4} t_{5}=p q \frac{S_{l^{\prime}, m^{\prime}}}{S_{l, m}}, \quad t^{n-1} t_{2} t_{3} t_{6} t_{7}=p q \frac{S_{l, m}}{S_{l^{\prime}, m^{\prime}}} \tag{9.35}
\end{equation*}
$$

If we rewrite this integral in terms of $D^{(n)}(t ; p, q)$ and $\Delta^{(n)}(; t ; p, q)$ and apply self-adjointness of $D_{l, m}^{(n)}(t ; p, q)$, the resulting composition of difference operators can be transformed by the commutation relation. The result is of the same form, and we thus obtain the following identity.

$$
\begin{align*}
\int & \left(D_{l, m}^{(n)}\left(t_{0}, t_{1}, t_{4} ; t ; p, q\right) f\right)\left(D_{l^{\prime}, m^{\prime}}^{(n)}\left(t_{2}, t_{3}, t_{6} ; t ; p, q\right) g\right) \Delta^{(n)}\left(t_{0}, t_{1}, t_{2}, t_{3}, t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)  \tag{9.36}\\
& =\int\left(D_{l^{\prime}, m^{\prime}}^{(n)}\left(t_{0}^{\prime}, t_{1}^{\prime}, t_{4}^{\prime} ; t ; p, q\right) f\right)\left(D_{l, m}^{(n)}\left(t_{2}^{\prime}, t_{3}^{\prime}, t_{6}^{\prime} ; t ; p, q\right) g\right) \Delta^{(n)}\left(t_{0}^{\prime}, t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, t_{4}^{\prime}, t_{5}^{\prime}, t_{6}^{\prime}, t_{7}^{\prime} ; t ; p, q\right)
\end{align*}
$$

where

$$
t_{r}^{\prime}= \begin{cases}\left(S_{l, m} / S_{l^{\prime}, m^{\prime}}\right)^{1 / 2} t_{r} & r \in\{0,1,4,5\}  \tag{9.37}\\ \left(S_{l^{\prime}, m^{\prime}} / S_{l, m}\right)^{1 / 2} t_{r} & r \in\{2,3,6,7\}\end{cases}
$$

If we set

$$
\begin{align*}
& f=\mathcal{R}_{\boldsymbol{\lambda}}^{*(n)}\left(; S_{l, m}^{1 / 2} t_{0}, S_{l, m}^{1 / 2} t_{1} ; t ; p, q\right)  \tag{9.38}\\
& g=\mathcal{R}_{\boldsymbol{\mu}}^{*(n)}\left(; S_{l^{\prime}, m^{\prime}}^{1 / 2} t_{2}, S_{l^{\prime}, m^{\prime}}^{1 / 2} t_{3} ; t ; p, q\right), \tag{9.39}
\end{align*}
$$

we obtain the special case $u^{2}=S_{l^{\prime}, m^{\prime}} / S_{l, m}$ of the theorem. Since this set is dense, the theorem holds in general.

Remark 1. Note that the above proof did not use Theorem 8.13 and is thus independent of the results of [19]. In fact, one can use this result to prove Theorem 8.13 as follows. Connection coefficients for interpolation functions can be obtained from the special case $t_{2} t_{7}=p q$ (essentially Theorem9.5), by comparing the result to that of Theorem 9.2 One can then reverse the first proof of Theorem 9.2 to show that the functions given by the binomial formula are indeed biorthogonal; Theorem 8.13 then follows via Theorem 9.1
Remark 2. In the special case $t^{n-1} t_{0} t_{2}=1 / S_{l, m}$, we recover Theorem 9.6 Also, the univariate case $n=1$ is precisely the case $n=m=1$ of the $A_{n}$ transformation.
Remark 3. Similarly, using our integral operators, one can give a direct proof for the case $u^{2}=t$, which presumably only extends to an argument valid for $u^{2} \in t^{\mathbb{Z}}$. This is, however, probably the simplest proof in the univariate case (since then the integral is independent of $t$ ).

We can simplify this transformation somewhat by adding an appropriate normalization factor. Define a meromorphic function

$$
\begin{equation*}
\tilde{I}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right):=Z_{\boldsymbol{\lambda} \boldsymbol{\mu}} I I_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(t^{1 / 2} t_{0}, t^{1 / 2} t_{1}: t^{1 / 2} t_{2}, t^{1 / 2} t_{3}: t^{1 / 2} t_{4}, t^{1 / 2} t_{5}, t^{1 / 2} t_{6}, t^{1 / 2} t_{7} ; t ; p, q\right) \tag{9.40}
\end{equation*}
$$

where

$$
\begin{align*}
Z_{\boldsymbol{\lambda} \boldsymbol{\mu}} & =\prod_{0 \leq r<s \leq 7} \Gamma^{+}\left(t t_{r} t_{s} ; t, p, q\right) Z_{\boldsymbol{\lambda}} Z_{\boldsymbol{\mu}}  \tag{9.41}\\
Z_{\boldsymbol{\lambda}} & =\mathcal{C}_{\boldsymbol{\lambda}}^{0}\left(t^{n}, p q / t t_{1} t_{2}, p q / t t_{1} t_{3} ; t ; p, q\right) \prod_{4 \leq r \leq 7} \mathcal{C}_{\boldsymbol{\lambda}}^{0}\left(p q / t t_{1} t_{r} ; t ; p, q\right)  \tag{9.42}\\
Z_{\boldsymbol{\mu}} & =\mathcal{C}_{\boldsymbol{\mu}}^{0}\left(t^{n}, p q / t t_{0} t_{3}, p q / t t_{1} t_{3} ; t ; p, q\right) \prod_{4 \leq r \leq 7} \mathcal{C}_{\boldsymbol{\mu}}^{0}\left(p q / t t_{3} t_{r} ; t ; p, q\right) \tag{9.43}
\end{align*}
$$

and the condition on the parameters is now $t^{2 n+2} t_{0} t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}=p^{2} q^{2}$. Here $\Gamma^{+}(x ; t, p, q)$ is defined by

$$
\begin{equation*}
\Gamma^{+}(x ; t, p, q):=\prod_{i, j, k \geq 0}\left(1-t^{i} p^{j} q^{k} x\right)\left(1-t^{i+1} p^{j+1} q^{k+1} / x\right) \tag{9.44}
\end{equation*}
$$

so for instance

$$
\begin{equation*}
\Gamma^{+}(t x ; t, p, q)=\Gamma^{+}(x ; t, p, q) \Gamma(x ; p, q) \tag{9.45}
\end{equation*}
$$

Note that for generic $p, q, t$, the integer $n$ can be deduced from the balancing condition on the parameters, and thus could in principle be omitted from the notation for $\tilde{I I}$. Note that it follows from the residue calculus of the appendix that $\tilde{I}_{00}$ is holomorphic for each $n$; it may very well be holomorphic for $\boldsymbol{\lambda}, \boldsymbol{\mu} \neq 0$, but this would require a deeper understanding of the singularities of $\mathcal{R}^{*}$ as a function of the parameters.

