Geometric Mechanics, Stability and Control

Jerrold E. Marsden*
Department of Mathematics
University of California, Berkeley, CA 94720
and The Fields Institute

April 22, 1992

Springer Applied Mathematical Sciences Series, **100**, 1994, 265–291

Abstract

This paper gives an overview of selected topics in mechanics and their relation to questions of stability, control and stabilization. The mechanical connection, whose holonomy gives phases and that plays an important role in block diagonalization, provides a unifying theme.

1 The Mechanical Connection

The mechanical connection is a basic object that links mechanics and gauge theory. This point of view has matured over the last decade to the stage where we now see enough applications and intrinsic beauty to elevate it to a special status. This topic owes much to Smale [1970], Meyer [1973], Marsden and Weinstein [1974], Abraham and Marsden [1978], Kummer [1981], Guichardet [1984], Montgomery, Marsden and Ratiu [1984], Montgomery [1986,1989,1990], Iwai [1987], Wilczeck and Shapere [1989], Marsden, Montgomery and Ratiu [1990], Simo, Posbergh and Marsden [1990], Simo, Lewis and Marsden [1991] and others.

We start with four basic ingredients:

- Q, the configuration manifold of a mechanical system,
- $\langle\langle\, , \rangle\rangle$, a Riemannian metric (usually derived from the kinetic energy),
- G, a Lie group of symmetries acting by isometries on Q,
- V, a G-invariant potential energy.

*Research partially supported by a Fairchild fellowship at Caltech and by DOE. We thank W. Sluis for providing a draft of the lectures given at the Fields Institute, March 1992.

Examples

1 ***The Spherical Pendulum*** Here $Q = S^2$ a sphere of radius l, where l is the length of the pendulum, $\langle\langle\, , \rangle\rangle$ is the inner product such that $K(q, v) = \frac{1}{2}\|v\|^2$ is the kinetic energy, $G = S^1$ acts by rotations around the vertical axis, and V is the gravitational potential energy. See Figure 1.1.

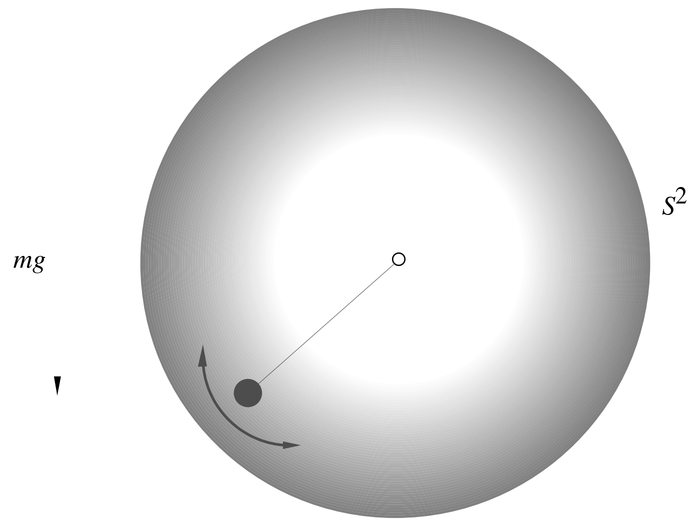


Figure 1.1: The spherical pendulum

2 ***The Ozone Molecule*** In this case, $Q = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, again $\langle\langle\, , \rangle\rangle$ gives the kinetic energy, G is the Euclidean group, and V models the interaction potential between the oxygen atoms. This problem also has an interesting group of discrete symmetries associated with the identity of the three oxygen atoms. See Figure 1.2.

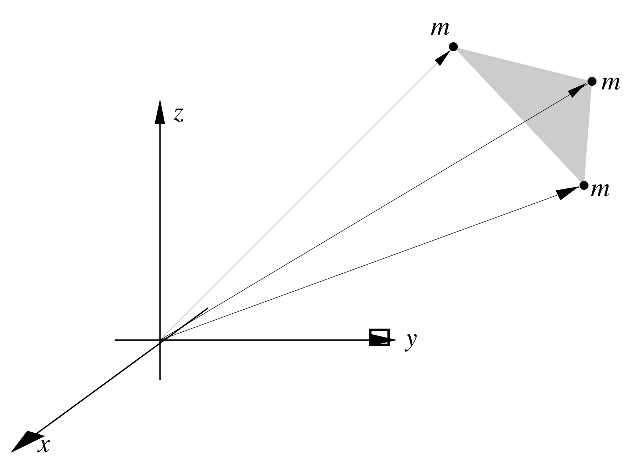


Figure 1.2: The ozone molecule.

