On singularity formation in three-dimensional vortex sheet evolution

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It is shown that if a doubly-infinite vortex sheet has cylindrical shape and strength distributions at some initial time, then this property is retained in its subsequent evolution. It is also shown that in planes normal to the generator of the cylindrical sheet geometry, the nonlinear evolution of the sheet is the same as that of an equivalent strictly two-dimensional sheet motion. These properties are used to show that when an initially flat vortex sheet is subject to a finite-amplitude, three-dimensional normal mode perturbation, weak singularities develop along lines which are oblique to the undisturbed velocity jump vector at a time that can be inferred from an extension of Moore's [Proc. R. Soc. A **365**, 105 (1979)] result for two-dimensional motion. © *1999 American Institute of Physics.* [S1070-6631(99)01010-7]

The nonlinear evolution of a doubly-infinite, uniform vortex sheet subject to small out-of-plane disturbances is a classic problem in fluid mechanics. It has long been known that the linearized Kelvin–Helmholtz instability of the vortex sheet subject to two-dimensional shape disturbances results in an ill-posed initial value problem[1,2]—see Saffman[3] for a discussion. Moore[4,5] studied the nonlinear problem by constructing small-disturbance solutions of the exact Birkhoff–Rott equation describing vortex sheet evolution in two space dimensions. Using approximate asymptotic methods, Moore demonstrated the possible formation of a finite-time singularity in the form of an infinite sheet curvature and a cusp in the sheet strength (local velocity jump). Subsequent numerical studies[6–9] of vortex sheet evolution up to the time at which the singularity forms have strongly supported this prediction. These results have practical relevance to the pointwise convergence of numerical solutions to the compressible Euler equations, where slip surfaces (vortex sheets) are produced at triple points.[10]

There is some evidence for the formation of singularities in the evolution of vortex sheets in axisymmetric geometry[11,12] and for certain three-dimensional perturbations to a plane vortex sheet.[13] In the present note we consider the vortex sheet evolution when subject to a disturbance which is a finite-amplitude, three-dimensional (3D) normal mode of the linear stability problem. The 3D normal mode is cylindrical, in a defined sense. It is shown that the full nonlinear sheet motion then remains cylindrical and that it is identical to a suitably chosen, strictly two-dimensional (2D) vortex sheet evolution. It follows that the singularity formation process is the same as for a 2D normal mode disturbance, allowing a straightforward extension of Moore's result to the case of an initial disturbance consisting of a finite amplitude, 3D normal mode.

Consider the motion of a doubly-infinite vortex sheet in three dimensions. In Cartesian coordinates (x,y,z) the undisturbed sheet shape lies in the $(x-y)$ plane with uniform x-velocities of $-\frac{1}{2}U$ for $z>0$ and $\frac{1}{2}U$ for $z<0$, where U is the velocity jump magnitude across $z=0$ in the x-direction. For a general sheet shape we employ a parametric surface description

$$\mathbf{X}(\xi,\eta,t)=\begin{pmatrix} X(\xi,\eta,t)\\ Y(\xi,\eta,t)\\ Z(\xi,\eta,t)\end{pmatrix},\quad \Gamma=\Gamma(\xi,\eta), \tag{1}$$

where (ξ,η) are Lagrangian parameters defining the sheet position $\mathbf{X}$, and the velocity potential jump Γ. Unless otherwise specified the dependence on time t will be suppressed. The vortex lines on the sheet are given by

$$\boldsymbol{\omega}=\mathbf{W}\Bigg/\left|\frac{\partial\mathbf{X}}{\partial\xi}\times\frac{\partial\mathbf{X}}{\partial\eta}\right|,\quad \mathbf{W}=\frac{\partial\Gamma}{\partial\xi}\frac{\partial\mathbf{X}}{\partial\eta}-\frac{\partial\Gamma}{\partial\eta}\frac{\partial\mathbf{X}}{\partial\xi}. \tag{2}$$

At a field point $\mathbf{x}$, the velocity induced by the sheet is

$$\mathbf{u}(\mathbf{x})=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathbf{W}\times(\mathbf{x}-\mathbf{X})}{|\mathbf{x}-\mathbf{X}|^3}\,d\xi\,d\eta. \tag{3}$$

When $\mathbf{x}\rightarrow\mathbf{X}$ on the sheet, the left hand side of Eq. (3) can be replaced by $\partial\mathbf{X}/\partial t$ resulting in a Lagrangian equation[14] for the sheet evolution from given initial conditions $\mathbf{X}(\xi,\eta,t=0)$, $\Gamma(\xi,\eta)$. The integral must then be interpreted as a Hadamard principal-part integral; thus sheet velocities are the average of the sheet-induced velocity when $\mathbf{x}\rightarrow\mathbf{X}$ from either side. In the sheet evolution $\Gamma(\xi,\eta)$ is invariant with time.

