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Chaos in Dynamical Systems by the Poincaré-Melnikov-Arnold Method

Jerrold E. Marsden*

Abstract. Methods for proving the existence of chaos in the sense of Poincaré-Birkhoff-Smale horseshoes are presented. We shall concentrate on explicitly verifiable results that apply to specific examples such as the ordinary differential equations for a forced pendulum, and for superfluid ^{3}He and the partial differential equation describing the oscillations of a beam. Some discussion of the difficulties the method encounters near an elliptic fixed point is given.

1. An Introductory Example. Consider the equation for a forced pendulum

$$\ddot{\phi} + \sin\phi = \varepsilon \cos \omega t \qquad (1.1)$$

where ω is a constant angular forcing frequency, and ε is a small parameter. For ε small but non-zero, (1.1) possess no analytic integrals of the motion. In fact, it possesses transversal intersecting stable and unstable manifolds (separatrices); that is, the Poincaré maps $P_{t_0}: \mathbb{R}^2 \to \mathbb{R}^2$ that advance solutions by one period $T = 2\pi/\omega$ starting at time t_0 possess transversal homoclinic points. This type of dynamic behavior has several consequences, besides precluding the existence of analytic integrals, that lead one to use the term 'chaotic'. For example, the equation (1.1) has infinitely many periodic solutions of arbitrarily high period. Also, using the shadowing lemma, one sees that given any bi-infinite sequence of zero and ones (for example, use the binary expansion of e or π), there exists a corresponding solution of (1.1) that successively crosses the plane $\phi = 0$ (the pendulum's vertically downward configuration) with $\dot{\phi} > 0$ corresponding to a zero and $\dot{\phi} < 0$ corresponding to a one. The origin of this chaos on an

*Research Group in Nonlinear Systems and Dynamics and Department of Mathematics, University of California, Berkeley, CA 94720. Research partially supported by DOE contract DE-AT03-82ER12097.

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intuitive level lies in the motion of the pendulum near its unperturbed homoclinic orbit -- the orbit that does one revolution in infinite time. Near the top of its motion (where $\phi = \pm\pi$) small nudges from the forcing term can cause the pendulum to fall to the left or right in a temporally complex way.

The dynamical systems theory needed to justify all of the preceding statements is now readily available in Smale [1967], Moser [1973] and Guckenheimer and Holmes [1983]. The key people responsible for the development of the basic theory are Poincare, Birkhoff and Smale. The idea of transversal intersecting separatrices comes from Poincaré's famous 1890 paper on the three body problem. His goal -- not quite achieved for reasons we shall comment on later -- was to prove the nonintegrability of the restricted three body problem and that various series expansions used up to that point diverged (he invented the theory of asymptotic expansions in the course of this work).

Although Poincaré had all the essential tools needed to prove that equations like (1.1) are not integrable (in the sense of having no analytic integrals) his interests lay with harder problems and he did not develop the easier basic theory very much. Important contributions were made by Melnikov [1963] and Arnold [1964] which leads to a very simple procedure for proving (1.1) is not integrable. The Poincare-Melnikov method was recently revived by Chirikov [1979], Holmes [1980] and Chow, Hale and Mallet-Paret [1980]. (For related work and more references and examples, see also Kozlov [1983].)

The procedure is as follows: rewrite (1.1) in abstract form as

$$\dot{x} = X_0(x) + \varepsilon X_1(x,t) \qquad (1.2)$$

where $x \in \mathbb{R}^2$, X_0 is a Hamiltonian vector field with energy H_0, X_1 is periodic with period T and is Hamiltonian with energy H_1. Assume X_0 has a homoclinic orbit $\bar{x}(t)$ so $\bar{x}(t) \to x_0$, a hyperbolic saddle point, as $t \to \pm\infty$. Compute the "Melnikov function"

$$M(t_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}(\bar{x}(t-t_0), t)\, dt \qquad (1.3)$$

where $\{\ ,\ \}$ denotes the Poisson bracket. If $M(t_0)$ has simple zeros as a function of t_0, then (1.2) has transversal intersecting separatrices (in the sense of Poincare maps as mentioned above).