Corollary 9.11. We have

$$
\begin{equation*}
\tilde{I I}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)=\tilde{I}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(t_{0} / u, t_{1} / u: u t_{2}, u t_{3}: t_{4} / u, t_{5} / u, u t_{6}, u t_{7} ; t ; p, q\right) \tag{9.46}
\end{equation*}
$$

where $u$ is chosen so that

$$
\begin{equation*}
u^{2}=\sqrt{\frac{t_{0} t_{1} t_{4} t_{5}}{t_{2} t_{3} t_{6} t_{7}}}=\frac{p q t^{-n-1}}{t_{2} t_{3} t_{6} t_{7}}=\frac{t_{0} t_{1} t_{4} t_{5}}{p q t^{-n-1}} . \tag{9.47}
\end{equation*}
$$

Since $\tilde{I I}$ is also invariant under permutations of $t_{4}, t_{5}, t_{6}, t_{7}$, it is in fact invariant under an action of the Weyl group $D_{4}$. Since there are three double cosets $S_{4} \backslash D_{4} / S_{4}$, there is one other type of nontrivial transformation, namely:

$$
\begin{equation*}
\tilde{I}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)=\tilde{I}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(u / t_{1}, u / t_{0}: u / t_{3}, u / t_{2}: v / t_{4}, v / t_{5}, v / t_{6}, v / t_{7} ; t ; p, q\right) \tag{9.48}
\end{equation*}
$$

where $u^{2}=t_{0} t_{1} t_{2} t_{3}, v^{2}=t_{4} t_{5} t_{6} t_{7}$, and $t^{n+1} u v=p q$. In terms of the unnormalized integral, this reads

$$
\begin{align*}
I I_{\boldsymbol{\mu}, \boldsymbol{\nu}}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)= & \prod_{1 \leq j \leq n} \prod_{0 \leq r \leq 3,4 \leq s \leq 7} \Gamma^{+}\left(t^{n-j} t_{r} t_{s} ; t, p, q\right) \\
& \Delta_{\boldsymbol{\mu}}^{0}\left(t^{n-1} t_{0} / t_{1} \mid t^{n} t_{0} t_{4}, t^{n} t_{0} t_{5}, t^{n} t_{0} t_{6}, t^{n} t_{0} t_{7} ; t ; p, q\right) \\
& \Delta_{\boldsymbol{\nu}}^{0}\left(t^{n-1} t_{2} / t_{3} \mid t^{n} t_{2} t_{4}, t^{n} t_{2} t_{5}, t^{n} t_{2} t_{6}, t^{n} t_{2} t_{7} ; t ; p, q\right) \\
& I_{\boldsymbol{\mu}, \boldsymbol{\nu}}^{(n)}\left(u / t_{1}, u / t_{0}: u / t_{3}, u / t_{2}: v / t_{4}, v / t_{5}, v / t_{6}, v / t_{7} ; t ; p, q\right) . \tag{9.49}
\end{align*}
$$

The reason for the factors of $t^{1 / 2}$ in the definition of $\tilde{I}^{(n)}$ is that the integral satisfies a further identity.
Theorem 9.12. Let $n \geq m \geq 0$ be nonnegative integers such that $\ell(\lambda), \ell(\boldsymbol{\mu}) \leq m$, and suppose the parameters satisfy $t^{n-m} t_{0} t_{2}=1$. Then

$$
\begin{equation*}
\tilde{I}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)=\tilde{I}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^{(m)}\left(1 / t_{2}, t_{1}: 1 / t_{0}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right) \tag{9.50}
\end{equation*}
$$

Proof. To compute the left-hand side, we must take a limit (as the condition on the contour cannot be satisfied); what we find is that we must take residues in $n-m$ of the variables, effectively setting those variables to $t^{1 / 2} t_{0}, \ldots, t^{n-m-1 / 2} t_{0}$, or equivalently (taking reciprocals) to to $t^{n-m-1 / 2} t_{2}, \ldots, t^{1 / 2} t_{2}$. The result is the desired $m$-dimensional instance of $\tilde{I} I$.

Remark. Note that the requirement that $\ell(\boldsymbol{\lambda}), \ell(\boldsymbol{\mu}) \leq m$ and $n \geq m \geq 0$ with $n, m \in \mathbb{Z}$ is equivalent to a requirement that both sides be well-defined.

We thus find that we have a formal symmetry under a larger group, isomorphic to the Weyl group $A_{1} D_{4}$.
If one of the partition pairs is trivial, the effective symmetry group becomes larger. To be precise, define

$$
\begin{equation*}
I I_{\boldsymbol{\lambda}}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}, t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right):=I I_{\lambda, 0}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}: t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right) \tag{9.51}
\end{equation*}
$$

and similarly for $\tilde{I}_{\boldsymbol{\lambda}}^{(n)}$. This function is now manifestly symmetric under permutations of $t_{2}$ through $t_{7}$; together with the symmetry of Theorem 9.7 this gives rise to the Weyl group $D_{6}$. Since $S_{6} \backslash D_{6} / S_{6}$ has four double cosets, we thus obtain a further transformation.

