Relative Equilibria for the Generalized Rigid Body

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Abstract

This paper gives necessary and sufficient conditions for the (n-dimensional) generalized free rigid body to be in a state of relative equilibrium. The conditions generalize those for the case of the three-dimensional free rigid body, namely that the body is in relative equilibrium if and only if its angular velocity and angular momentum align, that is, if the body rotates about one of its principal axes. For the n-dimensional rigid body in the Manakov formulation, these conditions have a similar interpretation. We use this result to state and prove a generalized Saari's Conjecture (usually stated for the N-body problem) for the special case of the generalized rigid body.

Contents

1 Introduction

Introductory Remarks. The notion of a relative equilibrium (that is, a dynamical orbit for a mechanical system that also is the orbit of a one-parameter symmetry group) is a key ingredient for mechanical systems with symmetry, an idea that goes back to Routh and Poincaré in the 1800's. Relative equilibria played an important role in some of the founding works of modern geometric mechanics, such as Arnold [1966], Smale [1970], and Marsden and Weinstein [1974], and is now an important ingredient in the general theory. For example, much modern research in geometric mechanics involves notions of stability and bifurcation of relative equilibria. For the planar N-body problem, relative equilibria are uniformly rotating rigid solutions, and therefore, such configurations have a moment of inertia that is constant in time.

The present paper grew out of recent research activity on ***Saari's Conjecture*** (Saari [1970]): if a solution of the N-body problem of celestial mechanics has a constant moment of inertia, then it must be a relative equilibrium. Attempts to answer this conjecture sparked a number of interesting works in the context of the N-body problem. Since the notion of the inertia tensor makes sense for general mechanical systems with symmetry, where it is called the *locked inertia tensor*, and since it is an important ingredient in stability theory (see Simo, Lewis, and Marsden [1991]), it is natural to investigate the validity of the Saari Conjecture in the more general context of geometric mechanics.

A Relative Equilibrium Criterion for Mechanics on Lie Groups. Consider a configuration manifold Q and a Lie group G that acts freely and properly on the left on Q. A Lagrangian simple mechanical system with symmetry consists of a Lagrangian $L : TQ \to \mathbb{R}$ that has the form of kinetic minus potential energy and that is invariant under the tangent lifted action. Suppose that $(q_e, \dot{q}_e) \in TQ$ is a relative equilibrium that is generated by a Lie algebra element $\xi \in \mathfrak{g}$, the Lie algebra of G. That is, the group orbit $\exp(t\xi) \cdot q_e$ is a solution of the Euler–Lagrange equations. Letting $\mu := \mathbf{J}(q_e, \dot{q}_e)$, where $\mathbf{J} : TQ \to \mathfrak{g}^*$ is the standard equivariant momentum mapping associated to the action of G, it is a simple fact following from conservation and equivariance of $\mathbf{J}$ that $\xi \in \mathfrak{g}_\mu$, the Lie algebra of the isotropy subgroup G_μ.

Consider the specific case of *mechanics on Lie groups*, that is, the case when $Q = G$, a (finite-dimensional) Lie group G acting on itself by left multiplication and a kinetic energy Lagrangian that is left-invariant under the natural lift of the action to TG. This case goes by the name of *Euler-Poincaré theory* (see Marsden and Ratiu [1999] for a general discussion and background).

This paper establishes, in this case, a converse to the fact stated in the previous paragraph. Namely, *if g_e is in G, $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$ satisfy the conditions $\xi \in \mathfrak{g}_\mu$ and*

$\mu = \mathbf{J}(\xi_G(g_e))$, *then* $(g_e, \xi_G(g_e))$ *is a relative equilibrium.* By group invariance, it is of course enough to prove such a statement at the identity element of the group.

If $G = \mathrm{SO}(n)$ (the *real* proper orthogonal group), this result has an interesting interpretation in the context of the Manakov formulation of the rigid body in $\mathbb{R}^n$. (See, for example, Bloch, Crouch, Marsden, and Ratiu [2002] for a recent discussion of the left-invariant Manakov equations for $\mathrm{SO}(n)$ and references.) For $n = 3$ the result specializes to the well-known fact that a necessary and sufficient condition for relative equilibrium of a rigid body in $\mathbb{R}^3$ is the alignment of the angular velocity and the angular momentum, both measured in the spatial frame of reference.

The Status of the Classical Saari Conjecture. The classical N-body problem concerns the dynamics of particles with masses $m_A, 1 \leq A \leq N$ at positions $q_A(t) \in \mathbb{R}^n$ (usually $n = 2$ or 3), relative to a fixed inertial frame, interacting by a pairwise mutual Newtonian gravitational attraction (that is, the Newtonian $1/r$ attractive potential). As was mentioned above, Saari's Conjecture states that a solution of the N-body system has constant moment of inertia if and only if the system is in relative equilibrium, that is, if the system is in uniform rotation with a constant angular velocity about a fixed axis through the center of mass. The necessity of the condition of constant moment of inertia for the system to be in relative equilibrium is obvious for the planar problem. Saari's Conjecture asks that one prove the converse.

Saari's Conjecture has been proven for the planar three-body problem. McCord [2004] has proved this in the case of three equal masses, while Llibre and Piña [2002] and Moeckel [2004] have proved it for three unequal masses using computer-assisted methods. In addition, Diacu, Pérez-Chavela, and Santoprete [2004] have proved Saari's Conjecture for the case of collinear relative equilibria in the N-body problem.

However, the general conjecture is still open for $N \geq 4$.

