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HAMILTONIAN SYSTEMS WITH SPIN

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In this note we give a brief exposition of the mathematical foundations of the theory of spin for both classical and quantum mechanical systems on oriented Riemannian manifolds. We shall use freely the notations and theory developed in Abraham [1] and Marsden [2, 3]. From the physical point of view nothing new appears. The whole purpose of the note is to explain how the theory fits in the spirit of [1].

In view of [3], we can handle the classical and quantum mechanical cases simultaneously, as both are Hamiltonian systems (the latter being on an infinite dimensional symplectic manifold).

We first recall the definition of a spin manifold, second define a spin Hamiltonian system and thirdly, give the appropriate conservation law for spin angular momentum. The classical case seems to be of little physical interest other than theoretical. Quantum mechanical examples are the two component Schrodinger equation and the Dirac equation. (One could also use the coupled Dirac-Maxwell system as a non-linear example; see [3]).

1. Spin Manifolds. We begin then with the definition of spin manifold following Palais' exposition [4, p. 91]. Other standard references for spinors are Milnor [5, 6], and Cartan [7]. Further references may be found in [7], and some references to the vast literature from physics are found in [10].

Let $SO(n)$ be the rotation group on R^n. We let $Spin(n)$ be the universal (2 fold) covering group of $SO(n)$. For an explicit construction for $n \geq 3$, in terms of Clifford algebras, see [4], [6, p. 14] and [8, p. 367], and for covering groups see any standard text such as [9. pp. 22-27].

There exists an (irreducible complex) $Spin(n)$ module denoted S_n, called the n-dimensional spinors. If $n = 2k$ or $2k+1$, S_n has complex dimension 2^k. For example, one checks that $\mathrm{Spin}(3) = SU(2)$; and $S_3 = \mathbf{C}^2$.

Let A be an oriented Riemannian manifold. To define a spin bundle over A we first define a local spin bundle and then globalize in the spirit of Eilenberg Cartan; see [1, §4].

A vector bundle $\pi : E \longrightarrow A$ with fibre S_n is a spin bundle if there is a covering U_α of A and bundle charts $\phi_\alpha : TU_\alpha \subset TA \longrightarrow U_\alpha \times \mathbf{R}^n$; of TA and $\phi^*_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times S_n$ of E such that (i) ϕ_α preserves the metric and orientation and (ii) the overlap maps $\phi^*_\beta \circ \phi^{*\,-1}_\alpha$ have the form $(x, s) \longmapsto (x, g_{\alpha\beta}(x) \cdot s)$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}(n)$ and and $\rho(g_{\alpha\beta}(x)) = \phi_\beta \circ \phi_\alpha^{-1}$ (restricted to x), where $\rho : \mathrm{Spin}(n) \longrightarrow SO(n)$ is the canonical projection.

Thus a spin bundle over A is a (vector) bundle $\pi : E \longrightarrow A$ whose local charts are local spin bundles and transition maps are local spin bundle isomorphisms. Roughly, when we have a coordinate change, the fibers "transform like" spinors rather than vectors; i.e. according to $\mathrm{Spin}(n)$ rather than $SO(n)$.

We shall also regard the restriction of π to a submanifold of A as a spin bundle.

2. Hamiltonian Systems with Spin. We define a spin Hamiltonian system in the following way. First, let A be an oriented Riemannian manifold and $\pi : E \longrightarrow A$ a spin bundle over A. Regarding E as a manifold M, put on T^*M (the cotangent bundle) the natural symplectic structure. A classical spin Hamiltonian system is a Hamiltonian system on T^*E. A spin quantum mechanical system is a quantum mechanical system over E (see [3] for the definition of a quantum mechanical system).

In other words, a classical spin system is a Hamiltonian system which depends on the spin coordinates and momenta and a quantum mechanical spin system depends just on the spin coordinates.

3. The Conservation Theorem. Let G be a Lie group which acts on a manifold M. If X is an infinitesimal generator of G on M, then the function P_X called the momentum of X defined by $P_X : T^*M \longrightarrow \mathbf{R}$; $P_X(\alpha_m) = \alpha_m(X(m))$ is invariant for any Hamiltonian system whose Hamiltonian function is invariant under the induced action of G on T^*M.

Similarly if Ω is a volume for M and a quantum mechanical system is invariant under the action, the function $< P_X > : D \subset L^2(M, \mathbf{C}) \longrightarrow \mathbf{R}$ defined by $< P_X > (\psi) = \int < \psi, iL_X \psi > d\Omega$ (for say D the smooth functions) is a constant of the motion.

These are the basic conservation laws of mechanics. For proofs, see [2, 3]. What we wish to do is simply to determine the corresponding conserved quantities in case M is a spin bundle over A and G acts on A.

Let $\pi : E \longrightarrow A$ be a spin bundle and suppose G acts on A, by $\phi_g : A \longrightarrow A$ and that this action lifts to E. That is, there is an action $\psi_g : E \longrightarrow E$ on E such that (i) $\pi \circ \psi_g = \phi_g \circ \pi$ and (ii), there are chart coverings ϕ_α, ϕ^*_α as above, such that over $x \in A$, $\phi^*_\beta \circ \psi_g \circ \phi^{*\,-1}_\alpha \in \mathrm{Spin}(n)$ and $\rho(\phi^*_\beta \circ \psi_g \circ \phi^{*\,-1}_\alpha) = \phi_\beta \circ (T\psi_g) \circ \phi_\alpha^{-1}$. $T\psi_g$ is the tangent (derivative) of ψ_g.