3 ***The Double Spherical Pendulum*** (Marsden and Scheurle [1993]). Here $Q = S^2_{l_1} \times S^2_{l_2}$, the product of two spheres, with radii l_1 and l_2, the lengths of the two pendula. The metric $\langle\langle\, , \rangle\rangle$ again gives the kinetic energy (it is not the standard metric!), $G = S^1$ acts by rotations about the vertical axis, and V is the gravitational potential. See Figure 1.3. ♦

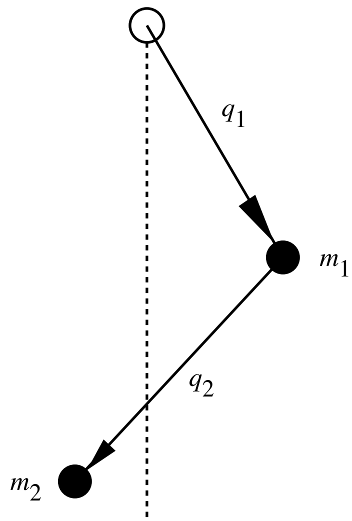


Figure 1.3: The double spherical pendulum.

Two Instances of the Mechanical Connection Before giving the abstract definition of the mechanical connection, which depends only on Q, $\langle\langle\, , \rangle\rangle$,and G, we give a glimpse of how it appears in two specific contexts.

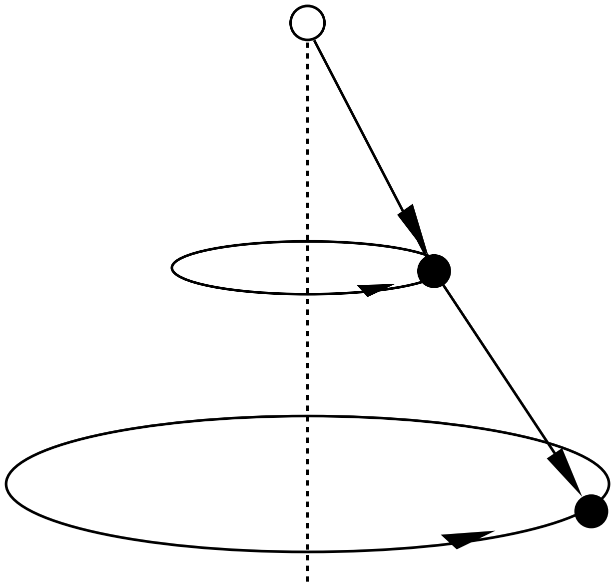
First consider the double spherical pendulum. This problem has some interesting special explicit solutions of the form $q(t) = \exp(t\xi)q(0)$ for some $\xi \in \mathbb{R}$, where $\exp(t\xi)q(0)$ denotes the S^1 action, and $q = (q_1, q_2) \in S^2_{l_1} \times S^2_{l_2}$ gives the configuration. The curve $q(t)$ satisfies the Euler-Lagrange equations for the Lagrangian $L =$ kinetic-potential energies for special choices of $q(0)$. Solutions of this sort are called ***relative equilibria***.

The general appearance of two of these special solutions is shown in Figure 1.4. The solution on the left is called the ***stretched out*** solution, while the one on the right is the ***cowboy*** solution.

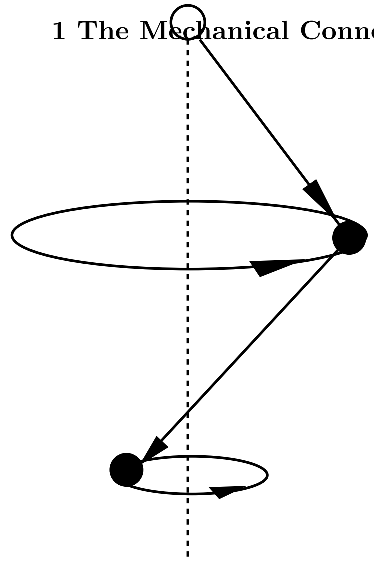
We will explain in §**2** the principles by which one determines these shapes. In a study of stability and bifurcation, one often linearizes the Euler-Lagrange equations about a given solution. We now describe the form of the equations linearized about either of the above relative equilibrium.