We shall give a proof of this result (essentially the one indicated by Arnold [1964] in §2. To apply it to equation (1.1) one proceeds as follows. Let $x = (\phi, \dot{\phi})$ so (1.1) becomes

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\dot\phi\\ -\sin\phi\end{pmatrix} + \varepsilon\begin{pmatrix}0\\ \cos\omega t\end{pmatrix}. \tag{1.4}$$

The homoclinic orbits for $\varepsilon = 0$ are computed to be given by

$$\bar{x}(t) = \begin{pmatrix}\phi(t)\\ \dot\phi(t)\end{pmatrix} = \begin{pmatrix}\pm 2\tan^{-1}(\sinh t)\\ \pm 2\,\mathrm{sech}\,t\end{pmatrix} \tag{1.5}$$

and one has

$$\left.\begin{aligned} H_0(\phi,\dot\phi) &= \tfrac{1}{2}\dot\phi^2 - \cos\phi \\ H_1(\phi,\dot\phi,t) &= \phi\cos\omega t\ .\end{aligned}\right\} \tag{1.6}$$

Hence (1.3) gives

$$\begin{aligned} M(t_0) &= \pm\int_{-\infty}^{\infty}\left(\frac{\partial H_0}{\partial\phi}\frac{\partial H_1}{\partial\dot\phi} - \frac{\partial H_0}{\partial\dot\phi}\frac{\partial H_1}{\partial\phi}\right)dt \\ &= \mp\int_{-\infty}^{\infty}\dot\phi\cos\omega t\,dt \\ &= \mp\int_{-\infty}^{\infty}[2\,\mathrm{sech}(t-t_0)\cos\omega t]\,dt\ . \end{aligned}$$

Changing variables and using the fact that sech is even and sin is odd, we get

$$M(t_0) = \mp 2\left(\int_{-\infty}^{\infty}\mathrm{sech}\,t\cos\omega t\,dt\right)\cos(\omega t_0)\ .$$

The integral is evaluated by residues:

$$M(t_0) = \mp 2\pi\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right)\cos(\omega t_0) \tag{1.7}$$

which clearly has simple zeros.

2. A Proof of the Poincaré-Melnikov Theorem. There are two convenient ways of visualizing the dynamics of (1.2). One can introduce the Poincaré map $P_\varepsilon^s:\mathbb{R}^2\to\mathbb{R}^2$, which is the time T map for (2.1) starting at time s. For $\varepsilon = 0$, the point x_0 and the homoclinic orbit are invariant under P_0^s, which is independent of s. The hyperbolic saddle x_0 persists as a nearby family of saddles x_ε for $\varepsilon > 0$, small, and we are interested in whether or not the stable and unstable manifolds of the point x_ε for the map P_ε^s intersect transversally (if this holds for one s, it holds for all s). If so, we say (1.2) <u>admits horseshoes for</u> $\varepsilon > 0$.

The second way to study (1.2) is to look directly at the suspended system on $\mathbb{R}^2\times S^1$, where S^1 stands for the circle, elements of which are regarded as the T-periodic variable θ. Then (1.2) becomes the autonomous suspended system

$$\left.\begin{aligned}\dot x &= f_0(x) + \varepsilon f_1(x,\theta)\\ \dot\theta &= 1\end{aligned}\right\}\ . \tag{2.1}$$

From this point of view the curve

$$\gamma_0(t) = (x_0, t)$$

is a periodic orbit for (2.1), whose stable and unstable manifolds $W_0^s(\gamma_0)$ and $W_0^u(\gamma_0)$ are coincident. For $\varepsilon > 0$ the hyperbolic closed orbit γ_0 perturbs to a nearby hyperbolic closed orbit which has stable and unstable manifolds $W_\varepsilon^s(\gamma_\varepsilon)$ and $W_\varepsilon^u(\gamma_\varepsilon)$. If $W_\varepsilon^s(\gamma_\varepsilon)$ and $W_\varepsilon^u(\gamma_\varepsilon)$ intersect transversally, we again say that (1.2) admits horseshoes. These two definitions of admitting horseshoes are readily seen to be equivalent.

