Further Results on Performance Analysis for Compressive Sensing Using Expander Graphs

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***Abstract*— Compressive sensing is an emerging technology which can recover a sparse signal vector of dimension n via a much smaller number of measurements than n. In this paper, we will give further results on the performance bounds of compressive sensing. We consider the newly proposed expander graph based compressive sensing schemes [31] and show that, similar to the l_1 minimization case, we can exactly recover any k-sparse signal using only $O(k\log(n))$ measurements, where k is the number of non-zero elements. The number of computational iterations is of order $O(k\log(n))$, while each iteration involves very simple computational steps.**

I. Introduction

Compressive sensing has recently received a great amount of attention in the applied mathematics and signal processing community. The theory of compressive sensing, as developed over the past few years, attempts to perform sampling and compression simultaneously, thus significantly reducing the sampling rate. What allows this theory is the fact that, in many applications, signals of interest have a "sparse" representation over an appropriate basis. In fact, compressive sampling is intimately related to solving underdetermined systems of linear equations with sparseness constraints. The work of Candes, Romberg and Tao [1], [2] and Donoho [4] came as a major breakthrough in that they rigorously demonstrated, for the first time, that, under some very reasonable assumptions, the solution could be found using simple linear programming—thus rendering the solution practically feasible. The method is essentially constrained l_1 minimization, which for a long time was empirically known to perform well for finding sparse solutions and has been known in the literature as "basis pursuit" [5], [7].

While solving the linear program resulting from l_1 optimization can be done in polynomial-time (often $O(n^3)$, where n is the number of unknowns), this may still be infeasible in applications where n is quite large (e.g., in current digital cameras the number of pixels is of the order $n = 10^6$ or more) [8]. Therefore there is a need for methods and algorithms that are more computationally efficient. Also, in many of the previous works, random measurement matrices are used where a successful signal recovery can not be always guaranteed although it succeeds with a high probability. So it is also desirable to have an explicit construction of measurement matrix for compressive sensing. made in addressing these two problems for compressing sensing [3] [10] [11] [16] [18] [21] [24] .

With the exception of the method in [11], the group testing methods in [16] and the Vandermonde measurement matrix based methods in [21], all the results described above hold with "high probability" either over the random measurement matrix or over some assumptions on the input signals [17]. While the methods in [11], [16] can guarantee sparse signal recovery deterministically with explicit measurement matrices, they suffer from the fact that they only work in the supersparse case where k can not be kept as a constant fraction of n. But recovering a constant fraction of n non-zero elements via a small number of measurements is of great practical interest [6]. If $k = \Theta(n)$, the complexity of the methods of [20], [21] are of order $O(n^3)$ and $O(n^2)$ respectively, which will still be unpractical for problems of large dimensions. Sometimes, it is also required that the recovery schemes are applicable to approximately sparse signals and robust to the noise in the measurements and numerical errors.

In[31], the authors of the current paper proposed a new scheme for compressive sensing with deterministic performance guarantees based on bipartite expander graphs and show that with $k = \Theta(n)$ (a constant fraction of n), the recovery complexity of our algorithm is $O(n)$ while saving a constant fraction of n measurements. Moreover, it was shown in [31] that the new method is applicable to approximately sparse signal and robust to measurement noise. But in the literature of bipartite expander graphs, only the case of k being a constant fraction of n was discussed, where the number of measurements is at least a linear fraction of n. It remains unknown how many measurements can be saved in the case where the sparsity level k remains constant and the dimension of the signal vector n grows large. In this paper, both lower bounds and upper bounds are established for the number of measurements needed in the expander graph based schemes. By showing the existence of bipartite expander graphs with $O(k\log(n))$ measurement nodes, we prove that using the same techniques in [31], any k-sparse signal of dimension n can be recovered efficiently with only $O(k\log(n))$ measurements, even when k may not be proportional to n.