To describe the pendula configurations, project the position vectors q_1 and q_2 to the horizontal plane, producing planar vectors $q_1^\perp$ and $q_2^\perp$. These vectors have polar coordinates (r_1, θ_1) and (r_2, θ_2) relative to a fixed inertial frame. Notice that $(r_1, \theta_1), (r_2, \theta_2)$ give local coordinates on $Q = S^2_{l_1} \times S^2_{l_2}$.

As a result of conservation of angular momentum μ, one of the velocities (or



(a) Stretched out solution



(b) cowboy solution

Figure 1.4: The shape of two relative equilibria of the double spherical pendulum.

momenta) can be eliminated. The resulting system, being S^1 invariant, can be written in terms of the ***reduced variables*** (r_1, r_2, φ), where

$$\varphi = \theta_2 - \theta_1.$$

We shall return later to some general comments on reduction. However, it should already be clear that in terms of the variables (r_1, r_2, φ), the equations have a *fixed point* at a relative equilibria, so we can linearize the equations in standard fashion. (Notice that this process does ***not*** involve passing to a rotating frame of reference.)

In terms of the linearized variables $x = (\delta r_1, \delta r_2, \delta\varphi)$, the linearized equations have the form (as shown in Marsden and Scheurle [1993]):

$$M\ddot{x} + S\dot{x} + \Lambda x = 0,$$

where

$$M = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{12} & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix}$$

is symmetric and positive definite (depending on the masses, pendulum lengths, the shape of the relative equilibrium and the angular momentum μ),

$$S = \begin{pmatrix} 0 & 0 & S_{13} \\ 0 & 0 & S_{23} \\ -S_{13} & -S_{23} & 0 \end{pmatrix}$$

is skew, and

$$\Lambda = \begin{pmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

is symmetric. The zeros in these matrices are due to discrete symmetry. At equilibrium, $\varphi = 0$, which means the two pendula lie in a uniformly rotating vertical plane. Reflection of the system in this plane is a symmetry, and it is its action on M, S and Λ that forces the above zeros.

The term $S\dot{x}$ is called the ***Coriolis***, ***magnetic***, or ***gyroscopic*** term. It has appeared through the reduction process, even though the original Euler-Lagrange (or Hamilton) equations had no such terms explicitly. This term is a direct manifestation of the mechanical connection; in fact, S may, on the linearized level, be interpreted as the *curvature* of the mechanical connection.

We observe also that equations of the form $M\ddot{q} + S\dot{q} + \Lambda q = 0$ as above, can be interpreted as either Euler-Lagrange or Hamilton equations, but not in a completely standard way. We demonstrate this using structures that reflect the general nonlinear theory.

For the Hamiltonian structure, let $p = M\dot{q}$ and let

$$H = \frac{1}{2}p^T M^{-1} p + \frac{1}{2}q^T \Lambda q,$$

the sum of the kinetic and potential energies, and let the Poisson bracket of two functions $F(q,p)$ and $K(q,p)$ be given (using the summation convention) by

$$\{F, K\} = \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i} - S_{ij}\frac{\partial F}{\partial p_i}\frac{\partial K}{\partial p_j}.$$

One checks that this is a Poisson structure and that the equations $M\ddot{x}+S\dot{x}+\Lambda X = 0$ are equivalent to Hamilton's equations in Poisson bracket form: $\dot{F} = \{F, H\}$ for all F. Notice that the "curvature" S_{ij} enters directly in the bracket, but not in the Hamiltonian.

If one writes $S = \frac{1}{2}(A - A^T)$ for a matrix A (notice that A is not unique, which reflects a "gauge invariance"), then by replacing p by $\mathfrak{p} = p - Aq$, the bracket becomes the *canonical bracket* in $(q, \mathfrak{p})$ while the Hamiltonian becomes dependent on A. *Momentum shifts* of this sort play a key role in reduction theory, as we shall see shortly.

