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QUALITATIVE METHODS IN BIFURCATION THEORY

Bifurcation decreases entropy
... Helen Petard

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Classical bifurcation theory is undergoing a revitalization with the infusion of ideas from singularities of mappings and structural stability. The situation now is similar to that a quarter century ago when Krasnosel'skiĭ introduced topological methods, especially degree theory, into the subject (see Krasnosel'skiĭ [**1964**]). Like degree theory, the theory of singularities of mappings is playing a fundamental role in the development of the subject.

Our goal is to give a few examples of how qualitative ideas can give insight into bifurcation problems. The literature and full scope of the theory is too vast to even attempt to survey here. On the classical bifurcation theory side, the survey article of Sather [**1973**] is valuable, and for the theory of singularities of mappings, we refer to Golubitsky and Guillemin [**1973**]. On the overlap, the article of Hale [**1977**] is recommended.

The credit for using ideas of singularities of mappings and structural stability in bifurcation theory is often attributed to Thom (see Thom [**1972**]) and on the engineering side, to Thompson and Hunt [**1973**], Roorda [**1965**] and Sewell [**1966**]. However, to penetrate the classical bifurcation circuit is another matter. For this, there are a number of recent articles, notably, Chillingworth [**1975**], Chow, Hale and Mallet-Paret [**1975**], Magnus and Poston [**1977**], Holmes [**1977**] and Potier-Ferry [**1977**]. The literature on this interaction is in an explosive state and we merely refer to the above articles, Chillingworth [**1976**], Golubitsky [**1978**], Marsden and McCracken [**1976**], Abraham and Marsden [**1978**] and Poston and Stewart [**1978**] for further references.

1. The definition of bifurcation point. The very definition of bifurcation point varies from author to author, although in any specific situation there is usually no doubt about what should be called a bifurcation point.

The "classical" definition is typified by the following discussion in Matkowski and Reiss [**1977**]:

> "Bifurcation theory is a study of the branching of solutions of nonlinear equations $f(x, \lambda) = 0$ where f is a nonlinear operator, x is the solution vector and λ is a

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parameter. It is of particular interest in bifurcation theory to study how the solutions $x(\lambda)$ and their multiplicities change as λ varies. Thus we refer to λ as the bifurcation parameter. A *bifurcation point* of a solution branch $x(\lambda)$ is a point $(\lambda_0, x(\lambda_0))$ from which another solution $x_1(\lambda)$ branches. That is, $x(\lambda_0) = x_1(\lambda_0)$ and $x(\lambda) \neq x_1(\lambda)$ for all λ in an interval about λ_0." (See Figure 1.)

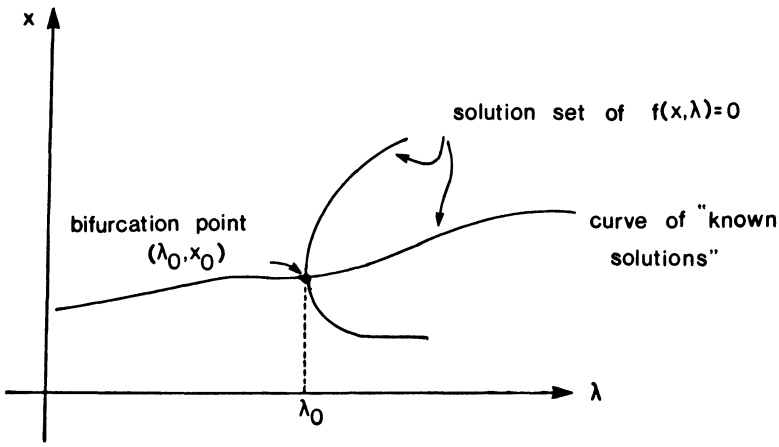

FIGURE 1

For studying bifurcation from known solutions, this is an appropriate definition. However, solutions can appear spontaneously or need not be connected to a "known" solution, as occurs in the saddle-node bifurcation (Figure 2) or in subtle dynamical bifurcations such as a global saddle connection or the bifurcation to the strange Lorentz attractor (Marsden [1977]).

Thus it seems wise to take a wider view. In doing so, we interpret the equation $f(x, \lambda) = 0$ liberally, to include dynamic (i.e. evolution) equations

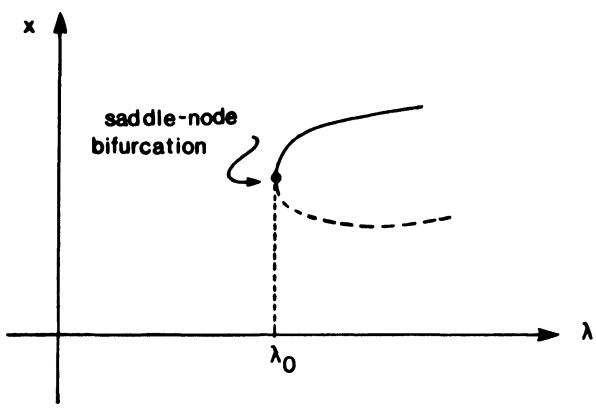

FIGURE 2

as well as static equations and allow the parameter λ to be multidimensional. Any more general definition of bifurcation ought to reduce to the above definition for bifurcation from a known branch.

A way to eliminate having a "known branch" is by means of the following definition (see Chow, Hale and Mallet-Paret [**1975**])

> "Suppose τ is a family of mappings from one Banach space X into another Banach space Z and suppose there is a norm on the members of τ. Let $T \in \tau$ be given and suppose there is an $x_0 \in X$ such that $Tx_0 = 0$. The operator T is said to be a *bifurcation point* for τ at x_0 if for every neighborhood U of T and V of x_0, there is an $S \in U$ and $x_1, x_2 \in V$, $x_1 \neq x_2$ such that $Sx_1 = Sx_2 = 0$."

This definition still fails to apply when the solution sets for fixed T are not locally isolated, but clearly a bifurcation occurs. In Figure 3, every point on Σ_1 would be a bifurcation point according to this definition, which is not what we want. Such situations occur in bifurcations of Hamiltonian systems (see Weinstein [**1978**]).

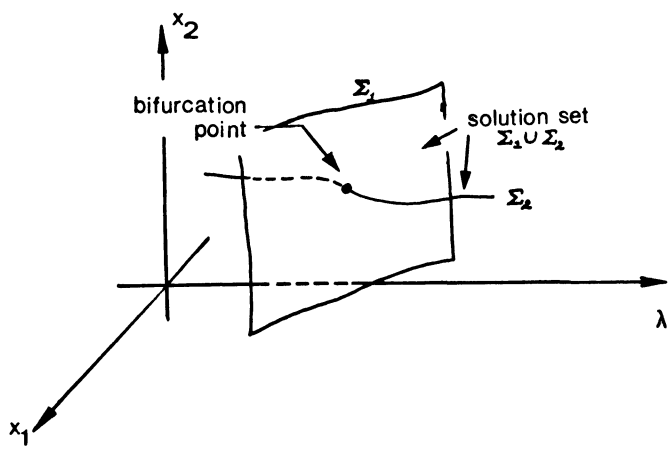

FIGURE 3

Crandall and Rabinowitz [**1971**] were amongst the first to emphasize the technical importance of supressing the parameters and treating the map $f(x, \lambda)$ as a whole and de-emphasizing the special role of the parameter. In fact, this is exactly what is done in singularities of mappings and global analysis. (However, in studying perturbations of bifurcation diagrams the special role played by the control (or bifurcation) parameter λ is crucial; see e.g. Golubitsky and Schaeffer [**1978**].) One usually adopts a definition like the following (see Smale [**1970**] for example).

