WELL-POSEDNESS OF THE EQUATIONS OF A

NON-HOMOGENEOUS PERFECT FLUID

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## Abstract

The Euler equations for a non-homogeneous, non-viscous compressible fluid are shown to be well-posed for a short time interval, using techniques of infinite dimensional geometry and a weighted Hodge theorem. Regularity and other properties of these solutions are pointed out as well.

1.  Introduction.

In [2], D. Ebin and the author introduced a technique for solving the Euler equations for a perfect (homogeneous, non-viscous, incompressible) fluid based on the use of the group $\mathcal{D}_\mu^s$ of Sobolev class $H^s$ (or $W^{s,p}$ or Hölder class $C^{k+\alpha}$) volume preserving diffeomorphisms. This method originated in an idea of V. Arnold [1].

In the present paper we shall use similar methods to solve the equations with the additional complication of non-homogeneity. These equations are (on a Riemannian manifold $M$ ), cf. [3]:

$$\frac{\partial v}{\partial t} + \nabla_v v = \frac{\text{grad } p}{\rho} \quad (\nabla = \text{covariant derivative})$$

$(\underline{1})$
$$\frac{\partial \rho}{\partial t} + \text{grad } \rho \cdot v = 0$$

$$\text{div } v = 0$$

$$v \text{ parallel to } \partial M = \text{boundary of } M$$

where $v$ is the velocity field of the fluid, $\rho(x,t) > 0$ is the mass density and $p(x,t)$ is the pressure. One given $v(x,0)$ , $\rho(x,0)$ and the problem is to find $v(x,t)$ , $p(x,t)$ , $\rho(x,t)$ satisfying $(\underline{1})$ .

The key differences with the case of constant $\rho$ are: first, $(\underline{1})$ is a coupled system between $\rho$ and $v$ and second, the corresponding equations on $\mathfrak{D}_\mu^s$ are no longer right invariant (as we shall see below). Therefore the equations $(\underline{1})$ are not derivable using the methods of Arnold (cf. [1,4]), although the methods of [2] do apply when suitably modified.

One also gets, as in [2], solutions for non-homogeneous viscous flow (in case of manifolds with no boundary) and strong convergence in $H^s$ or $W^{s,p}$ of the solutions to solutions of $(\underline{1})$ as the viscosity $\nu \to 0$ .

## 2. Statement of the Results.

Concerning equations ($\underline{1}$) we have the following which is our main result:

Theorem. *Let  M  be a compact,  $C^\infty$ , n-dimensional manifold possibly with  ($C^\infty$)  boundary, oriented and having a  ($C^\infty$)  Riemannian metric.*

*Let  $H^s(TM)$  [resp. $W^{s,p}(TM)$ , $C^{k+\alpha}(TM)$] denote the vector fields of Sobolev class  $H^s$ [resp. $W^{s,p}$ , Hölder class  $C^{k+\alpha}$] and suppose  $s > \frac{n}{2} + 1$  [resp. $s > \frac{n}{p} + 1$ , $k \geq 1$ , $0 < \alpha < 1$]. Similarly define  $H^s(M) = H^s(\Lambda^\circ(M))$  to be the real valued functions of class  $H^s$ .*

*For  $v_0 \in H^s(TM)$ , $\rho_0 \in H^s(M)$  [resp. $v_0 \in W^{s,p}(TM)$  $C^{k+\alpha}(TM)$ , $\rho \in W^{s,p}(M)$  $C^{k+\alpha}(M)$] with  div $v_0 = 0$  and  $v_0$  parallel to  $\partial M$ , and  $\rho_0(x) > 0$  for all  $x \in M$ , there is an  $\varepsilon > 0$  and a unique solution*

$$v(x,t) = v_t(x) \in H^s(TM)  \text{ [resp. } W^{s,p}(M) \text{ } C^{k+\alpha}(TM)\text{]}$$

$$\rho(x,t) = \rho_t(x) \in H^s(M)  \text{ [resp. } W^{s,p}(M) \text{ } C^{k+\alpha}(M)\text{]}$$

*of the equations ($\underline{1}$) for  $-\varepsilon < t < \varepsilon$ ; v  and  $\rho$  are at least  $C^1$  jointly in  t,x  so the solution is a classical solution.  The pressure  p  is  $H^{s+1}$  [resp. $W^{s+1,p}$ , $C^{k+\alpha}$] .  Also, the*

*equations (1) define on* $\{v, \rho \in H^s(TM) \times H^s(M) | \text{div } v=0$ ,

