The Maxwell–Vlasov equations in Euler–Poincaré form

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Low's well-known action principle for the Maxwell–Vlasov equations of ideal plasma dynamics was originally expressed in terms of a mixture of Eulerian and Lagrangian variables. By imposing suitable constraints on the variations and analyzing invariance properties of the Lagrangian, as one does for the Euler equations for the rigid body and ideal fluids, we first transform this action principle into purely Eulerian variables. Hamilton's principle for the Eulerian description of Low's action principle then casts the Maxwell–Vlasov equations into Euler–Poincaré form for right invariant motion on the diffeomorphism group of position-velocity phase space, $\mathbb{R}^6$. Legendre transforming the Eulerian form of Low's action principle produces the Hamiltonian formulation of these equations in the Eulerian description. Since it arises from Euler–Poincaré equations, this Hamiltonian formulation can be written in terms of a Poisson structure that contains the Lie–Poisson bracket on the dual of a semidirect product Lie algebra. Because of degeneracies in the Lagrangian, the Legendre transform is dealt with using the Dirac theory of constraints. Another Maxwell–Vlasov Poisson structure is known, whose ingredients are the Lie–Poisson bracket on the dual of the Lie algebra of symplectomorphisms of phase space and the Born–Infeld brackets for the Maxwell field. We discuss the relationship between these two Hamiltonian formulations. We also discuss the general Kelvin–Noether theorem for Euler–Poincaré equations and its meaning in the plasma context. © *1998 American Institute of Physics.* [S0022-2488(98)00506-4]

I. INTRODUCTION

A. Reduction of action principles

Due to their wide applicability, the Maxwell–Vlasov equations of ideal plasma dynamics have been studied extensively. In 1958 Low[1] wrote down an action principle for them in preparation for studying stability of plasma equilibria. Low's action principle is expressed in terms of a mixture of Lagrangian particle variables and Eulerian field variables.

Following the initiative of Arnold[2] and its later developments (see Ref. 3 for background), we start with a purely Lagrangian description of the plasma and investigate the invariance properties of the corresponding action. Using this setup and recent developments in the theory of the Euler–

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Poincaré equations[4,5] due to Holm, Marsden, and Ratiu,[6] we are able to cast Low's action principle into a purely Eulerian description.

In this paper, we start with the *standard* form of Hamilton's variational principle (in the Lagrangian representation) and *derive* the new Eulerian action principle by a systematic reduction process, much as one does in the corresponding derivation of Poisson brackets in the Hamiltonian formulation of the Maxwell–Vlasov equations starting with the *standard canonical brackets* and proceeding by symmetry reduction (as in Ref. 7). In particular, the Eulerian action principle we obtain in this way is different from the ones found in Ye and Morrison[8] by *ad hoc* procedures. We also mention that the method of reduction of variational principles we develop naturally justifies constraints on the variations of the so-called ''Lin constraint'' form, well known in fluid mechanics.

The methods of this paper are based on reduction of variational principles, that is, on Lagrangian reduction (see Refs. 9–12). These methods have also been useful for systems with nonholonomic constraints. This has been demonstrated in the work of Bloch *et al.*,[13] who derived the reduced Lagrange d'Alembert equations for nonholonomic systems, which also have a constrained variational structure. The methods of the present paper should enhance the applicability of the Lagrangian reduction techniques for even wider classes of continuum systems.

B. Passage to the Hamiltonian formulation

The Hamiltonian structure and nonlinear stability properties of the equilibrium solutions for the Maxwell–Vlasov system have been thoroughly explored. Some of the key references are Iwínski and Turski,[14] Morrison,[15] Marsden and Weinstein,[7] and Holm, Marsden, Ratiu, and Weinstein.[16] See also the introduction and bibliography of Marsden *et al.*[17] for a guide to the history and literature of this subject.

In our approach, Lagrangian reduction leads to the Euler–Poincaré form of the equations, which is still in the Lagrangian formulation. Using this setup, one may pass from the Lagrangian to the Hamiltonian formulation of the Maxwell–Vlasov equations by Legendre transforming the action principle in the Eulerian description at either the level of the group variables (the level that keeps track of the particle positions), or at the level of the Lie algebra variables. One must be cautious in this procedure because the relevant Hamiltonian and Lagrangian are degenerate. We deal with this degeneracy by using a version of the Dirac theory of constraints.

Legendre transforming at the group level leads to a canonical Hamiltonian formulation and the latter leads to a new Hamiltonian formulation of the Maxwell–Vlasov equations in terms of a Poisson structure containing the Lie–Poisson bracket on the dual of a semidirect product Lie algebra. This new formulation leads us naturally to the starting point for Hamiltonian reduction used by Marsden and Weinstein[7] (see also Refs. 15 and 18).

C. Stability and asymptotics

The new Hamiltonian formulation of the Maxwell–Vlasov system places these equations into a framework in which one can use the energy-momentum and energy-Casimir methods for studying nonlinear stability properties of their relative equilibrium solutions. This is directly in line with Low's intended program, since the study of stability was Low's original motivation for writing his action principle. Sample references in this direction are Holm, Marsden, Weinstein, and Ratiu,[16] Morrison,[19] Morrison and Pfirsch,[20] Wan,[21] Batt and Rein[22] and Batt, Morrison, and Rein.[23] Other historical references for the Lagrangian approach to the Maxwell–Vlasov equations include Sturrock,[24] Galloway and Kim,[25] and Dewar.[26]

The Eulerian formulation of Low's action principle also casts it into a form that is amenable to asymptotic expansions and creation of approximate theories (such as guiding center theories) possessing the same mathematical structure arising from the Euler–Poincaré setting. See, for example, Ref. 27 for applications of this approach of Hamilton's principle asymptotics in geophysical fluid dynamics.

D. Comments on the Maxwell–Vlasov system

The rest of this paper will be concerned with variational principles for the Maxwell–Vlasov system of equations for the dynamics of an ideal plasma. These equations have a long history dating back at least to Jeans,[28] who used them in a simpler form known as the Poisson–Vlasov system to study structure formation on stellar and galactic scales. Even before Jeans, Poincaré[4,29] had investigated the stability of equilibrium solutions of the Poisson–Vlasov system for the purpose of determining the stability conditions for steller configurations. The history of the efforts to establish stellar stability conditions using the Poisson–Vlasov system is summarized by Chandrasekhar.[30] The Poisson–Vlasov system is also used to describe the self-consistent dynamics of an electrostatic collisionless plasma, whereas the Maxwell–Vlasov system is used to describe the dynamics of a collisionless plasma evolving self-consistently in an electromagnetic field.

E. Organization of the paper

The paper is organized as follows. In Sec. II we introduce the Maxwell–Vlasov equations. In Sec. III we state the Euler–Poincaré theorem for Lagrangians depending on parameters along with the associated Kelvin–Noether theorem. This general theorem plays a key role in our analysis. In Sec. V we reformulate these equations in a purely Eulerian form and show how they satisfy the Euler–Poincaré theorem. The following section reviews some aspects of the Legendre transformation for degenerate Lagrangians. In Sec. IV we reprise Low's action principle for the Maxwell–Vlasov equations. In Sec. VII we cast the Euler–Poincaré formulation of the Maxwell–Vlasov equations into Hamiltonian form possessing a Poisson structure that contains a Lie–Poisson bracket. In Sec. VIII we summarize our conclusions.

II. THE MAXWELL–VLASOV EQUATIONS

The Maxwell–Vlasov system of equations describes the single particle distribution for a set of charged particles of one species moving self-consistently in an electromagnetic field. In this description, the Boltzmann function $f(\mathbf{x},\mathbf{v},t)$ is viewed as the instantaneous probability density function for the particle distribution, i.e., given a region Ω of phase space, the probability of finding a particle in that region is

$$\int_\Omega d\mathbf{x}\, d\mathbf{v} f(\mathbf{x},\mathbf{v},t), \tag{2.1}$$

where $\mathbf{x}$ and $\mathbf{v}$ are the current positions and velocities of the plasma particles. Thus, if the phase-space domain Ω is the whole $(\mathbf{x},\mathbf{v})$ space, the value of this integral at a certain time t is normalized to unity.

As is customary, we assume that the particles of the plasma obey dynamical equations and that the plasma density f is advected as a scalar along the particle trajectories in phase space, i.e.,

$$\frac{\partial f}{\partial t}+\dot{\mathbf{x}}\cdot\nabla_{\mathbf{x}}f+\dot{\mathbf{v}}\cdot\nabla_{\mathbf{v}}f=0. \tag{2.2}$$

In this equation, and in the sequel, an overdot refers to a time derivative along a phase space trajectory, and $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{v}}$ denote the gradient operators with respect to position and velocity, respectively. For pressureless motion in the electromagnetic field of the charged particle distribution, the acceleration of a particle is given by

$$\ddot{\mathbf{x}}=-\frac{q}{m}\left[\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}-\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{A})\right], \tag{2.3}$$

where (q/m) denotes the charge-to-mass ratio of an individual particle, Φ is the electric potential, and $\mathbf{A}$ is the magnetic vector potential. Substituting this expression for $\dot{\mathbf{v}}$ in Eq. (2.2) yields

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}f-\frac{q}{m}\left[\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}-\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{A})\right]\cdot\nabla_{\mathbf{v}}f=0. \tag{2.4}$$

This is the ***Vlasov equation*** (also called the collisionless Boltzmann, or Jeans equation). The system is completed by the Maxwell equations with sources:

$$\nabla_{\mathbf{x}}\cdot\mathbf{E}=\rho, \quad \nabla_{\mathbf{x}}\times\mathbf{B}=\frac{\partial\mathbf{E}}{\partial t}+\mathbf{j}, \tag{2.5}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field variables, respectively, ρ is the charge density, and $\mathbf{j}$ is the current density. These quantities are expressed in terms of the Boltzmann function f and the Maxwell scalar and vector potentials Φ and $\mathbf{A}$ by

$$\mathbf{E}=-\nabla_{\mathbf{x}}\Phi-\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B}=\nabla_{\mathbf{x}}\times\mathbf{A},$$
$$\rho(\mathbf{x},t)=q\int d\mathbf{v}\ f(\mathbf{x},\mathbf{v},t), \quad \mathbf{j}(\mathbf{x},t)=q\int d\mathbf{v}\ \mathbf{v}f(\mathbf{x},\mathbf{v},t). \tag{2.6}$$

By their definitions, $\mathbf{E}$ and $\mathbf{B}$ satisfy the kinematic Maxwell equations

$$\nabla_{\mathbf{x}}\cdot\mathbf{B}=0, \quad \nabla_{\mathbf{x}}\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}. \tag{2.7}$$

Equations (2.4)–(2.7) comprise the ***Maxwell–Vlasov equations***. When $\mathbf{A}$ is absent, the field is electrostatic and one obtains the Poisson–Vlasov equations. The Poisson–Vlasov system can also be used to describe a self-gravitating collisionless fluid, and so it forms a model for the evolution of galactic dynamics (see, e.g., Ref. 31).