DEFINITION (GLOBAL BIFURCATION). *If h: $M \to N$ is a continuous map between topological spaces then $y_0 \in N$ is a* **bifurcation point** *of h if, for every neighborhood U of y_0, not all the sets $h^{-1}(y)$, $y \in U$, are homeomorphic; i.e. $h^{-1}(y)$ changes topological type at y_0.*

(LOCAL BIFURCATION). *If h: $M \to N$ is a continuous map, a point $x_0 \in M$ is*

a **bifurcation point** *for h if for every neighborhood U of* $y_0 = h(x_0)$ *and V of* x_0, *the sets* $h^{-1}(y) \cap V$, $y \in U$, *are not all homeomorphic; i.e.* $h^{-1}(y)$ *locally changes topological type at* (x_0, y_0).

This definition is related to the previous ones for the equation $f(x, \lambda) = 0$, where $x \in X$, $\lambda \in \Lambda$, by letting

$$\Sigma = \text{solution set of } f = \{(x, \lambda) | f(x, \lambda) = 0\}$$

and h: $\Sigma \to \Lambda$, $(x, \lambda) \mapsto \lambda$. The local definition applied to this map captures what we want in the general case and slightly extends the first definition for bifurcation from known solutions. For global bifurcations (Figure 4), the global definition is appropriate. This definition also suggests that one should make direct use of algebraic-topological invariants to detect a change in topological type.

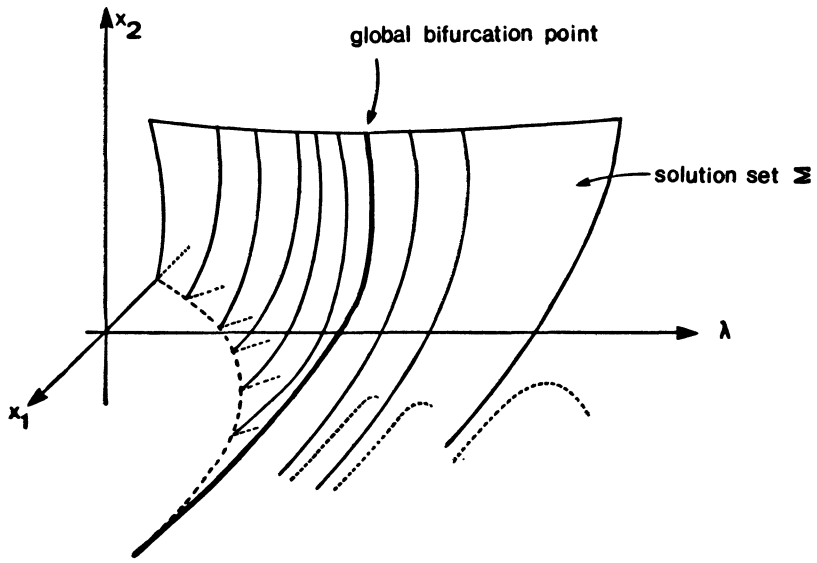

FIGURE 4

This definition is also desirable because in many problems there are no parameters, yet bifurcation techniques are called for. We shall give an illustration of this in §3 below.

In the definition we can modify the relation "homeomorphic" to other relations appropriate to the context. For instance let M be a family of vector fields on a manifold and let h be the identity map on M. If the relation is that of having topologically conjugate flows, then the general definition reduces to that in Thom [1972] i.e. a vector field is a bifurcation point (of a family) if it is in the complement of the structurally stable vector fields (relative to the given family). See Abraham and Marsden [1978] for additional details.

There are deep connections between bifurcation and symmetry that are only beginning to be understood. For example, Sattinger [1976] and others have suggested that bifurcation is closely related to symmetry breaking and that bifurcation points necessarily have a degree of symmetry. We shall see this borne out in some examples below. Although systems with symmetries

are nongeneric, they nevertheless play pivotal roles as bifurcation points. (This is one reason why Hamiltonian systems with symmetry are so important within the class of all Hamiltonian systems.)

2. Topological methods in bifurcation theory.[2] The use of degree theory in bifurcation problems is well known and is described in Krasnosel'skiĭ **[1964]**. Other topological or differential-topological methods can be useful as well. For example, in Duistermaat **[1974]** and Nirenberg **[1974]** the Morse lemma is shown to yield quite directly some main results on bifurcation at simple eigenvalues (cf. Crandall and Rabinowitz **[1971]**, Cesari **[1976]** and references therein). We shall now recall part of their argument.

Let X and Y be Banach spaces and $f\colon X \times \mathbf{R}^p \to Y$ a C^k map, $k \geqslant 3$. Let $D_x f(x, \lambda)$ be the (Fréchet) derivative of f with respect to x, a continuous linear map of X to Y. Let $f(x_0, \lambda_0) = 0$ and

$$X_1 = \ker D_x f(x_0, \lambda_0).$$

Assume X_1 is finite dimensional with a complement X_2 so that $X = X_1 \oplus X_2$. Also, assume

$$Y_1 = \text{Range } D_x f(x_0, \lambda_0)$$

is closed and has a finite-dimensional complement Y_2. In other words, $D_x f(x_0, y_0)$ is a Fredholm operator. Write $Y = Y_1 \oplus Y_2$ and let $P\colon Y \to Y_1$ be the projection. By the implicit function theorem, if $D_x f(x_0, \lambda_0)$ is surjective, then (x_0, λ_0) is not a bifurcation point. In general, by the same theorem

$$Pf(x_1 + x_2, \lambda) = 0$$

has a unique solution $x_2 = u(x_1, \lambda)$ near x_0, λ_0, where $x = x_1 + x_2 \in X = X_1 \oplus X_2$. Thus, the equation $f(x, \lambda) = 0$ is equivalent to the *bifurcation equation*

$$(I - P)f(x_1 + u(x_1, \lambda), \lambda) = 0,$$

a system of dim Y_2 equations in dim X_1 unknowns. This reduction is usually called the *Liapunov-Schmidt procedure*.

Let dim X_1 = dim $Y_2 = 1$ (a simple eigenvalue[3]), $p = 1$ and suppose that

$$\frac{\partial f}{\partial \lambda}(x_0, \lambda_0) = 0, \quad \frac{\partial^2 f}{\partial \lambda^2}(x_0, \lambda_0) \in Y_1, \quad \frac{\partial^2 f}{\partial \lambda \partial x}(x_0, \lambda_0)x_1 \notin Y_1.^4$$

Then (x_0, λ_0) is a bifurcation point for $f(x, \lambda) = 0$; moreover the directions of bifurcation are those of the zeros of the quadratic form associated to

$$(I - P)D^2 f(x_0, \lambda_0),$$

restricted to $X_1 \times \mathbf{R}$.

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[2]This section is based on Buchner, Marsden and Schecter **[1978]**.

[3]The terminology "eigenvalue" derives from the case $f(x, \lambda) = Lx - \lambda x +$ higher order terms, where L is a linear operator of X to x; then $\ker D_x f(x_0, \lambda_0)$ is the eigenspace of L with eigenvalue λ_0.

[4]These are sample hypotheses relevant for the case in which a trivial solution to an equation like the one in the preceding footnote, loses stability. A more general result is proven in the theorem below.

This result follows by letting l be a linear functional on Y orthogonal to Y_1, letting φ: $\mathbf{R}^2 \to \mathbf{R}$ be defined by

$$\varphi(x_1, \lambda) = l(f(x_1 + u(x_1, \lambda), \lambda))$$

and applying the Morse lemma to φ. Our hypotheses imply that $((x_0)_1, \lambda_0)$ is a nondegenerate critical point of index 1, so the set of zeros of φ consist of two intersecting curves (Figure 5).