We define a strictly 2D vortex sheet evolution by

$$\mathbf{X}(\xi,\eta,t)=\begin{pmatrix} g_1(\xi,t)\\ \eta\\ g_2(\xi,t)\end{pmatrix},\quad \Gamma=g_3(\xi),\quad \mathbf{W}=\frac{\partial g_3}{\partial\xi}\begin{pmatrix}0\\1\\0\end{pmatrix}. \tag{4}$$

The field velocity is then

$$\mathbf{u}(\mathbf{x})_{2\mathrm{D}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\partial g_3}{\partial\xi}\right)\frac{1}{(x-g_1)^2+(z-g_2)^2}\begin{pmatrix} z-g_2\\ 0\\ -x+g_1\end{pmatrix}d\xi. \tag{5}$$

It follows from Eqs. (4) to (5) that the vortex sheet shape is invariant in the y-direction and that vortex lines remain parallel to the y-axis. When $\mathbf{x}\rightarrow\mathbf{X}$ on the sheet and the left-side of Eq. (5) is replaced by $\partial \mathbf{X}/\partial t$, a version of the Birkhoff–Rott equation for 2D vortex sheet evolution results.

Moore[4] showed that the 2D evolution of a vortex sheet from initial conditions given, in our notation by

$$g_1(\xi)=\xi, \quad g_2(\xi)=\epsilon \sin k\xi, \quad g_3(\xi)=-U\xi, \tag{6}$$

where $k=2\pi/\lambda$, λ is the wavelength and ϵ is the initial disturbance height, results in a possible curvature singularity in the sheet shape at a time t_c given by

$$1+\pi\tilde{t}_c+\ln(2\pi\tilde{t}_c)=\ln(2/\pi\tilde{\epsilon}), \tag{7}$$

where $\tilde{\epsilon}=\epsilon/\lambda$ and $\tilde{t}_c=t_c U/\lambda$. This result requires that $\tilde{\epsilon} \ll 1$. The mechanism of singularity formation is the focusing of vorticity onto the singularity point by the nonlinear sheet dynamics for $t>0$. This choice of initial disturbance does not correspond to a normal mode of the linearized Birkhoff–Rott evolution equation. However, a modification of Moore's result for a normal mode disturbance was given by Krasny,[6] who showed that the initial sheet disturbance of wavelength λ given by

$$g_1(\xi)=\xi-\epsilon\sin(k\xi), \quad g_2(\xi)=\epsilon\sin(k\xi), \\ g_3(\xi)=-U\xi, \tag{8}$$

produced Moore's curvature singularity at a time t_c given by

$$1+\pi\tilde{t}_c+\ln(\pi\tilde{t}_c)=\ln(1/2\pi\tilde{\epsilon}). \tag{9}$$

We now introduce and define a "cylindrical" sheet. We work in Cartesian coordinates (x',y',z') and use Lagrangian sheet coordinates (ξ',η'). For present purposes, a vortex sheet is said to be cylindrical at some time instant if the shape geometry and vorticity distribution are of the form

$$\mathbf{X}'(\xi',\eta')=\begin{pmatrix} f_1(\xi') \\ \eta' \\ f_2(\xi') \end{pmatrix}, \quad \Gamma=f_3(\xi')+c_1\eta', \\ \mathbf{W}=\frac{\partial f_3}{\partial \xi'}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}-c_1\begin{pmatrix} \partial f_1/\partial\xi' \\ 0 \\ \partial f_2/\partial \xi' \end{pmatrix}, \tag{10}$$