The Generalized Saari Conjecture. The third author of this paper orally conjectured, at the Cincinnati Midwest Dynamical Systems meeting in October 2002, that the Saari Conjecture should have a generalization to more general mechanical systems with symmetry. However, the examples in Chenciner [2002, 2003], as well as in Roberts [2004] and Santoprete [2004], show that a generalized Saari Conjecture in the context of N-body systems in $\mathbb{R}^2$ or $\mathbb{R}^3$ with power-law potential functions and rotational ($\mathrm{SO}(2)$) symmetry about a fixed axis does not hold. Chenciner's counterexamples involve a Hamiltonian (or Lagrangian) with a Jacobi ($1/r^2$) potential, and Roberts' counterexamples use either a Jacobi potential or a class of homogeneous potentials with "masses" of opposite sign. Santoprete's counterexample uses four equal masses in a harmonic oscillator potential. In these examples, the $\mathrm{SO}(2)$-invariance of the problem makes it obvious that a relative equilibrium necessarily has constant moment of inertia.

The Naive and Refined Saari Conjecture. In this paper we also provide a counterexample to the naively stated generalized Saari Conjecture simply by using the dynamics of a free rigid body in $\mathbb{R}^3$, a problem with $SO(3)$ symmetry. This counterexample reveals that, even for a free generalized rigid body, the condition of having a relative equilibrium is not sufficient to ensure a constant-in-time spatial moment of inertia tensor, unlike the case of the planar N-body problem. With this counterexample in mind we propose and prove a refined generalized Saari Conjecture for a generalized rigid body; the key to the refinement is to consider more carefully which *components* of the locked inertia tensor should be constant.

Outline. The flow of this paper is as follows. In the following section, we set up the notation from geometric mechanics for a simple mechanical system with symmetry and interpret the concepts of moment of inertia and relative equilibrium in this context. In the third section we review geometric mechanics on Lie groups. In the fourth section we prove the main result and apply it to the rigid body in $\mathbb{R}^3$ and to the Manakov formulation of the rigid body in $\mathbb{R}^n$. In the fifth section we show the counterexample to a generalized Saari's Conjecture for the generalized rigid body and refine the conjecture appropriately, and then we conclude with suggestions for future investigations.

2 Simple Mechanical Systems with Symmetry

This section recalls the geometric mechanical interpretation of the moment of inertia and of a relative equilibrium in the context of a generic simple mechanical system with symmetry. This exposition borrows both the notation and the results of Marsden [1992]. We begin with some general facts about simple mechanical systems from the Lagrangian viewpoint.

Simple Mechanical Systems. A ***Lagrangian simple mechanical system with symmetry*** on a configuration manifold Q consists of a Lagrangian $L : TQ \to \mathbb{R}$ of the form kinetic energy minus potential energy that is invariant under the natural lift to TQ of the free and proper left action of a Lie group G on Q. The configuration space Q possesses, correspondingly, a metric $\langle\langle\ ,\ \rangle\rangle$ whose quadratic form is the kinetic energy, and thus G acts by isometries (that is, the metric is invariant under the action of G). The ***momentum mapping*** corresponding to the action of G on TQ is the map $\mathbf{J} : TQ \to \mathfrak{g}^*$ (as before, $\mathfrak{g}$ is the Lie algebra of G), given by the formula

$$\mathbf{J}(v_q)(\xi) = \langle\langle v_q, \xi_Q(q)\rangle\rangle\ ,$$

where ξ_Q is the infinitesimal generator of the action on Q corresponding to $\xi \in \mathfrak{g}$. Of course, Noether's Theorem guarantees that $\mathbf{J}$ is conserved along solutions of the Euler–Lagrange equations.

Locked Inertia Tensor. The ***locked inertia tensor*** is defined to be the mapping $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$, for $q \in Q$, given by

$$\langle \mathbb{I}(q)\eta, \zeta\rangle := \langle\langle \eta_Q(q), \zeta_Q(q)\rangle\rangle = \mathbf{J}(\eta_Q(q))(\zeta)\,,$$

where $\langle\ ,\ \rangle$ is the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$.

The name comes about from the fact that if one has, for example, two freely spinning rigid bodies, connected by a ball-in-socket joint, then given a configuration q, the locked inertia tensor is the inertia tensor for the rigid body obtained by locking, or welding, the joint in this configuration.

We begin with the following equivariance result, which is an important ingredient in establishing the needed equivariance property of the *mechanical connection* and whose proof may be found in Simo, Lewis, and Marsden [1991] (see also Marsden [1992], §3.3), or which may be readily supplied by the reader with a little definition chasing; we just remark that the proof makes use of the following identity (see Marsden and Ratiu [1999], Chapter 9):

$$(\mathrm{Ad}_g\,\xi)_Q(q) = \Phi^*_{g^{-1}}(\xi_Q)(q) := T\Phi_g(\xi_Q)(\Phi_{g^{-1}}(q)), \tag{2.1}$$

where $\Phi_g : Q \to Q; q \mapsto g \cdot q$ denotes the action by the group element g.

Lemma 2.1 (Equivariance of the Locked Inertia Tensor). *Under the preceding assumptions, for each $g \in G, q \in Q$, and $\eta, \zeta \in \mathfrak{g}$, we have*

$$\langle \mathbb{I}(\Phi_g(q))\eta, \zeta\rangle = \left\langle \mathbb{I}(q)\,\mathrm{Ad}_{g^{-1}}\,\eta, \mathrm{Ad}_{g^{-1}}\,\zeta \right\rangle .$$

Notice in particular that if G is abelian then $\mathbb{I}$ is literally invariant under the group action.

Relative Equilibria. Let $(q_e, \dot{q}_e) \in TQ$ be a relative equilibrium, and $\mu := \mathbf{J}(q_e, \dot{q}_e)$. By definition of a relative equilibrium, there is a $\xi \in \mathfrak{g}$ such that the solution curve in TQ with initial condition $(q_e, \dot{q}_e)$ is given by the one-parameter family

$$t \mapsto \exp(t\xi) \cdot (q_e, \dot{q}_e)\,. \tag{2.2}$$

By Noether's Theorem, equivariance of $\mathbf{J}$, and the basic fact that elements of the group G that leave the set $\mathbf{J}^{-1}(\mu)$ invariant are necessarily in the isotropy subgroup G_μ (see, for example, Marsden and Weinstein [1974]), it also follows that $\xi \in \mathfrak{g}_\mu$, where $\mathfrak{g}_\mu$ is the Lie subalgebra of G_μ.

