APPENDIX TO LECTURE I: BIFURCATIONS, SEMIFLOWS, AND NAVIER-STOKES EQUATIONS

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As was pointed out in J. Marsden's talk, the Ruelle-Takens picture for turbulence assumes that the motion of the fluid is inherently chaotic, that the flow obtained for $Re = 0$ (solutions of the Stokes equations) gets more and more complicated as the Reynolds number Re increases, due to bifurcation phenomena until it eventually gets trapped into a "strange" attractor which has chaoticness as one of its main features. In this talk I shall summarize the mathematical results involved in this machinery, trying to back up with exact statements of theorems many exciting ideas presented in Marsden's exposition. The main source of this talk is Marsden-McCracken [1].

The leading idea is to obtain a model born out of the Navier-Stokes equations for homogeneous, incompressible, viscous fluids:

$$\begin{cases} \frac{\partial v}{\partial t} + (v\cdot\nabla)v - \nu\Delta v = -\text{grad } p + f, \quad \nu = 1/Re \\ \text{div } v = 0 \\ v = \text{prescribed on } \partial M, \text{ possibly depending on } \nu \end{cases}.$$

Everything takes place in a compaot Riemannian manifold M with smooth boundary ∂M, v representing the velocity field of the fluid, p the pressure and f the external force exercised on the moving fluid. As already mentioned, Euler's equations for an ideal fluid are obtained by setting $\nu = 0$ in the above equations; it is a theorem that the solutions to the Euler equations are obtained as a strong limit in the H^s-topology for $s > (\dim M)/2+1$ (see Ebin-Marsden [1]). Also notice that in Euler's equations we have to change the boundary conditions to $v \parallel \partial M$. The intuitive reason why this is so is that our fluid, being ideal, has no friction at all on the walls; however, a much more subtle mathematical analysis of the above described limit process yields formally the same result, cf. Marsden [2], Ebin-Marsden [1].

Now we would like to write our Euler and Navier-Stokes equations in the form of a system of evolution equations

$$\frac{dv}{dt} = X_\nu(v) \quad , \ v(0) = \text{given} \quad ,$$

where X_ν is a densely defined nonlinear operator on a function space picked in such a way that our boundary conditions and $\operatorname{div} v = 0$ should be automatically satisfied. The answer to this question is given by the Hodge Decomposition Theorem.

Denote by $W^{s,p}$ the completion of the normed vector space of vector-valued C^∞-functions on M under the norm

$$\|f\|_{s,p} = \sum_{0\le t\le s} \|D^t f\|_{L^p} ;$$

here $D^t f$ denotes the differential of f , $s \geq 0$ and $1 < p < \infty$. $W^{s,p}(M)$ is the set of vector fields of class $W^{s,p}$ on M . Note that a function is of class $W^{s,p}$ if and only if all its derivatives up to order s are in L^p .

<u>Hodge Decomposition Theorem</u>. Let M be a compact Riemannian manifold with boundary and $X \in W^{s,p}(M)$, $s \geq 0$, $1 < p < \infty$. Then X has a unique decomposition

$$X = Y + \text{grad } f$$

where $\text{div } Y = 0$, $Y \| \partial M$, $Y \in W^{s,p}(M)$ and f is of class $W^{s+1,p}$.

Denote $\tilde{W}^{s,p}(M) = \{X \in W^{s,p}(M) | \text{div } X = 0, X \| \partial M\}$. Apply now the Hodge Theorem and get a map $P: W^{s,p}(M) \to \tilde{W}^{s,p}(M)$ via $X \mapsto Y$. Let us now reformulate the Euler equations: suppose $s > n/p$; find $v: (a,b) \to \tilde{W}^{s+1,p}(M)$ such that

$$\frac{dv(t)}{dt} + P((v(t)\cdot\nabla)v(t)) = 0$$

(plus initial data). We need to assume $s > n/p$ in order to insure that the product of two elements of $W^{s,p}$ is in $W^{s,p}$ (see Adams [1], page 115). In this way, if $v \in \tilde{W}^{s+1,p}(M)$,

$(v\cdot\nabla)v \in W^{s,p}(M)$ and we can apply the Hodge Theorem. In doing this we tacitly assume that the external force is a gradient.