Corollary 9.13. We have

$$
\begin{align*}
I I_{\lambda}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}, t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)= & \Delta_{\lambda}^{0}\left(t^{n-1} t_{0} / t_{1} \mid t^{n-1} t_{0} t_{2}, \ldots, t^{n-1} t_{0} t_{7} ; t ; p, q\right) \prod_{1 \leq i \leq n} \prod_{0 \leq r<s \leq 7} \Gamma\left(t^{n-i} t_{r} t_{s} ; p, q\right) \\
& I_{\lambda}^{(n)}\left(u / t_{1}, u / t_{0}: u / t_{2}, u / t_{3}, u / t_{4}, u / t_{5}, u / t_{6}, u / t_{7} ; t ; p, q\right) \tag{9.52}
\end{align*}
$$

where $u^{2}=\sqrt{t_{0} t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}}=p q / t^{n-1}$.

Remark. In the limit $t_{6} t_{7}=p^{l+1} q^{m+1}$, the right-hand side becomes a sum; taking $\boldsymbol{\lambda}=0$ and reparametrizing, we obtain the following integral representation for a Warnaar-type sum:

$$
\begin{align*}
I I^{(n)}\left(p q / u_{0}, t_{1}, t_{2}, t_{3}, t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)= & \prod_{1 \leq i \leq n} \frac{\Gamma\left(t^{n-i} t, t^{n-i} u_{0}^{2} ; p, q\right) \prod_{1 \leq r<s \leq 7} \Gamma\left(t^{n-i} t_{r} t_{s} ; p, q\right)}{\prod_{1 \leq r \leq 7} \Gamma\left(t^{n-i} u_{0} t_{r} ; p, q\right)} \\
& \sum_{\mu \subset(l, m)^{n}} \Delta_{\mu}\left(t^{n-1} u_{0}^{2} / p q \mid t^{n}, u_{0} / t_{1}, u_{0} / t_{2}, \ldots, u_{0} / t_{7} ; t ; p, q\right), \tag{9.53}
\end{align*}
$$

assuming $t_{1}=p^{l} q^{m} u_{0}$ and

$$
\begin{equation*}
t^{2 n-2} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}=p^{1-l} q^{1-m} \tag{9.54}
\end{equation*}
$$

Of course, other, less-symmetric, integral representations can be obtained from transformations of the left-hand side.

The formal group (adding in the dimension-changing transformation) now becomes the Weyl group $D_{7}$; it is, however, unclear what significance this has, since we cannot in general compose such transformations. Thus rather than obtaining the full Weyl group, we only obtain a union of two $D_{6} \backslash D_{7} / D_{6}$ double cosets (out of three). This gives rise to several new dimension altering transformations, some of which correspond to well-defined integrals. Thus for instance, we find that

$$
\begin{equation*}
\tilde{I}_{\lambda}^{(n)}\left(t_{0}, t_{1}: t_{2}, t_{3}, t_{4}, t_{5}, t_{6}, t_{7} ; t ; p, q\right)=\tilde{I}_{\lambda}^{(n+m)}\left(t_{0} / u, t_{1} u: t_{2} / u, t_{3} / u, t_{4} / u, t_{5} / u, t_{6} / u, t_{7} u ; t ; p, q\right) \tag{9.55}
\end{equation*}
$$

where

$$
\begin{equation*}
u^{2}=\sqrt{t_{0} t_{2} t_{3} t_{4} t_{5} t_{6} / t_{1} t_{7}}=\frac{p q t^{-n-1}}{t_{1} t_{7}}=\frac{t_{0} t_{2} t_{3} t_{4} t_{5} t_{6}}{p q t^{-n-1}} \tag{9.56}
\end{equation*}
$$

such that $u^{2}=t^{m}$ with $m \in \mathbb{Z}, n, n+m \geq \ell(\boldsymbol{\lambda})$. (In all, there are essentially 9 distinct dimension altering transformations, coming to the 12 legal $S_{6} \backslash D_{7} / S_{6}$ double cosets not in $D_{6}$ (modulo inverses))

If $\boldsymbol{\lambda}=0$, the group enlarges even further; in that case, the main group is the Weyl group $E_{7}$, while the "formal" group is the Weyl group $E_{8}$. Moreover, the action of $E_{8}$ comes from the usual root system, with roots of the form

$$
\begin{equation*}
\left( \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\right) \tag{9.57}
\end{equation*}
$$

(with an even number of - signs) and permutations of

$$
\begin{equation*}
(1,1,0,0,0,0,0,0),(1,-1,0,0,0,0,0,0) \tag{9.58}
\end{equation*}
$$

(Thus, for instance, Corollary 9.11 corresponds to the reflection in the root $\left(\frac{1}{2}, \frac{1}{2},-\frac{1}{2},-\frac{1}{2}, \frac{1}{2}, \frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$.) The subgroup $E_{7}$ is then the stabilizer of the root $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, corresponding to $\sqrt{t_{0} t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}}=$ $p q / t^{n+1}$. Since there are again four double cosets $S_{8} \backslash E_{7} / S_{8}$, we do not obtain any new forms of the main transformation (and similarly for the dimension altering transformation). The various subgroups considered
above are related to $E_{8}$ as follows:

$$
\begin{align*}
E_{7} & =\operatorname{Stab}_{E_{8}}\left(\sqrt{t_{0} t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}}\right)  \tag{9.59}\\
D_{7} & =\operatorname{Stab}_{E_{8}}\left(\sqrt{t_{1}^{3} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7} / t_{0}}\right)  \tag{9.60}\\
D_{6} & =\operatorname{Stab}_{E_{8}}\left(t_{1} / t_{0}, \sqrt{t_{0} t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}}\right)  \tag{9.61}\\
A_{1} D_{4} & =\operatorname{Stab}_{E_{8}}\left(\sqrt{t_{1}^{3} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7} / t_{0}}, \sqrt{t_{0} t_{1} t_{3}^{3} t_{4} t_{5} t_{6} t_{7} / t_{2}}, t_{1} t_{3}\right)  \tag{9.62}\\
D_{4} & =\operatorname{Stab}_{E_{8}}\left(t_{1} / t_{0}, t_{3} / t_{2}, t_{1} t_{3}, \sqrt{t_{0} t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}}\right) \tag{9.63}
\end{align*}
$$

If $t=q$, the integrand can be expressed as a product of two determinants, and is itself expressible as a determinant. This gives rise to a system of three-term quadratic recurrences, which turn out to be a form of Sakai's elliptic Painlevé equation [29] ; this generalizes the result of [13] for the univariate case (the authors of which also observed the existence of an $E_{7}$ symmetry in that case). This also generalizes results of [9] at the Selberg level (showing that certain Selberg-type integrals give solutions of the ordinary Painlevé equations). Also of interest are the cases $t=q^{2}, t=\sqrt{q}$, when the integral can be expressed as a pfaffian, and thus satisfies a system of four-term quadratic recurrences. See [20] for more details.