We also need to recall that the ***augmented potential*** is defined to be

$$V_\xi(q) := V(q) - \frac{1}{2}\langle \mathbb{I}(q)\xi, \xi\rangle\ .$$

A powerful tool for the identification of relative equilibria is the "augmented potential proposition," which says that $(q_e, \xi_Q(q_e))$ is a relative equilibrium if and only if q_e is a critical point of V_ξ. There is a similar and also very useful criterion for the ***amended potential***, which is given by

$$V_\mu(q) = V(q) + \frac{1}{2}\langle \mu, \mathbb{I}(q)^{-1}\mu\rangle\,.$$

3 Mechanics on Lie Groups

In this section we explore some general criteria for relative equilibria in the specific case of mechanics on Lie groups; that is, when $Q = G$. This topic, which really started in the classical 1901 work of Poincaré, was revived in modern form in the important paper of Arnold [1966]. One can find most of what we will need in Abraham and Marsden [1978], §4.4, and in Marsden and Ratiu [1999], Chapter 13.

Some General Facts. Consider a Lagrangian simple mechanical system with symmetry on G. In other words, assume a Lagrangian on TG of the simple-mechanical form that is left invariant under the natural lift of the left action of G on itself. Since the potential must be constant, the Lagrangian will be assumed to have only a kinetic energy term.

Recall that ***body coordinates*** are defined by the map $\lambda : TG \to G \times \mathfrak{g}$ given by

$$\lambda(v_g) = (g, T_g L_{g^{-1}}(v_g)),$$

and ***spatial coordinates*** are defined by the map $\rho : TG \to G \times \mathfrak{g}$ given by

$$\rho(v_g) = (g, T_g R_{g^{-1}}(v_g)).$$

We note the identities

$$\lambda \circ TL_g \circ \lambda^{-1}(h, \xi) = (gh, \xi)$$

and

$$\rho \circ TL_g \circ \rho^{-1}(h, \xi) = (gh, Ad_g \xi).$$

As emphasized in Simo, Lewis, and Marsden [1991], the locked inertia tensor is naturally identified with the *spatial moment of inertia tensor* as the following calculations show:

$$\begin{aligned}
\langle \mathbb{I}(g)\eta, \zeta \rangle &= \langle\langle \eta_G(g), \zeta_G(g) \rangle\rangle \\
&= \left\langle\!\!\left\langle \left.\frac{d}{ds}\right|_{s=0} \exp(s\eta)g, \left.\frac{d}{dt}\right|_{t=0} \exp(t\zeta)g \right\rangle\!\!\right\rangle \\
&= \langle\langle T_e R_g(\eta), T_e R_g(\zeta) \rangle\rangle \\
&= \left\langle\!\left\langle \rho^{-1}(g, \eta), \rho^{-1}(g, \zeta) \right\rangle\!\right\rangle . \qquad (3.1)
\end{aligned}$$

The Three-dimensional Rigid Body. The standard free rigid body is of course the case in which $G = \mathrm{SO}(3)$. Recall the Lie algebra isomorphism $(\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\cdot, \cdot])$ given by

$$\Theta = \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} \mapsto \hat{\Theta} = \begin{pmatrix} 0 & -\Theta^3 & \Theta^2 \\ \Theta^3 & 0 & -\Theta^1 \\ -\Theta^2 & \Theta^1 & 0 \end{pmatrix}. \qquad (3.2)$$

Consider a curve $R(t) : \mathbb{R}^3 \to \mathbb{R}^3$ in $\mathrm{SO}(3)$ that gives the transformation between a reference configuration of the body and the current (spatial) configuration. Recall the definition of the spatial and body coordinates for velocity via the following calculations,

$$\lambda(\dot{R}(t)) = (R(t), T_{R(t)}L_{R(t)^{-1}}\dot{R}(t)) = (R(t), R(t)^{-1}\dot{R}(t)),$$

which leads one to define the ***body angular velocity*** to be $\hat{\Omega} := R^{-1}\dot{R} \in \mathfrak{so}(3)$. Similarly, $\rho(\dot{R}(t)) = (R(t), \dot{R}(t)R(t)^{-1})$ motivates the definition of ***spatial angular velocity*** $\hat{\omega} = \dot{R}R^{-1} \in \mathfrak{so}(3)$. To transform from body to spatial coordinates, observe that $\hat{\omega} = \mathrm{Ad}_R\,\hat{\Omega}$; that is, in $\mathbb{R}^3$, $\omega = R\Omega$.

The kinetic energy Lagrangian is

$$L = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\|\dot{R}\mathbf{X}\|^2 d^3\mathbf{X}.$$

where ρ is the material density, $\mathcal{B}$ is the body reference configuration in $\mathbb{R}^3$, and $\mathbf{X} \in \mathcal{B}$. One easily checks that this Lagrangian is left invariant as a function on $T\,\mathrm{SO}(3)$. The metric on SO(3) induced by this Lagrangian is also left invariant and may be defined at the identity as

$$\begin{aligned}\left\langle\!\!\left\langle \hat{\Theta}, \hat{\Xi} \right\rangle\!\!\right\rangle_e &= \int_{\mathcal{B}} \rho(\mathbf{X})(\Theta \times \mathbf{X}) \cdot (\Xi \times \mathbf{X}) d^3\mathbf{X} \\ &= \Theta \cdot \mathbb{J}\Xi,\end{aligned}$$

where

$$\mathbb{J} = \int_{\mathcal{B}} \rho(\mathbf{X})(\|\mathbf{X}\|^2\,\mathrm{Id} - \mathbf{X} \otimes \mathbf{X}) d^3\mathbf{X}.$$

The formula for the matrix $\mathbb{J}$ is easily obtained using the vector identity

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c),$$

and $\mathbb{J}$ represents the inertia tensor (mass matrix) in body coordinates.