Note that the integral in (2.1) is independent of time (as the region and the function f evolve), since the vector field defining the motion of particles [see Eq. (2.3)] is divergence free with respect to the standard volume element on velocity phase space. Thus, one may interpret f either as a density or as a scalar. For our purposes later, we will need to be careful with the distinction, since the volume-preserving nature of the flow of particles will be a consequence of our variational principle and will not be imposed at the outset.

III. THE EULER–POINCARÉ EQUATIONS, SEMIDIRECT PRODUCTS, AND KELVIN'S THEOREM

A. The general Euler–Poincaré equations

Here we recall from Ref. 6 the general form of the Euler–Poincaré equations and their associated Kelvin–Noether theorem. In the next section, we will immediately specialize these statements for a general invariance group G to the case of plasmas when G is the diffeomorphism group, $\mathrm{Diff}(T\mathbb{R}^3)$. We shall state the general theorem for right actions and right invariant Lagrangians, which is appropriate for the Maxwell–Vlasov situation. The notation is as follows.

(i) There is a *right* representation of the Lie group G on the vector space V and G acts in the natural way from the *right* on $TG\times V^*$: $(v_g,a)h=(v_gh,ah)$.

(ii) $\rho_v:\mathfrak{g}\to V$ is the linear map given by the corresponding right action of the Lie algebra on V: $\rho_v(\xi)=v\xi$, and $\rho_v^*:V^*\to\mathfrak{g}^*$ is its dual. The $\mathfrak{g}$–action on $\mathfrak{g}^*$ and V^* is defined to be *minus* the dual map of the $\mathfrak{g}$–action on $\mathfrak{g}$ and V, respectively, and is denoted by $\mu\xi$ and $a\xi$ for $\xi\in\mathfrak{g}$, $\mu\in\mathfrak{g}^*$, and $a\in V^*$. For $v\in V$ and $a\in V^*$, it will be convenient to write

$$v\diamond a=\rho_v^*a, \quad \text{i.e.,} \quad \langle v\diamond a,\xi\rangle=\langle a,v\xi\rangle=-\langle v,a\xi\rangle,$$

for all $\xi\in\mathfrak{g}$. Note that $v\diamond a\in\mathfrak{g}^*$.

(iii) Let $\mathcal{Q}$ be a manifold on which G acts *trivally* and assume that we have a function $L:TG\times T\mathcal{Q}\times V^*\to\mathbb{R}$ which is right G-invariant.

(iv) In particular, if $a_0\in V^*$, define the Lagrangian $L_{a_0}:TG\times T\mathcal{Q}\to\mathbb{R}$ by $L_{a_0}(v_g,u_q)=L(v_g,u_q,a_0)$. Then L_{a_0} is right invariant under the lift to $TG\times T\mathcal{Q}$ of the right action of G_{a_0} on $G\times\mathcal{Q}$.

(v) Right G-invariance of L permits us to define $l:\mathfrak{g}\times T\mathcal{Q}\times V^*\to\mathbb{R}$ by

$$l(v_gg^{-1},\ u_q,\ ag^{-1})=L(v_g,u_q,a).$$

Conversely, this relation defines for any $l:\mathfrak{g}\times T\mathcal{Q}\times V^*\to\mathbb{R}$ a right G-invariant function $L:TG\times T\mathcal{Q}\times V^*\to\mathbb{R}$.

(vi) For a curve $g(t)\in G$, let $\xi(t):=\dot{g}(t)g(t)^{-1}$ and define the curve $a(t)$ as the unique solution of the linear differential equation with time-dependent coefficients $\dot{a}(t)=-a(t)\xi(t)$

with initial condition $a(0)=a_0$. The solution can be equivalently written as $a(t)=a_0 g(t)^{-1}$.

Theorem 3.1: The following are equivalent:

(i) Hamilton's variational principle holds:

$$\delta\int_{t_1}^{t_2} L_{a_0}(g(t),\dot{g}(t),q(t),\dot{q}(t))\,dt=0, \tag{3.1}$$

for variations of g and q with fixed endpoints.

(ii) $(g(t),q(t))$ satisfies the Euler–Lagrange equations for L_{a_0} on $G\times\mathcal{Q}$.

(iii) The constrained variational principle,[32]

$$\delta\int_{t_1}^{t_2} l(\xi(t),q(t),\dot{q}(t),a(t))\,dt=0, \tag{3.2}$$

holds on $\mathfrak{g}\times\mathcal{Q}$, upon using variations of the form

$$\delta\xi=\frac{\partial\eta}{\partial t}-ad_\xi\eta=\frac{\partial\eta}{\partial t}-[\xi,\eta],\qquad \delta a=-a\eta, \tag{3.3}$$

where $\eta(t)\in\mathfrak{g}$ vanishes at the endpoints and $\delta q(t)$ is unrestricted except for vanishing at the endpoints.

(iv) The following system of Euler–Poincaré equations (with a parameter) coupled with Euler–Lagrange equations holds on $\mathfrak{g}\times T\mathcal{Q}\times V^*$:

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta\xi}=-\mathrm{ad}_\xi^*\frac{\delta l}{\delta\xi}+\frac{\delta l}{\delta a}\diamond a \tag{3.4}$$

and

$$\frac{\partial}{\partial t}\frac{\partial l}{\partial\dot{q}^i}-\frac{\partial l}{\partial q^i}=0. \tag{3.5}$$

The strategy of the proof is simple: one just determines the form of the variations on the reduced space $\mathfrak{g}\times\mathcal{Q}\times V^*$ that are induced by variations on the unreduced space $TG\times T\mathcal{Q}$ and includes the relation of $a(t)$ to a_0. One then carries the variational principle to the quotient. See Ref. 6 for details. Here we have included the extra factor of $\mathcal{Q}$ which is needed in the present application; this will be the space of potentials for the Maxwell field. This extra factor does not substantively alter the arguments.

B. The Kelvin–Noether Theorem

We start with a Lagrangian L_{a_0} depending on a parameter $a_0\in V^*$ as above and introduce a manifold $\mathcal{C}$ on which G acts. We assume this is also a right action and suppose we have an equivariant map $\mathcal{K}:\mathcal{C}\times V^*\rightarrow\mathfrak{g}^{**}$.

In the case of continuum theories, the space $\mathcal{C}$ is chosen to be a loop space and $\langle\mathcal{K}(c,a),\mu\rangle$ for $c\in\mathcal{C}$ and $\mu\in\mathfrak{g}^*$ will be a circulation. This class of examples also shows why we *do not* want to identify the double dual $\mathfrak{g}^{**}$ with $\mathfrak{g}$.

Define the ***Kelvin–Noether quantity*** $I:\mathcal{C}\times\mathfrak{g}\times T\mathcal{Q}\times V^*\rightarrow\mathbb{R}$ by

$$I(c,\xi,q,\dot{q},a)=\left\langle\mathcal{K}(c,a),\,\frac{\delta l}{\delta\xi}(\xi,q,\dot{q},a)\right\rangle. \tag{3.6}$$

Theorem 3.2 (Kelvin–Noether): Fixing $c_0\in\mathcal{C}$, let $\xi(t),q(t),\dot{q}(t),a(t)$ satisfy the Euler–Poincaré equations and define $g(t)$ to be the solution of $\dot{g}(t)=\xi(t)g(t)$ and, say, $g(0)=e$. Let $c(t)=g(t)^{-1}c_0$ and $I(t)=I(c(t),\xi(t),q(t),\dot{q}(t),a(t))$.

Then

$$\frac{d}{dt}I(t)=\left\langle \mathscr{K}(c(t),a(t)),\frac{\delta l}{\delta a}\diamond a\right\rangle. \tag{3.7}$$

The proof of this theorem is relatively straightforward; we refer to Ref. 6. We shall express the relation (3.7) explicitly for Maxwell–Vlasov plasmas at the end of Sec. VII.

IV. AN ACTION FOR THE MAXWELL–VLASOV EQUATIONS

A typical element of $T\mathbb{R}^3 \cong \mathbb{R}^3\times\mathbb{R}^3$ will be denoted $\mathbf{z}=(\mathbf{x},\mathbf{v})$. We let $\pi_s:T\mathbb{R}^3\to\mathbb{R}^3$ and $\pi_v:T\mathbb{R}^3\to\mathbb{R}^3$ be the projections $\pi_s(\mathbf{z})=\mathbf{x}$ and $\pi_v(\mathbf{z})=\mathbf{v}$ onto the first and second factors, respectively.

A. Spaces of fields

We let Diff($T\mathbb{R}^3$) denote the group of C^∞-diffeomophisms from $T\mathbb{R}^3$ onto itself. An element $\psi\in$ Diff($T\mathbb{R}^3$) maps plasma particles having initial position and velocity $(\mathbf{x}_0,\mathbf{v}_0)$ to their current position and velocity $(\mathbf{x},\mathbf{v})=\psi(\mathbf{x}_0,\mathbf{v}_0)$. This is the particle evolution map. We shall sometimes abbreviate $(\mathbf{x}_0,\mathbf{v}_0)=\mathbf{z}_0$, $(\mathbf{x},\mathbf{v})=\mathbf{z}$, etc. The spatial components of $\psi(\mathbf{x}_0,\mathbf{v}_0)$ are written as $\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0)$ and the velocity components as $\mathbf{v}(\mathbf{x}_0,\mathbf{v}_0)$. We shall also use the following notation:

(i) $\mathscr{V}=C^\infty(\mathbb{R}^3,\mathbb{R})$ is the space of electric potentials $\Phi(\mathbf{x})$;
(ii) $\mathscr{A}$ is the space of magnetic potentials $A(\mathbf{x})$;
(iii) $\mathscr{F}=C^\infty(T\mathbb{R}^3,\mathbb{R})$ is the space of plasma densities $f(\mathbf{x},\mathbf{v})$;
(iv) $\mathscr{F}_0=C_0^\infty(T\mathbb{R}^3,\mathbb{R})$ is the space of plasma densities with compact support; and
(v) $\mathscr{D}_0=C_0^\infty(\mathbb{R}^3,\mathbb{R})$ is a space of test functions, denoted $\varphi(\mathbf{x})$.