Finally, to obtain a reasonable degeneration of the integral in the limit $p \rightarrow 0$, we would need two "upper" parameters, of order $O(p)$, while the remaining parameters would have order $O(1)$; we would then use the fact that $\Gamma(p q / x ; p, q)=\Gamma(x ; p, q)^{-1}$ to move the upper parameters to the denominator. This property is in fact not invariant under $E_{8}$, or even under the above $E_{7}$; instead we obtain a different instance of $E_{7}$ (as the stabilizer of the root $(0,0,0,0,0,0,1,1)$, assuming the upper parameters are $t_{6}$ and $t_{7}$ ) from the $E_{8}$ action, while the $E_{7}$ action reduces to $E_{6}$. (The one-dimensional instance of the resulting integral identity is a trivial consequence of the hypergeometric series representation of Rahman [17]). If we further degenerate the integral to the multivariate Askey-Wilson case (a.k.a. the Koornwinder density), the symmetry group reduces to $D_{5}$, and the corresponding identity was proved in 21].

## 10 Appendix: Meromorphy of integrals

In the above work, we have made heavy use of the fact that the various contour integrals we consider are meromorphic functions of the parameters. This does not quite follow from the meromorphy of the integrands, as can be seen from the following two examples:

$$
\begin{align*}
& \int_{|z|=1} \frac{1}{1+t(z+1 / z)} \frac{d z}{2 \pi \sqrt{-1} z}=\left(1-4 t^{2}\right)^{-1 / 2},|t|<1 / 4  \tag{10.1}\\
& \int_{|z|=1} e^{1 /(z-2)} \frac{d z}{2 \pi \sqrt{-1}(z-t)}=e^{1 /(t-2)},|t|<1 \tag{10.2}
\end{align*}
$$

In both cases, the integrand is meromorphic in a neighborhood of the contour, but there are obstructions to meromorphically continuing the integral. (The second integrand, of course, has an essential singularity, but so do the integrands of interest to us.) It turns out, however, that these are typical of the only two such obstructions: an initial contour that separates branches of a component of the polar divisor of the integrand, or such a component that leaves the domain of meromorphy.

We will prove this fact in Theorem 10.2 below, but first need a lemma about meromorphy of residues. Note that with $g$ as described in the hypotheses of the lemma, the polar divisor of $g$ is a codimension 1 analytic subvariety of $D \times P$, and thus each component is either of the form $D \times \chi_{0}$, or can be viewed as a family of point sets in $D$ parametrized by $P$.

Lemma 10.1. Let $D$ be a nonempty open subset of $\mathbb{C} P^{1}$, and let $P$ be an irreducible normal holomorphic variety. Let $g$ be a meromorphic function on $D \times P$, and let $\chi$ be a component of the polar divisor of $g$ which is closed in $\bar{D} \times P$. Let $P^{\prime}$ be the subset of $P$ on which the fibers of $\chi$ are disjoint from the other polar divisors of $g$ (the complement of a codimension 1 subvariety), and define a function $f(p)$ on $P^{\prime}$ by

$$
\begin{equation*}
f(p)=\sum_{d:(d, p) \in \chi} \operatorname{Res}_{z=d} g(z, p) \tag{10.3}
\end{equation*}
$$

Then $f$ extends uniquely to a meromorphic function on all of $P$.
Proof. Note that on any compact subset of $P, \chi$ is bounded away from the complement of $D$, and thus its fibers lie in a compact subset of $D$, so are finite in number. In particular, the above sum is thus well-defined, and gives a holomorphic function on $P^{\prime}$.

Now, by Levi's theorem, we can freely remove any codimension 2 subvariety $W$ of $P$ without affecting the extension of $f$; in particular, we may assume that $P$ is regular (since its singular locus is codimension 2 by normality). We can then further restrict to a neighborhood of a general regular point, to reduce to the case $P \subset \mathbb{C}^{n}$ an open polydisc; we can also then write $g(d, p)=g_{1}(d, p) / g_{2}(d, p)$ for $g_{1}, g_{2}$ holomorphic. Let $Z \subset P$ be the locus for which $g_{2}(d, p)$ is identically 0 as a function of $d$; then by multiplying $g$ by a suitable function of $p$ alone, we can remove all codimension 1 components of $Z$, leaving a codimension 2 locus which can be removed by another application of Levi's theorem.

Now, consider a point $p_{0} \in P$. Reducing $P$ as necessary, we can assume that the fiber of $\chi$ over $p_{0}$ consists of a single point $d_{0}$; we can then reduce $D$ to assure that $g_{2}\left(d, p_{0}\right)$ also vanishes only at $d_{0}$. But then by the Weierstrass preparation theorem, $g_{2}(d, p)$ is the product of a monic polynomial in $d$ with holomorphic coefficients and a holomorphic function nowhere vanishing on $D$, which can be absorbed into $g_{1}$. Moreover, $g_{2}(d, p)$ factors as $h_{1}(d, p) h_{2}(d, p)$ where the monic polynomial $h_{1}(d, p)$ vanishes precisely along $\chi$, and the monic polynomial $h_{2}(d, p)$ is relatively prime to $h_{1}(d, p)$. We can thus write

$$
\begin{equation*}
g(d, p)=\frac{i_{1}(d, p)}{h_{1}(d, p)}+\frac{i_{2}(d, p)}{h_{2}(d, p)}+i_{0}(d, p) \tag{10.4}
\end{equation*}
$$

where $i_{0}$ is holomorphic in $d$, and $i_{1}, i_{2}$ are polynomials with meromorphic coefficients of degree less than $\operatorname{deg}\left(h_{1}\right), \operatorname{deg}\left(h_{2}\right)$ respectively. But the above sum of residues is then precisely the leading coefficient of $i_{1}(d, p)$, and is thus meromorphic in a neighborhood of $p_{0}$.

Remark. In fact, $i_{1}(d, p) \Delta(p)$ is holomorphic, where $\Delta(p)$ is the resultant of the polynomials $h_{1}(d, p)$ and $h_{2}(d, p)$.