In order to be able to write in a similar way the Navier-Stokes equations, we change the function space to $\tilde{W}_0^{s,p} = \{X \in W^{s,p}(M) | \operatorname{div} X = 0, X|\partial M = 0\}$. Then the Navier-Stokes equations can be reformulated: find $v: (a,b) \to \tilde{W}_0^{s+1,p}$ such that

$$\frac{dv(t)}{dt} - \nu P(\Delta v(t)) + P((v(t)\cdot\nabla)v(t)) = 0 \quad .$$

The following theorem is proved in Section 9 of Marsden-McCracken.

__Theorem__. *The Navier-Stokes equations in dimensions* 2 *or* 3 *define a smooth local semiflow on* $\tilde{W}_0^{s,2}$ *, i.e., we have a collection of maps* $\{F_t^\nu\}$ *for* $t \geq 0$ *satisfying:*

(a) F_t^ν *is defined on an open subset of* $[0,\infty) \times \tilde{W}_0^{s,2}$ *;*

(b) $F_{t+s}^\nu = F_t^\nu \circ F_s^\nu$ *;*

(c) F_t^ν *is separately (hence, jointly)** *continuous;*

(d) for each fixed t,ν , F_t^ν *is a* C^∞*-map, i.e.,* $\{F_t^\nu\}$ *is a __smooth semigroup__. More, our semiflow* $\{F_t^\nu\}$ *satisfies the so called __continuation assumption__, namely, if* $F_t(x)$ *lies in a bounded set of* $\tilde{W}_0^{s,2}$ *for each fixed* x *and for all* t *for which* $F_t^\nu(x)$ *is defined, then* $F_t(x)$ *is defined for all* $t \geq 0$.

Also, $F_t^\nu(x)$ *is jointly smooth in* t,x,ν *for* $t>0$.

*See Chernoff-Marsden [1], Chapter 3, or Marsden-McCracken [1], Section 8A, for the proof of the fact that separate continuity ⇒ joint continuity.

This result which goes back to Ladyzhenskaya [1] encourages us to not work with the Navier-Stokes equations under their classical form, but rather with the evolution equations in $\tilde{W}_0^{s,2}$ which they define and to analyze more closely their semiflow which has such pleasant properties.

Following the idea of chaotic dynamics, we may try to show that turbulence occurs after successive bifurcations of the solutions of the Navier-Stokes equations. Hence a first question is how much of the classical bifurcation theory can be obtained for semiflows. The work of Marsden shows that almost everything works, if one mimics the conditions on the semiflow from those, one usually has for vector fields. We shall summarize these results below.

Hence we have to cope with a system of evolution equations of the general form

$$\frac{dx}{dt} = X_\mu(x) \quad , \; x(0) = \text{given} \quad ,$$

where X_μ is a nonlinear densely defined operator on an appropriate Banach space E , usually -- as we already saw -- a function space and μ is a parameter. We assume that our system defines unique local solutions generating a semiflow F_t^μ for $t \geq 0$. The assumptions made on the semiflow are (a), (b), (c) and (d) above. We also ask for the continuation assumption described before. It may seem that we force our assumptions on the semiflow such as to suit our particular problem. In reality it is exactly the other way around: one usually has these conditions satisfied and checks them for the Navier-Stokes equations -- and this is hard work involving a serious mathematical machinery (see Section 9 of Marsden-McCracken). It is true that the continuation assumption might seem strong; but it merely says that we have at our disposal a "good" local existence theorem, so "good" as to insure the fact that an orbit fails to be defined only if it tends to infinity in a finite time. That makes sense physically, looking at expected solutions of the governing equations of the law of motion of a fluid (Navier-Stokes): a solution fails to exist only if it "blows up". Another remark is of mathematical character and concerns the generator X_μ ; this is not a smooth map from E to E , hence we cannot expect smoothness of $F_t^\nu(x)$ in t . The fact is that the trouble is actually only at $t = 0$, as can be seen from the theorem on the Navier-Stokes semiflow from before, and exactly the derivative at $t = 0$ gives the generator. The next group of assumptions regards the spectrum of the linearized semiflow relevant for the Hopf bifurcation.