We may express the locked inertia tensor $\mathbb{I}(g) : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$ as a 3×3 matrix $\tilde{\mathbb{I}}$, which has direct interpretation as the inertia tensor (mass matrix) in spatial coordinates.

$$\begin{aligned}\Theta \cdot (\tilde{\mathbb{I}}(g)\Xi) &:= \left\langle \hat{\Theta}, \mathbb{I}(g)\hat{\Xi} \right\rangle \\ &= \left\langle\!\!\left\langle \hat{\Theta}_{SO(3)}(g), \hat{\Xi}_{SO(3)}(g) \right\rangle\!\!\right\rangle_g \\ &= \left\langle\!\!\left\langle \hat{\Theta}g, \hat{\Xi}g \right\rangle\!\!\right\rangle_g \\ &= \left\langle\!\!\left\langle g^{-1}\hat{\Theta}g, g^{-1}\hat{\Xi}g \right\rangle\!\!\right\rangle_e \\ &= \langle\!\langle (g^{-1}\Theta)^\wedge, (g^{-1}\Xi)^\wedge \rangle\!\rangle_e \\ &= (g^{-1}\Theta) \cdot \tilde{\mathbb{I}}(e)(g^{-1}\Xi) \\ &= \Theta \cdot (g\tilde{\mathbb{I}}(e)g^{-1}\Xi).\end{aligned}$$

Observing that $\tilde{\mathbb{I}}(e) = \mathbb{J}$, we obtain the formula

$$\tilde{\mathbb{I}}(g) = g\mathbb{J}g^{-1}\,.$$

A one-parameter group orbit has the form $R(t) = \exp(t\hat{\Omega})R(0)$. Excluding the trivial case $\hat{\Omega} = 0$, a necessary condition for this curve to be a relative equilibrium is that $\Omega \in \mathbb{R}^3$ correspond to a principal axis of rotation of the rigid body; in other words, that Ω is an eigenvector of the matrix $\mathbb{J}$. Observe that if $\mathbb{J}\Omega = \alpha\Omega$ for $\alpha \in \mathbb{R}$ then $\hat{\Omega}$ is an eigenvector of the linear operator on $\mathfrak{so}(3)$ defined by

$$\hat{\Theta} \mapsto \mathbb{J}\hat{\Theta} + \hat{\Theta}\mathbb{J}\,,$$

with eigenvalue $(\mathrm{Tr}\,\mathbb{J} - \alpha)$. Entities are expressed this way to link the discussion to the n-dimensional rigid body, which is given later.

4 Relative Equilibria for Mechanics on Lie Groups

In this section we develop a characterization of relative equilibria for the case of mechanics on Lie groups.

Conditions for a Relative Equilibrium. If $(g_e, \dot{g}_e) \in TG$ is a relative equilibrium then there exists a unique $\xi \in \mathfrak{g}$ such that if $g(t)$ is the solution of the (second-order) Euler–Lagrange equations with initial conditions $(g_e, \dot{g}_e)$ then $g(t) = (\exp t\xi)g_e$, and in particular, $\dot{g}|_{t=0} = \xi_G(g_e)$.

The following proposition gives necessary and sufficient conditions for a relative equilibrium of the free generalized rigid body.

Proposition 4.1. *Let G act on itself by left multiplication and assume a left-invariant kinetic energy Lagrangian on TG. Let $\xi \in \mathfrak{g}$ and $g_e \in G$. Then $(g_e, \xi_G(g_e)) \in T_{g_e}G$ is a relative equilibrium if and only if $\xi \in \mathfrak{g}_\mu$ where $\mu = \mathbf{J}(\xi_G(g_e))$.*

Proof. We saw one direction of the argument before: If $(g_e, \xi_G(g_e))$ is a point of relative equilibrium then the solution of the Euler–Lagrange equations with the initial conditions $(g_e, \xi_G(g_e))$ is given by $g(t) = \exp(t\xi)g_e$. Since, by Noether's Theorem, $(g(t), \dot{g}(t)) \in \mathbf{J}^{-1}(\mu)$, and $\mathbf{J}$ is equivariant, it follows that $\exp(t\xi) \in G_\mu$, or equivalently, $\xi \in \mathfrak{g}_\mu$.

Conversely, notice that the augmented potential associated with the Lie algebra element ξ is given by

$$V_\xi(g) = -\frac{1}{2}\left\langle \mathbb{I}(g)\xi, \xi \right\rangle =: -\frac{1}{2}\left\langle \mathbb{I}(\cdot)\xi, \xi \right\rangle(g)\,.$$

Let δg be an arbitrary vector in $T_{g_e}G$ and write $\delta g = \zeta_G(g_e)$ for $\zeta \in \mathfrak{g}$. With the help of Lemma 2.1, the derivative of V_ξ in the direction δg is given by

$$
\begin{aligned}
d_{g_e} V_\xi \cdot \delta g &= d_{g_e} V_\xi \cdot \zeta_G(g_e) \\
&= -\frac{1}{2} d_{g_e} \left\langle \mathbb{I}(\cdot)\xi, \xi \right\rangle \cdot \zeta_G(g_e) \\
&= -\frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} \left\langle \mathbb{I}(\exp(t\zeta) g_e)\xi, \xi \right\rangle \\
&= -\frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} \left\langle \mathbb{I}(g_e)\mathrm{Ad}_{\exp(-t\zeta)}\xi, \mathrm{Ad}_{\exp(-t\zeta)}\xi \right\rangle \\
&= -\left\langle \mathbb{I}(g_e)\xi, \mathrm{ad}_\xi \zeta \right\rangle \\
&= -\left\langle \mathbf{J}(\xi_G(g_e)), \mathrm{ad}_\xi \zeta \right\rangle \\
&= -\left\langle \mu, \mathrm{ad}_\xi \zeta \right\rangle = -\left\langle \mathrm{ad}_\xi^* \mu, \zeta \right\rangle = 0\,,
\end{aligned}
$$

the last equality holding because $\xi \in \mathfrak{g}_\mu$. Therefore, g_e is a critical point of V_ξ, and hence, by the augmented potential proposition, $(g_e, \xi_G(g_e))$ is a relative equilibrium. ■

It is a general fact that relative equilibria come in sets: they are unions of group orbits. This can also be seen directly in the present context and the proof is somewhat instructive, so we include it.