$v$ *parallel to* $\partial M$ *and* $\rho(x) > 0$ *for all* $x \in M\}$

*[resp.* $W^{s,p}(TM) \times W^{s,p}(M)$ *,* $C^{k+\alpha}(TM) \times C^{k+\alpha}(M)]$ *a*

*strongly continuous (non-linear) local one*

*parameter group (this domain of definition is an*

*open set in a Banach space).*

　*Further, the (time dependent) flow* $\eta_t$ *of*

$v_t$ *is an* $H^s$ *[resp.* $W^{s,p}$ *,* $C^{k+\alpha}]$ *diffeomorphism*

*of* $M$ *and we have*

$$(\underline{2}) \qquad \rho(x,t) = \rho_0(\eta_t^{-1}(x))$$

　Further properties of the solutions to equations (1)
are as follows:

　(i)　(<u>Conservation of energy</u>):

$\displaystyle\int_M$ $<v_t,v_t>\rho_t \, d\mu$　is independent of　$t$　(where

$\mu$　is the measure on　$M$ )

　(ii)　(<u>Conservation of angular momentum</u>):　if　$Y$　is a

vector field on　$M$　whose flow　$F_t$　is an isometry of　$M$

(i.e.,　$L_Y g = 0$　where　$g$　is the given metric on　$M$　and

$L_Y$　is the Lie derivative), and if　$\rho_0$　is invariant under

$F_t$ , then

$$\int_M <v_t,Y>\rho_t \, d\mu \quad \text{is independent of} \quad t$$

　(iii)　(<u>Covering Theorem</u>):　there is an　$H^s$　[resp.

$W^{s,p}$ ,　$C^{k+\alpha}]$　neighborhood　U of the identity diffeomorphism

and an $H^s$ [resp. $W^{s,p}$ , $C^{k+\alpha}$]neighborhood $V$ of $0$ in $H^s(TM)$ [resp. $W^{s,p}(TM)$ , $C^{k+\alpha}(TM)$] such that for $\eta \in U$ there is a unique $v_0 \in V$ such that if $\eta_t$ is the flow for the solution $v_t$ of ($\underline{1}$) then $\eta_1 = \eta$ .

(iv) (<u>Variational Theorem</u>): for $U,V$ as in (iii), and any curve $\zeta_t$ of volume preserving diffeomorphisms joining the identity to $\eta$ and $\zeta_t \in U$ , then

$$\int_0^1 \int_M <v_t,v_t>\rho_t \, d\mu dt \leq \int_0^1 \int_M < \frac{d\zeta_t}{dt} \circ \zeta_t^{-1} , \frac{d\zeta_t}{dt} \circ \zeta_t^{-1}>\rho_t \circ \zeta_t^{-1} d\mu dt$$

<u>Remarks</u>. 1.. In (iv) we only have minimizing among curves in a sufficiently small $H^s$ neighborhood $U$ . (Examples of Hilbert manifolds with weak Riemannian metrics are known such that curves of shorter length than the geodesic curves are obtained if these curves are allowed to go outside such a neighborhood.)

2. One can solve equations ($\underline{1}$) if a force term $f(x,t)$ is added. This force $f_t(x)$ should be in $H^{s+\ell}(TM)$ for $\ell \geq 1$ and must be divergence free and parallel to $\partial M$ . We can also allow $\ell = 0$ although the proof is harder (requiring appendix B of [2]).

3. All of the results (i)-(iv) are easily proved from our methods below and the results of [2], [4], [5].

4. In [5], a priori estimates are established which enable one to give an alternative Galerkin type proof of existence and uniqueness for the Euler equations (see [7]

for details).  A similar situation holds here. However, the
present method has the advantage that the other properties
of the solution are obtained automatically  and naturally;
these require considerably more effort using the method of
[7]. This is the case, in particular, for the well-posedness.
We note that the solutions are continuous but probably not
locally Lipschitz functions of the initial data.  (This is
true only in Lagrangian coordinates.)

In §3, 4 below we will outline the proof of the main
theorem.  The idea is similar to that used in [2]:  We
replace the problem with one of finding geodesics with
respect to a weak Riemannian metric on an infinite dimen-
sional manifold of diffeomorphisms.  We stick to the $H^s$
case,  $W^{s,p}$  and  $C^{k+\alpha}$  being similar.

## 3.  A Generalized Hodge Decomposition.

See §7 of [2] and Chapter 7 of [6] for the usual Hodge
decomposition.  We use the notations of [2].