<u>Poincaré-Melnikov Theorem</u>. <u>Define the Melnikov function by</u> (1.3). <u>Assume</u> $M(t_0)$ <u>has simple zeros as a T-periodic function of</u> t_0. <u>Then</u> (1.2) <u>has horseshoes</u>.

<u>Proof</u>. In the suspended picture, we use the energy function H_0 to measure the first order movement of $W_\varepsilon^s(\gamma_\varepsilon)$ at $\bar x(0)$ at time t_0 as ε is varied. Note that points of $\bar x(t)$ are regular points for H_0 since H_0 is constant on $\bar x(t)$ and $\bar x(0)$ is not a fixed point. Thus, the values of H_0 give an accurate measure of the

distance from the homoclinic orbit. If $(x_\epsilon^s(t,t_0),t)$ is the curve on $W_\epsilon^s(\gamma_\epsilon)$ that is an integral curve of the suspended system (2.1) and has an initial condition $x^s(t_0,t_0)$ which is the perturbation of $W_0^s(\gamma_0)$ (the plane $t = t_0$) in the normal direction to the homoclinic orbit, then $H_0(x_\epsilon^s(t_0,t_0))$ measures this normal distance. But

$$H_0(x_\epsilon^s(T,t_0)) - H_0(x_\epsilon^s(t_0,t_0)) = \int_{t_0}^{T} \frac{d}{dt} H_0(x_\epsilon^s(t,t_0))\, dt \quad (2.2)$$

From (2.2), we get

$$H_0(x_\epsilon^s(T,t_0)) - H_0(x_\epsilon^s(t_0,t_0)) = \int_{t_0}^{T} \{H_0, H_0 + \epsilon H_1\}(x^s(t,t_0),t)\,dt \quad (2.3)$$

Since $x_\epsilon^s(T,t_0)$ is ϵ-close to $\bar{x}(t-t_0)$ (uniformly as $T \to +\infty$), and $d(H_0 + \epsilon H^1)(x^\epsilon(t,t_0),t) \to 0$ exponentially as $t \to +\infty$, and $\{H_0,H_0\} = 0$, (2.3) becomes

$$H_0(x_\epsilon^s(T,t_0)) - H_0(x_\epsilon^s(t_0,t_0)) = \epsilon \int_{t_0}^{T} \{H_0,H_1\}(\bar{x}(t-t_0),t)\, dt + O(\epsilon^2). \quad (2.4)$$

Similarly,

$$\begin{aligned} H_0(x_\epsilon^u(t_0,t_0)) &- H_0(x_\epsilon^u(-S,t_0)) \\ &= \epsilon \int_{-S}^{t_0} \{H_0,H_1\}(\bar{x}(t-t_0),t)\, dt + O(\epsilon^2) \end{aligned} \quad (2.5)$$

Now $x_\epsilon^s(T,t_0) \to \gamma_\epsilon$, a periodic orbit for the perturbed system as $T \to +\infty$. Thus, we can choose T and S such that $H_0(x_\epsilon^s(T,t_0)) - H_0(x_\epsilon^u(-S,t_0)) \to 0$ as $T,S \to \infty$. Thus, adding (2.4) and (2.5), and letting $T,S \to \infty$, we get

$$H_0(x_\epsilon^u(t_0,t_0)) - H_0(x_\epsilon^s(t_0,t_0)) = \epsilon \int_{-\infty}^{\infty} \{H_0,H_1\}(\bar{x}(t-t_0),t)\, dt + O(\epsilon^2) \quad (2.6)$$

It follows that if $M(t_0)$ has a simple zero in time t_0, then $x_\epsilon^u(t_0,t_0)$ has $x_\epsilon^s(t_0,t_0)$ must intersect transversally near the point $\bar{x}(0)$ at time t_0.