The test functions $\varphi(\mathbf{x})$ are used to localize the variational principle. Thus, once one obtains Euler–Lagrange equations depending on f_0 and φ_0, if their validity can be naturally extended for any f_0 and φ_0, which will happen in our case, then we shall consider those extended equations to be the Euler–Lagrange equations of the system. We will usually be interested in the Euler–Lagrange equations for $f_0>0$ and $\varphi_0=1$.

B. The Lagrangian and the action

For each choice of the initial plasma distribution function f_0 and the test function φ_0, we define the Lagrangian

$$\begin{aligned}L_{f_0,\varphi_0}(\psi,\dot\psi,\Phi,\dot\Phi,\mathbf{A},\dot{\mathbf{A}})=\int d\mathbf{x}_0\, d\mathbf{v}_0\, f_0(\mathbf{x}_0,\mathbf{v}_0)\Bigg(&\frac{1}{2}m|\dot{\mathbf{x}}(\mathbf{x}_0,\mathbf{v}_0)|^2+\frac{1}{2}m|\dot{\mathbf{x}}(\mathbf{x}_0,\mathbf{v}_0)-\mathbf{v}(\mathbf{x}_0,\mathbf{v}_0)|^2\\ &+q\dot{\mathbf{x}}(\mathbf{x}_0,\mathbf{v}_0)\cdot\mathbf{A}(\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0))-q\Phi(\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0))\Bigg)\\ &+\frac{1}{2}\int d\mathbf{r}\,\varphi_0(\mathbf{r})\left(\left|\nabla_{\mathbf{r}}\Phi+\frac{\partial\mathbf{A}}{\partial t}(\mathbf{r})\right|^2-|\nabla_{\mathbf{r}}\times\mathbf{A}(\mathbf{r})|^2\right).\end{aligned} \tag{4.1}$$

This Lagrangian is the natural generalization of that for an N-particle system, with terms corresponding to kinetic energy, electric and magnetic field energy, the usual magnetic coupling term with coupling constant q (the electric charge), and a constraint that ties the Eulerian fluid velocity $\mathbf{v}$ to $\dot{\mathbf{x}}$, the material derivative of the Lagrangian particle trajectory. Here $\mathbf{x}$ and $\mathbf{v}$ are Lagrangian phase space variables, while $\mathbf{A}$ and Φ are Eulerian field variables. Thus, there should be no confusion created by the slight abuse of notation in abbreviating $\partial\mathbf{A}/\partial t$ and $\partial\Phi/\partial t$ as $\dot\Phi$ and $\dot{\mathbf{A}}$, respectively, in the arguments of the Lagrangian. This Lagrangian is inspired by Low.[1] However, we have added the term

$$\tfrac{1}{2}m|\dot{\mathbf{x}}(\mathbf{x}_0,\mathbf{v}_0)-\mathbf{v}(\mathbf{x}_0,\mathbf{v}_0)|^2,$$

which allows $\mathbf{v}$ to be varied independently in the variational treatment.

Consider the action

$$\mathfrak{S}=\int dt\ L_{f_0,\varphi_0}(\psi,\dot{\psi},\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}}),$$

defined on the family of curves $(\psi(t),\Phi(t),\mathbf{A}(t))$ satisfying the usual fixed-endpoint conditions $(\psi(t_i),\Phi(t_i),\mathbf{A}(t_i))=(\psi_i,\Phi_i,\mathbf{A}_i)$, $i=1,2$. One now applies the standard techniques of the calculus of variations. In particular, integration by parts can be performed since f_0 and φ_0 have compact support. Moreover, once the Euler–Lagrange equations have been obtained, their validity can be easily extended in a natural way for $f_0>0$ and $\varphi_0=1$.

C. Derivation of the equations

To write the equations of motion, we need some additional notation. Consider the evolution map $\psi_t(\mathbf{x}_0,\mathbf{v}_0)=(\mathbf{x},\mathbf{v})$ so that ψ_t relates the initial positions and velocities of fluid particles to their positions and velocities at time t. Let $\mathbf{u}$ be the corresponding vector field:

$$\mathbf{u}(\mathbf{x},\mathbf{v}):=\dot{\psi}_t\circ\psi_t^{-1}(\mathbf{x},\mathbf{v})=:\dot{\mathbf{x}}\frac{\partial}{\partial\mathbf{x}}+\dot{\mathbf{v}}\frac{\partial}{\partial\mathbf{v}},$$

so the components of $\mathbf{u}$ are $(\dot{\mathbf{x}},\dot{\mathbf{v}})$. Recall that the transport of f_0 as a scalar is given by $f(\mathbf{x},\mathbf{v},t)=f_0\circ\psi_t^{-1}(\mathbf{x},\mathbf{v})$, which satisfies

$$\frac{\partial f}{\partial t}+\mathbf{u}\cdot\nabla_{\mathbf{z}}f=0,\tag{4.2}$$

where $\nabla_{\mathbf{z}}=(\nabla_{\mathbf{x}},\nabla_{\mathbf{v}})$ is the six-dimensional gradient operator in $(\mathbf{x},\mathbf{v})$ space. Let J_ψ be the Jacobian determinant of the mapping $\psi\in\mathrm{Diff}(T\mathbb{R}^3)$, that is, the determinant of the Jacobian matrix $\partial(\mathbf{x},\mathbf{v})/\partial(\mathbf{x}_0,\mathbf{v}_0)$.

Define $F(\mathbf{x},\mathbf{v},t)$ to be f_0, transported as a *density*:

$$F(\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0),\mathbf{v}(\mathbf{x}_0,\mathbf{v}_0),t)J_\psi(\mathbf{x}_0,\mathbf{v}_0)=f_0(\mathbf{x}_0,\mathbf{v}_0),$$

so that

$$\frac{\partial F}{\partial t}+\nabla_{\mathbf{z}}\cdot(F\mathbf{u})=0.\tag{4.3}$$

Taking variations in our Lagrangian (4.1) and making use of the preceding equation for F, we obtain the following equations (taking $\varphi_0=1$):

$$\begin{gathered}\delta\mathbf{x}:\ \ m\ddot{\mathbf{x}}+m(\ddot{\mathbf{x}}-\dot{\mathbf{v}})=-q\nabla_{\mathbf{x}}\Phi-q\frac{\partial\mathbf{A}}{\partial t}+q\dot{\mathbf{x}}\times(\nabla_{\mathbf{x}}\times\mathbf{A}),\\ \delta\mathbf{v}:\ \ \dot{\mathbf{x}}-\mathbf{v}=0,\\ \delta\Phi:\ \ \nabla_{\mathbf{x}}\cdot\left(\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)=-q\int d\mathbf{v}\ F(\mathbf{x},\mathbf{v},t),\\ \delta\mathbf{A}:\ \ \nabla_{\mathbf{x}}\times(\nabla_{\mathbf{x}}\times\mathbf{A})=-\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)+q\int d\mathbf{v}\ \mathbf{v}F(\mathbf{x},\mathbf{v},t).\end{gathered}\tag{4.4}$$

The second equation in (4.4) treats the Eulerian velocity $\mathbf{v}$ as a Lagrange multiplier, and ties its value to the fluid velocity $\dot{\mathbf{x}}$, hence $\dot{\mathbf{v}}=\ddot{\mathbf{x}}$ as well. The first two variational equations in the set (4.4) provide the desired relation for particle acceleration and the last two equations are the Maxwell equations with source terms. Thus, Hamilton's principle with Low's action provides the equations for self-consistent particle motion in an electromagnetic field, as required, and the description is completed by substituting

$$\left(\mathbf{v},-\frac{q}{m}\left[\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}-\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{A})\right]\right)$$

for the components of $\mathbf{u}$ in the transport equation (4.2) to give the Vlasov equation (2.4).

V. THE MAXWELL–VLASOV SYSTEM AS EULER–POINCARÉ EQUATIONS

We will now specialize the general Euler–Poincaré theorem to the case of plasmas. The Lagrangian $L_{f_0,\varphi_0}(\psi,\dot{\psi},\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}})$ in Eq. (4.1) has a right $\mathrm{Diff}(T\mathbb{R}^3)$-symmetry. Let $\eta\in\mathrm{Diff}(T\mathbb{R}^3)$, $F\in\mathscr{F}$, and define the action of η on F by $F\eta=(F\circ\eta)J_\eta$ where, as above, J_η is the Jacobian determinant of η.

The symmetry of $L_{f_0,\varphi_0}(\psi,\dot{\psi},\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}})$ is the property

$$L_{f_0\eta,\varphi_0}(\psi\eta,\dot{\psi}\eta,\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}})=L_{f_0,\varphi_0}(\psi,\dot{\psi},\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}}),$$

for all $\eta\in\mathrm{Diff}(T\mathbb{R}^3)$.