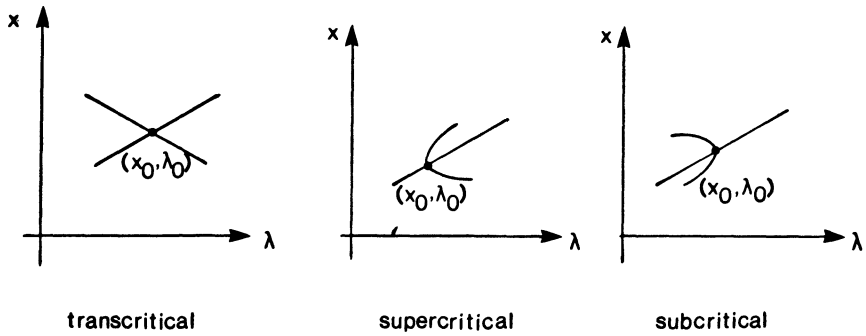

Figure 5. Bifurcation at Simple Eigenvalues

For elementary applications of this result we refer to standard sources such as Keller and Antman [**1969**], Nirenberg [**1974**] and Pimbley [**1969**].

This theorem can be generalized in many ways. The appropriate generalization of the Morse lemma is the list of elementary catastrophes of Thom and Zeeman. It is also useful to generalize in a different direction, namely to the case in which there is a multiple eigenvalue, but the singularity is still quadratic. We now give such a result following the methods of Buchner, Marsden and Schecter [**1978**].

Let B: $Z \times Z \to W$ be a continuous symmetric bilinear form on a Banach space Z with values in a Banach space W and let Q: $Z \to W$, $z \mapsto B(z, z)/2$ be the associated quadratic form. Set $C = Q^{-1}(0)$, the cone of zeros of Q. We say Q is in *general position on* C if for each $v \in C$, $v \neq 0$, the map $z \mapsto B(z, v)$ from Z to W is surjective.

THEOREM. *Let f: $X \times \mathbf{R}^p \to Y$ be C^k, $k \geqslant 3$, with $f(x_0, \lambda_0) = 0$ and $D_x f(x_0, \lambda_0)$ Fredholm, as above. Assume*

$$\frac{\partial f}{\partial \lambda}(x_0, \lambda_0) = 0$$

and let $B = (I - P)D^2 f(x_0, \lambda_0)$ restricted to $(X_1 \times \mathbf{R}^p) \times (X_1 \times \mathbf{R}^p)$ with Q the associated quadratic form. Assume Q is in general position on $C = Q^{-1}(0)$.

Then the set of solutions of $f(x, \lambda) = 0$ near (x_0, λ_0) is homeomorphic to C near 0; the set of solutions consists of C^{k-2} curves through (x_0, λ_0) tangent to elements of C.

For the proof, we use the method of blowing up a singularity.

LEMMA. *Let g: $\mathbf{R}^n \to \mathbf{R}^m$ be C^k, $k \geqslant 3$, and $g(0) = 0$, $Dg(0) = 0$. Set $B = D^2 g(0)$ and Q its quadratic form. Assume Q is in general position on*

$C = Q^{-1}(0)$. *Then near* 0, $g^{-1}(0)$ *is homeomorphic to* C *near* 0 *and is a union of* C^{k-2} *curves through* 0 *tangent to* C.

PROOF. Let $S \subset \mathbf{R}^n$ be the unit sphere, a codimension one smooth submanifold. We proceed in a number of steps.

(1) $Q|S$ *has* 0 *as a regular value.*

First note that $DQ(x)v = B(x, v)$. Therefore, if $x \in Q^{-1}(0)$, then $DQ(x)$ is surjective. Now for $x \in Q^{-1}(0) \cap S$, $T_x\mathbf{R}^n = T_xS + \mathbf{R}x$. Since $DQ(x)|\mathbf{R}x = 0$, $DQ(x)|T_xS = D(Q|S)(x)$ is surjective, i.e., 0 is a regular value.

Next we "blow up" the singularity at 0 as follows. Let φ: $S \times \mathbf{R} \to \mathbf{R}^n$, $\varphi(x, r) = rx$, and note that φ is C^∞. Observe that $\varphi(X \times \{0\}) = 0$, $D\varphi(x, r)$ is invertible if $r \neq 0$, and $\ker D\varphi(x, 0) = T_xS \times \{0\}$.

By Taylor's theorem,

$$\begin{aligned} g(x) &= \int_0^1 (1-t)D^2g(tx)(x, x)\,dt \\ &= \int_0^1 (1-t)D^2g(0)(x, x)\,dt + \int_0^1 (1-t)\big[D^2g(tx) - D^2g(0)\big](x, x)\,dt \\ &= Q(x) + \int_0^1 (1-t)\big[D^2g(tx) - D^2g(0)\big](x, x)\,dt. \end{aligned}$$

Therefore $g \circ \varphi(x, r) = r^2Q(x) + r^2h(x, r)$, where h is C^{k-2} and $h|S \times \{0\} = 0$. (In fact, $h(x, r) = \int_0^1(1-t)[D^2g(trx) - D^2g(0)](x, x)\,dt$.) Define $\tilde{g}$: $S \times \mathbf{R} \to \mathbf{R}^n$ by

$$\tilde{g}(x, r) = r^{-2}g \circ \varphi(x, r) = Q(x) + h(x, r).$$

Then $\tilde{g}$ is of class C^{k-2} and is C^k away from $r = 0$. Notice that $\tilde{g}^{-1}(0)$ and $(g \circ \varphi)^{-1}(0)$ coincide away from $S \times \{0\}$. While $g \circ \varphi$ is degenerate on $S \times \{0\}$, $\tilde{g}$ is not, so we will be able to describe explicitly the zero set of $\tilde{g}$.

(2) *For* $\varepsilon > 0$ *sufficiently small if* $|\varepsilon'| < \varepsilon$ *then* $\tilde{g}|S \times \{\varepsilon'\}$ *has* 0 *as a regular value.*

Indeed, $\tilde{g}|S \times \{0\} = Q|S$ (identifying S and $S \times \{0\}$) which has 0 as a *regular* value by (1). Since S is compact, (2) follows.

From (2) and the fact that $\tilde{g}$ is C^{k-2} we get

(3) $\tilde{g}^{-1}(0) \cap S \times (-\varepsilon, \varepsilon)$ *is a* C^{k-2} *manifold that intersects* $S \times \{0\}$ *transversally.*

Moreover,

(4) *For* $\varepsilon > 0$ *sufficiently small,* $\tilde{g}^{-1}(0) \cap (S \times (-\varepsilon, \varepsilon))$ *is* C^{k-2} *diffeomorphic to*

$$\big[\tilde{g}^{-1}(0) \cap (S \times \{0\})\big] \times (-\varepsilon, \varepsilon) = \big(Q^{-1}(0) \cap S\big) \times (-\varepsilon, \varepsilon)$$

by a diffeomorphism which is C^k *away from* $S \times \{0\}$.

The C^{k-2} diffeomorphism is obtained by noting that by (2) and (3) the map r: $\tilde{g}^{-1}(0) \cap S \times (-\varepsilon, \varepsilon) \to \mathbf{R}$ is C^{k-2} and has no critical points (see Milnor [1963]). The diffeomorphism thus obtained is C^k away from $S \times \{0\}$ because $\tilde{g}^{-1}(0) \cap S \times (-\varepsilon, \varepsilon)$ is a C^k manifold away from $S \times \{0\}$, so the function r is C^k away from $S \times \{0\}$.