Corollary 4.2. *If $(e, \xi_G(e))$ is a relative equilibrium and $g \in G$, then the left translation of $(e, \xi_G(e))$ by g is also a relative equilibrium.*

Proof. From equation (2.1), the tangent of left translation by the group element g is given by $TL_g \xi_G(e) = \left(\mathrm{Ad}_g \xi\right)_G(g)$. Therefore, it suffices to show that g is a critical point of $V_{\mathrm{Ad}_g \xi}$. In fact,

$$
\begin{aligned}
d_g V_{\mathrm{Ad}_g \xi}(\zeta_G) &= -\frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} \left\langle \mathbb{I}(\exp(t\zeta) g)\mathrm{Ad}_g \xi, \mathrm{Ad}_g \xi \right\rangle \\
&= -\frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} \left\langle \mathbb{I}(e)\mathrm{Ad}_{g^{-1}\exp(-t\zeta) g}\xi, \mathrm{Ad}_{g^{-1}\exp(-t\zeta) g}\xi \right\rangle \\
&= \left\langle \mathbb{I}(e)\xi, \mathrm{ad}_{\mathrm{Ad}_{g^{-1}}\zeta}\xi \right\rangle = \mu(\mathrm{ad}_{\mathrm{Ad}_{g^{-1}}\zeta}\xi) \\
&= -\mu(\mathrm{ad}_\xi(\mathrm{Ad}_{g^{-1}}\zeta)) = -\mathrm{ad}_\xi^* \mu(\mathrm{Ad}_{g^{-1}}\zeta) = 0\,,
\end{aligned}
$$

since, by hypothesis, $\xi \in \mathfrak{g}_\mu$. ■

The n-dimensional Rigid Body. For the case of the rigid body in $\mathbb{R}^n$, $G = \mathrm{SO}(n)$. We now recall the basic set up of this system following Manakov [1976] and Ratiu [1980]. First of all, choose, on the Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$, the following inner product, which is a multiple of the Killing form:

$$
\langle \xi, \eta \rangle = -\frac{1}{2}\,\mathrm{Tr}(\xi\eta)\,. \tag{4.1}
$$

(The factor $-\frac{1}{2}$ is chosen so that (4.1) agrees with the Euclidean inner product when $n = 3$.)

The locked inertia tensor at the identity, that is, the kinetic energy inner product on $\mathfrak{so}(n)$, is represented by a symmetric positive-definite linear operator $\mathcal{J}$ on $\mathfrak{so}(n)$; that is,

$$\langle\!\langle A, B \rangle\!\rangle = \langle A, \mathcal{J} B \rangle \,.$$

Assuming all of the eigenvalues of $\mathcal{J}$ are distinct, it may be represented as

$$\mathcal{J}(\xi) = \Lambda\xi + \xi\Lambda \,,$$

for a diagonal matrix $\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$, and $\Lambda_i + \Lambda_j > 0$ if $i \neq j$. If E_{ij} are the standard basis vectors for $\mathfrak{gl}(n)$, then a basis of eigenvectors of $\mathcal{J}$ is given by $\{E_{ji} - E_{ij} \mid i < j\}$ and the corresponding eigenvalues are $\{\Lambda_i + \Lambda_j \,|\, i < j\}$. This representation is generally known as the n-dimensional Manakov rigid body. (See, for example, Bloch, Crouch, Marsden, and Ratiu [2002] for further information and references.)

As before, let $\mathbf{J}$ be the standard tangent lifted momentum mapping corresponding to the action of $\mathrm{SO}(n)$ on itself; if $\mu := \mathbf{J}(\xi_{SO(n)}(e))$ then

$$\langle \mu, \zeta \rangle = -\frac{1}{2} \operatorname{Tr}(\mathcal{J}(\xi)\zeta) \,. \tag{4.2}$$

Corollary 4.3. *Let $\xi \in \mathfrak{so}(n) \setminus \{0\}$ and $\mu = \mathbf{J}(\xi_{SO(n)}(e))$. Then ξ is an eigenvector of $\mathcal{J}$ if and only if $\xi \in \mathfrak{so}(n)_\mu$.*

Proof. It is easy to see directly that ξ is an eigenvector of $\mathcal{J}$ if and only if $(e, \xi_G(e))$ is a relative equilibrium, and so the corollary follows from Proposition 4.1. However, a proof worked out using equation (4.2) is also instructive.