A. Ingredients for Euler–Poincaré

Now we apply the general Euler–Poincaré Theorem 3.1, taking $G=\mathrm{Diff}(T\mathbb{R}^3)$ and $\mathscr{Q}=\mathscr{V}\times\mathscr{A}$ and the parameter $a_0=f_0$. As we have explained before, φ_0 is an auxiliary quantity that will ultimately take the value unity. In the general Euler–Poincaré Theorem 3.1 we take

$$\delta\mathbf{u}=\frac{\partial\mathbf{w}}{\partial t}-\mathrm{ad}_{\mathbf{u}}\mathbf{w},\qquad\delta a=-\pounds_{\mathbf{w}}a,\tag{5.1}$$

where $\mathbf{w}\in\mathfrak{g}$ is a vector field on $T\mathbb{R}^3$, $\pounds_{\mathbf{w}}$ is the Lie derivative, and $\mathrm{ad}_{\mathbf{u}}\mathbf{w}=-[\mathbf{u},\mathbf{w}]$ defines $\mathrm{ad}_{\mathbf{u}}\mathbf{w}$ in terms of the Lie bracket of vector fields, $[\mathbf{u},\mathbf{w}]$. The Euler–Poincaré equations (3.4) are

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta\mathbf{u}}=-ad^*_{\mathbf{u}}\frac{\delta l}{\delta\mathbf{u}}+\frac{\delta l}{\delta a}\diamondsuit a,\tag{5.2}$$

where $\mathrm{ad}^*_{\mathbf{u}}$ is the dual of $\mathrm{ad}_{\mathbf{u}}$ and $\delta l/\delta\mathbf{u}$ is a one-form density. The one-form density $(\delta l/\delta a)\diamondsuit a$ is defined by

$$\left\langle\frac{\delta l}{\delta a}\diamondsuit a,\mathbf{w}\right\rangle=-\int\frac{\delta l}{\delta a}\cdot\pounds_{\mathbf{w}}a.\tag{5.3}$$

When the quantities a are tensor fields, $\delta l/\delta a$ will be elements of the dual space under the natural pairing.

We shall apply this result to obtain the Maxwell–Vlasov system (2.4)–(2.7) as Euler–Poincaré equations. We begin by recording a formula that will be needed later. Let $\mathbf{u},\mathbf{w}$ be two elements of $\mathfrak{g}$, the Lie algebra of vector fields for the diffeomorphism group on a manifold $\mathscr{M}$. Choose the one-form density $\mathbf{c}\in\mathfrak{g}^*$, and let the pairing $\langle\mathbf{c},\mathbf{u}\rangle:\mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ be given by

$$\langle\mathbf{c},\mathbf{u}\rangle=\int_{\mathscr{M}}d\mathbf{z}\;\mathbf{c}\cdot\mathbf{u}=\int_{\mathscr{M}}d\mathbf{z}\;c_ju^j,\tag{5.4}$$

where c_j and u^j, $j=1,\dots,n$, are components of $\mathbf{c}$ and $\mathbf{u}$ in $\mathbb{R}^n$ and $d\mathbf{z}$ is the volume form on $\mathscr{M}$. Then we can write the desired formula,

$$\begin{aligned}\langle \mathrm{ad}^*_{\mathbf{u}}\mathbf{c},\mathbf{w}\rangle &= \int d\mathbf{z}\ \mathrm{ad}^*_{\mathbf{u}}\mathbf{c}\cdot\mathbf{w}\\ &= \int d\mathbf{z}\ \mathbf{c}\cdot \mathrm{ad}_{\mathbf{u}}\mathbf{w}\\ &= -\int d\mathbf{z}\ c_i\left(u^j\frac{\partial w^i}{\partial z^j}-w^j\frac{\partial u^i}{\partial z^j}\right)\\ &= \int d\mathbf{z}\ w^i\left(c_j\frac{\partial u^j}{\partial z^i}+c_i(\nabla\cdot\mathbf{u})+(\mathbf{u}\cdot\nabla)c_i\right)=\langle \pounds_{\mathbf{u}}\mathbf{c},\mathbf{w}\rangle. \end{aligned} \tag{5.5}$$

Here $\pounds_{\mathbf{u}}\mathbf{c}$ is the Lie derivative of the the one-form density $\mathbf{c}$ with respect to the vector field $\mathbf{u}$, z^j is the coordinate chart, and c_j, u^j, w^j are the components of vectors in $\mathbb{R}^n$. Unless otherwise stated, we sum repeated indices over their range, $i,j=1,\ldots,n$, where n is the dimension of $\mathscr{M}$. We assume that the vector fields and one-form densities are defined so that integration by parts gives no contribution at the boundary (inclusion of nonzero boundary terms is straightforward). Formula (5.5) for $\mathrm{ad}^*_{\mathbf{u}}\mathbf{c}$ will be useful later.

By definition, $\mathbf{u}=(\dot{\mathbf{x}},\dot{\mathbf{v}})$; we will denote $\mathbf{u}_s=\dot{\mathbf{x}}$, the spatial part of the phase space velocity field.

B. The reduced action

We may transform the action (4.1) into the Eulerian description as the reduced action

$$\begin{aligned}\mathfrak{S}_{\mathrm{red}} &= \int dt\ l(\mathbf{u},\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}})\\ &= \int dt \int d\mathbf{x}\ d\mathbf{v}\ F(\mathbf{x},\mathbf{v},t)\left(\frac{1}{2}m|\mathbf{u}_s|^2+\tfrac{1}{2}m|\mathbf{u}_s-\mathbf{v}|^2-q\Phi+q\mathbf{u}_s\cdot\mathbf{A}\right)\\ &\quad+\frac{1}{2}\int dt\int d\mathbf{x}\left|\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right|^2-|\nabla_{\mathbf{x}}\times\mathbf{A}|^2. \end{aligned} \tag{5.6}$$

We vary this action with respect to $\mathbf{u}_s$, F, Φ and $\mathbf{A}$:

$$\begin{aligned}\delta\mathfrak{S}_{\mathrm{red}} &= \int dt\int d\mathbf{x}\ d\mathbf{v}\{F[(m\mathbf{u}_s+m(\mathbf{u}_s-\mathbf{v})+q\mathbf{A})\cdot\delta\mathbf{u}_s-q\,\delta\Phi+\mathbf{u}_s\cdot\delta\mathbf{A}]\\ &\quad+\delta F[\tfrac{1}{2}m|\mathbf{u}_s|^2+\tfrac{1}{2}m|\mathbf{u}_s-\mathbf{v}|^2-q\Phi+q\mathbf{u}_s\cdot\mathbf{A}]\}\\ &\quad+\int dt\int d\mathbf{x}\left(\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\cdot\left(\nabla_{\mathbf{x}}\delta\Phi+\delta\frac{\partial\mathbf{A}}{\partial t}\right)-(\nabla_{\mathbf{x}}\times\mathbf{A})\cdot(\nabla_{\mathbf{x}}\times\delta\mathbf{A}). \end{aligned} \tag{5.7}$$

Stationary variations in Φ and $\mathbf{A}$ yield

$$\begin{aligned}\nabla_{\mathbf{x}}\cdot\left(\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)&=-q\int d\mathbf{v}\ F(\mathbf{x},\mathbf{v},t),\\ \nabla_{\mathbf{x}}\times(\nabla_{\mathbf{x}}\times\mathbf{A})&=-\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)+q\int d\mathbf{v}\ F(\mathbf{x},\mathbf{v},t)\mathbf{u}_s. \end{aligned} \tag{5.8}$$

Thus, Maxwell's equations for the electromagnetic field of the plasma are recovered by requiring $\delta l=0$ for all variations of the field potentials Φ and $\mathbf{A}$. To continue toward the Euler–Poincaré form of the Maxwell–Vlasov equations, one must determine the forms of the variations $\delta\mathbf{u}_s$ and δF in (5.7).

According to the general theory, variations in the particle evolution map ψ lead to variations in the phase space velocity $\delta\mathbf{u}$ of the form

$$\delta\mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} + [\mathbf{u},\mathbf{w}] \equiv \frac{\partial \mathbf{w}}{\partial t} - \mathrm{ad}_{\mathbf{u}}\mathbf{w}. \tag{5.9}$$

This Euler–Poincaré form of the variations may also be verified by a direct tensorial calculation, which is given in Ref. 6. The spatial part of this equation gives the variation of the spatial part of the field $\mathbf{u}$.

Variations of the field ψ also induce variations of the density F, in the same way as the parameter variations are induced in the general theory for the Euler–Poincaré equations [see Eq. (5.1)]. Either from that equation, or by direct calculations, these variations are computed to be

$$\delta F = -\nabla_{\mathbf{z}} \cdot (F\mathbf{w}), \tag{5.10}$$

which is equivalent to the formula

$$\delta(F\ d\mathbf{x}\ d\mathbf{v}) = -\pounds_{\mathbf{w}}(F\ d\mathbf{x}\ d\mathbf{v}).$$

C. Computation of the variations

With these formulas for $\delta\mathbf{u}$ and δF in place, we compute

$$\delta\mathfrak{S}_{\mathrm{red}} = \int dt \int d\mathbf{x}\ d\mathbf{v}\ F\left[(m\mathbf{u}_s + m(\mathbf{u}_s - \mathbf{v}) + q\mathbf{A}) \cdot \left(\frac{\partial}{\partial t}\mathbf{w} + [\mathbf{u},\mathbf{w}] \right) \right] - \nabla_{\mathbf{z}} \cdot (F\mathbf{w}) \left(\frac{1}{2}(m|\mathbf{u}_s|^2 + m|\mathbf{u}_s - \mathbf{v}|^2) + q\mathbf{u}_s \cdot \mathbf{A} - q\Phi \right). \tag{5.11}$$

Integrating by parts and dropping boundary terms gives

$$\delta\mathfrak{S}_{\mathrm{red}} = \int dt \int d\mathbf{x}\ d\mathbf{v}\ \mathbf{w} \cdot \left[-\frac{\partial}{\partial t}\left(Fm\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right) \right) - \mathrm{ad}^*_{\mathbf{u}}\left(Fm\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right) \right) + F\nabla_{\mathbf{z}}\left(\frac{1}{2}m|\mathbf{u}_s|^2 + \tfrac{1}{2}m|\mathbf{u}_s - \mathbf{v}|^2 + q\mathbf{u}_s \cdot \mathbf{A} - q\Phi \right) \right]. \tag{5.12}$$

Expanding the ad* term using formula (5.5) results in

$$\begin{aligned} \delta\mathfrak{S}_{\mathrm{red}} = \int dt \int d\mathbf{x}\ d\mathbf{v}\ \mathbf{w} \cdot \Bigg[& -\frac{\partial F}{\partial t} m\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right) - Fm\frac{\partial}{\partial t}\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right) \\ & - (\mathbf{u} \cdot \nabla_{\mathbf{z}})\left(Fm\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right) \right) - Fm\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right)(\nabla_{\mathbf{z}} \cdot \mathbf{u}) \\ & - \left(Fm\left(\mathbf{u}_{sj} + (\mathrm{u}_{sj} - \mathrm{v}_j) + \frac{q}{m}\ \mathrm{A}_j \right) \right)\nabla_{\mathbf{z}}u^j + F\nabla_{\mathbf{z}}\ (\tfrac{1}{2}m|\mathbf{u}_s|^2 + \tfrac{1}{2}m|\mathbf{u}_s - \mathbf{v}|^2 + q\mathbf{u}_s \cdot \mathbf{A} - q\Phi) \Bigg]. \end{aligned} \tag{5.13}$$