On the Lagrangian side, the analogue of the above structure is as follows. Let

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} - \frac{1}{2}q^T \Lambda q,$$

the difference of the kinetic and potential energies. The equations $M\ddot{q}+S\dot{q}+\Lambda q = 0$ are equivalent to the variational principle (over curves $q(t)$ with fixed endpoints):

$$\delta \int L(q, \dot{q})dt = \int \delta q^T S \dot{q},$$

as is readily checked. This variational principle is of the Lagrange-d'Alembert form with the gyroscopic forces (but that do no work!) appearing on the right hand side.

If the conserved energy quadratic form H is positive definite, one may conclude that the relative equilibrium is linearly *and* nonlinearly stable. Since M is positive definite, this can be tested by looking at the signature of Λ. For the straight out solution, the signature is $(+,+,+)$, so it is stable. For the cowboy solution, the signature is $(-,-,+)$, so the standard energy test for stability fails. In fact, the presence of the gyroscopic terms in the equations makes the conclusion of stability or instability more subtle. The system is (assuming no resonances) in fact, linearly stable for many system parameters, but most people would probably guess that the corresponding nonlinear system is unstable due to Arnold diffusion (being 3 degree of freedom, one cannot use KAM theory to conclude nonlinear stability). However, with the addition of internal (joint) dissipation, general theory tells us that the system becomes *linearly unstable*; see. Bloch, Krishnaprasad, Marsden and Ratiu [1992].

Something else quite interesting happens for the cowboy solution. As the angular momentum μ increases, eigenvalues (complex roots of $\det(\lambda^2 M + \lambda S + \Lambda) = 0$) split off the imaginary axis, creating a linear instability in a (generic) $1:1$ resonance bifurcation, as in Figure 1.5.

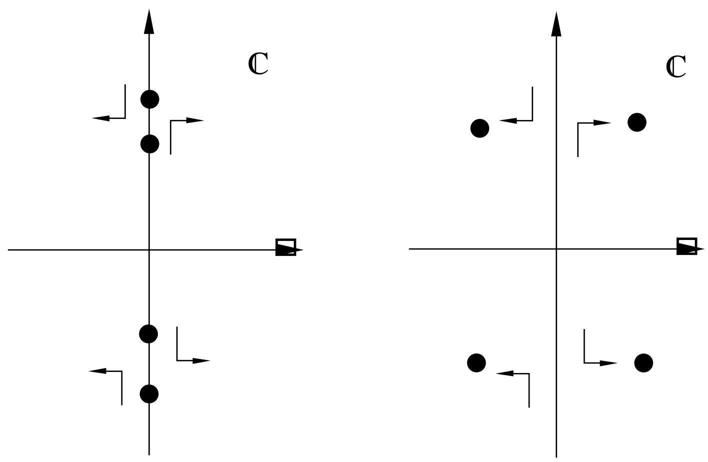


Figure 1.5: The eigenvalue movement in a 1 : 1 resonance bifurcation.

This type of bifurcation is in fact quite common and may be analyzed using normal forms, as in van der Meer [1985]. Keep in mind that this bifurcation occurs on the reduced space and that if one detects a periodic orbit (or a two torus) it yields a two torus (or a three torus) on the original phase space.

Next we turn to another aspect of the mechanical connection that comes up in rigid body dynamics. We consider a free rigid body and fix the center of mass at the origin. The position of the rigid body is given by a special orthogonal matrix $A \in SO(3)$ that maps the reference configuration $\mathcal{B}$ to the current configuration, as in Figure 1.6.

Letting $X \in \mathcal{B}$ and $x = AX$ be points in the fixed reference and moving current configurations respectively, we see that the velocity of a point is