If $\mathcal{J}(\xi) = \lambda\xi$ then $\langle \mu, \zeta \rangle = -\frac{\lambda}{2} \operatorname{Tr}(\xi\zeta)$, so

$$\begin{aligned} \left\langle \operatorname{ad}^*_\xi \mu, \zeta \right\rangle &= -\frac{\lambda}{2} \operatorname{Tr}(\xi \operatorname{ad}_\xi \zeta) = -\frac{\lambda}{2} \operatorname{Tr}(\xi(\xi\zeta - \zeta\xi)) \\ &= -\frac{\lambda}{2} \left(\operatorname{Tr}(\xi^2\zeta) - \operatorname{Tr}(\xi\zeta\xi) \right) = 0 \,, \end{aligned}$$

and therefore, $\xi \in \mathfrak{so}(n)_\mu$. Conversely, if $\xi \in \mathfrak{so}(n)_\mu$ then

$$\begin{aligned} 0 &= \operatorname{ad}^*_\xi \mu(\zeta) = \mu([\xi, \zeta]) \\ &= -\frac{1}{2} \operatorname{Tr}(\mathcal{J}(\xi)[\xi, \zeta]) \\ &= -\frac{1}{2} \operatorname{Tr}([\mathcal{J}(\xi), \xi], \zeta) \\ &= \langle [\mathcal{J}(\xi), \xi], \zeta \rangle \end{aligned}$$

for arbitrary ζ. Thus $[\mathcal{J}(\xi), \xi] = 0$. Therefore, $\mathcal{J}(\xi)$ is in a maximal abelian subalgebra of $\mathfrak{so}(n)$ containing ξ. But in $\mathfrak{so}(n)$ the dimension of such a subalgebra is necessarily one. Thus $\mathcal{J}(\xi) = \lambda\xi$ for some $\lambda \in \mathbb{R}$. Because $\mathcal{J}$ is positive definite, it follows that $\lambda \neq 0$. ∎

For $n = 3$, when $\mathfrak{so}(3)$ is identified with $\mathbb{R}^3$ via the Lie algebra isomorphism (3.2), $\mathcal{J}$ may be identified with the linear operator $\mathbb{J}$ on $\mathbb{R}^3$, whose matrix form represents the moment of inertia in body coordinates. If we assume that $\mathbb{J} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, where the λ_i are distinct and positive, then the eigenvectors of $\mathbb{J}$ are the standard orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and are also the principal axes of the rigid body. The eigenvalues λ_i of $\mathbb{J}$ (and of $\mathcal{J}$) are given by the relationship

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{pmatrix};$$

that is, $\lambda_i = \operatorname{Tr}(\Lambda) - \Lambda_i$. Corollary 4.3 now takes a special form.

Corollary 4.4. *Let $\hat{\Omega} \in \mathfrak{so}(3) \setminus \{0\}$ and $\hat{\Pi} = \mathbf{J}(\hat{\Omega}_{\mathrm{SO}(3)}(e))$. The corresponding nonzero vector $\Omega \in \mathbb{R}^3$ is a principal axis of the rigid body if and only if $\hat{\Omega} \in \mathfrak{so}(3)_\Pi$.*

Proof. This follows directly from Corollary 4.3, but we shall present a proof that emphasizes explicitly the vector algebra in $\mathbb{R}^3$.

If Ω is a principal axis, that is, if $\mathbb{J}\Omega = \lambda\Omega$ for some $\lambda \in \mathbb{R}^\times$, then for any $\Xi \in \mathbb{R}^3$,

$$\Pi(\Xi) = \hat{\Pi}(\hat{\Xi}) = \Omega^T \mathbb{J} \Xi = (\mathbb{J}\Omega)^T \Xi = \lambda \Omega^T \Xi .$$

In other words, $\Pi = \lambda\Omega^T \in \mathbb{R}^{3*}$, and

$$\operatorname{ad}^*_{\hat{\Omega}} \hat{\Pi}(\hat{\Xi}) = \lambda(\Omega^T(\Omega \times \Xi))^\wedge = 0 ,$$

or, abusing notation slightly,

$$\operatorname{ad}^*_\Omega \Pi = \lambda \operatorname{ad}^*_{\Omega^T} \Omega = \lambda \operatorname{ad}_\Omega \Omega = \lambda(\Omega \times \Omega) = 0 ,$$

so $\hat{\Omega} \in \mathfrak{so}(3)_\Pi$.

Conversely, if $\hat{\Omega} \in \mathfrak{so}(3)_\Pi$ then for arbitrary $\Xi \in \mathbb{R}^3$,

$$0 = \operatorname{ad}^*_{\hat{\Omega}} \hat{\Pi}(\hat{\Xi}) = \hat{\Pi}([\hat{\Xi}, \hat{\Xi}]) = \Omega^T \mathbb{J}(\Omega \times \Xi) = (\mathbb{J}\Omega)^T (\Omega \times \Xi) ,$$

and thus the collection of vectors $\{\Omega, \mathbb{J}\Omega, \Xi\}$ is linearly dependent. Because Ξ is arbitrary, $\mathbb{J}\Omega = \lambda\Omega$ for some $\lambda \in \mathbb{R}^\times$. ■

The usual criterion for a relative equilibrium is that the body angular velocity is an eigenvector of the inertia tensor, which is equivalent to the body angular velocity and the body angular momentum being parallel. What we have shown in the case of the n-dimensional rigid body is that this criterion is consistent with our general criterion: $\xi \in \mathfrak{g}_\mu$.

5 A Generalized Saari's Conjecture

Motivated by Proposition 4.1, we seek to generalize Saari's Conjecture, focusing on the case of mechanics on Lie groups. First we state an appropriate generalization of the original conjecture in the context of simple mechanical systems with symmetry.

A Proposed Generalization. For a simple mechanical system with symmetry, we could interpret the moment of inertia to be the locked inertia tensor. Therefore a logical generalization of Saari's Conjecture could state:

> ***Naive Saari Conjecture***: *A simple mechanical system with symmetry is at a point of relative equilibrium if and only if the locked inertia tensor is constant along the integral curve that passes through that point.*

Note that for the planar N-body problem with SO(2) symmetry this naive conjecture reduces to the original Saari Conjecture. However, this conjecture is naive for two reasons: First, as discussed earlier, this conjecture is false for more general gravitational potentials than the Newtonian potential, so the conjecture is too broadly stated, even for planar N-body problems. Second, one must be more careful with what is meant by the constancy of the locked inertia tensor for nonabelian groups.