We expand the products to obtain

$$\begin{aligned} \delta\mathfrak{S}_{\mathrm{red}} = \int dt \int d\mathbf{x}\ d\mathbf{v}\ \mathbf{w} \cdot \Bigg\{ & -m\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right)\left(\frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{z}}F \right) \\ & - Fm\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right)(\nabla_{\mathbf{z}} \cdot \mathbf{u}) - Fm\left[\left(\frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla_{\mathbf{z}}) \right)\left(\mathbf{u}_s + (\mathbf{u}_s - \mathbf{v}) + \frac{q}{m}\mathbf{A} \right) + \frac{q}{m}\nabla_{\mathbf{z}}\Phi \right] \\ & - Fm\left(\mathrm{u}_{sj} + (\mathrm{u}_{sj} - \mathrm{v}_j) + \frac{q}{m}\ \mathrm{A}^j \right)\nabla_{\mathbf{z}}u^j + Fm\mathrm{u}_{sj}\nabla_{\mathbf{z}}\mathrm{u}_s^j + Fq\mathrm{A}^j\nabla_{\mathbf{z}}\mathrm{u}_{sj} \\ & + Fm(\mathrm{u}_{sj} - \mathrm{v}_j)\nabla_{\mathbf{z}}(\mathrm{u}_s^j - \mathrm{v}_s^j) + Fq\mathrm{u}_{sj}\nabla_{\mathbf{z}}\mathrm{A}^j \Bigg\}. \end{aligned} \tag{5.14}$$

Consider the last two lines of Eq. (5.14). Upon writing $\mathbf{w}=(\mathbf{w}_1,\mathbf{w}_2)$, where $\mathbf{w}_1,\mathbf{w}_2\in\mathbb{R}^3$, these lines reduce to

$$\begin{aligned}
&-Fm(\mathbf{u}_s+(\mathbf{u}_s-\mathbf{v}))\cdot(\mathbf{w}_1\cdot\nabla_{\mathbf{x}}+\mathbf{w}_2\cdot\nabla_{\mathbf{v}})\mathbf{u}+Fm(\mathbf{u}_s+(\mathbf{u}_s-\mathbf{v}))\cdot(\mathbf{w}_1\cdot\nabla_{\mathbf{x}}+\mathbf{w}_2\cdot\nabla_{\mathbf{v}})\mathbf{u}_s\\
&\quad -Fq\mathbf{A}\cdot(\mathbf{w}_1\cdot\nabla_{\mathbf{x}}+\mathbf{w}_2\cdot\nabla_{\mathbf{v}})\mathbf{u}+Fq\mathbf{A}\cdot(\mathbf{w}_1\cdot\nabla_{\mathbf{x}}+\mathbf{w}_2\cdot\nabla_{\mathbf{v}})\mathbf{u}_s+Fq\mathbf{u}_s\cdot(\mathbf{w}_1\cdot\nabla_{\mathbf{x}}+\mathbf{w}_2\cdot\nabla_{\mathbf{v}})\mathbf{A}\\
&\quad -Fm(\mathbf{u}_s-\mathbf{v})\cdot(\mathbf{w}_1\cdot\nabla_{\mathbf{x}}+\mathbf{w}_2\cdot\nabla_{\mathbf{v}})\mathbf{v} \;=\; Fq\mathbf{u}_s(\mathbf{w}_1\cdot\nabla_{\mathbf{x}})\mathbf{A}-Fm(\mathbf{u}_s-\mathbf{v})\cdot\mathbf{w}_2. \qquad (5.15)
\end{aligned}$$

The first three lines cancel to zero because they only involve spatial velocity projections, where $\mathbf{u}=\mathbf{u}_s$. The last line follows upon using $\nabla_{\mathbf{x}}\mathbf{v}=0$ and $\nabla_{\mathbf{v}}\mathbf{A}=0$, which hold, respectively, because $\mathbf{v}$ is an independent coordinate and $\mathbf{A}$ is a function of space alone. Similarly, and under the additional observation that $\nabla_{\mathbf{z}}\Phi=(\nabla_{\mathbf{x}}\Phi,0)$ because the potential Φ also does not depend on velocity, the other three lines of Eq. (5.14) are purely spatial, i.e., the projection onto the last three coordinates would give zero, and hence the contribution to the variation of the action $\delta\mathfrak{S}_{\mathrm{red}}$ from $\mathbf{w}_2$ comes only from the calculation in Eq. (5.15). Stationarity of the action under the velocity components of the variation, $\mathbf{w}_2$, then implies

$$Fm(\mathbf{u}_s-\mathbf{v})=0, \quad \text{i.e.,} \quad \mathbf{u}_s=\mathbf{v}. \qquad (5.16)$$

Consequently, in Eq. (5.14) we can write $\mathbf{u}$ as $(\mathbf{v},\mathbf{a})$ where $\mathbf{a}$ is yet to be determined, and we can also replace $\mathbf{u}_s-\mathbf{v}$ with zero. On doing this, the contribution to the variation of the action from $\mathbf{w}_1$ becomes

$$\begin{aligned}
\delta\mathfrak{S}_{\mathrm{red}}=\int dt\int d\mathbf{x}\,d\mathbf{v}\,\mathbf{w}_1\cdot\Bigg[&-(m\mathbf{v}+q\mathbf{A})\left(\frac{\partial F}{\partial t}+\nabla_{\mathbf{z}}\cdot(F\mathbf{u})\right)\\
&-F\left(m\frac{\partial\mathbf{v}}{\partial t}+m(\mathbf{v}\cdot\nabla_{\mathbf{x}})\mathbf{v}+m(\mathbf{a}\cdot\nabla_{\mathbf{v}})\mathbf{v}+q\frac{\partial\mathbf{A}}{\partial t}+q\nabla_{\mathbf{x}}\Phi+q\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{A})\right)\Bigg]. \qquad (5.17)
\end{aligned}$$

Here, we have used standard vector identities in obtaining the result

$$\mathbf{w}\cdot qF(\mathrm{u}_{sj}\nabla_{\mathbf{z}}\mathrm{A}_s^j-(\mathbf{u}\cdot\nabla_{\mathbf{z}})\mathbf{A})=qF\mathbf{w}_1\cdot(\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{A})). \qquad (5.18)$$

Referring to the continuity equation (4.3) for F and using the identities $\partial\mathbf{v}/\partial t=0$ and $\nabla_{\mathbf{x}}\mathbf{v}=0$ reduces Eq. (5.17) to

$$\delta\mathfrak{S}_{\mathrm{red}}=-\int dt\int d\mathbf{x}\,d\mathbf{v}\,\mathbf{w}_1\cdot F\left(m\mathbf{a}+q\nabla_{\mathbf{x}}\Phi+q\frac{\partial\mathbf{A}}{\partial t}-q\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{A})\right).$$

Therefore, $\delta\mathfrak{S}_{\mathrm{red}}=0$ implies that

$$m\mathbf{a}=-q\nabla_{\mathbf{x}}\Phi-q\frac{\partial\mathbf{A}}{\partial t}+q\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{A}). \qquad (5.19)$$

Now consider what the invariance of the Boltzmann function f implies. By Eq. (4.2) and substitution for $\mathbf{u}=(\mathbf{v},\mathbf{a})$ we obtain

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}f-\frac{q}{m}\left[\left(\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)-\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{A})\right]\cdot\nabla_{\mathbf{v}}f=0, \qquad (5.20)$$

and so, along with Eqs. (5.8), we have recovered the full Maxwell–Vlasov system from stationarity of the action (5.6) entirely in the Eulerian description.

VI. THE GENERALIZED LEGENDRE TRANSFORMATION

A. Introduction

Before passing to the Hamiltonian description of the Maxwell–Vlasov equations, we pause to explain the theoretical background of how one does this when there are degeneracies. This section can be skipped if one is willing to simply take on faith that one should *do the Legendre transformation slowly and carefully* when there are degeneracies.

As explained in Ref. 3, one normally thinks of passing from Euler–Poincaré equations on a Lie algebra $\mathfrak{g}$ to Lie–Poisson equations on the dual $\mathfrak{g}^*$ by means of the Legendre transformation. In some situations involving the Euler–Poincaré equations, one starts with a Lagrangian on $\mathfrak{g}\times V^*$ and performs a *partial* Legendre transformation, in the variable ξ only, by writing

$$\mu=\frac{\delta l}{\delta\xi},\quad h(\mu,a)=\langle\mu,\xi\rangle-l(\xi,a). \tag{6.1}$$

Since

$$\frac{\delta h}{\delta\mu}=\xi+\left\langle\mu,\frac{\delta\xi}{\delta\mu}\right\rangle-\left\langle\frac{\delta l}{\delta\xi},\frac{\delta\xi}{\delta\mu}\right\rangle=\xi, \tag{6.2}$$

and $\delta h/\delta a=-\delta l/\delta a$, we see that the Euler–Poincaré equations (3.4) for $\xi\in\mathfrak{g}$ and $\dot{a}(t)=-a(t)\xi(t)$ imply the Hamiltonian semidirect-product Lie–Poisson equations for $\mu\in\mathfrak{g}^*$. Namely,

$$\frac{\partial}{\partial t}\mu=-\mathrm{ad}^*_{(\delta h/\delta\mu)}\mu-\frac{\delta h}{\delta a}\diamond a=\{\mu,h\}_{LP},\quad \frac{\partial}{\partial t}a=-a\frac{\delta h}{\delta\mu}=\{a,h\}_{LP}, \tag{6.3}$$

with $(+)$ Lie–Poisson bracket on $\mathfrak{g}^*\times V^*$ given by

$$\{g,h\}_{LP}=-\left\langle\mu,\mathrm{ad}_{(\delta h/\delta\mu)}\frac{\delta g}{\delta\mu}\right\rangle+\left\langle a,\frac{\delta g}{\delta a}\frac{\delta h}{\delta\mu}-\frac{\delta h}{\delta a}\frac{\delta g}{\delta\mu}\right\rangle. \tag{6.4}$$

If the Legendre transformation (6.1) is invertible, then one can also pass Lie–Poisson equations to the Euler–Poincaré equations together with the equations $\dot{a}(t)=-a(t)\xi(t)$.