The rest of this paper is organized as follows. In the next section we give the background knowledge and problem formulation for compressive sensing. For reference, we introduce expander graphs in Section III and review how they can be used to develop efficient recovery methods with deterministic performance guarantees. Lower bounds and upper bounds on the number of measurements are given in the section IV, where the

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main result of this paper, the existence of good expanders with only $O(k\log(n))$ measurement nodes, is given.

II. Background and Problem Formulation

In compressive sensing the starting point is an n-dimensional signal vector which admits a sparse representation in some particular basis. Since the basis is not of primary concern to us, we may, without loss of generality, assume that it is the standard basis. In other words, we shall assume that we have an n-dimensional vector $x \in \mathcal{R}^n$, such that no more than k entries are non-zero. Clearly, $k < n$.

The vector x itself is not directly observable. What is observable are *measurements* of x that correspond to linear combinations of the form

$$\sum_{j=1}^{n} a_j x_j. \tag{1}$$

We often have control over what measurements to employ, and this may turn out to help us. In any event, assuming we have m ($k < m < n$) measurements of this form, we may collect them in a $m \times n$ matrix A so that

$$y = Ax, \tag{2}$$

or, in other words,

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \qquad i = 1, \ldots, m. \tag{3}$$

The system of equations (2) is, of course, underdetermined. However, the fact that a sparse solution exists, allows us to be able to find the solution. It was a significant result when it was rigorously shown by Candes, Romberg and Tao [1], [2] and Donoho [4] that, under the sparsity assumption, the solution could be found via solving the l_1 optimization problem

$$\min_{x, Ax=y} \|x\|_1, \tag{4}$$

where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ is the l_1-norm of the vector x. The above problem is equivalent to a certain linear program whose solution can be found in roughly $O(n^3)$ time. While the use of l_1 minimization for finding sparse solutions has been around for a while (see [5], [7]), where it has been called Basis Pursuit, these were the first results to rigorously establish that the method could work exactly. The upshot is that something that appeared to be practically infeasible can now be potentially computed. For example, in [6], it was shown that if the measurement matrix A satisfies the restricted isometry conditions, then the l_1 minimization can recover a vector with up to k nonzero elements, where k is a constant fraction of n.

Since the introduction of compressive sensing, there has been a surge of research activity in finding explicitly constructed measurement matrices and faster recovery algorithms. In [31], we proposed to use expander graph based measurement matrix and gave an algorithm with $O(n)$ complexity which can recover any k-sparse signal with k as a linear fraction of n. There are explicitly constructed measurement matrix for the proposed expander graph based schemes. In the next section, we will introduce the proposed bipartite expander graph based schemes.

III. Expander Graphs and Efficient Recovery Algorithms

A. Expander Graphs

Expander graphs can be defined for arbitrary graphs, however, here we shall restrict ourselves to *bipartite* graphs. For a bipartite graph, we have two types of nodes. Following coding theory parlance, we will call one type left *variable nodes* of which there are n and which correspond to the entries of x, and right *parity check nodes* (or *measurement nodes*) of which there are m and which correspond to the entries of y (or the measurements). We assume that $n \geq m$. In a bipartite graph connections within the variable nodes and within the parity check nodes are not allowed. The existence of edges between the different variables and parity check nodes are represented by a $m \times n$ matrix A. In particular,

$$A_{ij} = \begin{cases} 1 & \text{if right node } i \text{ connected to left node } j \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. In what follows we shall use the matrix thus obtained from a suitably chosen bipartite graph as the measurement matrix for compressive sampling.

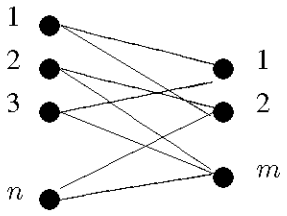

Fig. 1. A bipartite graph.

A bipartite graph will be said to have *regular left degree* c if the number of edges emanating from each variable node is c.