Given a closed contour $C$ in $\mathbb{C} P^{1}$, every point not in $C$ of course has an associated winding number; we extend this by linearity to formal linear combinations of contours.

Theorem 10.2. Let $D$ be a nonempty connected open subset of $\mathbb{C} P^{1}$, let $C$ be a finite complex linear combination of contours in $D$, and let $P$ be an irreducible normal holomorphic variety. Let $g$ be a function meromorphic on $D \times P$, and suppose the function $f$ is defined on an open subset $U$ of $P$ by

$$
\begin{equation*}
f(p)=\int_{C} g(z, p) d z \tag{10.5}
\end{equation*}
$$

(Thus, in particular, we assume that the polar divisor of $g$ in $D \times U$ is disjoint from $C \times U$ ).
Suppose that each irreducible component $\chi$ of the polar divisor of $g$ is either of the form $D \times \chi_{0}$ or satisfies the assumptions:

1. For every point $u \in U$, every point $d \in D$ such that $(d, u) \in \chi$ has the same winding number with respect to $C$; call this the winding number of $\chi$.
2. If $(d, p)$ is a limit point of $\chi$ in $\bar{D} \times P$ outside $D \times P$, then the winding number of $d$ with respect to $C$ is the same as that of $\chi$ itself.

Then $f(p)$ extends uniquely to a meromorphic function on all of $P$.
Proof. Again, we may as well assume that $P$ is an open polydisc in $\mathbb{C}^{n}$ for some $n$. We may then assume that the polar divisor of $g$ contains no components of the form $D \times \chi_{0}$, since we can in that case simply multiply $g$ by a holomorphic function to remove that pole.

Now, let $U^{\prime}$ be an open subset of $P$, and consider a component $\chi$ of the polar divisor of $g$ on $D \times U^{\prime}$. This is contained in a unique component of the full polar divisor, with winding number $w_{0}$, say; on the other hand, if $U^{\prime}$ is not contained in $U, \chi$ can easily intersect $C$ or have well-defined winding number different from the "true" winding number $w_{0}$, in which case we call it "problematical". We claim that every point $p \in P$ has a neighborhood with only finitely many problematical polar components. Indeed, by condition (2) above, we can choose a bounded neighborhood $U^{\prime}$ of $p$ such that the problematical components of $g$ on $D \times U^{\prime}$ are bounded away from the complement of $D$, and are thus contained in $D^{\prime} \times U^{\prime}$ for some compact subset of $D^{\prime}$, in which $g$ can support only finitely many poles.

If $p$ is such that we can choose $U^{\prime}$ so that the problematical components are disjoint from all components with different "true" winding number (which will hold for $p$ away from a codimension 1 subvariety), then we can obtain a new contour $C^{\prime}$ by deforming $C$ and adding small circles in such a way that the integralp

$$
\begin{equation*}
\int_{C}^{\prime} g(z, p) d z \tag{10.6}
\end{equation*}
$$

is well-defined on $U^{\prime}$, and such that on the intersection of two such components, the functions agree. Indeed, we can clearly deform $C$ in such a way that the problematical components have well-defined winding numbers w.r.to the new contour; by adding small circles around the problematical components (shrinking $U^{\prime}$ as necessary to allow these circles to be fixed) we can make these winding numbers equal to the "true" winding numbers. Any two such contours will give the same integral, by Cauchy's theorem, and thus these functions agree on intersections.

At a general point, we can still deform $C$ to give well-defined winding numbers to the problematical components, but now have the difficulty that they might intersect components with different winding numbers. Here, we can observe that the above analytic continuation can be written as

$$
\begin{equation*}
\int_{C}^{\prime} g(z, p) d z+\text { finite sum of residues } \tag{10.7}
\end{equation*}
$$

where instead of adding a small circle around the problematical components, we simply add the corresponding residue. The first term is certainly meromorphic (in fact, holomorphic near $p$ ); that the residue terms are meromorphic (and thus that the theorem holds) results from the lemma.

We need only the following special case. Here

$$
\begin{equation*}
(x ; p, q)_{\infty}:=\prod_{0 \leq i, j}\left(1-p^{i} q^{j} x\right) . \tag{10.8}
\end{equation*}
$$

Corollary 10.3. Let $F\left(z ; t_{0}, \ldots, t_{m-1} ; u_{0}, \ldots, u_{n-1} ; p, q\right)$ be a function holomorphic on the domain

$$
\begin{equation*}
z, t_{0}, \ldots, t_{m-1}, u_{0}, \ldots, u_{n-1} \in \mathbb{C}^{*}, 0<|p|,|q|<1 \tag{10.9}
\end{equation*}
$$

Then the function defined for $\left|t_{r}\right|,\left|u_{r}\right|<1$ by

$$
\begin{equation*}
G\left(t_{0}, \ldots, t_{m-1} ; u_{0}, \ldots, u_{n-1} ; p, q\right)=\prod_{0 \leq r<m, 0 \leq s<n}\left(t_{r} u_{s} ; p, q\right)_{\infty} \int_{|z|=1} \Delta\left(z ; t_{0}, \ldots, t_{m-1} ; p, q\right) \frac{d z}{2 \pi \sqrt{-1} z} \tag{10.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta\left(z ; t_{0}, \ldots, t_{m-1} ; p, q\right)=\frac{F\left(z ; t_{0}, \ldots, t_{m-1} ; p, q\right)}{\prod_{0 \leq r<m}\left(t_{r} z ; p, q\right)_{\infty} \prod_{0 \leq r<n}\left(u_{r} / z ; p, q\right)_{\infty}} \tag{10.11}
\end{equation*}
$$

extends uniquely to a holomorphic function on the domain

$$
\begin{equation*}
t_{0}, \ldots, t_{m-1}, u_{0}, \ldots, u_{n-1} \in \mathbb{C}^{*}, 0<|p|,|q|<1 \tag{10.12}
\end{equation*}
$$

Away from the divisor of $\prod_{0 \leq r<m, 0 \leq s<n}\left(t_{r} u_{s} ; p, q\right)_{\infty}$, this extension can be obtained by replacing the unit circle by any (possibly disconnected) contour that contains the points $p^{i} q^{j} u_{r}$ and excludes the points $1 / p^{i} q^{j} t_{r}$, for $0 \leq i, j$.