$$\dot{x} = \dot{A}X = \dot{A}A^{-1}x,$$

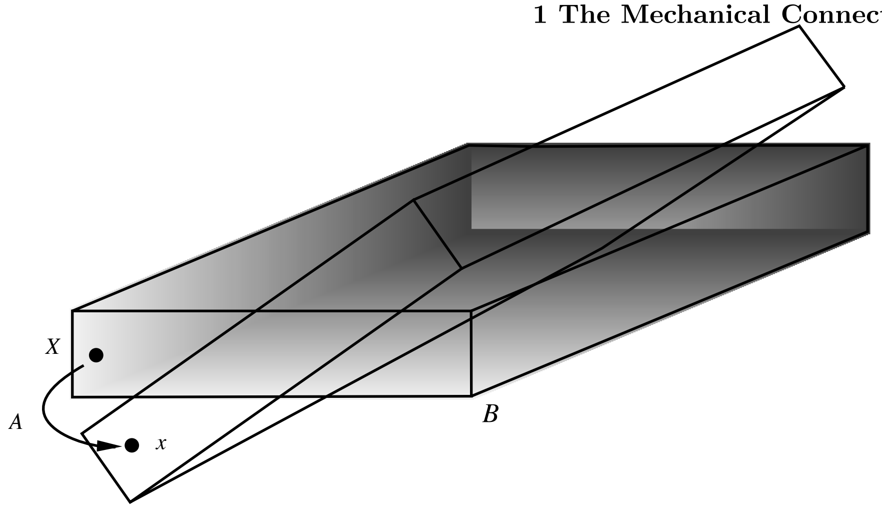


Figure 1.6: The rigid body attitude matrix.

where the dynamics is captured through the time dependence of A. Since A is orthogonal, $\dot{A}A^{-1}$ is skew, and so we can write, for all $v \in \mathbb{R}^3$,

$$\dot{A}A^{-1}v = \omega \times v,$$

defining the ***spatial angular velocity*** ω. The ***body angular velocity*** is defined by

$$\Omega = A^{-1}\omega.$$

The mass distribution of the reference body is captured by the moment of inertia tensor (defined as in calculus textbooks!), which we denote I. The ***body angular*** momentum is given by

$$\Pi = I\Omega$$

and the ***spatial angular momentum*** by

$$\pi = A\Pi.$$

Note that $\pi = (AIA^{-1})\omega$ and that $I_A = AIA^{-1}$ is the moment of inertia tensor of the current configuration.

Euler's famous equations for free rigid body dynamics are given by

$$\dot{\Pi} = \Pi \times \Omega.$$

These equations are again Hamiltonian in the sense that they are equivalent to $\dot{F} = \{F, H\}$, where the Hamiltonian is

$$H(\Pi) = \frac{1}{2}\Pi I^{-1}\Pi = \frac{1}{2}\pi I_A^{-1}\pi$$

and where the ***rigid body bracket*** is

$$\{F, K\}(\Pi) = -\Pi \cdot (\nabla F \times \nabla K).$$

This bracket is a special case of the ***Lie-Poisson*** bracket valid on the dual of any Lie algebra.

The spatial angular momentum is conserved in time:

$$\begin{aligned}\dot{\pi} &= (A\Pi)^{\cdot} = \dot{A}\Pi + A\dot{\Pi} \\ &= \dot{A}A^{-1}\pi + A(\Pi \times \Omega) \\ &= \omega \times \pi + \pi \times \omega = 0.\end{aligned}$$

In particular, the length $\|\pi\| = \|\Pi\|$ is conserved. In fact the functions

$$C(\Pi) = \varphi(\|\Pi\|^2)$$

for any function φ of one variable are not only conserved, they are ***Casimir functions*** for the rigid body bracket. That is,

$$\{C, K\} = 0$$

for any K.

The two invariants $\|\Pi\|$ and $H(\Pi)$ show that the trajectories of Euler's equations are given by the standard picture of intersecting a sphere and a family of ellipsoids, as in Figure 1.7.

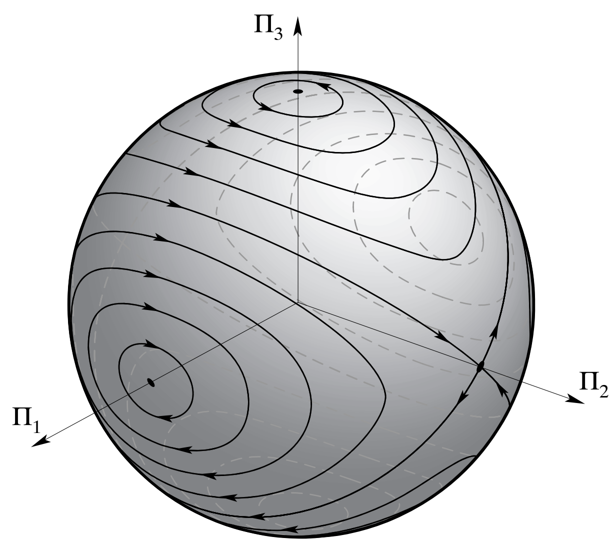


Figure 1.7: The rigid body angular momentum sphere.