Definition 1 (Expander)**.** *A bipartite graph with n variable nodes, m parity check nodes and regular left degree c will be called a $(\alpha n, \beta c)$ expander, for some $0 < \alpha, \beta < 1$, if for every subset of variable nodes $\mathcal{V}$ with cardinality less than or equal to αn, i.e., $|\mathcal{V}| \leq \alpha n$, the number of neighbors connected to $\mathcal{V}$ is larger than $\beta c|\mathcal{V}|$, i.e., $|\mathcal{N}(\mathcal{V})| > \beta c|\mathcal{V}|$, where $\mathcal{N}(V)$ is the set of neighbors of $\mathcal{V}$.*

Here we assume that each righthand side node also has a regular degree d, where $cn = md$. The existence of expander graphs has been known for quite some time since the work of Pinsker and Bassylago [27], who used probabilistic arguments to prove their existence. Expander graphs arise in many applications: fast, distributed routing algorithms [28], LDPC codes [26], storage schemes [29], and proof systems [30], to name a few. An explicit construction of constant regular left degree *lossless* (with β arbitrarily close to 1) expander

graph is recently given in [25]. The existence result for the case where α independent of n (namely αn is a constant fraction of n) is given in [26], [22].

Theorem 1. *Let $0 < \beta < 1$ and the ratio $r = \frac{m}{n}$ be given. Then for large enough n there exists a regular left degree c and a regular right degree d bipartite expander $(\alpha n, \beta c)$ for some $0 < \alpha < 1$ and some constant (not growing with n) c.*

B. The Main Algorithm

We are now in a position to describe our main algorithm. We begin with $\beta = \frac{3}{4}$ and some fixed $r = \frac{m}{n}$. (Thus, our number of measurements is $m = nr$. We can use the construction of [25], or any other recent one, to construct an expander with some $0 < \alpha < 1$ and constant c.) Denote the resulting measurement matrix by A. In particular, assuming $x \in \mathcal{R}^n$ is sparse with at most k nonzero entries, we perform the m measurements

$$y = Ax. \tag{6}$$

We will assume that

$$k \leq \frac{\alpha n}{2}. \tag{7}$$

We need one further notation: given an estimate $\hat{x}$ of x, we define as the *gap* in the i-th equation the quantity

$$g_i = y_i - \sum_{j=1}^{n} A_{ij}\hat{x}_j. \tag{8}$$

Algorithm 1. 1) *Start with $\hat{x} = 0_{n\times 1}$.*
2) *If $y = A\hat{x}$, declare $\hat{x}$ the solution and exit. Else, find a variable node, say $\hat{x}_j$, such that of the c measurement equations it participates in $c' > \frac{c}{2}$ of them have an identical* nonzero *gap g.*
3) *Set $\hat{x}_j = \hat{x}_j + g$. Go to 2.*

Algorithm 1 is incredibly simple. What is remarkable about it is that, in step 2 of the algorithm, if $y \neq A\hat{x}$ one can always find a variable node with the property that $c' > \frac{c}{2}$ among the measurement equations it participates in has *identical* nonzero gap g. Furthermore, the algorithm terminates in at most ck steps. These two claims are established via a series of lemmas in [31], which in turn give the following theorem about the performance guarantees of the expander graph based schemes.

Theorem 2 (Validity of Algorithm 1)**.** *Consider a regular left degree bipartite graph with n variable nodes and m parity check nodes. Assume further that the graph is an $(\alpha n, \frac{3}{4}c)$ expander and consider its corresponding A matrix. Let $x \in \mathcal{R}^n$ be an arbitrary vector with at most $k \leq \alpha n/2$ nonzero entries and consider the m measurements*

$$y = Ax. \tag{9}$$

Then Algorithm 1 finds the value of x in at most $kc \leq \frac{c}{2}\alpha n$ iterations. If we assume that the bipartite graph has a regular right degree, we will have a recovery algorithm with complexity linear in n.