Let  $\sigma \in H^s(M)$  and  $\sigma(x) > 0$  (if  s  is not larger
than  $n/2 + 1$  a Hodge theorem is also valid but we must
suppose  $\sigma$  is continuous in that case).

Define the operator  $d^\sigma: H^s(\Lambda^k) \to H^{s-1}(\Lambda^{k+1})$  by

$$d^\sigma(\alpha) = \frac{d(\sigma\alpha)}{\sigma}$$

$(H^s(\Lambda^k)$  stands for the k-forms of class  $H^s$ .)

Then it is easy to see that

$$(d^\sigma)^2 = 0$$

and that relative to the inner product

$$(\alpha,\beta)_\sigma = \int (\alpha \wedge *\beta)\sigma$$

that $\delta$ and $d^\sigma$ are duals:

$$(\alpha,\delta\beta)_\sigma = (d^\sigma\alpha,\beta)_\sigma$$

A form $\alpha$ is called $\sigma$-closed if $d^\sigma\alpha = 0$ and is $\sigma$-harmonic if $d^\sigma\alpha = 0$ and $\delta\alpha = 0$ . The notions of rallel to $\partial M$ and normal to $\partial M$ are the same as in [2]. Since there exists constants $M$ , $\delta > 0$ , such that $\delta \leq \sigma(x) \leq M$ , we see that the norm of $(\ ,\ )_\sigma$ is equivalent to the usual $H^0$ norm (with $\sigma = 1$ ).

Let $\mathcal{H}^s$ denote the space of $\sigma$-harmonic fields of class $H^s$ . We have:

## Generalized Hodge Decomposition:

$$(\underline{3}) \qquad H^s(\Lambda^k) = d^\sigma\left(\frac{H_n^{s+1}(\Lambda^{k-1})}{\sigma}\right) \oplus \delta(H_t^{s+1}(\Lambda^{k+1})) \oplus \mathcal{H}_\sigma^s$$

*where* $\oplus$ *denotes direct orthogonal sum in the inner product* $(\ ,\ )_\sigma$ . *Here* $H_n^s$ *denotes the space of* $H^s$

*forms which are normal to*  $\partial M$ .  *Also,*

$(\mathcal{K}_\sigma^s)_n$   *and*   $(\mathcal{K}_\sigma^s)_t$   *are finite dimensional.*

*There is a similar result for*  $W^{s,p}$  *and*  $C^{k+\alpha}$ .
*We also have the decomposition*

(4)             $$H^s(\Lambda^k) = d^\alpha \frac{H^{s-1}(\Lambda^{k-1})}{\sigma} \oplus \mathcal{C}_t^s (\Lambda^k)$$

*where*  $\mathcal{C}_t^s$  *denotes the coclosed forms*  $(\delta\alpha = 0)$
*tangent to*  $\partial M$ .

The proof of this follows from the methods of [6],
Chapter 7.  (The proof of the differentiability of the
members of the decomposition follows from the fact that

$$d^\sigma\!\left(\frac{H_n^{s+1}}{\sigma}\right) = \frac{1}{\sigma} \, d(H_n^{s+1})$$

is closed in the  $H^0$  topology on  $H^s$  as is  $\delta(H_t^{s+1})$ , by
the usual Hodge theorem.)  We shall omit the detailed proof.

As a corollary of decomposition (4), note that any  $H^s$
vector field  $X$  can be written  $X = Y + \dfrac{\text{grad } p}{\sigma}$  where
$\delta Y = 0$ , and  $Y$  is parallel to  $\partial M$ .

4.   Outline of the Proof of the Main Result.

We shall consider the  $H^s$  case; that for  $C^{k+\alpha}$  is
similar.  Let  $\mathcal{D}_\mu^s$  denote the  $H^s$  diffeomorphisms of  M

which preserve the standard volume element $\mu$ . This is a
Hilbert manifold, and a topological group; right multipli-
cation is $C^\infty$ . Define the following weak metric on $\mathfrak{D}_\mu^s$ :

$$(X,Y)_{\rho_0} = \int_M <X(x),Y(x)>_{\eta(x)} \rho_0(x) \, d\mu(x)$$

where $X,Y \in T_\eta \mathfrak{D}_\mu^s$ , the tangent space at $\eta$ which can be
identified with $H^s$ sections of $TM$ covering $\eta \in \mathfrak{D}_\mu^s$ .
That $( \, , \, )_{\rho_0}$ is smooth in $\eta$ follows because the
metric on $M$ is $C^\infty$ as in [2]. However, the metric
$( \, , \, )_{\rho_0}$ is not right invariant (unless $\rho_0$ is constant).