Notice that most trajectories on the sphere are periodic. Now we ask the following question: *if* $\Pi(t)$ *goes through a periodic motion with period* T, *so that* $\Pi(T) = \Pi(0)$, *what is* $A(T)A(0)^{-1}$?

A beautiful answer to this question was given by Montgomery [1991]; see also Marsden, Montgomery and Ratiu [1990]. (The proof of Montgomery is based on Stokes theorem in the given phase space, but that of Marsden, Montgomery and

Ratiu is based directly on holonomy formulas from differential geometry). Namely, from the definitions and conservation of π, we see that

$$A(T)A(0)^{-1}\pi = \pi,$$

so $A(t)A(0)^{-1}$ is a rotation about the axis π through some angle, say $\Delta\theta$. The formula for $\Delta\theta$ is

$$\Delta\theta = -\Lambda + \frac{2ET}{\|\pi\|}$$

where Λ is the solid angle enclosed by the loop $\Pi(t)$, E is the energy and T is the period.

This formula can be interpreted in terms of holonomy. In fact, we can regard the set $\pi = \mu =$ constant as being a subset of $TSO(3)$ (containing elements $(A, \dot{A})$), or of its dual $T^*SO(3)$. As such, this set is diffeomorphic to $SO(3)$ (it is the graph of the right invariant one form whose value at the identity is μ). If we call $\mathbf{J} : T^*SO(3) \to \mathbb{R}^3$ the map whose value is π, then the set in question is $\mathbf{J}^{-1}(\mu)$, and we have a map

$$\rho : \mathbf{J}^{-1}(\mu) \to S^2$$

given by taking $(A, \dot{A}) \in \mathbf{J}^{-1}(\mu)$ and producing the corresponding Π. The map ρ is, in fact, the ***Hopf fibration***. There is a natural connection on the bundle ρ, namely (a multiple of) the canonical one form on $T^*SO(3)$, restricted to $\mathbf{J}^{-1}(\mu)$. Its curvature gives the area form on the sphere.

The holonomy of the loop $\Pi(t)$ for this connection is exactly the term Λ in the formula for $\Delta\theta$.

These two instances of the connection are in fact related — one can view the rigid body bracket as ***all curvature*** whereas the bracket in the reduced double spherical pendulum reflects the more general case of a mixture of canonical and curvature. Reduction theory sorts out these special cases into a unified scheme. The mechanical connection is the concept that puts these two instances of a connection into a common framework.

We also wish to point out that the formula for $\Delta\theta$ is useful for a variety of other problems, such as attitude shifts due to internal moving parts (falling cats, satellites with internal rotors or flexible appendages, etc.). It also is very useful in the framework of attitude control, as we shall hint at in §**3**.

The Definition of the Mechanical Connection First of all, let $\mathfrak{g}$ denote the Lie algebra of G, and for $\xi \in \mathfrak{g}$, let ξ_Q denote the infinitesimal generator on Q, so ξ_Q is a vector field on Q. In coordinates $q^i, i = 1, \ldots, n$ on Q, write $\xi_Q = \xi^i_Q \partial/\partial q^i$ and

$$\xi^i_Q(q) = A^i{}_a(q)\xi^a$$

where $\xi = \xi^a e_a$ relative to a choice of basis for $\mathfrak{g}$. The ***locked inertia tensor*** $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ is defined by

$$\langle \mathbb{I}(q) \cdot \xi, \eta \rangle = \langle\langle \xi_Q(q), \eta_Q(q) \rangle\rangle.$$

Note that as a bilinear form $\mathbb{I}(q)$ is positive definite when $\xi \mapsto \xi_Q(q)$ is injective; *i.e.,* the action is locally free. In this case also, $\mathbb{I}(q)$ is invertible. In coordinates,

$$\mathbb{I}_{ab} = g_{ij} A^i{}_a A^j{}_b$$

where g_{ij} is the metric tensor.

