On the Geometry of the Liapunov-Schmidt Procedure†

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1. Introduction

One of the most commonly used procedures in bifurcation theory is the Liapunov-Schmidt procedure (which we review in §2). However, in differential topology this procedure is also regularly used, but of course, under a different name, namely "transversality." The aim of this short note is to make this link explicit.

There are two reasons for geometrizing the Liapunov-Schmidt procedure. First of all, it is useful in some applications the author, A. Fischer and V. Moncrief have made to relativity (work in progress). Secondly, the dynamic analogue of the procedure, namely center manifold theory, already has a geometric flavor (i.e. it makes intrinsic sense on manifolds), so it is natural to bring the classical Liapunov-Schmidt procedure in line with it.

2. Review of the Liapunov-Schmidt Procedure

Let X and Y be Banach spaces and $f: X \times R^p \to Y$ a C^k map, $k \geqslant 1$. Let $D_x f(x,\lambda)$ be the (Fréchet) derivative of f with respect to x, a continuous linear map of X to Y. Let $f(x_0,\lambda_0) = 0$ and let

$$X_1 = \ker D_x f(x_0,\lambda_0).$$

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†The lectures presented by the author are not reproduced here since that material is available in J. Marsden, Qualitative Methods in Bifurcation Theory, Bull. Am. Math. Soc. 84 (1978), 1125-1148, R. Abraham and J. Marsden, *Foundations of Mechanics* , Second Edition, Addison Wesley (1978), and in J. Marsden and M. McCracken, The Hopf Bifurcation and its Applications, Springer Applied Math Sciences #19 (1976).

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Assume X_1 is finire dimensional with a complement X_2 so that $X = X_1 \oplus X_2$. Also, assume

$$Y_1 = \text{Range } D_x f(x_0,\lambda_0)$$

is closed and has a finite-dimensional complement Y_2. In other words, $D_x f(x_0,y_0)$ is a Fredholm operator. Write $Y = Y_1 \oplus Y_2$ and let $P: Y \to Y_1$ be the projection. By the implicit function theorem, the equation

$$Pf(x_1 + x_2,\lambda) = 0$$

has a unique solution $x_2 = u(x_1,\lambda)$ near x_0,λ_0, where $x = x_1 + x_2 \in X = X_1 \oplus X_2$. Thus, the equation $f(x,\lambda) = 0$ is equivalent to the *bifurcation equation*

$$(I - P)f(x_1 + u(x_1,\lambda),\lambda) = 0,$$

a system of dim Y_2 equations in dim X_1 unknowns. This reduction of $f(x,\lambda) = 0$ to the bifurcation equation is the *Liapunov-Schmidt procedure.*

For purposes of this procedure alone, the assumption that X_1 and Y_2 are finite dimensional is, of course, irrelevant. This is made with the theory of Fredholm operators waiting in the wings. Similarly, the parameter space R^p may be replaced by a Banach space Z. In fact, the parameter is just "along for the ride."

3. A Topological Procedure

Let X and Z be Banach manifolds, Y a Banach space and $f: X \times Z \to Y$ be a C^1 map. (More generally, one can replace $X \times Z$ by a fiber bundle over Z). We are interested in solving the equation

$f(x,\lambda) = 0$ for $(x,\lambda) \in X \times Z$. Let (x_0,λ_0) be a known solution and let

$$X_1 = \ker D_x f(x_0,\lambda_0)$$

and assume X_1 splits; i.e. $T_{x_0} X = X_1 \oplus X_2$ for a closed subspace $X_2 \subset T_{x_0} X$. Let

$$Y_1 = \text{Range } D_x f(x_0,\lambda_0)$$

and assume Y_1 is closed and splits; i.e. $Y = Y_1 \oplus Y_2$ for a closed subspace Y_2. (This, of course, involves a choice of Y_2, as it did above.) Let $P: Y \to Y_1$ be the projection.

The map f is, for fixed λ_0, transversal to the subspace Y_2 at (x_0,λ_0). Therefore, in a neighborhood of (x_0,λ_0),

$$S_P = \{(x,\lambda) \mid Pf(x,\lambda) = 0\}$$

is a smooth submanifold of $X \times Z$ tangent to $X_1 \times T_{\lambda_0} Z$ at (x_0,λ_0). [In the notation of §2, $S_P = \{(x_1 + u(x_1,\lambda),\lambda)\}$].

Let f_P denote the restriction of f to S_P. Clearly $f(x,\lambda) = 0$ iff $(I - P)f_P(x,\lambda) = 0$ iff $f_P(x,\lambda) = 0$. The later condition is the geometric version of the bifurcation equation. It has proven to be useful, at least to the author.

4. More General

We can allow Y to be an arbitrary Banach manifold to clarify the choices involved. (I don't know any examples where this is necessary or useful.) Now we fix $y_0 \in Y$ and attempt to solve $f(x,\lambda) = y_0$ near a known solution (x_0,y_0). Let X_1, X_2 be as above and let $T_{y_0} Y = Y_1 \oplus Y_2$ where $Y_1 = \text{Range } D_x f(x_0,\lambda_0)$.