IV. Recovering k-Sparse Signal with $O(k \log n)$ Measurements

In the previous parts, we assume that the number of nonzero elements k in a sparse signal vector grows linearly with n and the number of measurements needed in compressive sensing need to grow linearly with n, too. However, in some cases, the number of nonzero elements k remains fixed while the dimension of the signal vector n can grow arbitrarily large. For the l_1-minimization framework, it has been shown that $O(k \log n)$ measurements suffice for perfectly recovering a sparse signal vector of dimension n with no more than k non-zero elements. In this part, we will show that only $O(k \log n)$ measurements are needed in order to perfect recovering all k-sparse signal when n goes large while requiring much lower recovery complexity. Before going to the precise statement and formal proof, we should notice that in Section III, the signal recovery mechanism still works as long as the parameters α β and c remain fixed for a fixed n even if they are a function of n as n grows. From the results of previous parts, to recover any k-sparse signal, we need a $(k, \frac{3}{4}c)$ bipartite expander graph with m measurements. So by showing the existence of such an expander graph with $m = O(k \log n)$, we actually show that for any k, $O(k \log n)$ measurements are enough for recovering any k-sparse signal with deterministic guarantees even as n grows. Before showing this, in the following theorem, we will give a lower bound on the number of measurements, namely m, in order to make an expander graph possible. Please note that this lower bound is a general result in the sense that it is also true for expander graphs with irregular right degrees.

Theorem 3 (Lower Bound on the Number of Measurements to Make an Expander Graph)**.** *Consider a bipartite graph with n variable nodes and m measurement nodes. Assume further that the graph is a $(k, \frac{3}{4}c)$ expander graph with regular left degree c. Then m must satisfy $\binom{m}{\frac{3}{4}ck} / \binom{m-c}{\frac{3}{4}ck-c} > n/k$.*

Proof: We prove this theorem by 'double counting'. In order for a bipartite graph to be a $(k, \frac{3}{4}c)$ expander, every $\frac{3}{4}ck$ measurement vertices must 'dominate' less than k variable nodes. Here we say a measurement set Ω dominates a variable node v if v is not connected to measurement nodes outside Ω. We now double count the number of 2-tuple pairs (Ω, v), where Ω is any set of measurement nodes of cardinality $\frac{3}{4}ck$ and v is a variable vertex dominated by the set Ω.

Notice that there are in total $\binom{m}{\frac{3}{4}ck}$ measurement node set Ω with cardinality $\frac{3}{4}ck$ and for the j-th ($1 \leq j \leq \binom{m}{\frac{3}{4}ck}$) such set Ω_j, we denote the set of variable nodes that are dominated by Ω_j as V_j. So the total number of 2-tuple pairs (Ω, v) is $\sum_{j=1}^{\binom{m}{\frac{3}{4}ck}} |V_j|$. Now let us count the number of 2-tuple pairs (Ω, v) from the perspective of variable nodes. For the i-th variable node v_i, there are $\binom{m-l_i}{\frac{3}{4}ck-l_i}$ measurement node sets Ω of cardinality $\frac{3}{4}ck$ that dominate v_i, where l_i ($1 \leq l_i \leq c$) is the number of measurement nodes that the variable node v_i is connected to. So the total number of 2-tuple pairs

(Ω, v) is also equal to $\sum_{i=1}^{n} \binom{m-l_i}{\frac{3}{4}ck-l_i}$, which is no smaller than $\binom{m-c}{\frac{3}{4}ck-c}n$. For an $(k, \frac{3}{4}c)$ expander graph, $\sum_{j=1}^{\binom{m}{\frac{3}{4}ck}} |V_j| < k \times \binom{m}{\frac{3}{4}ck}$ because each set Ω dominates less than k variable nodes. By combining the results of double counting, we have $\binom{m-c}{\frac{3}{4}ck-c}n < k \times \binom{m}{\frac{3}{4}ck}$. This proves Theorem 3. ∎

Lemma 1 (Constant Left Degree not Achieving the $O(k\log(n))$ bound)**.** *Consider a bipartite graph with n variable nodes and m measurement nodes. Assume further that the graph is a $(k, \frac{3}{4}c)$ expander graph with regular left degree c. If $m = O(k\log(n))$, then c can not be a constant independent of n.*

Proof: It is straightforward from Theorem 1 that $m \geq (\frac{n}{k})^{\frac{1}{c}}$, which is a polynomial over n. ∎

In the following part, we will give the main result of this section, the sufficiency of $O(k\log(n))$ measurements to make a $(\frac{k}{2}, \frac{3}{4}c)$ regular expander graph for n variable nodes. We prove that a randomly generated bipartite graphs will be the desirable expander with high probability. Different from the case where k is a linear fraction of n, as proved in the previous lemma, we need to carefully choose the left degree c and m.