where c_1 is a constant equal to the gradient of Γ in the y' direction. The functions f_1 and f_2 are such that $f_1(\xi)=\xi+\hat{f}_1(\xi)$, with $|\hat{f}_1(\xi)/\xi|\to 0$ as $\xi\to\infty$, and $|f_2(\xi)|$ is bounded. A line parallel to the y'-axis is the generator of the cylindrical surface. Substituting Eq. (10) into Eq. (3) gives the velocity at a general field point $\mathbf{x}'$ as the sum of two components

$$\mathbf{u}(\mathbf{x}')=\mathbf{u}(\mathbf{x}')_{2D}+\mathbf{u}(\mathbf{x}')_{y'}, \tag{11}$$

where

$$\mathbf{u}(\mathbf{x}')_{2D}=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial f_3}{\partial \xi'}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\times(\mathbf{x}'-\mathbf{X}')\frac{d\xi'\,d\eta'}{|\mathbf{x}'-\mathbf{X}'|^3}, \tag{12}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\partial f_3}{\partial \xi'}\right)\frac{1}{(x'-f_1)^2+(z'-f_2)^2} \\ \times\begin{pmatrix} z'-f_2 \\ 0 \\ -x'+f_1 \end{pmatrix}d\xi', \tag{13}$$

and

$$\mathbf{u}(\mathbf{x}')_{y'}=-\frac{c_1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} \partial f_1/\partial\xi' \\ 0 \\ \partial f_2/\partial\xi' \end{pmatrix}\times(\mathbf{x}'-\mathbf{X}')\frac{d\xi'\,d\eta'}{|\mathbf{x}'-\mathbf{X}'|^3}, \tag{14}$$

$$=\frac{c_1}{2\pi}\int_{-\infty}^{\infty}\frac{(\partial f_1/\partial\xi')(z'-f_2)-(\partial f_2/\partial\xi')(x'-f_1)}{(x'-f_1)^2+(z'-f_2)^2} \\ \times\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}d\xi'. \tag{15}$$

This last integral can be evaluated analytically to give

$$\mathbf{u}(\mathbf{x}')_{y'}=\begin{pmatrix} 0 \\ \pm\frac{1}{2}c_1 \\ 0 \end{pmatrix}, \tag{16}$$

where the plus sign holds for a field point which lies above the sheet and the minus sign applies if the field point lies below the sheet. Thus the sheet-induced velocity in the y'-direction is constant and nonzero at all points above and below the sheet, when $c_1\neq 0$, irrespective of the sheet shape in the $(x'-z')$ plane. Moreover, comparison of Eqs. (13) and (5) shows that the components of the induced velocity in the $(x'-z')$ plane are identical to an equivalent 2D sheet, in the sense described above. When $\mathbf{x}'\to\mathbf{X}'$ on the sheet, the component of the velocity in the y'-direction, taken as the average of the constant values from above and below, is zero at all points on the sheet. It therefore follows that:

(i) a vortex sheet which is cylindrical at some initial time in the sense of Eq. (10) remains cylindrical for all time, and

(ii) the nonlinear evolution of a cylindrical sheet is identical to that of an equivalent $(c_1=0)$ 2D vortex sheet with the same initial conditions.

The third of equations (10) shows that when $c_1\neq 0$ there are spatially nonuniform vorticity components on the sheet in both the x' and the y' directions. This vorticity is passive and has no dynamical effect on the sheet evolution. These results are reminiscent of Stuart's procedure[15] for constructing 3D solutions of the Euler equations from a given 2D solution.

We now obtain the shape and vorticity distributions for a 3D normal mode disturbance to the vortex sheet. The spatial parts of the normal modes obtained from a linear analysis in terms of perturbation potentials above and below the flat-sheet with uniform velocity jump U in the x direction, are

$$h(x,y)=A\exp[i(mx+ny)], \tag{17}$$

$$\phi_{\pm}(x,y)=B_{\pm}\exp(\mp kz)\exp[i(mx+ny)]\mp\tfrac{1}{2}Ux, \tag{18}$$

where h is the disturbance in the z direction, $\phi_\pm$ are velocity potentials above and below the sheet, m,n are wave numbers in the x,y directions respectively and

$$k^2=m^2+n^2, \quad B_\pm=\tfrac{1}{2}UA\,(m/k)\,(i\mp 1). \tag{19}$$

At an initial time $t=0$ this corresponds to a potential jump function

$$\Gamma\equiv\phi_+-\phi_-=-Ux-UA\,(m/k)\exp[i(mx+ny)]. \tag{20}$$

We refer to Eqs. (17) and (20) as normal modes of the 3D vortex sheet.