Notice that $\varphi(\tilde{g}^{-1}(0) \cap S \times (-\varepsilon, \varepsilon)) = g^{-1}(0) \cap U$ where U is the ball of

radius ε and that $\varphi\{(Q^{-1}(0) \cap S) \times (-\varepsilon, \varepsilon)\} = Q^{-1}(0) \cap U$. It follows that the diffeomorphism of (4) induces a homeomorphism of these two sets which is a C^k diffeomorphism away from the origin. Next, let $v \in Q^{-1}(0)$, $\|v\| = 1$. Then $(v, 0) \in \tilde{g}^{-1}(0)$. By (3) there is a C^{k-2} curve $\beta(s)$ through $(v, 0)$ that is transverse to $S \times \{0\}$ and lies in $\tilde{g}^{-1}(0)$. Assume $\beta(0) = (v, 0)$. Since $\beta(s)$ is transverse to $S \times \{0\}$, we may assume $\beta'(0) = (w, 1)$ where $w \in T_v S$. Let $\alpha(s) = \varphi \circ \beta(s)$. Then $\alpha'(0) = D\varphi(v, 0) \circ \beta'(0) = v$. □

PROOF OF THEOREM. We can assume $x_0 = 0$, $\lambda_0 = 0$. Using the Liapunov-Schmidt procedure, let

$$g(x_1, \lambda) = (I - P)f(x_1 + u(x_1, \lambda), \lambda)$$

a C^k map on $X_1 \times \mathbf{R}^p$ with $g(0, 0) = 0$ and $Dg(0, 0) = 0$. One calculates easily that $Du(0, 0) = 0$ and $D^2g(0, 0) = (I - P)D^2f(0, 0)$ restricted to $X_1 \times \mathbf{R}^P$. The lemma now applies to g. Since the zeros of f comprise the graph of u over the zeros of g, the theorem follows. □

In Buchner, Marsden and Schecter [**1978**] a number of extensions of this result are given. For instance the cases in which the leading nonzero derivative is higher than quadratic and in which there may be directions of degeneracy are considered. Related results are given in Shearer [**1978**]. Another generalization which is important for the example in §3, treats the case where $D_x f(x_0, \lambda_0)$ is not Fredholm, and has an infinite-dimensional kernel. The proof uses the Morse lemma on Banach spaces developed by Tromba [**1976a**], but the basic idea is already contained in the proof given.

We would like to point out two other applications of topological methods, both of which are due to Alan Weinstein. In the first, one considers a Hilbert space $\mathcal{H}$ and a C^2 map V: $\mathcal{H} \to \mathbf{R}$ with $V(0) = 0$, $DV(0) = 0$. We let $p = 1$ and

$$f(x, \lambda) = \nabla V(x) - \lambda x.$$

Problems of this type are considered by Krasnosel'skiĭ [**1964**] and many other authors. See Rabinowitz [**1977**] for references and the version given here. A main result for these equations states that if λ_0 is an isolated eigenvalue of $L = D^2V(0)$ (regarded as a linear mapping of $\mathcal{H}$ to $\mathcal{H} \approx \mathcal{H}^*$) of finite multiplicity, then there are three alternatives

(i) $(0, \lambda_0)$ is not an isolated solution of $f(x, \lambda) = 0$ in $X \times \{\lambda_0\}$,

(ii) there is a one-sided neighborhood U of λ_0 such that for all $\lambda \in U \setminus \{\lambda_0\}$, there are at least two distinct nonzero solutions of $f(x, \lambda) = 0$,

(iii) there is a neighborhood I of λ_0 such that for all $\lambda \in I \setminus \{\lambda_0\}$, $f(x, \lambda) = 0$ has at least one nontrivial solution.

As in Rabinowitz [**1977**], this reduces to a finite-dimensional problem via the Liapunov-Schmidt procedure. Instead of using degree theory one can use the relative homology groups in Gromoll and Meyer [**1969**]; see also Kuiper [**1971**]. Since λ_0 is an eigenvalue of finite multiplicity and if alternative (i) does not hold, then these groups must change as λ passes λ_0 and so $(0, \lambda_0)$ is a bifurcation point. Depending on whether 0 is an extremum in $X \times \{\lambda_0\}$ of V or not (after reduction) determines case (ii) or (iii). (If V is even one can say more; see Fadell and Rabinowitz [**1977**] . . . the same information may be deduced from the relative homology groups for even functions.)

Weinstein [1978] describes a bifurcation theorem for the zero sets of a closed one form under a nondegeneracy condition and based on a variational principle. It differs from the result just given in the nondegeneracy condition assumed and in the fact that bifurcation occurs off a whole manifold. In this respect it is similar to the results of Buchner, Marsden, and Schecter [1978] on bifurcation from zero sets. Techniques used in the latter may be useful in generalizing the result to infinite dimensions. Weinstein's result applies to periodic solutions of Hamiltonian systems under resonance and is of considerable importance. Some related infinite dimensional problems have been considered by Rabinowitz [1978].

3. Bifurcations in geometry and general relativity. In this and the following section we briefly describe two bifurcation problems in which there are no parameters given a priori. In fact, the initial investigations of each problem were done in a context outside of bifurcation theory.

Let V_4 be a four manifold and $\mathfrak{L}$ the set of (time oriented) Lorentz metrics on V_4. Consider the map Ein from $\mathfrak{L}$ to the symmetric two tensors on V_4 which maps ${}^{(4)}g \in \mathfrak{L}$ to its Einstein tensor (in coordinates $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$). We are interested in bifurcation points for the map Ein at solutions of $\mathrm{Ein}({}^{(4)}g) = 0$.

The situation is complicated by the hyperbolic nature of the equations $\mathrm{Ein}({}^{(4)}g) = 0$. For this reason we shall not specify our function spaces precisely. However, we must assume that we are working with globally hyperbolic spacetimes which have a compact Cauchy surface M.

THEOREM (FISCHER, MARSDEN AND MONCRIEF). *Let* ${}^{(4)}g_0$ *satisfy* $\mathrm{Ein}({}^{(4)}g) = 0$. *Then* ${}^{(4)}g_0$ *is a bifurcation point of the Einstein equations if and only if* ${}^{(4)}g_0$ *has a nonzero Killing field.*

The number of Killing fields is, roughly speaking, the multiplicity of the eigenvalue.

One can describe the directions of bifurcation in terms of conserved quantities of Taub and show that the singularity is quadratic.

The proof has a number of interesting features. First of all one reduces the problem to one for elliptic constraint equations on the hypersurface M. Here one uses the blowing up technique in §2 to analyze the singularities. The situation is further complicated by directions of degeneracy for the second deriviative and by the fact that the first derivative is not Fredholm; i.e. the constraint equations are underdetermined. However, as was mentioned, these difficulties can be overcome. The relationship between Killing fields on V_4 and bifurcation directions on M is a long story for which we refer the reader to Fischer and Marsden [1975a], Moncrief [1975], Fischer, Marsden and Moncrief [1978] and Arms and Marsden [1978].

The original motivation for this problem came from perturbation theory. One is interested in when a perturbation series accurately represents, to first order, a solution i.e. when an equation is *linearization stable*. The above results may be phrased by saying that, for relativity at least, linearization stability fails if and only if we have a bifurcation and moreover, a perturbation series near a spacetime with symmetries has to be readjusted to

second (but no higher) order, in order to be tangent to a curve of exact solutions.

There are corresponding questions in Riemannian geometry. Fix a compact n-manifold M and consider the problem

$$R(g) - \rho = 0$$

where g is a Riemannian metric on M, $R(g)$ is its scalar curvature and ρ is a scalar function on M. (Everything being in suitable Sobolev spaces or C^∞.) Regard ρ as a parameter. (Note that the parameter space is now infinite dimensional.)

If $\rho_0/(n-1)$ is not a constant in the spectrum of the Laplace-Beltrami operator for g_0, then $DR(g_0)$ is surjective by a result of Bourguignon, Fischer and Marsden (see Fischer and Marsden [**1975b**]), and so (g_0, ρ_0) is not a bifurcation point.