It is important in this paper to give a detailed explanation that incorporates the degeneracy of the parameter-dependent system together with the role of symmetry. Unlike the examples considered in Ref. 6 such as compressible flow or MHD, in the case of the Maxwell–Vlasov system or even the Vlasov–Poisson system, the Lagrangian L_{a_0} corresponding to the action in Eq. (5.6) is *degenerate*, since it does not depend on the variables $\dot{\Phi}$ and $\dot{\mathbf{v}}$. In other words, *the degeneracy and corresponding constraints that appear in Vlasov plasmas are more serious than for fluids or the heavy top, etc.* To deal with this degeneracy, we shall use the generalized Legendre transformation in the context of Lagrangian submanifolds, as described in Ref. 32. This is also related to the Dirac theory of constraints (see Ref. 33). In particular, we shall take special care to ensure that the Hamiltonian formulation of the Maxwell–Vlasov system preserves the constraints associated with the degeneracy of its Lagrangian.

B. The general construction

Let Q be a manifold and $\pi:T^*Q\to Q$ be the cotangent bundle of Q. Then TT^*Q is a symplectic manifold with a symplectic form that can be written in two distinct ways as the exterior derivative of two intrinsic one-forms. These two one-forms are denoted λ and χ and are given in coordinates by

$$\lambda=\dot{p}\;dq+p\;d\dot{q} \tag{6.5}$$

and

$$\chi=\dot{p}\;dq-\dot{q}\;dp, \tag{6.6}$$

where (q,p) are coordinates for T^*Q and $(q,p,\dot{q},\dot{p})$ are the corresponding coordinates for TT^*Q. For the intrinsic definitions of these forms, see Ref. 33.

Let $L:J\to\mathbb{R}$ be a Lagrangian defined on a submanifold $J\subset TQ$ called the ***Lagrangian constraint***. The Legendre transformation is a procedure to obtain a Hamiltonian $H:K\to\mathbb{R}$ defined on a submanifold $K\subset T^*Q$, called the ***Hamiltonian constraint***. The Euler–Lagrange equations are

$$\lambda = dL \quad \text{on} \quad J, \tag{6.7}$$

while the Hamilton equations are

$$\chi = -dH \quad \text{on} \quad K. \tag{6.8}$$

The abbreviated expressions (6.7) and (6.8) stand for

$$\lambda = d(L\circ T\pi) \quad \text{on} \quad (T\pi)^{-1}(J) \tag{6.9}$$

and

$$\chi = -d(H\circ\tau^{-1}) \quad \text{on} \quad (\tau)^{-1}(K), \tag{6.10}$$

where τ is the canonical projection $\tau:TT^*Q\to T^*Q$, given in coordinates by $\tau(q,p,\dot{q},\dot{p})=(q,p)$. The map $T\pi$ is given by $T\pi(q,p,\dot{q},\dot{p})=(q,\dot{q})$.

Both the Euler–Lagrange and Hamilton equations define the same Lagrangian submanifold D of TT^*Q. The Lagrangian and Hamiltonian L and H are the generating functions with respect to the one-forms λ and χ, respectively.

The ***generalized Legendre transformation*** consists of the following steps:

Step 1: For each $(q,p)\in T^*Q$ define

$$K(q,p)=\left\{(q,\dot{q})\in T_qQ \,\middle|\, \frac{\partial}{\partial\dot{q}}(p\dot{q}-L(q,\dot{q}))=0\right\}, \tag{6.11}$$

and let

$$K=\{(q,p)\in T^*Q \,|\, K(q,p)\neq\emptyset\}. \tag{6.12}$$

Assumption: Assume that for each $(q,p)\in K$, the submanifold $K(q,p)$ is connected. This implies that the stationary value

$$\mathrm{stat}_{\dot{q}}(p\dot{q}-L(q,\dot{q})) \tag{6.13}$$

of $p\dot{q}-L(q,\dot{q})$ on $K(q,p)$ is uniquely defined; that is, it does not depend on $\dot{q}$.

Step 2: Define $H:K\to\mathbb{R}$ as follows:

$$H(q,p)=\mathrm{stat}_{\dot{q}}(p\dot{q}-L(q,\dot{q})). \tag{6.14}$$

C. The generalized Legendre transformation with parameters and symmetry

Now we adapt this methodology to the case of parameter-dependent Lagrangians with symmetry. Let $L_{a_0}:TG\times T\mathcal{Q}\to\mathbb{R}$ be a Lagrangian depending on a parameter $a_0\in V^*$. Assume that G acts on V^* on the right and denote by ag the action of $g\in G$ on $a\in V^*$. Assume also the following invariance property:

$$L_{ah}(gh,\dot{g}h,q,\dot{q})=L_a(g,\dot{g},q,\dot{q}), \tag{6.15}$$

for all $g,h\in G$, all $(q,\dot{q})\in T\mathcal{Q}$, and all $a\in V^*$. A typical element of $T^*G\times T^*\mathcal{Q}$ will be denoted (g,α_g,q,ν_q) or simply (g,α,q,ν). For each $a_0\in V^*$ and $(g,\alpha)\in T^*G$, define

$$K_{a_0}(g,\alpha,q,\nu)=\left\{(g,\dot{g},q,\dot{q})\left|\frac{\partial}{\partial \dot{g}}(\alpha\dot{g}+\nu\dot{q}-L_{a_0}(g,\dot{g},q,\dot{q}))=0\right.\right.$$
$$\left.\text{and }\frac{\partial}{\partial \dot{q}}(\alpha\dot{g}+\nu\dot{q}-L_{a_0}(g,\dot{g},q,\dot{q}))=0\right\}. \tag{6.16}$$

One can immediately check for any $a_0\in V^*$, $h\in G$, and $(g,\alpha,q,\nu)\in T^*G\times T^*\mathcal{Q}$ that $K_{a_0h}(gh,\alpha h,q,\nu)=K_{a_0}(g,\alpha,q,\nu)h$. Define

$$K_{a_0}=\{(g,\alpha,q,\nu)|K_{a_0}(g,\alpha,q,\nu)\neq\emptyset\}. \tag{6.17}$$

Then one can easily prove for any $h\in G$ that $K_{a_0h}=K_{a_0}h$. Define

$$K=\{(g,\alpha,q,\nu,a)|K_a(g,\alpha,q,\nu)\neq\emptyset\}. \tag{6.18}$$

Then $K\subset T^*G\times T^*\mathcal{Q}\times V^*$ is an invariant subset under the action of G given by $(g,\alpha,q,\nu,a)h=(gh,\alpha h,q,\nu,ah)$. Now for each $a_0\in V^*$ we define $H_{a_0}:K_{a_0}\to\mathbb{R}$ by

$$H_{a_0}(g,\alpha,q,\nu)=\alpha\dot{g}+\nu\dot{q}-L_{a_0}(g,\dot{g},q,\dot{q}), \tag{6.19}$$

for any $(g,\dot{g},q,\dot{q})\in K_{a_0}(g,\alpha,q,\nu)$. Then, according to the general theory explained above, Hamilton's equations are, for each $a_0\in V^*$, $-dH_{a_0}=\chi$ on K_{a_0}, where

$$\chi=\dot{\alpha}\ dg-\dot{g}\ d\alpha+\dot{\nu}\ dq-\dot{q}\ d\nu. \tag{6.20}$$

One can also easily prove, using the previous equalities, that $H_{a_0}(g,\alpha,q,\nu)$ has the following invariance property:

$$H_{a_0h}(gh,\alpha h,q,\nu)=H_{a_0}(g,\alpha,q,\nu). \tag{6.21}$$

Let $\mathfrak{s}^*$ be the dual of the semidirect product Lie algebra $\mathfrak{s}=\mathfrak{g}\circledS V$. Then define $\mathcal{K}\subset\mathfrak{s}^*\times T^*\mathcal{Q}$ by

$$\mathcal{K}=\{(\alpha,q,\nu,a)\in\mathfrak{s}^*\times T^*\mathcal{Q}|(e,\alpha,q,\nu,a)\in K\},$$

and the Hamiltonian $h_{\mathcal{K}}:\mathcal{K}\to\mathbb{R}$ by $h_{\mathcal{K}}(\alpha,a,q,\nu)=H_a(e,\alpha,q,\nu)$. Thus, $h_{\mathcal{K}}$ is the restriction to $\mathcal{K}\subset\mathfrak{s}^*$ of the right invariant Hamiltonian $H:K\to\mathbb{R}$ given by $H(g,\alpha,q,\nu,a)=H_a(g,\alpha,q,\nu)$. Then, by a natural generalization of semidirect product theory to include constrained Hamiltonian systems, we have that Hamilton's equations on $\mathcal{K}\subset\mathfrak{s}^*$ generated by $h_{\mathcal{K}}$ give the evolution of the system on $\mathcal{K}$ determined by the Poisson–Hamilton equations $\dot{f}=\{f,h_{\mathcal{K}}\}$ on the Poisson submanifold $\mathcal{K}\subset\mathfrak{s}^*\times T^*\mathcal{Q}$, where the Poisson structure is defined in a natural way. More precisely, we have the Dirac brackets on K (see, for instance, Ref. 34 or 3) which, by reduction, give the brackets on $\mathcal{K}$. This is the abstract procedure underlying the computations we do in the specific case of plasmas given in the next section.

VII. HAMILTONIAN FORMULATION

We now pass to the corresponding Hamiltonian formulation of the Maxwell–Vlasov system (2.4) and (2.5) in the Eulerian description by taking the Legendre transform of the reduced action (5.6).