Let X be an infinitesimal generator of ϕ on A (a vectorfield on A) and Y the corresponding one for ψ on E. Then an easy computation shows that locally, $Y = X + Y_s$ where Y_s, at each point lies in the Lie algebra of Spin(n), i.e. $T_e\mathrm{Spin}(n)$. If the ϕ_α above have the form Tf_α for charts f_α on A, then $T_e\rho(Y_s) = TX$ (in the chart).

In summary then, the conserved quantities consist of the ordinary conserved momentum X plus the <u>spin angular momentum</u> Y_s.

If G is the rotation group and $A = R^3$ these yield the usual conservation laws. See [3]. For the Schrodinger equation with spin we use A directly (two component spinors) while for the Dirac equation we use the spin structure derived from A as a submanifold of R^4 (four component spinors). For a general Dirac system A would be a space-like 3 surface with spin structure derived from $A \times R$. Note that a pseudo-Riemannian manifold does not define a spin structure as we have described it.

We summarize the results in these two theorems:

THEOREM 1. <u>Let</u> A <u>be an oriented Riemannian manifold and</u> $\pi : E \longrightarrow A$ <u>a spin bundle over</u> A. <u>Let</u> $H : T^*E \longrightarrow R$ <u>be a Hamiltonian system on</u> T^*E (H is usually a C^∞ function but distributions are also acceptable; see [2]). <u>Let</u> ϕ <u>be a smooth action of</u> G <u>on</u> A <u>which lifts to an action on</u> E, <u>and under which</u> H <u>is invariant.</u> <u>Let</u> X <u>be an infinitesimal generator of</u> ϕ <u>on</u> A. <u>Then the function</u> $P : T^*E \longrightarrow R$ <u>defined by</u> $P = P_X \circ \pi + P_s$ <u>is a constant of the motion</u> (invariant under the flow of H) <u>where</u> $P_s : T^*E \longrightarrow R$ <u>is such that</u> $P_s = P_{Y_s}$ <u>where</u> Y_s <u>is a vectorfield on</u> E <u>linear along the fibers of</u> π <u>determined above.</u>

For example, take $A = R^3$ and two component spinors $\alpha = (\alpha_1, \alpha_2)$ over A. We identify T^*E with $(R^3 \times R^3) \times (C^2 \times C^2)$. Under the rotation group, the corresponding conserved quantities are components of the following vector function (using the obvious notations)

$$P(\vec{x}, \vec{p}_x, \alpha, p_\alpha) = \vec{x} \times \vec{p}_x + 1/2 \; p_\alpha \cdot \vec{\sigma} \cdot \alpha$$

where the components of $\vec{\sigma}$ are the standard Pauli spin matrices.

Spin theory is usually neglected classically but there is surely no theoretical reason for it, only physical precedence.

THEOREM 2. Let A, E and ϕ be as in Theorem 1. Let $H_{op} : D \subset L^2 \longrightarrow L^2$ be a Hamiltonian operator on a domain D where L^2 is the complex Hilbert space of functions $\psi : E \longrightarrow C$ with respect to a volume Ω on A and some fiber inner product and ϕ preserves that volume (again distribution valued operators are also acceptable; see [3]). If H_{op} is invariant under ϕ, then the expectation of $X + Y_s$ (defined above) is a constant of the motion.

In the same example as above, the conserved momentum about the z axis is the expectation

$$\sum_i \int_A \bar{\psi}_i (x) \, i \left(x \frac{\partial \psi_i}{\partial y} - y \frac{\partial \psi_i}{\partial x}\right) dx + 1/2 \int_A \bar{\psi} \cdot \sigma_x \cdot \psi \, dx$$

where $\psi_1(x) = \psi(x, 1, 0)$ and $\psi_2(x) = \psi(x, 0, 1)$. The first term is the usual angular momentum of ψ and the second is the spin angular momentum.

Lifting an action from $SO(n)$ to $Spin(n)$ was trivial in the Euclidean examples above. However, in the general case this may be a topologically non-trivial process.

4. Rigid Body with Spin. The motion of a rigid body can be regarded as a geodesic on $SO(3)$ with respect to a given left invariant metric (moment of inertia tensor) c.f. [12]. However the analogous thing on $Spin(n)$ is not a rigid body with spin. It is easily checked that the corresponding geodesics are just lifts of geodesics on $SO(3)$, so is just another description of rigid body motion (i.e. the rigid body can be regarded on either $SU(2)$ or on $SO(3)$).

The correct description of a rigid body with spin is as follows. Let $A = SU(2)$ and I a left invariant metric (the given moment of inertia).

Let $E = SU(2) \times \mathbb{C}^2$ and as a chart, use $\phi : TA \longrightarrow A \times T_e SU(2) = A \times \mathbb{R}^3$ defined by $\phi(v_x) = (x, T_x L_x^{-1} v_x)$, L_x being left translation, and $\phi^* : SU(2) \times \mathbb{C}^2 \longrightarrow SU(2) \times \mathbb{C}^2$; $\phi^*(g, c) = (g, g^{-1} c)$. With the obvious product metric on the spin bundle E, namely I times the standard inner product, we define geodesics on E to be the motion of a <u>rigid body with spin</u>. That the natural action of SU(2) on itself by left translation lifts to E is easily checked, by setting $L^*_g(h, c) = (gh, gc)$. By the conservation theorem 1, the conserved functions are given by (on TE):

Total Angular Momentum $(g, P_g; c, P_c)$

$$= \text{rigid body angular momentum}$$

$$- \frac{1}{2} P_c \cdot i\vec{\sigma} \cdot c$$

For more detailed proofs of the results sketched in this paper, see Marsden, Chang, Robinson, <u>Hamiltonian Mechanics</u>, <u>Infinite Dimensional Lie Groups</u>, <u>Geodesic Flows and Hydrodynamics</u> (Berkeley lecture notes).

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