Theorem 4 (The Sufficiency of $O(k\log(n))$ Measurements)**.** *Consider regular bipartite graphs with n variable nodes and m measurement nodes. Assume that they have regular left-degree c and regular right-degree d. For any k, if n is large enough, there exists a regular $(k, \frac{3}{4}c)$ expander bipartite graph with $m = O(k\log(n))$ for some number c (Note that the left-degree c depends on n). Let $x \in \mathcal{R}^n$ be an arbitrary vector with at most $\frac{k}{2}$ nonzero entries and consider the m measurements*

$$y = Ax. \tag{10}$$

Then Algorithm 1 finds the value of x in at most $\frac{kc}{2}$ iterations.

Proof: We show the existence of the expander graphs stated in Theorem 4. Then the signal recovery performance statement in Theorem 4 follows from the existence of such an expander graph and Theorem 2. In proving the existence of such an expander graph, we show that a regular bipartite graph randomly generated in a certain way will be a $(k, \frac{3}{4}c)$ expander graph with probability approaching 1 as n goes large.

Here we take $c = C\log(n)$ and $m = Dk\log(n)$, where C and D are constants independent of k and n and will be specified later. Consider the bipartite graph as shown in Figure 1. For the time being, we assume that $C \leq D$. So, in total, we have

$$T_E = (C\log(n)) \times n \tag{11}$$

edges emanating from the n variable nodes. We generate a random permutation of these $(C\log(n)) \times n$ emanating edges with a uniform distribution (over all the possible permutations) and connect ('plug') these $(C\log(n)) \times n$ edges to the $(C\log(n)) \times n$ 'sockets' on the $Dk\log(n)$ measurement nodes according to the randomly generated permutation. So the number of edges each measurement node connects is

$$d = (C\log(n)) \times n/m = \frac{Cn}{Dk}. \tag{12}$$

Take an arbitrary variable node set S of cardinality k and consider the random variable Y, which is the number of check nodes connected to S in this randomly generated graph. Obviously,

$$Y = \sum_{i=1}^{kC\log(n)} I_i, \tag{13}$$

where I_i is the indicator function of whether the i-th edge is connected to a check node which is not connected to any of the previous $(i-1)$ edges. Suppose the previous $(i-1)$ edges are connected to L_{i-1} measurement nodes, then I_i takes the value '1' with probability

$$\frac{T_E - d \times L_{i-1}}{T_E - L_{i-1}}, \tag{14}$$

whatever measurement nodes the previous $(i-1)$ edges are connected to. Since any $(i-1)$ edges are connected to at most $(i-1)$ measurement nodes and $i \leq C\log(n)$, we have

$$\frac{T_E - d \times L_{i-1}}{T_E - L_{i-1}} \geq \frac{T_E - d \times (k \times C\log(n))}{T_E - (k \times C\log(n))}. \tag{15}$$

So the probability that $(1 - I_i)$ takes the value '1' is at most

$$1 - \frac{T_E - d \times (k \times C\log(n))}{T_E - (k \times C\log(n))} = \frac{\frac{Cn}{D} - k}{n - k} \leq \frac{C}{D}, \tag{16}$$

whatever $I_j, 1 \leq j \leq (i-1)$, are.