A. The role of the general theory

From the geometrical point of view, we simply apply the generalized Legendre transformation described abstractly in Sec. VI to the degenerate Lagrangian

$$L_{f_0,\varphi_0}(\psi,\dot{\psi},\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}}).$$

This Lagrangian is degenerate because it does not depend on the variables $\dot{\Phi}$ and $\dot{\mathbf{v}}$. The theory described in Sec. VI may be applied to this action on $T(\mathscr{F}\times\mathscr{V}\times\mathscr{A})$. The action of the group $\mathrm{Diff}(T\mathbb{R}^3)$ on the factor $\mathscr{F}$ for this Lagrangian is given as before, while the actions on the factors $\mathscr{V}$ and $\mathscr{A}$ are trivial. It is easy to see that the Hamiltonian constraint for each f_0 is $K_{f_0}\subset T^*(\mathrm{Diff}(T\mathbb{R}^3)\times\mathscr{V}\times\mathscr{A})$, defined by the conditions

$$\Psi=\frac{\delta L}{\delta\dot{\Phi}}=0 \quad \text{and} \quad \mathbf{m}_v=\frac{\delta L}{\delta\dot{\mathbf{v}}}=0.$$

These conditions impose constraints, which for consistency must be dynamically preserved.

B. Calculation of the transformed equations

We will perform the calculations in detail, working with the reduced Lagrangian rather than the Lagrangian

$$L_{f_0,\varphi_0}(\psi,\dot{\psi},\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}})$$

and setting $\varphi_0=1$ as usual.

We start with the action (5.6) for the Maxwell–Vlasov system in the Eulerian description,

$$\mathfrak{S}_{\mathrm{red}}(\mathbf{u},\Phi,\dot{\Phi},\mathbf{A},\dot{\mathbf{A}})=\int dt\int d\mathbf{x}\ d\mathbf{v}F(\mathbf{x},\mathbf{v},t)\left(\frac{1}{2}m|\mathbf{u}_s|^2+\frac{1}{2}m|\mathbf{u}_s-\mathbf{v}|^2-q\Phi+q\mathbf{u}_s\cdot\mathbf{A}\right)$$
$$+\frac{1}{2}\int dt\int d\mathbf{x}\left|\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right|^2-|\nabla_{\mathbf{x}}\times\mathbf{A}|^2. \tag{7.1}$$

This leads immediately to

$$\frac{\delta l}{\delta\dot{\mathbf{A}}}=\nabla_{\mathbf{x}}\Phi+\frac{\partial\mathbf{A}}{\partial t}=-\mathbf{E}, \tag{7.2}$$

and so (minus) the electric field variable $\mathbf{E}$ is the field momentum density canonically conjugate to the magnetic potential. Let us define the material momentum density in six dimensions,

$$\mathbf{m}\equiv\frac{\delta l}{\delta\mathbf{u}}. \tag{7.3}$$

We write $\mathbf{m}=(\mathbf{m}_s,\mathbf{m}_v)$, where $\mathbf{m}_s$ is the projection of $\mathbf{m}$ onto the first three coordinate positions, and $\mathbf{m}_v$ is the projection onto the last three places. We think of $\mathbf{m}_s$ and $\mathbf{m}_v$ also as vectors in six dimensions. From the Lagrangian we see that

$$\mathbf{m}_s=F(m\mathbf{u}_s+m(\mathbf{u}_s-\mathbf{v})+q\mathbf{A}) \quad \text{and} \quad \mathbf{m}_v=0. \tag{7.4}$$

Proceeding with the Legendre transform of our action (7.1) results in a corresponding (reduced) Hamiltonian function written in terms of the velocities,

$$h=\int d\mathbf{x}\ d\mathbf{v}\ F\left(m|\mathbf{u}_s|^2-\frac{1}{2}m|\mathbf{v}|^2+q\Phi\right)+\mathbf{m}_v\cdot\mathbf{a}+\frac{1}{2}\int d\mathbf{x}(|\mathbf{E}|^2+|\nabla_{\mathbf{x}}\times\mathbf{A}|^2+2\mathbf{E}\cdot\nabla_{\mathbf{x}}\Phi), \tag{7.5}$$

where $\mathbf{a}$ denotes the projection of $\mathbf{u}$ onto its last three entries. Transforming this to the momentum variables gives

$$h=\int d\mathbf{x}\,d\mathbf{v}\,\frac{1}{4Fm}|\mathbf{m}_s+mF\mathbf{v}-qF\mathbf{A}|^2-\frac{1}{2}mF|\mathbf{v}|^2+qF\Phi+\mathbf{m}_v\cdot\mathbf{a}$$
$$+\frac{1}{2}\int d\mathbf{x}\left(|\mathbf{E}|^2+|\nabla_\mathbf{x}\times\mathbf{A}|^2+2\mathbf{E}\cdot\nabla_\mathbf{x}\Phi\right). \tag{7.6}$$

The variation of this Hamiltonian with respect to **m**, **a**, **E**, **A**, F, and Φ is given by

$$\delta h=\int d\mathbf{x}\,d\mathbf{v}\left[\mathbf{u}\cdot\delta\mathbf{m}+\mathbf{m}_v\cdot\delta\mathbf{a}-qF\mathbf{u}_s\cdot\delta\mathbf{A}+qF\,\delta\Phi-\left(\frac{1}{2}m|\mathbf{u}_s|^2+\frac{1}{2}m|\mathbf{u}_s-\mathbf{v}|^2+q\mathbf{u}_s\cdot\mathbf{A}\right.\right.$$
$$\left.\left.-q\Phi\right)\delta F\right]+\int d\mathbf{x}(\mathbf{E}+\nabla_\mathbf{x}\Phi)\cdot\delta\mathbf{E}-(\nabla_\mathbf{x}\cdot\mathbf{E})\,\delta\Phi+\nabla_\mathbf{x}\times(\nabla_\mathbf{x}\times\mathbf{A})\cdot\delta\mathbf{A}. \tag{7.7}$$

This expression allows one to read off the evolution equations for the electromagnetic field:

$$\frac{\partial\mathbf{A}}{\partial t}=-\frac{\delta h}{\delta\mathbf{E}}=-\mathbf{E}-\nabla_\mathbf{x}\Phi,\quad\text{i.e.,}\quad\mathbf{E}=-\nabla_\mathbf{x}\Phi-\frac{\partial\mathbf{A}}{\partial t},$$
$$\frac{\delta h}{\delta\Phi}=0=-\nabla_\mathbf{x}\cdot\mathbf{E}+q\int d\mathbf{v}\,F,\quad\text{i.e.,}\quad\nabla_\mathbf{x}\cdot\mathbf{E}=q\int d\mathbf{v}F:=\rho, \tag{7.8}$$
$$\frac{\partial\mathbf{E}}{\partial t}=\frac{\delta h}{\delta\mathbf{A}}=\nabla_\mathbf{x}\times(\nabla_\mathbf{x}\times\mathbf{A})-q\int d\mathbf{v}\,F\mathbf{u}_s,\quad\text{i.e.,}\quad\frac{\partial\mathbf{E}}{\partial t}=\nabla_\mathbf{x}\times\mathbf{B}-\mathbf{j}.$$

Note that the constraint $\delta h/\delta\Phi=0$ (Gauss' law) arises from the absence of $\dot{\Phi}$ dependence in l.

The general theory of Sec. VI shows that F is an element of the second factor of the semidirect product and so its evolution is given by Lie dragging as a density. Likewise, f is Lie dragged as a scalar and m_i satisfies a Lie–Poisson evolution equation:

$$\frac{\partial F}{\partial t}=-\nabla_\mathbf{z}\cdot(F\mathbf{u}),\quad\frac{\partial f}{\partial t}=-\mathbf{u}\cdot\nabla_\mathbf{z}\,f, \tag{7.9}$$
$$\frac{\partial m_i}{\partial t}=-\frac{\partial}{\partial z^j}m_iu^j-m_j\frac{\partial}{\partial z^i}u^j-F\frac{\partial}{\partial z^i}\frac{\delta h}{\delta F}.$$

The first two of these equations reflect the assumptions that were made in the definitions of f and F, while the last equation encodes the dynamics for the system. We first consider the case where the momentum component i takes the values 4,5,6. In this case,

$$-\frac{\partial m_i}{\partial t}=m_{sj}\frac{\partial}{\partial z^i}u^j+m_{vj}\frac{\partial}{\partial z^i}u^j-F\frac{\partial}{\partial z^i}\left(\tfrac{1}{2}m|\mathbf{u}_s|^2+\tfrac{1}{2}m|\mathbf{u}_s-\mathbf{v}|^2+q\mathbf{u}_s\cdot\mathbf{A}-q\Phi\right)$$
$$=m_{vj}\frac{\partial}{\partial z^i}u^j+Fm(u_{sj}-v_j)\frac{\partial u^j}{\partial z^i}-Fm(u_{sj}-v_j)\frac{\partial}{\partial z^i}(u_s^j-v_s^j)-qFu_{sj}\frac{\partial A_s^j}{\partial z^i}+Fq\frac{\partial}{\partial z^i}\Phi, \tag{7.10}$$

where $i=4,5,6$. In the second line of Eq. (7.10), we have substituted for $\mathbf{m}_s$ from Eq. (7.4) and rearranged terms. Here $\mathbf{m}_v=0$, because l does not depend on $\dot{\mathbf{v}}$. Setting $\mathbf{m}_v=0$ initially in Eq. (7.10) ensures that $\mathbf{m}_v\equiv 0$ persists throughout the ensuing motion, for potentials Φ and **A** that are independent of **v** and provided the constraint holds that $\mathbf{u}_s=\mathbf{v}$, as in Eq. (5.16). Likewise, the Gauss' law constraint imposed by $\delta h/\delta\Phi=0$ also persists during the ensuing motion, as seen from the last equation of (7.8) and the first equation of (7.9), provided the constraint $\mathbf{u}_s=\mathbf{v}$ holds and F vanishes in the limit as $|\mathbf{v}|\to\infty$.