If S^n is the standard sphere of radius r_0 in $\mathbf{R}^{n+1}$ with metric g_0, $\rho_0 = n(n-1)/r_0^2$, then (g_0, ρ_0) is a bifurcation point and the singularity is quadratic. This is proved by the results of §2 adapted to the non-Fredholm context.

If T^n is the standard flat torus, then (g_0, ρ_0) is also a bifurcation point. However, there is a difference: for S^n, $R^{-1}(\rho_0)$ is not a manifold near g_0 and carries the singularities: for T^n, $R^{-1}(\rho_0)$ is a manifold near g_0, and in fact consists of the flat metrics (a result of Fischer and Marsden [**1975b**]). However, the equation $R(g) = \rho_0$ (for ρ_0 fixed now) is *not* linearization stable in either case, although g_0 is a bifurcation point for S^n, but not T^n (again fixing ρ_0).

4. A bifurcation problem in elastostatics. Elasticity is a parent of bifurcation theory and continues to provide many of the most challenging problems. See, for example, the papers of Antman listed in the bibliography and the articles in Keller and Antman [**1969**]. For different qualitative techniques, see Chillingworth [**1975**] and Zeeman [**1976**].

A problem which goes back to Signorini in the 1930s is to describe the solutions of the traction problem in elastostatics. Signorini was interested in this for reasons similar to those for the relativity example discussed in §3, namely he wanted to study perturbation expansions and the relationship between the linearized and nonlinear theories. We now know that he found difficulties with linearization precisely because he was working at a bifurcation point.

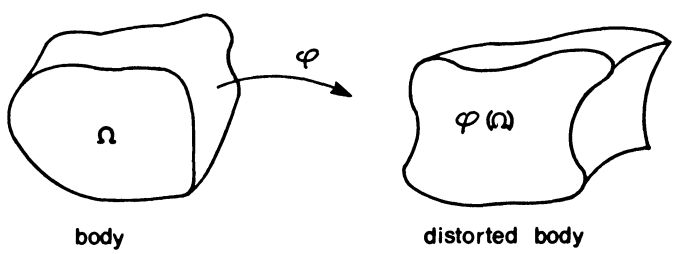

FIGURE 6

To describe the problem, let $\Omega \subset \mathbf{R}^3$ be a bounded open region with smooth boundary $\partial\Omega$ and let $\mathcal{C}$ be the collection of all diffeomorphisms of $\bar{\Omega}$ into $\mathbf{R}^3$. Then $\mathcal{C}$ is the configuration space of the problem and $u \in \mathcal{C}$ represents a possible configuration of the body Ω (Figure 6).

Write $\mathbf{F} = Du$, the deformation gradient. Let W be a given (stored energy) function of $\mathbf{F}\mathbf{F}^T$ ($\mathbf{F}^T = \mathbf{F}$ transpose) with $W(0) = 0$, and let

$$\mathbf{T} = \partial W / \partial \mathbf{F}$$

the Piola-Kirchhoff stress tensor and $\mathbf{A} = \partial^2 W / \partial \mathbf{F} \partial \mathbf{F}$ the elasticity tensor. We assume that the strong ellipticity condition holds i.e.

$$\mathbf{A} \cdot ((\xi, \xi), (\eta, \eta)) = A^{ij}_{kl} \xi_i \xi_j \eta^k \eta^l \geqslant \varepsilon |\xi|^2 |\eta|^2$$

for some $\varepsilon > 0$ and all $\xi, \eta \in \mathbf{R}^3$.

Let $\mathbf{B}$: $\Omega \to \mathbf{R}^3$ be a given body force and τ: $\partial\Omega \to \mathbf{R}^3$ be given boundary tractions. The traction problem for elastostatics is the problem of finding $u \in \mathcal{C}$ such that

$$\begin{aligned} \operatorname{div} \mathbf{T} + \rho_0 \lambda \mathbf{B} &= 0 && \text{on } \Omega, \\ \mathbf{T} \cdot \mathbf{N} &= \lambda \tau && \text{on } \partial\Omega \end{aligned}$$

where $(\operatorname{div} \mathbf{T})^i = \sum_{i=1}^3 \partial T^{ij} / \partial x^j$, ρ_0 is a given mass density on Ω, $\mathbf{N}$ is the unit outward normal to $\partial\Omega$ and λ is a parameter. It readily follows that a necessary condition for a solution is that the total force and moment of the force acting on the body be zero; i.e. the forces must be equilibrated.

This problem is equivalent to finding critical points of the mapping

$$f: \mathcal{C} \to \mathbf{R},$$

$$f(u) = \int_\Omega W(\mathbf{F})\, dx - \int_\Omega \rho_0 \lambda \mathbf{B} \cdot u\, dx - \int_{\partial\Omega} \tau \cdot \lambda u\, dx.$$

In Stoppelli [1958] (see also van Buren [1968] and Wang and Truesdell [1973]) it was proved that if $\mathbf{B}$, τ do not have an axis of equilibrium (i.e. a vector $\mathbf{e}$ such that rotations of the body about $\mathbf{e}$ leaving the loads $\mathbf{B}$, τ fixed, maintains the equilibration of forces), then for λ sufficiently small there is a unique solution $u \in \mathcal{C}$. On the other hand, if $\mathbf{B}$, τ do have an axis of equilibrium, then there may be several solutions (0, 1, 2, or 3 depending on the hypotheses) growing as a certain power of λ (λ, $\lambda^{1/2}$ or $\lambda^{1/3}$).

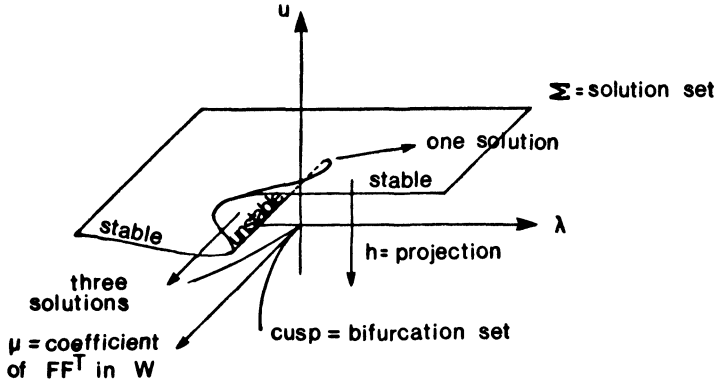

FIGURE 7

We now know that Stoppelli was looking at sections of a cusp or a fold. (See Figure 7.) Indeed this is rather easy to see if we solve the problem by looking for critical points of f, follow the usual Liapunov-Schmidt procedure, and make a generic hypothesis on W.

In more degenerate circumstances one can expect a double cusp, with up to nine solutions. Details of all this will be given in Chillingworth and Marsden **[1978]**.

5. Generic finiteness of solutions of elliptic equations. An important question for nonlinear elliptic boundary value problems is whether or not the set of solutions is generically finite and if so, how this number changes with the data of the problem. This problem is not always phrased in terms of bifurcation theory, but it seems desirable to do so.

Three interesting recent works on this problem are those of Uhlenbeck **[1975]**, Foiaş and Temam **[1977]** and Tromba **[1978]**. Their idea is to use transversality techniques. The following abstract result of Tromba **[1976b]** which is a consequence of the Sard-Smale theorem, seems to be representative.