Spectrum Hypotheses. Let $F_t^\mu(x)$ be jointly continuous in t,μ,x for $t > 0$ and μ in an interval around $0 \in \mathbb{R}$. Suppose in addition that:

(i) 0 is a fixed point of F_t^μ , i.e., $F_t^\mu(0) = 0$, $\forall \mu,t$;

(ii) for $\mu < 0$, the spectrum of $G_t^\mu = DF_t^\mu(0)$ is contained inside the unit disc $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$;

(iii) for $\mu = 0$ (resp. $\mu < 0$) the spectrum of G_1^μ at the origin has two isolated simple eigenvalues $\lambda(\mu)$ and

$\overline{\lambda(\mu)}$ with $\lambda(\mu) = 1$ (resp. $\lambda(\mu) > 1$) and the rest of the spectrum is in D and remains bounded away from the unit circle;

(iv) $\left.\frac{d|\lambda(\mu)|}{dt}\right|_{\mu=0} > 0$, i.e., the eigenvalues move steadily across the unit circle.

Sometimes we look at these hypotheses but with (iii) changed to:

(iii') for $\mu = 0$ (resp. $\mu < 0$) the spectrum of G_1^μ at the origin has one isolated simple real eigenvalue $\lambda(\mu) = 1$ (resp. $\lambda(\mu) > 1$) and the rest of the spectrum is in D and remains bounded away from the unit circle;

(v) for $\mu = 0$ the origin is asymptotically stable.

We won't go into the technical details of this last hypothesis here and say only that it involves an algorithm of checking if a certain displacement function obtained via Poincaré map has strictly negative third derivative.

Bifurcation to Periodic Orbits: *Under the above hypotheses (i)-(v) there is a fixed neighborhood* V *of* 0 *in* E *and an* $\varepsilon > 0$ *such that* $F_t^\mu(x)$ *is defined for all* $t \geq 0$ *for* $\mu \in [-\varepsilon,\varepsilon]$ *and* $x \in V$. *There is a one-parameter family of closed orbits for* F_t^μ *for* $\mu > 0$ *, one for each* $\mu > 0$ *varying continuously with* μ . *They are locally attracting and* *hence stable. Solutions near them are defined for all* $t \geq 0$ *. There is a neighborhood* U *of the origin such that any closed orbit in* U *is one of the above orbits.*

Bifurcation to Fixed Points: *Same hypothesis with (iii) and (iii') interchanged. Then the same result holds, replacing the words "closed orbit" with "two fixed points".*

I shall not go into the proof of these theorems but will give the two crucial facts behind the formal proof. One is the Center Manifold Theorem and the other is a theorem of Chernoff-Marsden regarding smooth semiflows on finite-dimensional manifolds. Coupling these two results reduces the whole problem to the classical Hopf Bifurcation Theorem in 2 dimensions, which is relatively simple and goes back to Poincaré. Here are the statements:

Center Manifold Theorem for Semiflows: *Let* Z *be a Banach space admitting a* C^∞*-norm away from zero, and let* F_t *be a continuous semiflow defined in a neighborhood of zero for* $0 \leq t \leq z$ *. Assume* $F_t(0) = 0$ *and that for* $t > 0$ *,* $F_t(x)$ *is jointly* C^{k+1} *in* t *and* x *. Assume that the spectrum of the linear semigroup* $DF_t(0): Z \to Z$ *is of the form* $e^{t(\sigma_1 \cup \sigma_2)}$ *where* $e^{t\sigma_1}$ *lies on the unit circle (i.e.,* σ_1 *lies on the imaginary axis) and* $e^{t\sigma_2}$ *lies in the unit circle at non-zero distance from it for* $t > 0$ *(i.e.,* σ_2 *is in the left half* *plane). Let* Y *be the generalized eigenspace corresponding to the spectrum on the unit circle; assume* $\dim Y = d < +\infty$ *. Then there exists a neighborhood of* 0 *in* Z *and a* C^k*-submanifold* $M \subseteq V$ *of dimension* d *passing through* 0 *and tangent to* Y *at* 0 *such that:*

(a) *Local Invariance: if* $x \in M$, $t > 0$ *and* $F_t(x) \in V$, *then* $F_t(x) \in M$;

(b) *Local Attractivity: if* $t > 0$ *and* $F_t^n(x)$ *remains defined and in* V *for all* $n = 0,1,2,\ldots$, *then* $F_t^n(x) \to M$ *as* $n \to \infty$.