Let $\tilde{M}$ be the double of $M$ and consider the manifold
$H^s(M,\tilde{M})$ and the map

$$\tilde{Z}: TH^s(M,\tilde{M})|\mathfrak{D}_\mu^s = H^s(M,TM)|\mathfrak{D}_\mu^s \rightarrow T(H^s(M,TM)|\mathfrak{D}_\mu^s)$$

defined by $X \mapsto Z \circ X$ where $Z: TM \rightarrow T^2M$ is the spray
derived from the given metric on $M$ . Then $\tilde{Z}$ is $C^\infty$ .

Now define a map

$$P^{\rho_0}: H^s(M,TM)|\mathfrak{D}_\mu^s \rightarrow T\mathfrak{D}_\mu^s$$

by its action $P_\eta^{\rho_0}$ on the fiber over $\eta \in \mathfrak{D}_\mu^s$ , as

$$P_\eta^{\rho_0}(X) = R_\eta \circ P_e^{\rho_0 \circ \eta^{-1}} \circ R_{\eta^{-1}}$$

where $P_e^{\rho_0 \circ \eta^{-1}}$ is the projection of a vector field onto
its divergence free part parallel to the boundary with
respect to $\sigma = \rho_0 \circ \eta^{-1}$ (in the generalized Hodge
decomposition described in §3 above) and where $R_\eta Y = Y \circ \eta$
is right translation by $\eta$ .

Main Technical Lemma. $P^{\rho_0}$ *is a* $C^\infty$ *mapping.*

This is proved as in [2], appendix A, with the
following modifications. We define

$$\overline{d}^{\rho_0} : H^s(M, \Lambda^k) | \mathcal{D}^s \to H^{s-1}(M, \Lambda^{k+1}) | \mathcal{D}^s$$

by

$$\overline{d}_\eta^{\rho_0} = R_\eta \circ d^{\rho_0 \circ \eta^{-1}} \circ R_{\eta^{-1}}$$

(compare [2, appendix A, lemma 2]).
Then $\overline{d}^{\rho_0}$ is a $C^\infty$ mapping (as is $\overline{\delta}_\eta = R_\eta \circ \delta \circ R_{\eta^{-1}}$) .
Indeed, for $\alpha \in H^s(M, \Lambda^k)$ covering $\eta$ ,

$$\overline{d}^{\rho_0}(\alpha) = R_\eta \left( \frac{d(\rho_0 \circ \eta^{-1} \cdot \alpha \circ \eta^{-1})}{\rho_0 \circ \eta^{-1}} \right) = \frac{\overline{d}(\rho_0 \alpha)}{\rho_0}$$

so $\overline{d}^{\rho_0}$ is smooth (by lemma 2 of [2]).

The proof of lemma 3 of [2] does not carry over to the present context because for one thing $(\mathcal{H}^s_{\rho_0 \circ \eta^{-1}})_t$ does not consist of $C^\infty$ elements, and depends on $\rho_0 \circ \eta^{-1}$. However, we know from [2] that $\ker \bar{\delta}$ and $\operatorname{im} \bar{\delta}$ are subbundles in $H^s_t(M, \Lambda^k) \not{\mathcal{D}}^s$. Also, $\operatorname{im} \bar{d}^{\rho_0} = \dfrac{\operatorname{im} \bar{d}^{\rho_0}}{\rho_0}$ so $\operatorname{im} \bar{d}^{\rho_0}$ is a subbundle. Hence ([2], lemma 1]) $\ker \bar{d}^{\rho_0}$ is also a subbundle. Therefore, $\ker \bar{d}^{\rho_0} \cap \ker \bar{\delta}$ is a subbundle since $\bar{d}^{\rho_0}$ restricted to $\ker \delta$ is still onto the subbundle $\operatorname{im} \bar{d}^{\rho_0}$ by the Hodge decomposition. Therefore the analogue of lemma 3 is valid.

With these modifications, the proof of the technical lemma is the same as [2, appendix A].

Now we define the map

$$S^{\rho_0}: T\mathcal{D}^s_\mu \to T^2\mathcal{D}^s_\mu$$

by

$$S^{\rho_0}(X) = TP^{\rho_0}(Z \circ X) = TP^{\rho_0} \circ \bar{Z}(X)$$

(where $TP$ = tangent or derivative of $P^{\rho_0}$ ).