<u>Remark</u>. Since $dH_0 \to 0$ exponentially at the saddle points, the integrals involved in this criterion are automatically convergent.

3. <u>An Extension to Include Damping</u>. There are a number of extensions and applications of this technique that have been developed, some of which we describe here and in the next few sections. The literature in this area is growing very quickly and we make no claim to be comprehensive (the reader can track down many additional references by consulting the references cited).

If in (1.2), X_0 is Hamiltonian but X_1 is not, the same conclusion holds if (1.3) is replaced by

$$M(t_0) = \int_{-\infty}^{\infty} (X_0 \times X_1)(\bar{x}(t-t_0),t)\, dt \quad (3.1)$$

where $X_0 \times X_1$ is the (scalar) cross product for planar vector fields. In fact, X_0 need not even be Hamiltonian if a volume expansion factor is inserted.

For example, this applies to the forced damped Duffing equation

$$\ddot{u} - \beta u + \alpha u^3 = \epsilon(\gamma \cos \omega t - \delta \dot{u}) \quad (3.2)$$

Here the homoclinic orbits are given by

$$\bar{u}(t) = \pm\sqrt{\frac{\beta}{\alpha}}\, \mathrm{sech}(\sqrt{\beta}t) \quad (3.3)$$

and (1.1) becomes, after a residue calculation,

$$M(t_0) = 2\gamma\, \pi\omega\sqrt{\frac{2}{\alpha}}\, \mathrm{sech}\left(\frac{\pi\omega}{2\sqrt{\beta}}\right) \sin(\omega t_0) + \frac{4\delta\beta^{3/2}}{3\alpha} \quad (3.4)$$

so one has simple zeros and hence chaos of the horseshoe type if

$$\frac{\gamma}{\delta} > \frac{\sqrt{2}\,\beta^{3/2}}{3\omega\sqrt{\alpha}} \cosh\left(\frac{\pi\omega}{2\sqrt{\beta}}\right) \tag{3.5}$$

and ϵ is small.

Another interesting example, due to Montgomery [1984] concerns the equations for superfluid ^3He. These are the Leggett equations and we shall confine ourselves to the A phase for simplicity (see Montgomery's paper for additional results). The equations are

$$\begin{aligned} \dot{s} &= -\frac{1}{2}\left(\frac{\chi\Omega^2}{\gamma^2}\right)\sin 2\theta \\ \dot{\theta} &= \left(\frac{\gamma^2}{\chi}\right)s - \epsilon\left(\gamma h \sin \omega t + \frac{1}{2}\Gamma \sin 2\theta\right) \end{aligned} \tag{3.6}$$

Here s is the spin, θ the angle describing the order parameter and γ, χ, ... are physical constants. The homoclinic orbits for $\epsilon = 0$ are given by

$$\begin{aligned} \bar{\theta}_\pm &= 2\tan^{-1}(e^{\pm\Omega t}) - \pi/2 \\ \bar{s}_\pm &= \pm 2\,\frac{\Omega e^{\pm 2\Omega t}}{1 + e^{\pm 2\Omega t}} \end{aligned} \tag{3.7}$$

One calculates using (3.6) and (3.7) in (3.1) that

$$M_\pm(t_0) = \mp \frac{\pi\chi\omega h}{8\gamma} \operatorname{sech}\left(\frac{\omega\pi}{2\Omega}\right)\cos \omega t - \frac{2}{3}\frac{\chi}{\gamma^2}\Omega\Gamma \tag{3.8}$$

so that (3.6) has chaos in the sense of horseshoes if

$$\frac{\gamma h}{\Gamma} > \frac{16}{3\pi}\frac{\Omega}{\omega}\cosh\left(\frac{\pi\omega}{2\Omega}\right) \tag{3.9}$$

and if ϵ is small.