The spatial part of the evolution equation of **m** will produce the required single-particle dynamics. From Eq. (7.9), we have

$$\frac{\partial m_i}{\partial t} = -\frac{\partial}{\partial z^j} m_i u^j - m_j \frac{\partial}{\partial z^i} u^j - F \frac{\partial}{\partial z^i} \frac{\delta h}{\delta F}. \tag{7.11}$$

Setting $i=1,2,3$, in Eq. (7.11), then substituting for $\delta h/\delta F$ and using the relations

$$\mathbf{m}_s = F(m\mathbf{u}_s + m(\mathbf{u}_s - \mathbf{v}) + q\mathbf{A}), \quad \mathbf{m}_v \equiv 0,$$

and

$$\nabla_{\mathbf{v}} \Phi = 0, \quad \nabla_{\mathbf{v}} \mathbf{A} = 0,$$

yields the spatial components of the motion equation,

$$\frac{\partial m_{si}}{\partial t} = -\frac{\partial}{\partial z^j} m_{si} u^j - m_{sj} \frac{\partial}{\partial z^i} u^j - F \frac{\partial}{\partial z^i} \left(\frac{1}{2} m |\mathbf{u}_s|^2 + \frac{1}{2} m |\mathbf{u}_s - \mathbf{v}|^2 + q\mathbf{u}_s \cdot \mathbf{A} - q\Phi \right). \tag{7.12}$$

Substituting for $\mathbf{m}_s$ and then using the continuity relation $\partial F/\partial t + \nabla_{\mathbf{z}} \cdot (F\mathbf{u}) = 0$ gives

$$m \frac{\partial \mathrm{u}_{si}}{\partial t} + q \frac{\partial \mathrm{A}_{si}}{\partial t} = -u^j \frac{\partial}{\partial z^j} m \mathrm{u}_{si} - \mathrm{u}_s^j \frac{\partial}{\partial z^j} q \mathrm{A}_{si} - q \mathrm{A}_j \frac{\partial}{\partial z^i} u^j - q \frac{\partial \Phi}{\partial z^i} + q \frac{\partial}{\partial z^i} (\mathbf{u}_s \cdot \mathbf{A})$$
$$- \frac{1}{2} m \frac{\partial}{\partial z^i} |\mathbf{u}_s - \mathbf{v}|^2. \tag{7.13}$$

Rearranging this equation results in

$$m \left(\frac{\partial}{\partial t} + u^j \frac{\partial}{\partial z^j} \right) \mathbf{u}_s = q\mathbf{E} + q\mathbf{u}_s \times (\nabla_{\mathbf{x}} \times \mathbf{A}) - \frac{1}{2} m \frac{\partial}{\partial \mathbf{z}} |\mathbf{u}_s - \mathbf{v}|^2. \tag{7.14}$$

We may now evaluate this on the constraint set $\mathbf{u}_s = \mathbf{v}$ and thereby obtain the Lorentz force,

$$m\mathbf{a} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{7.15}$$

where $\mathbf{a}$ is the acceleration of a fluid parcel [the last three components in $\mathbf{u} = (\mathbf{v}, \mathbf{a})$]. As we have seen, in this Hamiltonian formulation of the Maxwell–Vlasov equations in the Eulerian description, the acceleration $\mathbf{a}$ in $\mathbf{u}$ is a vector Lagrange multiplier which imposes $\mathbf{m}_v = 0$. Equation (7.15) provides an expression for this Lagrange multiplier in terms of known dynamical variables and, as a consequence, we regain the equation for the acceleration of a charged particle in an electromagnetic field. The momentum constraint $\mathbf{m}_v = 0$ remains invariant when the electromagnetic potentials are independent of the phase space velocity coordinate $\mathbf{v}$ and the velocity constraint $\mathbf{u}_s = \mathbf{v}$ holds. Perhaps not unexpectedly, one finds that $\nabla_{\mathbf{z}} \cdot \mathbf{u} = 0$. Also, (minus) the electric field is canonically conjugate to the vector potential, and the electrostatic potential Φ plays the role of a Lagrange multiplier which imposes Gauss's law. Thus, our Hamiltonian formulation augments the usual Maxwell–Vlasov description of plasma dynamics by self-consistently deriving the particle acceleration by the Lorentz force $m\mathbf{a} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ instead of assuming it *a priori*.

C. The Poisson Hamiltonian structure

The general theory outlined briefly in Sec. VI also leads to the Poisson bracket structure for the Maxwell–Vlasov theory on the Hamiltonian side. However, our Hamiltonian description has a redundancy, namely the information for the particle trajectories can be recovered from the spatial plasma density. Explicitly, if we let $H(f) = (1/2)|\mathbf{v}|^2 + \Phi_f(\mathbf{x})$ be the single-particle Hamiltonian determined by the plasma density f, then the flow of this Hamiltonian function can be identified with the particle evolution map ψ. We can also think of this as a constraint on the level of equations of motion, as the Hamiltonian vector field of $H(f)$ must equal the time derivative of the map ψ, i.e., the particle velocity field in phase space. In other words, as is well known, the particle dynamics is completely determined by the plasma density dynamics. This may be regarded as a constraint on the system that leads to the elimination of the forward map as a dynamical variable. This ‘‘redundancy’’ is, of course, one of the sources of degeneracy of the Lagrangian and Hamiltonian structures.

Thus, the constraint of explicitly enforcing this consistency condition leads to a further ‘‘reduction’’ which again may be handled by the Dirac theory of constraints to arrive at the Hamiltonian structure in terms of the variables F (or equivalently f in view of the canonical nature of the particle transformations) and the electromagnetic potentials. The resulting Poisson bracket structure is given by the Lie–Poisson structure for the f's plus the canonical structure for the electromagnetic potentials, which was the starting point for Marsden and Weinstein,[7] who carried out the reduction of this bracket with respect to the action of the electromagnetic gauge group to obtain the final Maxwell–Vlasov bracket on the space with variables f, $\mathbf{E}$, and $\mathbf{B}$. This procedure was motivated by and corrected a bracket found by *ad hoc* methods in Ref. 15. We need not repeat this construction.

D. The Kelvin–Noether theorem

A final result worth mentioning is Kelvin's theorem for the Maxwell–Vlasov particle dynamics. These dynamics, given in the last equation in (7.9), may be rewritten as

$$\left(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\right)\left(\frac{1}{F}m_i dz^i\right)+d\,\frac{\delta h}{\delta F}=0, \tag{7.16}$$

so that

$$\frac{d}{dt}\oint_{\gamma(t)}\frac{1}{F}m_i\,dz^i=0, \tag{7.17}$$

for a loop $\gamma(t)$ which follows the particle trajectories in phase space. The Kelvin circulation integral in phase space,

$$I=\oint_{\gamma(t)}\frac{1}{F}m_i\,dz^i, \tag{7.18}$$

may be evaluated on the invariant constraint manifold $\mathbf{m}_v=0$ as

$$I=\oint_{\gamma(t)}(mu_{si}+qA_i)dx^i. \tag{7.19}$$

We recognize this integral as the *Poincaré invariant* for the single-particle motion in phase space.

The above result follows from the abstract Kelvin–Noether theorem by letting $\mathscr{C}:=\{\gamma:S^1\to T\mathbb{R}^3|\gamma\ \text{continuous}\}$ be the space of continuous loops in single-particle velocity phase space and letting the group $\mathrm{Diff}(T\mathbb{R}^3)$ act on $\mathscr{C}$ on the right by $(\eta,\gamma)\in\mathrm{Diff}(T\mathbb{R}^3)\times\mathcal{C}\mapsto\gamma\circ\eta\in\mathcal{C}$. The quantity $\mathscr{K}$ is chosen to be

$$\langle\mathscr{K}(\gamma,F),a\rangle=\oint_\gamma\frac{1}{F}a. \tag{7.20}$$

The abstract Kelvin–Noether theorem for the Maxwell–Vlasov equations in Euler–Poincaré form then reproduces the version of Kelvin's theorem given in (7.17).

VIII. CONCLUSION

In this paper we have cast Low's mixed Eulerian–Lagrangian action principle for Maxwell–Vlasov theory into a purely Eulerian description. In this description we find that Maxwell–Vlasov dynamics are governed by the Euler–Poincaré equations for right invariant motion on the diffeomorphism group of $\mathbb{R}^n$ ($n=6$ for three-dimensional Maxwell–Vlasov motion). These equations were recently discovered by Holm *et al.*[6] who investigated the class of Hamilton's principles which are right invariant under the subgroup of the diffeomorphisms which leaves invariant a set

$\mathscr{T}$ of tensor fields in the Eulerian variables. The Maxwell–Vlasov motions invariant under this subgroup are the steady Eulerian solutions, which, thus, are identified as relative equilibria. This identification of steady Eulerian Maxwell–Vlasov solutions as right invariant equilibria places these solutions into the Hamiltonian framework required for investigating their nonlinear stability characteristics using, e.g., the energy-Casimir method (see Ref. 16). It was this stated goal that first motivated Low to write his Lagrangian for Maxwell–Vlasov dynamics.

Thus, our formulation of a purely Eulerian action principle and its associated Euler–Poincaré equations and Hamiltonian framework advances Low's original intention of using his action principle for studying stability of plasma equilibria by placing the entire Maxwell–Vlasov equations (including the particle dynamics, field dynamics and probability distribution dynamics) into one self-consistent Hamiltonian picture in the Eulerian description. (As we discussed, Low used mixed aspects of both Eulerian and Lagrangian phase space descriptions in his action principle.)

Our Eulerian Hamilton's principle for Maxwell–Vlasov dynamics is constrained, and all of the corresponding Lagrange multipliers have been resolved. This Hamilton's principle is thus available for further approximations, e.g., by Hamilton's principle asymptotics (see, e.g., Ref. 27).

In summary, we have taken an existing action, due to Low,[1] for the Maxwell–Vlasov system of equations and demonstrated how to rederive this system as Euler–Poincaré equations. The Euler–Poincaré form emerges from Hamilton's principle for a system whose configuration space is a group and whose action is right invariant under a subgroup. This situation commonly appears in the Eulerian description of continuum mechanics. In the case of continuum mechanics, the dynamics takes place on the group of diffeomorphisms and the Eulerian variables are invariant under a subgroup of the diffeomorphism group. (This subgroup corresponds to steady Eulerian flows with nonzero velocity and vorticity.) We showed that this situation also occurs for the Maxwell–Vlasov equations of plasma dynamics in the Eulerian description, by showing that the variations considered take the appropriate form, and then deriving the Maxwell–Vlasov equations from the Hamilton's principle for the right invariant action (5.6) in Eulerian variables. We then passed to the Hamiltonian formulation of this system and found its Lie–Poisson structure.

As discussed in the Introduction, the Euler–Poincaré form of the dynamics is naturally adapted for applying Lagrange–D'Alembert methods for geometrical constraints and control as in Ref. 13. In future work, our Euler–Poincaré form of the Maxwell–Vlasov system shall be implemented to describe the control features of a plasma driven by an external antenna, following the lines of inquiry begun in the oscillation center approximation for plasmas by Similon *et al.*[35]

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