The above lemma shows that $S^{\rho_0}$ is a $C^\infty$ mapping; furthermore, by a straightforward modification of the proof in [2, §11], we find that $S^{\rho_0}$ is the spray associated to the weak metric $(\ ,\ )_{\rho_0}$ .

It follows immediately that $S^{\rho_0}$ has (unique) integral curves $u_t \in T\mathcal{D}_\mu^S$ for any initial conditions $u_0 \in T_{\eta_0}\mathcal{D}_\mu^S$ .

To prove the theorem, let $u_t$ be the integral curve on $T\mathcal{D}_\mu^S$ with $u_0$ covering the identity and $u_0 = v_0$ . Let

$$v_t = u_t \circ \eta_t^{-1}$$

where $u_t \in T_{\eta_t}\mathcal{D}_\mu^S$ so $\eta_t$ is the time dependent flow of the vector field $v_t$ , and $\eta_0 =$ identity.

Also let (cf. equation (2))

$$\rho_t(x) = \rho_0 \circ \eta_t^{-1}(x)$$

We claim that $v_t, \rho_t$ satisfy equations (1) (conversely given $v_t, \rho_t$ we get $\eta_t$ and hence $u_t$ so solutions are unique; cf. [2, §15]). The result will follow from this.

By the same proof as [2, theorem 14.4] and the fact that $P_e^{\rho_0 \circ \eta^{-1}}$ is the identity on the divergence free vector fields, we have

$$\frac{dv_t}{dt} = -P_e^{\rho_0 \circ \eta_t^{-1}} (\nabla_{v_t} v_t)$$

$$= -P_e^{\rho_t} (\nabla_{v_t} v_t)$$

$$= -\nabla_{v_t} v_t + \frac{\text{grad } \rho_t g_t}{\rho_t}$$

by the generalized Hodge decomposition. This is the
equation for $v_t$ if we set $P_t = \rho_t g_t$ . That for $\rho_t$
is seen as follows:  Since

$$\frac{d}{dt} \eta_t^{-1} = -T\eta_t^{-1} \circ \frac{d\eta_t}{dt} \circ \eta_t^{-1}$$

and  $\dfrac{d\eta_t}{\alpha t} = u_t$  (as  S  is a spray), we have

$$\frac{d}{dt} \rho_t = -d\rho_0 \cdot T\eta_t^{-1} \cdot v_t$$

$$= -d(\rho_0 \circ \eta_t^{-1}) \cdot v_t$$

$$= -d(\rho_t) \cdot v_t$$

where we have used the chain rule. This establishes the
equation for $\rho_t$ and completes the proof.


## 5.  Regularity of Solutions.

Because we do not have right invariance on $\mathcal{D}_\mu^s$ , the
regularity theorem ([2, theorem 12.1]), which states in
particular that if $v_0$ is $C^\infty$ , so is $v_t$ on the interior
of  M  as long as $v_t$ is defined in $H^s$ , is not immediate
in this case.  Nevertheless, the result is still true for
$\rho_0 \in C^\infty$ .

Indeed, an examination of the above proof shows that $S^{\rho_0}$ depends in a $C^\infty$ fashion on $\rho_0$. The associated exponential map

$$\tilde{E}^{\rho_0}: T\mathfrak{D}^s_\mu \to \mathfrak{D}^s_\mu$$

therefore also depends smoothly on $\rho_0$. If $\xi \in \mathfrak{D}^s_\mu$ we have

$$(X,Y)_{\rho_0} = (X\circ\xi, Y\circ\xi)_{\rho_0} \circ \xi$$

in place of right invariance. It follows that

$$\tilde{E}^{\rho_0\circ\xi}(X\cdot\xi) = \tilde{E}^{\rho_0}(X)\circ\xi$$

If $X$ and $\rho_0$ are $C^\infty$, the left side is smooth in $\xi$, (by the composition lemmas [2, §2]) so the right side is also. Now the argument in theorem 12.1 of [2] applies to yield the result.

Regularity, including at the boundary, can also be proved by the methods of [5].

6. Global Regularity.

It is amusing to note that while global (in time) regular solutions exist for Euler's equations in two dimensions (the Wolibner-Judovich-Kato theorem), this

problem remains open for non-homogenous flow in two
dimensions, and may well be false.  The key fact used
in the proof of Wolibner's theorem is conservation of
vorticity (Kelvin's circulation theorem).  That result
is not true for inhomogeneous fluids.

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