Define a new random variable

$$Z = kC\log(n) - Y = \sum_{i=1}^{kC\log(n)} (1 - I_i), \tag{17}$$

and consider another random variable

$$Z' = \sum_{i=1}^{kC\log(n)} b_i, \tag{18}$$

where b_i's are independent binary Bernoulli random variables of parameter $\frac{C}{D}$ (taking the value '1' with probability $\frac{C}{D}$ and taking the value '0' with probability $1 - \frac{C}{D}$). Then the probability that $Z \geq \frac{1}{4}kC\log(n)$ is always no larger than the probability that $Z' \geq \frac{1}{4}kC\log(n)$. This is because *whatever* $I_j, 1 \leq j \leq (i-1)$ are, the probability of $(1 - I_i)$ taking the value '1' is at most $\frac{C}{D}$ conditioned on $I_j, 1 \leq j \leq (i-1)$.

By the well-known Chernoff bound for the sum of independent Bernoulli random variables [32], we know that if $\frac{C}{D} < \frac{1}{4}$,

$$P(Z \geq \frac{1}{4}kC\log(n)) \leq e^{-\mathbf{H}(\frac{1}{4}\|\frac{C}{D})kC\log(n)}. \tag{19}$$

Here $\mathbf{H}(a\|b)$ is the Kullback-Leibler divergence between two Bernoulli random variables with parameter a and b, namely,

$$\mathbf{H}(a\|b) = a\log\frac{a}{b} + (1-a)\log\frac{1-a}{1-b}. \tag{20}$$

(Notes: Another way to get the exponentially decaying upper bound

$$P(Z \geq \frac{1}{4}kC\log(n)) \leq e^{-\mathbf{H}(\frac{1}{4}\|\frac{C}{D})kC\log(n)} \quad (21)$$

is through the direct application of the Chernoff bound for the probability $P(Z \geq \frac{1}{4}kC\log(n))$. Although the random variables I's are not independent, the conditions on the conditional probability still lead to the exponentially decaying upper bound on $P(Z \geq \frac{1}{4}kC\log(n))$.)

In summary, with probability no larger than $e^{-\mathbf{H}(\frac{1}{4}\|\frac{C}{D})kC\log(n)}$, a variable node set S of cardinality k is connected to no more than $\frac{3}{4}kC\log(n)$ measurement nodes. Since there are at most $\binom{n}{k} \leq e^{k\log(n)}$ variable node sets of cardinality k, by a simple union bound, we have with probability at least

$$P_k = 1 - e^{klog(n)} \times e^{-\mathbf{H}(\frac{1}{4}\|\frac{C}{D})kClog(n)}, \quad (22)$$

all variable node sets of cardinality of k are connected to more than $\frac{3}{4}kC\log(n)$ check nodes. If we take the constants C and D such that $\frac{C}{D}$ is sufficiently small, $e^{klog(n)} \times e^{-\mathbf{H}(\frac{1}{4}\|\frac{C}{D})kClog(n)}$ will go to zero exponentially in $kC\log n$. In fact, if $\frac{C}{D}$ is arbitrarily small, $\mathbf{H}(\frac{1}{4}\|\frac{C}{D})$ can go to infinity.

Similarly, for each integer $1 \leq l \leq k$, the probability that all variable sets of cardinality of l are connected to more than $\frac{3}{4}lC\log(n)$ check nodes is at least

$$P_l = 1 - e^{l\log(n)} \times e^{-\mathbf{H}(\frac{1}{4}\|\frac{C}{D})lC\log(n)}. \quad (23)$$

By union bound, the probability that any set of cardinality no larger than k has good expansion property satisfies be

$$P = 1 - \sum_{l=1}^{k} e^{l\log(n)} \times e^{-\mathbf{H}(\frac{1}{4}\|\frac{C}{D})lC\log(n)}, \quad (24)$$

which is positive given that n is large enough and if we choose $\frac{C}{D}$ sufficiently small. This shows that we only need $O(k\log(n))$ check nodes to make a bipartite graph a $(k, \frac{3}{4}c)$ expander. ∎

V. Conclusions

We consider the newly proposed expander graph based compressive sensing schemes and show that, similar to the l_1 minimization case, we can exactly recover any k-sparse signal using only $O(k\log(n))$ measurements, where k is the number of non-zero elements. The number of computational iterations is of order $O(k\log(n))$, while each iteration involves very simple computational steps.

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