We now rotate from (x,y,z) axes to (x',y',z') axes using

$$kx'=mx+ny, \quad ky'=-nx+my, \quad z'=z. \tag{21}$$

In the (x',y',z') axes, the sheet shape and potential jump distribution corresponding to the real parts of the linear normal mode Eqs. (17)–(20) can then be parameterized as

$$\mathbf{X}'(\xi',\eta')=\begin{pmatrix}\xi'\\ \eta'\\ \epsilon\cos(k\xi')\end{pmatrix}, \qquad \Gamma=-U\,(m/k)\,[\xi'+\epsilon\cos(k\xi')]+U(n/k)\,\eta', \tag{22}$$

where $\epsilon=A$. A re-parameterization $\tilde{\xi}'=\xi'+\epsilon\cos(k\xi')$, $\tilde{\eta}'=\eta'$ applied to Eq. (22) accompanied by a phase shift, and expansion of the result to $O(\epsilon)$ gives

$$\mathbf{X}'(\xi',\eta')=\begin{pmatrix}\xi'-\epsilon\sin(k\xi')\\ \eta'\\ \epsilon\sin(k\xi')\end{pmatrix}, \qquad \Gamma=-(m/k)U[\xi'-(n/m)\,\eta'], \tag{23}$$

where following the re-parameterization we have put $\tilde{\xi}'\rightarrow\xi'$, $\tilde{\eta}'\rightarrow\eta'$. When $n=0$ Eq. (23) agrees with Eq. (8). When $n\neq 0$ Eq. (23) is the same as Eq. (8) if U in Eq. (8) is replaced by mU/k and the term η' in the Γ equation of Eq. (23) is omitted. This is effectively a statement of Squires theorem for oblique disturbances. We remark that the difference in form of Eqs. (22) and (8) arises because the 2D normal mode was obtained from a linear analysis of the 2D Birkhoff–Rott evolution equation while our Eq. (17) is obtained from an analysis in terms of linear perturbation potentials; the expressions must agree to $O(\epsilon)$.

Comparing Eq. (10) to Eq. (23) shows that in the (x',y',z') axes, the geometry and vorticity for the 3D normal mode correspond to our definition of a cylindrical vortex sheet with the choices

$$f_1=\xi'-\epsilon\sin(k\xi'), \quad f_2=\epsilon\sin(k\xi'), \qquad f_3=-(m/k)U\xi', \quad c_1=(n/k)U. \tag{24}$$

If this is taken as a finite amplitude initial condition, then from (i) and (ii) following Eq. (16) we can conclude that

(a) the evolution of the vortex sheet from an initial condition corresponding to a finite-amplitude 3D normal mode disturbance remains cylindrical for all time, and

(b) in the $(x'-z')$ plane its nonlinear evolution is the same as that for an equivalent 2D normal mode with suitably modified velocity jump and ratio of disturbance amplitude to wavelength. When $n\neq 0$, the evolution proceeds with a uniform shear, with velocity difference nU/k, along the generator of the cylindrical sheet geometry. The vortex lines are then not parallel to the generator.

To the order of accuracy of Eq. (9), it follows that the evolution produces a curvature singularity at a time t_c obtained from

$$1+\pi\tilde{t}_c+\ln(\pi\tilde{t}_c)=\ln(1/2\pi\tilde{\epsilon}), \tag{25}$$

where we now have

$$\tilde{t}_c=t_c[(m/k)U]/\lambda=t_c mU/2\pi, \tag{26}$$

$$\tilde{\epsilon}=\epsilon/\lambda=(\epsilon/2\pi)\sqrt{m^2+n^2}, \tag{27}$$

$$\lambda=2\pi/k=2\pi/\sqrt{m^2+n^2}. \tag{28}$$

The singularity forms along the lines $mx+ny=b$, parallel to the generator of the 3D cylindrical sheet, where b is some constant. Equations (13), (16) and (25)–(28) summarize the main results of this note. They show that the singularity which forms following a finite amplitude, 3D normal mode perturbation to an undisturbed vortex sheet is the same as Moore's result for the 2D perturbation. When ϵ, k and U are held constant, it is clear that t_c is a minimum for the 2D case $n=0$.

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