In particular, our multidimensional integrals can all be expressed as iterated contour integrals of this form (in general restricted to a subvariety of parameter space), so are meromorphic by straightforward induction. This does, however, tend to grossly overestimate the polar divisor. This overestimation can easily occur even in the one-dimensional case, in the presence of symmetry.

In the case of the $B C_{1}$ integral, one role of the balancing condition, as we have seen, is to make the summation limits factor into $p$-abelian and $q$-abelian factors, which occurs because the density satisfies the relation

$$
\begin{equation*}
\Delta\left(p^{i} q^{j} z\right) \Delta(z)=\Delta\left(p^{i} z\right) \Delta\left(q^{j} z\right) \tag{10.13}
\end{equation*}
$$

for $i, j \in \mathbb{Z}$. As observed by Spiridonov (personal communication), this only determines the balancing condition up to a sign. However, one special case of this relation is the identity

$$
\begin{equation*}
\Delta\left( \pm p^{i / 2} q^{j / 2}\right) \Delta\left( \pm p^{-i / 2} q^{-j / 2}\right)=\Delta\left( \pm p^{i / 2} q^{-j / 2}\right) \Delta\left( \pm p^{-i / 2} q^{j / 2}\right) \tag{10.14}
\end{equation*}
$$

assuming both sides are defined; using the fact that $\Delta(z)=\Delta(1 / z)$, we conclude that

$$
\begin{equation*}
\Delta\left( \pm p^{i / 2} q^{j / 2}\right)^{2}=\Delta\left( \pm p^{i / 2} q^{-j / 2}\right)^{2} \tag{10.15}
\end{equation*}
$$

so that

$$
\begin{equation*}
\Delta\left(p^{i / 2} q^{j / 2}\right)= \pm \Delta\left(p^{i / 2} q^{-j / 2}\right), \quad \Delta\left(-p^{i / 2} q^{j / 2}\right)= \pm \Delta\left(-p^{i / 2} q^{-j / 2}\right) \tag{10.16}
\end{equation*}
$$

The balancing condition for the $B C_{1}$ integral then has the effect of choosing the sign in this identity:

$$
\begin{equation*}
\Delta\left( \pm p^{i / 2} q^{j / 2}\right)=-\Delta\left( \pm p^{i / 2} q^{-j / 2}\right) \tag{10.17}
\end{equation*}
$$

This motivates the hypotheses for the following result.
Lemma 10.4. Let $\Delta(z ; \mathbf{p})$ be a $B C_{1}$-symmetric function on $\mathbb{C}^{*} \times P$, with $P$ an irreducible normal subvariety of $\left\{t_{0}, t_{1}, \ldots, t_{d-1}, p, q \in \mathbb{C}^{*}:|p|,|q|<1\right\}$. Suppose furthermore that the following conditions are satisfied.

1. The function

$$
\begin{equation*}
\prod_{0 \leq r<d}\left(t_{r} z^{ \pm 1} ; p, q\right) \Delta(z ; \mathbf{p}) \tag{10.18}
\end{equation*}
$$

is holomorphic.
2. At a generic point of $P$, the denominator has only simple zeros; it has triple zeros only in codimension 2.
3. For any integers $i, j$,

$$
\begin{equation*}
\Delta\left( \pm p^{i / 2} q^{j / 2} ; \mathbf{p}\right)=-\Delta\left( \pm p^{i / 2} q^{-j / 2} ; \mathbf{p}\right) \tag{10.19}
\end{equation*}
$$

as an identity of meromorphic functions on $P$.
Then the function on $P$ defined for $\left|t_{r}\right|<1$ by

$$
\begin{equation*}
\prod_{0 \leq r<s<d}\left(t_{r} t_{s} ; p, q\right)_{\infty} \int_{|z|=1} \Delta(z) \frac{d z}{2 \pi \sqrt{-1} z} \tag{10.20}
\end{equation*}
$$

extends to a holomorphic function on $P$.
Proof. The integral extends meromorphically to this domain by Corollary 10.3 that it has at most simple poles along the subvarieties $t_{r} t_{s}=p^{-i} q^{-j}, i, j \geq 0$ follows immediately from the fact that at a generic point of such a subvariety, there are no higher-order collisions of poles. However, these considerations still leave open the possibility that the given function might have poles along the subvarieties $t_{r}^{2}=p^{-i} q^{-j}, i, j \geq 0$.

We thus need, without loss of generality, to show that the above function is holomorphic at a generic point $\mathbf{p}_{0}$ such that $t_{0}= \pm p^{-l / 2} q^{-m / 2}, i, j \geq 0$. Now, in a neighborhood of such a point, the analytic continuation is given by

$$
\begin{equation*}
\int_{C} \Delta(z) \frac{d z}{2 \pi \sqrt{-1} z} \tag{10.21}
\end{equation*}
$$

where $C=C^{-1}$ is a contour containing $p^{i} q^{j} t_{r}$ for $i, j \geq 0,0 \leq r<d$. Now, let $C^{\prime}$ be a modified symmetric contour that still contains $p^{i} q^{j} t_{r}$ for $r>0$ and $p^{i} q^{j} t_{0}$ for $i \geq l$ or $j \geq m$, but excludes $p^{i} q^{j} t_{0}$ for $0 \leq i \leq l$, $0 \leq j \leq m$. Then we claim that

$$
\begin{equation*}
\int_{C} \Delta(z) \frac{d z}{2 \pi \sqrt{-1} z}+\int_{C^{\prime}} \Delta(z) \frac{d z}{2 \pi \sqrt{-1} z} \tag{10.22}
\end{equation*}
$$

is holomorphic on a neighborhood of $\mathbf{p}_{0}$. Indeed, anywhere that two poles coalesce, the poles have the same overall winding number with respect to the two contours. Thus to show the first term is holomorphic, it suffices to prove that the difference of the two terms is holomorphic. But this is just a sum of residues; it is therefore sufficient to prove that

$$
\begin{equation*}
\sum_{0 \leq i \leq l} \sum_{0 \leq j \leq m} \operatorname{Res}_{z=t_{0} p^{i} q^{j}} \Delta(z ; \mathbf{p}), \tag{10.23}
\end{equation*}
$$