The Rigid Body Counterexample. Let us now show that the Naive Saari Conjecture is false even for the rigid body in $\mathbb{R}^3$. Again, assume that $\mathbb{J} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, where the λ_i are all distinct and nonzero. Let $R(t) \in \mathrm{SO}(3)$ represent rotation by t radians about the $\mathbf{e}_3$ axis. Clearly, $R(t)$ is an equilibrium curve through the identity of SO(3) and its body angular velocity is always parallel to $\mathbf{e}_3$. Recall that the time evolution of the locked inertia tensor is described by

$$\tilde{\mathbb{I}}(R(t)) = R(t)\mathbb{J}R^{-1}(t)\,.$$

Now, at $t = 0$,

$$\tilde{\mathbb{I}}(R(0))\mathbf{e}_2 := \mathbb{J}\mathbf{e}_2 = \lambda_2\mathbf{e}_2\,.$$

But at $t = \frac{\pi}{2}$,

$$\tilde{\mathbb{I}}\left(R\left(\frac{\pi}{2}\right)\right)\mathbf{e}_2 = R\left(\frac{\pi}{2}\right)\mathbb{J}R\left(-\frac{\pi}{2}\right)\mathbf{e}_2 = R\left(\frac{\pi}{2}\right)\mathbb{J}\mathbf{e}_1 = \lambda_1 R\left(\frac{\pi}{2}\right)\mathbf{e}_1 = \lambda_1\mathbf{e}_2\,,$$

and so $\mathbb{I}$ is not constant.

A Refined Conjecture and Proof. The above counterexample occurs in the so-called "easy" direction—the direction that was immediately true for Saari's original conjecture. We may modify the Naive Saari Conjecture slightly to accommodate this counterexample. The following proposition applies to a general configuration space Q with the free left action of a Lie group G on Q.

Proposition 5.1. *Consider a Lagrangian simple mechanical system with symmetry. If a solution curve $q_e(t)$ in Q is a relative equilibrium curve, that is, if there exists a $\xi \in \mathfrak{g}$ such that $q_e(t) = \exp t\xi \cdot q_e(0)$ for any $t \in \mathbb{R}$, then for any $\zeta \in \mathfrak{g}$*

$$\langle \mathbb{I}(q_e(t))\xi, \zeta\rangle$$

is constant along the curve $q_e(t)$; that is, $\mathbb{I}(q_e(t))\xi$, as a curve in $\mathfrak{g}^*$, is constant.

Proof. Rephrasing, the proposition states that the momentum mapping $\mathbf{J} : TQ \to \mathfrak{g}^*$ satisfies the relation

$$\mathbf{J}\left(\xi_Q(\exp(t\xi) \cdot q_e(0))\right) = \mathbf{J}\left(\xi_Q(q_e(0))\right) .$$

To see this, note that

$$\begin{aligned}
\xi_Q(\exp(t\xi) \cdot q_e(0)) &= \frac{d}{ds} \exp(s\xi) \cdot (\exp(t\xi) \cdot q_e(0))\bigg|_{s=0} \\
&= \frac{d}{ds} \exp(t\xi) \cdot (\exp(s\xi) \cdot q_e(0))\bigg|_{s=0} \\
&= TL_{\exp(t\xi)}\left(\xi_Q(q_e(0))\right) ,
\end{aligned}$$

and invoking the equivariance of $\mathbf{J}$,

$$\begin{aligned}
\mathbf{J}(\xi_Q(\exp(t\xi) \cdot q_e(0)) &= \mathbf{J}\left(TL_{\exp(t\xi)}(\xi_Q(q_e(0))\right) \\
&= \mathrm{Ad}^*_{\exp(-t\xi)}\,\mathbf{J}(\xi_Q(q_e(0))) \\
&= \mathbf{J}(\xi_Q(q_e(0))) ,
\end{aligned}$$

because, by Noether's Theorem and equivariance of $\mathbf{J}$, if $\mathbf{J}(\xi_Q(q_e(0)) = \mu$ then $\exp(t\xi) \in G_\mu$. ■

If $G = SO(2)$ (such as in Saari's original conjecture) then it follows immediately from Proposition 5.1 that a relative equilibrium necessarily has a constant scalar moment of inertia.

Proposition 5.1 establishes the "easy" direction of the

> ***Refined Saari Problem***: *Find classes of simple mechanical systems with symmetry such that a solution of the Euler–Lagrange equations $q(t)$ is a relative equilibrium if and only if $\mathbb{I}(q(t))\xi$ is constant as a curve in $\mathfrak{g}^*$, where $\xi_Q(q(0)) = \dot{q}(0)$.*

Of course the counterexamples for N-body problems with SO(2) symmetry show that while the solution to the Refined Saari Problem includes the classical three-body problem, it cannot include all simple mechanical systems with symmetry. However, we now show that it *does* include systems on Lie groups.

Proposition 5.2. *The solution to the Refined Saari Problem includes the class of simple mechanical systems with symmetry defined on Lie groups; that is, if $g(t)$ is a geodesic in G and if $\xi \in \mathfrak{g}$ is defined by $\xi := \dot{g}(0) \cdot g^{-1}(0)$ and if for each $\eta \in \mathfrak{g}$ the quantity $\langle \mathbb{I}(g(t))\xi, \eta\rangle$ is constant in t then $g(t)$ is a relative equilibrium.*

Remark. To motivate why ξ is taken to be $\dot g(0)\cdot g^{-1}(0)$ (the spatial velocity at $t=0$) and not $g^{-1}(0)\cdot\dot g(0)$ (the body velocity at $t=0$), note that if we have a relative equilibrium $g(t)=(\exp(t\xi))\cdot g(0)$, then $\dot g(0)=\xi\cdot g(0)$ and so $\xi=\dot g(0)\cdot g^{-1}(0)$.