THEOREM. *Let Λ be a manifold (the parameter space) and E, F vector bundles over Λ. Let f: $E \to F$ be a smooth bundle map and $\Sigma = f^{-1}(0)$; for fixed $\lambda \in \Lambda$, let f_λ be the restriction of f to E_λ, the fiber over λ. Assume*:

(i) *for each $\lambda \in \Lambda$, $x \in \Sigma \cap E_\lambda$. $Df_\lambda(x)$ is Fredholm of index zero,*

(ii) *for $x \in \Sigma$, $Df(x)$ is surjective,*

(iii) *f is Σ-proper i.e. if $x_n \in \Sigma \cap E_{\lambda_n}$ and λ_n converge, then x_n has a convergent subsequence.*

Then $\Sigma \subset E$ is a submanifold (by (ii)) *and there is an open dense set $U \subset \Lambda$ such that $\Sigma \cap E_\lambda$ is finite for $\lambda \in U$.*

This is related to bifurcation theory because the projection h: $\Sigma \to \Lambda$ will, in interesting examples, have bifurcation points.

For the stationary Navier-Stokes equations this result shows easily that for generic forces or boundary conditions, there is a finite number of solutions, a result of Foiaş and Temam **[1977]**. In Marsden and Tromba **[1978]** the same result is shown for the Navier-Stokes equations on generic regions. The methods of Tromba **[1978]**, developed for the Plateau problem, seem important for counting the solutions.

6. Dynamical bifurcations and a problem of flutter.[5] Many bifurcation problems study the bifurcation of fixed points of some dynamical system. In such cases it is important to tie these bifurcations up with any additional dynamical bifurcations and to consider questions of stability. Stability can be either dynamical or structural stability.

To illustrate what we mean by structural stability, we consider Euler buckling.

Viewed as a one parameter system with parameter the beam tension, one gets the traditional picture shown in Figure 8.

[5]This section was written jointly with Philip Holmes.

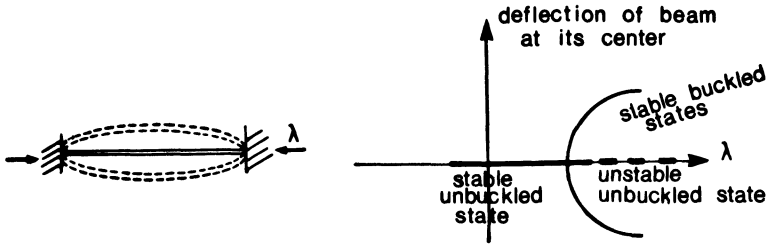

FIGURE 8

However, this bifurcation diagram is "unstable". It can be stabilized by adding a second parameter ε, which describes the asymmetry of the force λ; see Figure 9.

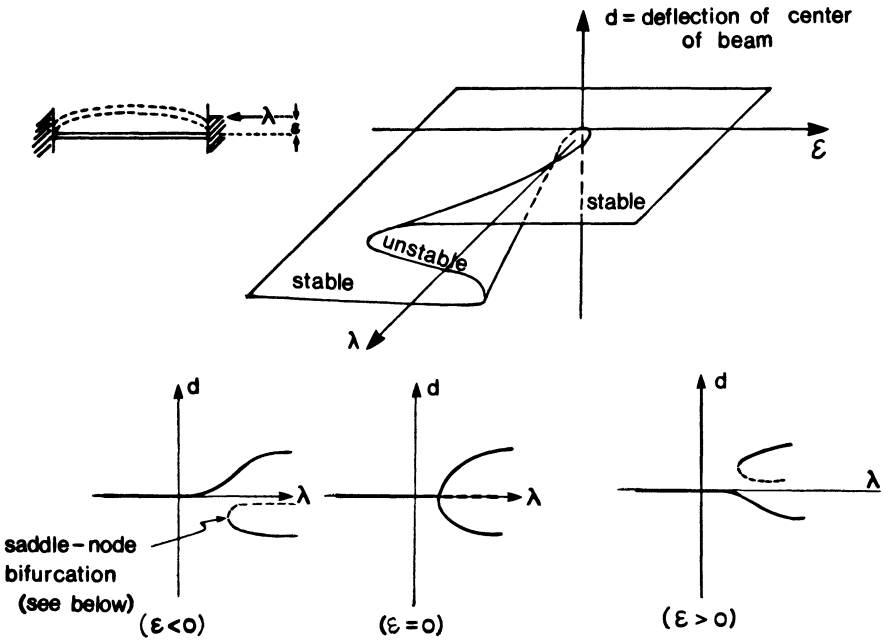

FIGURE 9

For additional details on this example, see Zeeman [1976], and, if one wishes to distinguish between the bifurcation and imperfection parameters, see Golubitsky and Schaeffer [1978].

The dynamical framework in which we operate is described as follows. Let $X \subset Y$ be Banach spaces (or manifolds) and let

$$f: X \times \Lambda \to Y$$

be a given C^k mapping. Here Λ is the parameter space and f may be defined only on an open subset of $X \times \Lambda$. The dynamics are given by

$$dx/dt = f(x, \lambda)$$

which defines a semiflow

$$F_t^\lambda: X \to X$$

by letting $F_t^\lambda(x_0)$ be the solution of $\dot{x} = f(x, \lambda)$ with initial condition $x(0) =$

x_0. We assume that this equation has, at least locally in time, unique solutions, which can be continued if they lie in a bounded set.

A *fixed point* is a point (x_0, λ) such that $f(x_0, \lambda) = 0$. Therefore, $F_t^\lambda(x_0) = x_0$ i.e. x_0 is an equilibrium point of the dynamics.

A fixed point (x_0, λ) is called *stable* if there is a neighborhood U_0 of x_0 on which $F_t^\lambda(x)$ is defined for all $t \geqslant 0$ and if for any neighborhood $U \subset U_0$, there is a neighborhood $V \subset U_0$ such that $F_t^\lambda(x) \in U$ if $x \in V$ and $t \geqslant 0$. The fixed point is called *asymptotically stable* if, in addition, $F_t^\lambda(x) \to x_0$ as $t \to +\infty$, for x in a neighborhood of x_0.

Many nonlinear partial differential equations of evolution type fall into this framework. Many semilinear hyperbolic and most parabolic equations satisfy an additional smoothness condition; we say F_t^λ is a *smooth semiflow* if for each t, λ, F_t^λ: $X \to X$ (where defined) is a C^k map and its derivatives are strongly continuous in t, λ.

For general conditions under which a semiflow is smooth, see Marsden and McCracken [**1976**]. One especially simple case occurs when

$$f(x, \lambda) = A_\lambda x + B(x, \lambda)$$

where A_λ: $X \to Y$ is a linear generator depending continuously on λ and B: $Y \times \mathbf{R}^p \to Y$ is a C^k map. This result is readily proved by the variation of constants formula

$$x(t) = e^{tA_\lambda} x_0 + \int_0^t e^{(t-s)A_\lambda} f(x(s), \lambda) ds.$$

(See Segal [**1962**] for details.)

Standard estimates and the classical proof for ordinary differential equations prove the following.

LIAPUNOV'S THEOREM. *Suppose F_t^λ is a smooth flow, (x_0, λ) is a fixed point and the spectrum of the linear semigroup*

$$U_t = D_x F_t^\lambda(x_0): X \to X.$$

(The Fréchet derivative with respect to $x \in X$) is $e^{t\sigma}$ where σ lies in the left half plane a distance $\geqslant \delta > 0$ from the imaginary axis. Then x_0 is asymptotically stable and for x sufficiently close to x_0 we have an estimate

$$\|F_t^\lambda(x) - x_0\|_X \leqslant (\text{Constant}) e^{-t\delta}.$$

If we are interested in the location of fixed points, then we solve the equation

$$f(x, \lambda) = 0,$$

and their stability will be determined by the spectrum σ of the linearization

$$A_\lambda = D_x f(x_0, \lambda).$$

(We assume the operator is nonpathological–e.g. has discrete spectrum–so $\sigma(e^{tA_\lambda}) = e^{t\sigma(A_\lambda)}$.) In critical cases where the spectrum lies on the imaginary axis, stability has to be determined by other means. It is at criticality where, for example, a curve of fixed points $x_0(\lambda)$, changes from stable to unstable, that a bifurcation can occur.