4. <u>An Extension to PDE's</u>. There is a version of the Poincaré-Melnikov theorem applicable to PDE's that is due to Holmes and Marsden [1981]. One basically still uses the formula (3.1) where

$X_0 \times X_1$ now is replaced by the symplectic pairing between X_0 and X_1. However, there are two new difficulties in addition to standard technical analytic problems that arise with PDE's. The first is that there is a serious problem with resonances. These can be dealt with using the aid of damping -- the undamped case would need an infinite dimensional version of Arnold diffusion -- see §6 below. Secondly, the problem is not reducible to two dimensions; the horseshoe involves all the modes. Indeed, the higher modes do seem to be involved in the physical buckling processes for the beam model discussed next.

A PDE model for a buckled forced beam is

$$\ddot{w} + w'''' + \Gamma w'' - \kappa\left(\int_0^1 [w']^2 dz\right)w'' = \epsilon(f \cos \omega t - \delta\dot{w}) \tag{4.1}$$

where $w(z,t)$ $0 \le z \le 1$ describes the deflection of the beam, $\dot{} = \partial/\partial t$, $' = \partial/\partial z$ and Γ, κ, ... are physical constants. For this case, the theory shows that if

(a) $\pi^2 < \Gamma < 4\rho^3$ (first mode is buckled)

(b) $j^2\pi^2(j^2\pi^2 - \Gamma^1) \neq \omega^2$, $j = 2, 3, \ldots$ (resonance condition)

(c) $\frac{f}{\delta} > \frac{\pi(\Gamma - \pi^2)}{2\omega\sqrt{\kappa}} \cosh\left(\frac{\omega}{2\sqrt{\Gamma - \omega^2}}\right)$ (transversal zeros for $M(t_0)$)

(d) $\delta > 0$

and ϵ is small, then (4.1) has horseshoes.

Experiments of F. Moon at Cornell which show chaos in a forced buckled beam provided the motivation which led to the study of (4.1).

This kind of result has recently been used by Slemrod and Marsden [1983] for a study of chaos in a van der Waal's fluid (see Slemrod's lecture in these proceedings) and by Moon, Birnir and Morrison for soliton equations. For example, in the damped, forced Sine-Gordon equation one has chaotic transitions between breathers and kink-antikink pairs and in the Benjamin-Ono equation one can have chaotic transitions between solutions with different numbers of poles.

5. <u>Autonomous Hamiltonian Systems</u>. For Hamiltonian systems with two degrees of freedom, Holmes and Marsden [1982a] show how the Melnikov method may be used to prove the existence of horseshoes on energy surfaces in two degree of freedom nearly integrable systems. The class of systems studied have a Hamiltonian of the form

$$H(q,p,\theta,I) = F(q,p) + G(I) + \epsilon H_1(q,p,\theta,I) + O(\epsilon^2) \qquad (5.1)$$

where (θ,I) are action angle coordinates for the oscillator G; $G(0) = 0$, $G' > 0$. It is assumed that F has a homoclinic orbit $\bar{x}(t) = (\bar{q}(t), \bar{p}(t))$ and that

$$M(t_0) = \int_{-\infty}^{\infty} \{F,H_1\}\, dt \qquad (5.2)$$

(the integral taken along $(\bar{x}(t-t_0), \Omega t, I))$ has simple zeros.

Then (5.1) has horseshoes on energy surfaces near the surface corresponding to the homoclinic orbit and small I; the horseshoes are taken relative to a Poincaré map strobed to the oscillator G. Holmes and Marsden 1982a also studies the effect of positive and negative damping. These results are related to that in §2 since one can often reduce a two degree of freedom Hamiltonian system to a one degree of freedom forced system.

For some systems in which the variables do not split as in 5.1, such as a nearly symmetric heavy top, one needs to exploit a symmetry of the system and this complicates the situation to some extent. The general theory for this is given in Holmes and Marsden [1983] and was applied to show the existence of horseshoes in the nearly symmetric heavy top; see also some closely related results of Ziglin [1980a].