Proof. Consider the curve $\gamma(t)=g(t)\cdot g^{-1}(0)$ which satisfies $\gamma(0)=e$ and $\dot\gamma(0)=\xi$. It follows that

$$\xi_G(g(0)) = \left.\frac{d}{dt}\right|_{t=0}\gamma(t)\cdot g(0) = \left.\frac{d}{dt}\right|_{t=0} g(t)=\dot g(0)\,.$$

By hypothesis, for arbitrary $\eta\in\mathfrak{g}$,

$$0=\frac{d}{dt}\left\langle \mathbb{I}(g(t))\xi,\eta\right\rangle\,,$$

and by Lemma 2.1,

$$0=\frac{d}{dt}\left\langle \mathbb{I}(e)\,\mathrm{Ad}_{g^{-1}(t)}\,\xi,\mathrm{Ad}_{g^{-1}(t)}\,\eta\right\rangle\,. \tag{5.1}$$

Now

$$\left.\frac{d}{dt}\right|_{t=0} g^{-1}(t) = -g^{-1}(0)\cdot\dot g(0)\cdot g^{-1}(0) = -g^{-1}(0)\cdot\xi = \left.\frac{d}{dt}\right|_{t=0} g^{-1}(0)\exp(-t\xi)\,,$$

so the curves $g^{-1}(t)$ and $g^{-1}(0)\exp(-t\xi)$ have tangent vectors that agree at $t=0$, and thus

$$\left.\frac{d}{dt}\right|_{t=0}\mathrm{Ad}_{g^{-1}(t)}\,\zeta = \left.\frac{d}{dt}\right|_{t=0}\mathrm{Ad}_{g^{-1}(0)\exp(-t\xi)}\,\zeta\,, \tag{5.2}$$

for any $\zeta\in\mathfrak{g}$. Let $\mu:=\mathbf{J}(g(0),\dot g(0))$. Using equations (5.1) and (5.2),

$$\begin{aligned}
0 &= \left\langle \mathbb{I}(e)\left.\frac{d}{dt}\right|_{t=0}\mathrm{Ad}_{g^{-1}(0)\exp(-t\xi)}\,\xi,\ \mathrm{Ad}_{g^{-1}(0)}\,\eta\right\rangle \\
&\quad + \left\langle \mathbb{I}(e)\,\mathrm{Ad}_{g^{-1}(0)}\,\xi,\ \left.\frac{d}{dt}\right|_{t=0}\mathrm{Ad}_{g^{-1}(0)\exp(-t\xi)}\,\eta\right\rangle \\
&= \left\langle \mathbb{I}(e)\,\mathrm{Ad}_{g^{-1}(0)}\,\mathrm{ad}_{(-\xi)}\,\xi,\mathrm{Ad}_{g^{-1}(0)}\,\eta\right\rangle + \left\langle \mathbb{I}(e)\,\mathrm{Ad}_{g^{-1}(0)}\,\xi,\mathrm{Ad}_{g^{-1}(0)}\,\mathrm{ad}_{(-\xi)}\,\eta\right\rangle \\
&= -\left\langle \mathbb{I}(g(0))\xi,\mathrm{ad}_\xi\,\eta\right\rangle \\
&= -\left\langle \mathbf{J}(\xi_G(g(0))),\mathrm{ad}_\xi\,\eta\right\rangle = -\left\langle \mu,\mathrm{ad}_\xi\,\eta\right\rangle = -\left\langle \mathrm{ad}^*_\xi\,\mu,\eta\right\rangle\,.
\end{aligned}$$

Therefore, $\xi\in\mathfrak{g}_\mu$, and so Proposition 4.1 tells us that $(g(0),\dot g(0))$ is a relative equilibrium. ■

Given ξ as defined in the preceding proposition, there may be other ξ' such that $\langle\mathbb{I}(g(t))\xi',\cdot\rangle$ is constant along $g(t)=\exp(t\xi)g(0)$. For example, if ξ' is in the maximal abelian subalgebra that contains ξ then this is the case.

One final comment: the general criterion $\xi\in\mathfrak{g}_\mu$ also holds in the case of ideal fluid mechanics, which was one of the original motivating examples of both Poincaré [1901] and Arnold [1966]; this condition states that one has a relative equilibrium when the stream function for the velocity field and the vorticity field are functionally dependent.

6 Conclusions

We have introduced a Lie-algebraic condition that is a necessary and sufficient condition for relative equilibria in simple mechanical systems with symmetry on Lie groups. This result led us to a proof of a "Refined Saari Conjecture" for this class of mechanical systems.

Our results leave much room for further investigation. We may want to consider cases where the eigenvalues of the inertia tensor in body coordinates are degenerate. We may also attempt to extend our results to actions of a Lie subgroup of the group and to more general simple mechanical systems with symmetry.

An interesting generalization of what we have done might be to examine a combination of the Newtonian gravitational problem with the rigid body problem, namely, does the *Refined Saari Problem* include the case of irregular rigid bodies interacting with each other through gravitational attraction? Problems such as this are of considerable astrodynamical interest and go by the name of *full body problems*; see, for example, Koon, Marsden, Ross, Lo, and Scheeres [2004].

Lawton and Noakes [2001] have shown that if a curve in $\mathbb{R}^3$ describes the angular velocity of a rigid body and that curve satisfies the condition that it not be contained in a two-dimensional subspace of $\mathbb{R}^3$, then the inertia operator may be computed up to a scaling factor. They provide two indirect methods of constructing the inertia operator and one direct method. The direct method is obtained by momentum mapping identities derived from symmetries in Euler's equation. This work is consistent with the present paper in that observations of the inertia tensor allows one to back out certain dynamical information.

It would also be interesting to see if there are any relations with the work of Fehér and Marshall [2003], who investigate the stability of equilibria of certain integrable Euler equations associated with $\mathrm{SO}(n)$.

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