The second major point we wish to make is that within the context of smooth semiflows, the usual invariant manifold theorems from ordinary differential equations carry over.

In bifurcation theory it is often useful to apply the invariant manifold theorems to the *suspended flow*

$$F_t: X \times \mathbf{R}^p \to X \times \mathbf{R}^p,$$
$$(x, \lambda) \mapsto (F_t^\lambda(x), \lambda).$$

The invariant manifold theorem states that if the spectrum of the linearization A at a fixed point (x_0, λ) splits into $\sigma_S \cup \sigma_C$, where σ_S lies in the left half plane and σ_C is on the imaginary axis, then the flow F_t leaves invariant manifolds M_S and M_C tangent to the eigenspaces corresponding to σ_S and σ_C respectively; M_S is the *stable* and M_C is the *center* manifold. (One can allow an *unstable* manifold too if that part of the spectrum is finite.) By Liapunov's theorem, orbits on M_S converge to (x_0, λ) exponentially. For suspended systems, note that we always have $1 \in \sigma_C$.

The idea of the proof is this: we apply invariant manifold theorems for smooth maps with a fixed point to each F_t separately. Then one shows that since F_t and F_s commute ($F_t \circ F_s = F_{t+s} = F_s \circ F_t$), these invariant manifolds can be chosen in common for all the F_t.

For bifurcation problems the center manifold theorem is the most relevant, so we summarize the situation. (See Marsden and McCracken [1976] for details.)

CENTER MANIFOLD THEOREM FOR FLOWS. *Let Z be a Banach space admitting a C^∞ norm away from* 0 *and let F_t be a C^0 semiflow defined on a neighborhood of* 0 *for $0 \leq t \leq \tau$. Assume $F_t(0) = 0$ and for each $t > 0$, F_t: $Z \to Z$ is a C^{k+1} map whose derivatives are strongly continuous in t. Assume that the spectrum of the linear semigroup $DF_t(0)$: $Z \to Z$ is of the form $e^{t(\sigma_S \cup \sigma_C)}$ where $e^{t\sigma_S}$ lies inside the unit circle a nonzero distance from it, for $t > 0$; i.e. σ_S is in the left half plane. Let Y be the generalized eigenspace corresponding to the part of the spectrum on the unit circle. Assume* $\dim Y = d < \infty$.

Then there exists a neighborhood V of 0 *in Z and a C^k submanifold $M_C \subset V$ of dimension d passing through* 0 *and tangent to Y at* 0 *such that*

(a) *If $x \in M_C$, $t > 0$ and $F_t(x) \in V$, then $F_t(x) \in M_C$.*

(b) *If $t > 0$ and $F_t^n(x)$ remains defined and in V for all $n = 0, 1, 2, \ldots$, then $F_t^n(x) \to M_C$ as $n \to \infty$.*

See Figure 10.

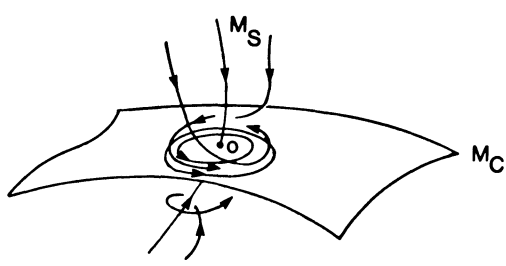

FIGURE 10

For dynamical bifurcations, the center manifold theorem plays the same role as the Liapunov-Schmidt procedure. It reduces the bifurcation problem to a finite-dimensional one. Using this gives, for example, an easy proof of the Hopf theorem (see Marsden and McCracken [1976]).

We shall now outline briefly how one can obtain a structurally stable bifurcation diagram for a particular two parameter problem. The method is taken from Holmes [1977] and Holmes and Marsden [1978] which should be consulted for details. Although one could probably obtain the same answers faster using, e.g. Boa and Cohen [1976], the present methods seem to fit into the theoretical framework of structural stability (or imperfection insensitivity) more satisfactorily.

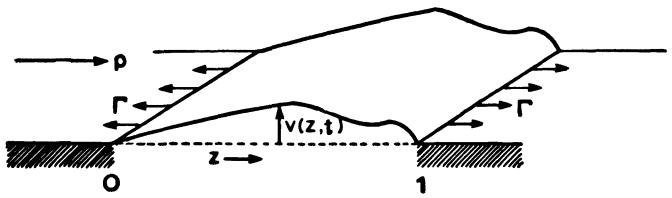

FIGURE 11

We consider the one-dimensional thin panel shown in Figure 11 and are concerned with bifurcations near the trivial (zero) solution. The equation of motion of such a panel, fixed at both ends and undergoing "cylindrical" bending can be written as (cf. Dowell [1975])

$$\alpha\dot{v}'''' + v'''' - \left\{\Gamma + \kappa\int_0^1 (v'(z))^2 dz + \sigma\int_0^1 (v'(z)\dot{v}'(z))\, dz\right\} v'' + \rho v' + \sqrt{\rho}\,\delta\dot{v} + \ddot{v} = 0. \tag{1}$$

Here $\cdot \equiv \partial/\partial t$, $' = \partial/\partial z$ and we have included viscoelastic structural damping terms α, σ as well as aerodynamic damping $\sqrt{\rho}\,\delta$. κ represents nonlinear (membrane) stiffness, ρ the dynamic pressure and Γ an in-plane tensile load. All quantities are nondimensionalized, and associated with (1) we have boundary conditions at $z = 0, 1$ which might typically be simply supported ($v = v'' = 0$) or clamped ($v = v' = 0$). We assume that $\alpha, \sigma, \delta, \kappa$ are fixed > 0 and let the control parameter $\mu = \{(\rho, \Gamma) | \rho > 0\}$ vary.

We redefine (1) as an ordinary differential equation on a Banach space, choosing as our basic space $X = H_0^2([0, 1]) \times L^2([0, 1])$, where H_0^2 denotes H^2 functions[6] on [0, 1] which vanish at 0, 1. Set $\|\{v, \dot{v}\}\|_X = (|\dot{v}|^2 + |v''|^2)^{1/2}$, where $|\ |$ denotes the L^2 norm and define the linear operator

$$A_\mu = \begin{pmatrix} 0 & I \\ C_\mu & D_\mu \end{pmatrix}; \qquad \begin{aligned} C_\mu v &= -v'''' + \Gamma v'' - \rho v', \\ D_\mu v &= \alpha v'''' - \sqrt{\rho}\,\delta v. \end{aligned} \tag{2}$$

The basic domain of A_μ, $D(A_\mu)$, consists of $\{v, \dot{v}\} \in X$ such that $\dot{v} \in H_0^2$ and $v + \alpha\dot{v} \in H^4$; particular boundary conditions necessitate further restrictions.

[6] H^2 is the Sobolev space of functions which, together with their first and second derivatives are in L^2.

After defining the nonlinear operator $B(v, \dot{v}) = (0, [\kappa|v'|^2 + \sigma(v', v')]v'')$, where (,) denotes the L^2 inner product, (1) can be rewritten as

$$dx/dt = A_\mu x + B(x) \equiv G_\mu(x); \qquad x = \{v, \dot{v}\}; x(t) \in D(A_\mu). \tag{3}$$

We next define an energy function $H: X \to \mathbf{R}$ by

$$H\{v, \dot{v}\} = \frac{1}{2}|\dot{v}|^2 + \frac{1}{2}|v''|^2 + \frac{\Gamma}{2}|v'|^2 + \frac{\kappa}{4}|v'|^4$$

and compute that

$$dH/dt = -\rho(v', \dot{v}) + \sqrt{\rho}\,\delta|\dot{v}|^2 - \alpha|v''|^2 - \sigma(v', \dot{v}')^2.$$

Using the methods of Segal [1962] one shows that (3) and hence (1) defines a unique smooth local semiflow F_t^μ on X. Using the energy function (4) and some arguments of Parks [1966], one shows that F_t^μ is in fact globally defined for all $t \geqslant 0$.