is holomorphic near $\mathbf{p}_{0}$, or in other words that

$$
\begin{equation*}
\lim _{\mathbf{p} \rightarrow \mathbf{p}_{0}} \sum_{0 \leq i \leq l} \sum_{0 \leq j \leq m} \operatorname{Res}_{z=t_{0} p^{i} q^{j}} \Delta(z ; \mathbf{p}) \tag{10.24}
\end{equation*}
$$

is well-defined. We claim in fact that

$$
\begin{equation*}
\lim _{\mathbf{p} \rightarrow \mathbf{p}_{0}}\left(1- \pm p^{l / 2} q^{m / 2} t_{0}\right)\left[\operatorname{Res}_{z=t_{0} p^{i} q^{j}} \Delta(z ; \mathbf{p})+\operatorname{Res}_{z=t_{0} p^{i} q^{m-j}} \Delta(z ; \mathbf{p})\right]=0 \tag{10.25}
\end{equation*}
$$

which then makes the poles of the summands cancel pairwise, giving the desired result. Now,

$$
\begin{equation*}
\lim _{\mathbf{p} \rightarrow \mathbf{p}_{0}}\left(1- \pm p^{l / 2} q^{m / 2} t_{0}\right) \operatorname{Res}_{z=t_{0} p^{i} q^{j}} \Delta(z ; \mathbf{p}) \propto \lim _{\mathbf{p} \rightarrow \mathbf{p}_{0}} \lim _{z \rightarrow t_{0} p^{i} q^{j}}\left(1- \pm p^{l / 2} q^{m / 2} t_{0}\right)\left(1-t_{0} p^{i} q^{j} / z\right) \Delta(z ; \mathbf{p}) \tag{10.26}
\end{equation*}
$$

and this limit is well-defined, again because at most two poles coalesce at any given point. Now, if we pull out the denominator factors $\left(t_{0} z^{ \pm 1} ; p, q\right)_{\infty}$ of $\Delta(z ; \mathbf{p})$, we can explicitly compute their contributions to the limit, and use the fact that limits of holomorphic functions can be exchanged to conclude that

$$
\begin{equation*}
\lim _{\mathbf{p} \rightarrow \mathbf{p}_{0}} \lim _{z \rightarrow t_{0} p^{i} q^{j}}\left(1- \pm p^{l / 2} q^{m / 2} t_{0}\right)\left(1-t_{0} p^{i} q^{j} / z\right) \Delta(z ; \mathbf{p})=\frac{1}{2} \lim _{\mathbf{p} \rightarrow \mathbf{p}_{0}}\left(1- \pm p^{l / 2} q^{m / 2} t_{0}\right)^{2} \Delta\left( \pm p^{i-l / 2} q^{j-m / 2} ; \mathbf{p}\right) . \tag{10.27}
\end{equation*}
$$

The claim follows.
Similarly, for higher dimensional integrals, the simple inductive argument leads to predictions of extremely high order poles along the divisors $t_{r} t_{s}=p^{-l} q^{-m}, l, m \geq 0$. That this does not occur for our integrals follows via a similar argument from the fact that

$$
\begin{equation*}
\Delta\left(p^{a} q^{b} z_{0}, p^{c} q^{d} z_{0}, z_{3}, \ldots, z_{n} ; \mathbf{p}\right)=-\Delta\left(p^{a} q^{d} z_{0}, p^{c} q^{b} z_{0}, z_{3}, \ldots, z_{n} ; \mathbf{p}\right) \tag{10.28}
\end{equation*}
$$

for our integrands; the consequence is that when moving the contour over a given collection of poles of the form $p^{i} q^{j} t_{0}, 0 \leq i \leq l, 0 \leq j \leq m$, the residues of residues that arise all cancel pairwise. Somewhat more generally, we have the following.

Lemma 10.5. Let $\Delta\left(z_{1}, \ldots, z_{n} ; \mathbf{p}\right)$ be a symmetric meromorphic function on $\left(\mathbb{C}^{*}\right)^{n} \times P$, where $P$ is an irreducible normal subvariety of the domain $\left\{t_{0}, t_{1}, \ldots, t_{d-1}, u_{0}, \ldots, u_{d-1}, p, q \in \mathbb{C}^{*}:|p|,|q|<1\right\}$. Suppose furthermore that the following conditions are satisfied.

1. The function

$$
\begin{equation*}
\prod_{1 \leq i \leq n} \prod_{0 \leq r<d}\left(t_{r} z_{i}, u_{r} / z_{i} ; p, q\right) \Delta\left(z_{1}, \ldots, z_{n} ; \mathbf{p}\right) \tag{10.29}
\end{equation*}
$$

is holomorphic.
2. At a generic point of $P$, the denominator has only simple zeros.
3. For any integers $a, b, c, d \geq 0$,

$$
\begin{equation*}
\Delta\left(p^{a} q^{b} z, p^{c} q^{d} z, z_{3}, \ldots, z_{n} ; \mathbf{p}\right)=-\Delta\left(p^{a} q^{d} z, p^{c} q^{b} z, z_{3}, \ldots, z_{n} ; \mathbf{p}\right) \tag{10.30}
\end{equation*}
$$

as an identity of meromorphic functions on $\left(\mathbb{C}^{*}\right)^{n-1} \times P$.
For generic $\mathbf{p} \in P$, choose a contour $C_{\mathbf{p}}$ containing all points of the form $u_{r} p^{i} q^{j}$ with $i, j \geq 0,0 \leq r<d$, and excluding all points of the form $\left(t_{r} p^{i} q^{j}\right)^{-1}$ with $i, j \geq 0,0 \leq r<d$. Let $C_{\mathbf{p}}^{\prime}$ be a similar contour that differs from $C_{\mathbf{p}}$ by excluding the points $u_{0} p^{i} q^{j}$ with $0 \leq i \leq l, 0 \leq j \leq m$. Then

$$
\begin{align*}
\int_{C_{\mathbf{p}}^{n}} \Delta(z ; \mathbf{p}) \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} & -\int_{C_{\mathbf{p}}^{\prime n}} \Delta(z ; \mathbf{p}) \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} \\
& =n \int_{C_{\mathbf{p}}^{\prime n-1}} \sum_{0 \leq(a, b) \leq(l, m)} \lim _{z_{n} \rightarrow p^{a} q^{b} t_{0}}\left(1-p^{a} q^{b} t_{0} / z_{n}\right) \Delta(z ; \mathbf{p}) \prod_{1 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} \tag{10.31}
\end{align*}
$$