This is applied to F_t^μ after suspending μ to obtain the semi-flow $F_t(x,\mu) = (F_t^\mu(x),\mu)$ on the original space x the parameter space

The version of this theorem for a C^{k+1} map is well known; however, this statement regarding semiflows -- although believable -- wasn't present in the literature before; the first time it appears is in Section 2 of Marsden-McCracken. Note that everything works out nicely in the theorem, even though the generator X of the semiflow is unbounded.

<u>Theorem (Chernoff-Marsden)</u>: *Let* F_t *be a local semiflow on a Banach manifold* N *jointly continuous and* C^k *in* $x \in N$*. Suppose that* F_t *leaves invariant a finite dimensional submanifolf* $M \subseteq N$ *. Then on* M , F_t *is locally reversible, is jointly* C^k *in* t *and* x *and is generated by a* C^{k-1} *vector field on* M.

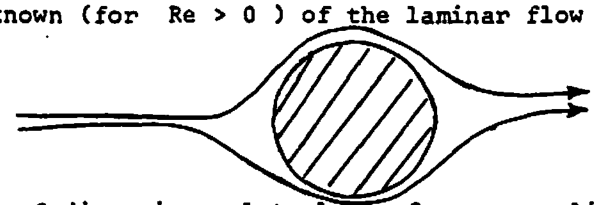
Some remarks are in order. Besides being one key factor in the proof of the bifurcation theorem, the center manifold theorem might justify some modal truncations of the Navier-Stokes equations to give a d-dimensional system (see Remark 8 of Lecture I by J. Marsden). Also, in Marsden-McCracken, Section 4A, an algorithm is described which enables us to check on the stability of the new born fixed points or closed orbits after bifuracations. Remark 4 of Lecture I hints toward that. The reduction to two dimensions appears as a corollary of the proof of the Bifurcation Theorem. The conclusion is that all the complexity in this case takes place only in a plane, even though we started off with an evolution equation on an infinite dimensional function space. This occurrence is characteristic when we work with semiflows; trying to prove a bifurcation, we reduce everything to a finite dimensional theorem for flows and this gives us then two things: the theorem itself and the reduction!

That's the way one approaches the next bifurcation to invariant tori. Here the Hopf Bifurcation Theorem for Diffeomorphisms will be needed and the idea of the proof is the same as before; one has to replace the argument of the Hopf Bifurcation Theorem in $\mathbb{R}^2$ with a similar argument using now the Hopf Bifurcation Theorem for Diffeomorphisms. I won't go into any technical details.

That would roughly solve the approach to the first two bifurcations. How about higher ones? The only leading idea is the Poincaré map, and the fact that something invariant for it, yields an invariant manifold of one higher dimension for the semiflow with the preservation of the attracting or repelling character: a fixed point -- attracting or repelling -- gave a closed orbit -- attracting or repelling -- a circle, an invariant torus, etc.

Let me mention that all these geometrical methods presented here are by no means the only ones with which one could attack bifurcation problems for the Navier-Stokes equations. An excellent reference is J. Sattinger [1], who in Chapters 4-7 does roughly the same thing, but using methods of eigenvalue problems, energy methods and Leray-Schauder degree theory. I prefer the above methods because I think they appeal more to one's geometrical intuition.

As a concluding remark, let me say that even if it seems that the first bifurcations can be attacked successfully with the above methods, the difficulties one faces might be very big. One has to start off with something known, namely a particular stationary solution, regard this as a fixed point of the generator of the semiflow and work his way through the conditions in the Bifurcation Theorem. In many cases we do not have even a stationary solution! In the research problem suggested in Lecture I about the flow behind a cylinder, the difficulty is exactly this one: there is no explicitly solution known (for $Re > 0$) of the laminar flow



in 2 or 3 dimensions, let alone of more complicated situations.

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