By making Galerkin approximations and taking $\delta = 0.1$, $\alpha = 0.005$ one finds that the operator A_μ has a double zero eigenvalue at $\mu = (\rho, \Gamma) \approx (107.8, -22.91)$ the remaining eigenvalues being in the left half plane. Thus around the zero solution we obtain a three-dimensional suspended center manifold. The eigenvalue evolution at the zero solution, obtained numerically, enables us to fill in the portions of the bifurcation diagram shown in Figure 12. In particular, a supercritical Hopf bifurcation occurs crossing B_h from I to II and a symmetrical saddle node on B_{s1} and B_{s2}.

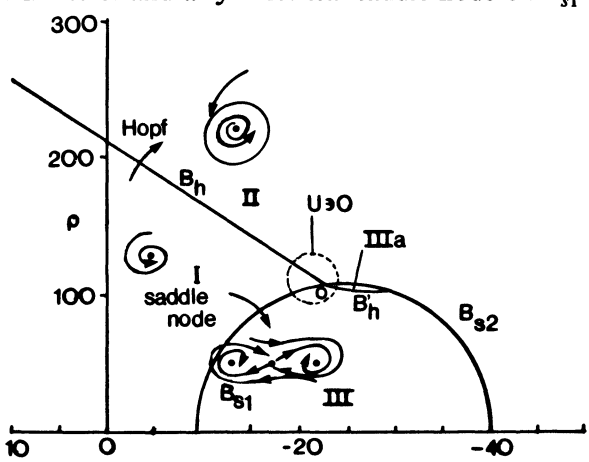

FIGURE 12. Partial bifurcation set for the panel ($\alpha = 0.005$, $\delta = 0.1$).

Moreover, computations for the two fixed points $\{\pm x_0\}$ appearing on B_{s1} and existing in region III shows that they are sinks below a curve B'_h originating at 0 which we also show on Figure 12. As μ crosses B'_h transversally from III to $\mathrm{III_a}$ $\{\pm x_0\}$ undergo simultaneous subcritical Hopf bifurcations.

(The stability formulas in Marsden and McCracken [1976] were used to make these calculations. They were confirmed by a stability program kindly computed by B. Hassard.) Now let μ cross $B_{s2} \setminus 0$ from region II to region $\mathrm{III_a}$. Here the closed orbits presumably persist, since they lie at a finite distance from the bifurcating fixed point $\{0\}$. In fact the new points $\{\pm x_0\}$ appearing on B_{s2} are saddles in region $\mathrm{III_a}$, with two eigenvalues of spectrum

$DG_\mu(\pm x_0)$ outside the unit circle and all others within it. Thus, crossing B_{s2} from II to III_a produces three fixed points surrounded by a stable closed orbit. We now have a *partial* picutre of the behavior near 0. They key to completing the analysis lies in the point 0, the "organizing center" of the bifurcation set at which B_{s2}, B_h and B_h' meet.

We now postulate that our bifurcation diagram near 0 is stable[7] to small perturbations in our (approximate) equations. We look in Taken's classification (with symmetry) and find that exactly one of them is consistent with the information found in Figure 12, namely the one shown in Figure 13 (see Takens [**1974b**]). Thus we are led to the complete bifurcation diagram shown in Figure 14 with the oscillations in various regions as shown in Figure 13.

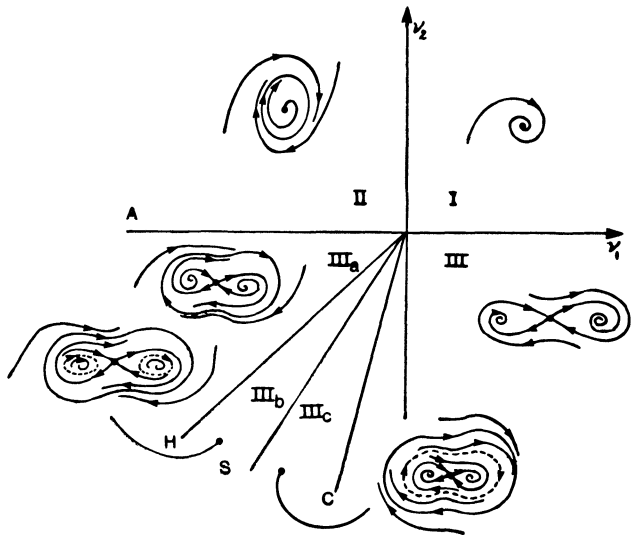

FIGURE 13

The Andronov-Takens "$m = 2$: $-$" normal form, with associated flows: $\ddot{x} = -\nu_2 x - \nu_1 \dot{x} - x^3 - x^2\dot{x}$.

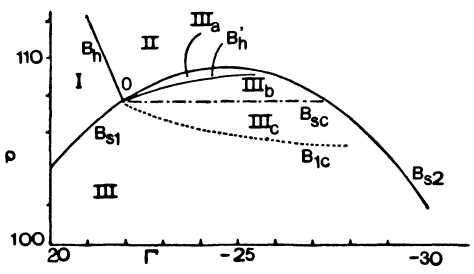

FIGURE 14

A local model for bifurcations of the panel near 0, $(\rho, \Gamma) \simeq (110, -220)$; $\alpha = 0.005$, $\delta = 0.1$.

[7]If the bifurcation was not structurally stable, this procedure would stabilize, or unfold it.

In this work we have used rather drastic semilinear approximations to the full equations of elastodynamics. It is an open problem to determine in what sense the solutions are approximations. (Presumably it is related to a high-frequency cut-off approximation.) See Ball [1974] for a related problem. More generally, the problem of embedding results from a bifurcation analysis of an elastostatics problem such as bifurcations of the plate equation into a dynamic picture is difficult. For instance it is still an open problem whether or not minima of the elastic energy give stable points dynamically. There are, in fact, unpleasant examples due to Knops and Payne; see Marsden and Hughes [1978] for a discussion.

Finally we mention that because the Andronov-Takens bifurcation in Figure 13 is structurally stable[8], it can be expected to arise in a number of problems; see, for instance Sel'kov [1968]. It seems to occur in some nonlinear oscillation problems for circuits as well. It also occurs in the population dynamics equations of Gurtin and MacCamy [1974]. Their equations (4.8) can be written, in the special case $\beta(P) = \beta_0 =$ constant, as

$$\dot{P} = Q, \qquad \dot{Q} = -\lambda(P)\dot{P} - \lambda'(P)P\dot{P} + (\beta_0 - \alpha - \lambda(P))(\dot{P} + \lambda(P)P).$$

If $\lambda(P) = \mu_0 + \mu_1 P^2 + \dots$, these equations have the form of those governing Figure 13 up to cubic terms (which are all that matter for bifurcations near the trivial solution) with

$$\nu_1 = 2\mu_0 - \beta_0 + \alpha, \quad \nu_2 = \mu_0^2 - \mu_0(\beta_0 - \alpha).$$

One concludes that if $2\mu_0 > \beta_0 - \alpha$, there are no closed (periodic) orbits, while if $2\mu_0 < \beta_0 - \alpha$, we are in the left half plane in Figure 13, so closed orbits are possible. In specific models studied by Gurtin and his co-workers $2\mu_0 < \beta_0 - \alpha$ does not occur (private communication).

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[8]We are not sure how one should break the symmetry in Figure 13 and produce an associated structurally stable unsymmetric bifurcation.

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