Proof. A straightforward induction using the symmetry of the integrand tells us that

$$
\begin{align*}
\int_{C_{\mathbf{p}}^{n}} \Delta(z ; \mathbf{p}) \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} & -\int_{C_{\mathbf{p}}^{\prime n}} \Delta(z ; \mathbf{p}) \prod_{1 \leq i \leq n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} \\
& =\sum_{1 \leq j \leq n} \int_{C_{\mathbf{p}}^{j-1} \times C_{\mathbf{p}}^{\prime n-j}} \sum_{0 \leq(a, b) \leq(l, m)} \lim _{z_{n} \rightarrow p^{a} q^{b} t_{0}}\left(1-p^{a} q^{b} t_{0} / z_{n}\right) \Delta(z ; \mathbf{p}) \prod_{1 \leq i<n} \frac{d z_{i}}{2 \pi \sqrt{-1} z_{i}} \tag{10.32}
\end{align*}
$$

indeed, the sum is simply the contributions from residues as we deform the contours in $z_{n}, z_{n-1}, \ldots, z_{1}$. It thus suffices to show that these $n$ terms all agree. But the difference between the $j$ th term and the $j+1$ st term is an $n$-2-dimensional integral of a sum of double residues:

$$
\begin{equation*}
\sum_{0 \leq(a, b),(c, d) \leq(l, m)} \lim _{z_{n} \rightarrow p^{a} q^{b} t_{0}} \lim _{z_{n-1} \rightarrow p^{c} q^{d} t_{0}}\left(1-p^{a} q^{b} t_{0} / z_{n}\right)\left(1-p^{a} q^{d} t_{0} / z_{n-1}\right) \Delta\left(z_{1}, \ldots, z_{n} ; \mathbf{p}\right) \tag{10.33}
\end{equation*}
$$

But again we can pull out the known pole factors and interchange limits of the resulting holomorphic function; we conclude that the $(a, b),(c, d)$ and $(a, d),(c, b)$ terms cancel.

Applying this to the type I integral gives the following result; similar results apply to the integral operators.
Theorem 10.6. The function

$$
\begin{equation*}
\prod_{0 \leq r<s \leq 2 m+2 n+3}\left(t_{r} t_{s} ; p, q\right)_{\infty} I_{B C_{n}}^{(m)}\left(t_{0}, \ldots, t_{2 m+2 n+3} ; p, q\right) \tag{10.34}
\end{equation*}
$$

extends to a holomorphic function on the domain $\prod_{r} t_{r}=(p q)^{m},|p|,|q|<1$.
Proof. Indeed, the integrand satisfies the hypotheses of the two lemmas; the second lemma readily shows that the integral has a simple pole along each subvariety $t_{r} t_{s} p^{a} q^{b}=1$ (with residue equal to a sum of $n$-1-dimensional integrals), while an induction using the first lemma shows that the potential singularites for $t_{r}^{2} p^{a} q^{b}=1$ are not present.

Remark. The situation for the $A_{n}$ integral is much more complicated, as we must integrate against a test function in $Z$ to allow the use of a product contour (which is legal since an inductive argument shows the $A_{n}$ integral to be meromorphic); but this extra integration, while preserving meromorphy, can easily remove singularities.

For the type II integral, a similar argument applies to contour deformations; the additional poles coming from the cross terms are sufficiently generic that the multidimensional lemma still holds. There is an important difference in that we have the additional constraint that the contour $C=C^{-1}$ should contain the contours $t p^{i} q^{j} C, i, j \geq 0$. Thus when deforming through the collection of points $t^{i} q^{j} p^{k} t_{0}, 0 \leq(i, j, k) \leq(a, b, c)$, it is necessary to first deform through the points with $i=0$, then those with $i=1$, and so forth; otherwise the contour constraint will be broken. With that caveat, however, the results still apply, and we obtain the following result.

Theorem 10.7. Let $I I_{n}^{(m)}\left(t_{0}, t_{1}, \ldots, t_{2 m+3} ; t ; p, q\right)$ be the $2 m+4$-parameter analogue of the type II integral, $m>0$. The function

$$
\begin{equation*}
\prod_{0 \leq i<n} \prod_{0 \leq r<s<2 m}\left(t^{i} t_{r} t_{s} ; p, q\right)_{\infty} I I_{n}^{(m)}\left(t_{0}, \ldots, t_{2 m+3} ; t ; p, q\right) \tag{10.35}
\end{equation*}
$$

extends to a holomorphic function on the domain $t^{2 n-2} \prod_{r} t_{r}=(p q)^{m},|t|,|p|,|q|<1$.
Proof. At a generic point with $t^{i} p^{j} q^{k} t_{r} t_{s}=1, r<s$, we can simply deform the contour through the points $t^{a} p^{b} q^{c} t_{r}, 0 \leq a \leq i 0 \leq b \leq j, 0 \leq c \leq k$ to obtain a holomorphic integral. We thus find that the desired integral is a sum of integrals over the new contour, with integrands given by multiple residues at a sequence of points with $a=0, a=1, \ldots$ The only such integrals that are singular at $t^{i} p^{j} q^{k} t_{r} t_{s}=1$ are those involving $i+1$-tuple residues, and those have simple poles. Since we can obtain at most $n$-tuple residues from an $n$-tuple integral, we conclude that we have at most simple poles, and those only when $i<n$.

For $r=s$, if we first deform through the points $t^{a} p^{b} q^{c} t_{r}$ with $0 \leq b \leq j, 0 \leq c \leq k, 0 \leq a<i / 2$, we find that the only integrals with possible singularities are those of $i / 2$-tuple residues; these are then generically holomorphic when $t^{i} p^{j} q^{k} t_{r}^{2}=1$ by induction.

Remark. A similar result applies to $I I_{\lambda \mu}^{(n)}$, with the caveat that the interpolation functions may have poles independent of $z_{1}, \ldots z_{n}$; these poles would then in general survive as poles of the integral.

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