This theory has been used, for example by Koiller and coworkers in a number of recent reprints on vortex dynamics (Koiller and Pinto de Carvalho [1983] seems to be the first to give a correct proof of the non-integrability of the restricted four vortex problem -- see §7 below). There have also been recent applications to the dynamics of general relativity showing the existence of horseshows in Bianchi IX models. See also Krishnaprasad [1983] for interesting applications to dual-spin spacecraft.

6. Arnold Diffusion. Arnold [1964] extended the Poincaré-Melnikov theory to systems with several degrees of freedom. In this case the transverse homoclinic manifolds are based on KAM tori and allow the possibility of chaotic drift from one torus to another. This drift, now known as Arnold diffusion is a basic ingredient in the study of chaos in Hamiltonian systems (see for instance, Chirikov [1979] and Lichtenberg and Lieberman [1983] and references therein). Instead of a single Melnikov function, one now has a Melnikov vector given schematically by

$$\vec{M} = \begin{pmatrix} \int_{-\infty}^{\infty} \{H_0,H_1\}\, dt \\ \vdots \\ \int_{-\infty}^{\infty} \{I_k,H_1\}\, dt \end{pmatrix} \qquad (6.1)$$

where I_k are integrals for the unperturbed (completely integrable) system and where $\vec{M}$ now depends on t_0 and on angles conjugate to $I_1, \ldots, I_n$. One now requires $\vec{M}$ to have transversal zeros in the vector sense. This result was given by Arnold for forced systems and was extended to the autonomous case by Holmes and Marsden [1982b]*, [1983].

These results apply to systems such as a pendulum coupled to several oscillators and the many vortex problem. It has also been used in power systems by Salam, Marsden and Varaiya [1984], building on the horseshoe case treated by Kopell and Washburn [1982]. See also the work of Salam and Sastry reported in these proceedings.

There have been a number of other directions of research on these techniques. For example, Grundler [1981] developed a multidimensional version applicable to the spherical pendulum and Greenspan and Holmes [1983] showed how it can be used to study subharmonic bifurcations.

7. Exponentially Small Melnikov Functions. There is a serious difficulty that arises when one uses the Melnikov method near an elliptic fixed point in a Hamiltonian system. The difficulty is closely related to the difficulty Poincaré encountered in trying to prove nonintegrability and the divergence of series expansions that occur in the restricted 3 body problem. Near elliptic points, one sees homoclinic orbits in normal form and after a temporal rescaling leads to a loss of analyticity and a rapidly oscillatory perturbation that is modelled by the following variation of (1.1):

$$\ddot{\phi} + \sin\phi = \epsilon \cos\left(\frac{\omega t}{\epsilon}\right) \qquad (7.1)$$

If one just blindly computes $M(t_0)$ one finds from (1.7),

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*As was pointed out by F.A. Salam and C. Robinson, one needs to interpret the integrals appearing here with care and correctly adjust the phases of orbits asymptotic to the tori.

$$M(t_0,\varepsilon) = \mp 2\pi \operatorname{sech}\left(\frac{\pi\omega}{2\varepsilon}\right) \cos\left(\frac{\omega t_0}{\varepsilon}\right) \qquad (7.2)$$

while this has simple zeros, the proof of the Poincaré-Melnikov theorem is no longer valid since $M(t_0,\varepsilon)$ is now of order $e^{-\pi/2\varepsilon}$ and the error analysis in the proof only gives errors of order ε^2. In fact no expansion in powers of ε can detect exponentially small terms like $e^{-\pi/2\varepsilon}$. (This is the sort of difficulty that seems to occur in the paper of Ziglin [1980b] on the four vortex problem; see also Sanders [1982].)

Recent work of Holmes, Marsden and Scheurle aims to show that indeed (7.1) has horseshoes for small ε. The idea is to expand expressions for the stable and unstable manifolds in a Perron type series whose terms are of order $\varepsilon^k e^{-\pi/2\varepsilon}$. To do so, the extension of the system to compelx time plays a crucial role.

One can hope that if such results for (7.1) can really be proven, then it may be possible to return to Poincaré's 1890 work and complete the arguments he left unfinished.

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