Now choose a submanifold $M \subset Y$ tangent to Y_2 at y_0. The procedure only depends on this choice; it is analogous to the choice in §2 of a *linear* complement to Y_1.

Again, f is transversal to M (with λ a parameter), so

$$S_M = \{(x,\lambda) \mid f(x,\lambda) \in M\}$$

is a submanifold of $X \times Z$. Let f_M be the restriction of f to S_M, regarded as a map of S_M to M. Then the obvious assertion that

$$f(x,\lambda) = y_0 \in Y \quad \text{iff} \quad f_M(x,\lambda) = y_0 \in M$$

is the abstract Liapunov-Schmidt procedure.

5. A Sample Calculation

In the usual Liapunov-Schmidt theory of §2, to analyze zeros of the map

$$g(x_1,\lambda) = (I - P)f(x_1 + u(x_1,\lambda),\lambda)$$

we need to compute its derivatives at (x_{10},λ_0) (x_{10} is the first component of x_0). It is usually assumed that $D_\lambda f(x_0,\lambda_0) = 0$. Thus (x_{10},λ_0) is a critical point for g. By implicit differentiation one finds that

$$D^2g(x_{10},\lambda_0) = (I - P)D^2f(x_0,\lambda_0)$$

(The right hand side appropriately restricted.)

In the context of §3, it is clear that if $D_\lambda f(x_0,\lambda_0) = 0$ then (x_0,λ_0) is a critical point (in the sense of zero derivative) of f_P. Moreover, it is now *obvious* from the fact that Hessians are well-defined at critical points that $D^2 f_P(x_0,\lambda_0) = (I - P)D^2 f(x_0,\lambda_0)$ restricted to $T_{(x_0,z_0)}S_P \times T_{(x_0,z_0)}S_P = (X_1 \times T_{\lambda_0}Z) \times (X_1 \times T_{\lambda_0}Z)$; i.e. we recover the same conclusion as above. The procedure can, of course, be repeated to obtain a formula for $D^k f_P$ if the k-1 jet of f_P vanishes, as is well-known in bifurcation theory. Note that once the structure of the zeros of g is found, this still has to be lifted via the graph of u to obtain the zeros of f. In the geometric setting this is not necessary; the zeros of f_P *are* the zeros of f.

While these results are essentially nothing more than new language for well-known material, the geometric setting seems to clarify and even simplify what is going on.

6. Potential Operators and Vector Fields

If X = Y are Hilbert spaces and the original map f in Section 2 is the gradient of a potential function ϕ in the variable x, then the reduced function $g(x_1,\lambda) = (1-P)f(x_1 + u(x_1,\lambda),\lambda))$ is also a gradient, with a modified potential $\bar{\phi}$ (depending on u); see, for instance. P. Rabinowitz, J. Funct. An. 25 (1977) 412-424. We wish to explain the geometric meaning of this. In such cases, the function f may be thought of as a vector field on a manifold X, depending on the parameters λ.

For vector fields the procedure in §3 can be refined somewhat. At a zero (x_0,λ_0) of a (parametrized) vector field $f: X \times Z \to TX$, the derivative $D_x f(x_0,\lambda_0)$ makes intrinsic sense as a linear map of $T_{x_0}X$ to itself. One can now seek an invariant manifold S_{λ_0} for f tangent to the subspace $X_1 = \ker D_x f(x_0,\lambda_0)$. To do this for nearby λ as well, it is convenient to suspend f to the vector field $\bar{f}: X \times Z \to T(X \times Z)$ by setting $\bar{f}(x,\lambda) = (f(x,\lambda),0)$ and find an invariant manifold $\bar{S}$ associated to $\ker D\bar{f}(x_0,\lambda_0)$; one can then take $S_\lambda = \bar{S} \cap X \times \{\lambda\}$, an invariant manifold for each λ. One may regard S_λ as implicitly defined in the same way as the function $u(x_1,\lambda)$ in the Liapunov-Schmidt procedure is implicitly defined.

Zeros of f near (x_0,λ_0) necessarily lie on $\bar{S}$, so the problem reduces to finding zeros of $f|\bar{S}$, the analogue of the bifurcation equation. For finding fixed points , this is a geometric formulation of the Liapunov-Schmidt procedure. The fact that we are dealing with vector fields entails that the choice of Y_2 (or M in Section 4) is now automatically made; both S_P and M are now replaced by $\bar{S}$.

In order to capture dynamic bifurcations as well as static ones, it is necessary to enlarge $\bar{S}$ to the full center manifold, as is explained in, for example, J. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications*, Springer Appl. Math. Sciences #19 (1976). (For operators with real eigenvalues, such as potential operators, $\bar{S}$ equals the center manifold.)

The fact that the reduction of a potential operator by the Liapunov-Schmidt procedure results in a potential operator is now clear. In fact, if one uses the space $\bar{S}$, a modification of ϕ is *not necessary*; one needs only to restrict it to $\bar{S}$. This is because of the following obvious fact: the restriction of a gradient vector field to an invariant submanifold is a gradient vector field whose potential is the restriction of the original one; i.e. $(\nabla\phi)|S = \nabla(\phi|S)$ if $\nabla\phi$ is tangent to S.