For systems such as coupled rigid bodies or rigid bodies with flexible attachments, where $G = SO(3), \mathbb{I}(q)$ is a 3×3 matrix, which represents the moment of inertia tensor for the rigid body obtained by locking the joints (or the internal degrees of freedom) in the configuration q. For the ordinary rigid body and using our earlier notation, $\mathbb{I}(A) = AIA^{-1}$.

The case in which $\mathbb{I}(q)$ is invertible will be studied here; the singular case (where $\mathbb{I}(q)$ is a singular matrix) requires special attention. For example, the straight down state of the double spherical pendulum is such a case.

The ***momentum map*** in our context is defined to be the map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ given by

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle$$

or in coordinates, by

$$J_a(q,p) = p_i A^i{}_a(q).$$

Recall that $\mathbf{J}$ will be a constant of the motion for our G-invariant system (Lagrangian on TQ or Hamiltonian on T^*Q), which is one version of Noether's theorem.

Define the ***locked angular velocity*** α by analogy with what we do for a rigid body:

$$\alpha(q,v) = \mathbb{I}(q)^{-1}\mu$$

where $\mu = \mathbf{J}(\mathbb{F}L(q,v))$ and where $\mathbb{F}L : TQ \to T^*Q$ is the Legendre transform. In coordinates,

$$\alpha^a = \mathbb{I}^{ab} g_{ij} A^i{}_b v^j,$$

where $\mathbb{I}^{ab}$ is the inverse matrix of $\mathbb{I}_{ab}$. View α as a map α: $TQ \to \mathfrak{g}$. As such, we also call α the ***mechanical connection***. It was Kummer [1981] who first pointed out that α indeed can be viewed as a connection on the bundle $Q \to Q/G$ regarded as a principal G-bundle, that is, α is equivariant and $\alpha(\xi_Q(q)) = \xi$. This latter property can also be interpreted as follows. Let $\mu \in \mathfrak{g}^*$ be fixed and define $\alpha_\mu(q) \in T^*Q$ by

$$\langle \alpha_\mu(q), v \rangle = \langle \alpha(v), \mu \rangle,$$

so α_μ is a one form on Q. Then $\alpha_\mu \in \mathbf{J}^{-1}(\mu)$. We shall need this remark below. In coordinates,

$$\alpha_\mu = \mu_a \mathbb{I}^{ab} g_{ij} A^i{}_b dq^j.$$

The ***magnetic term*** β_μ is defined on Q by

$$\beta_\mu = \mathbf{d}\alpha_\mu.$$

It plays an important role in the next section.

We refer to Smale [1970] and Abraham and Marsden [1978] for characterizations of α_μ that are equivalent to the one given here.

Here is a direct link with our discussion of the rigid body. If $G = Q$, then α_μ is independent of the metric, and equals the right invariant form on G whose value at the identity is μ. This is easily checked, and it was used in this form in Marsden and Weinstein [1974].

2 Reduction, Stability and Bifurcation

In the last section, we defined the mechanical connection $\alpha : TQ \to \mathfrak{g}$, which can be viewed as completing the commutative diagram

$$\begin{array}{ccc} T^*Q & \xrightarrow{\ \mathbf{J}\ } & \mathfrak{g}^* \\ \uparrow{\scriptstyle \mathbb{F}L} & & \uparrow{\scriptstyle \mathbb{I}(q)} \\ TQ & \xrightarrow[\ \alpha\ ]{} & \mathfrak{g} \end{array}$$

For example, for the double spherical pendulum, one checks that $\alpha : TQ \to \mathbb{R}$ is given by

$$\alpha = \frac{\mathbf{k} \cdot [m_1 q_1 \times v_1 + m_2(q_1 + q_2) \times (v_1 + v_2)]}{m_1 \|q_1^\perp\|^2 + m_2 \|(q_1 + q_2)^\perp\|^2}$$

where $(q_i, v_i) \in TS^2_{l_i}$ gives the positions and velocities of the two pendulum bobs, with masses m_1 and m_2, and where, as above, $q^\perp$ denotes the projection of the vector q to the horizontal plane.

For the ozone molecule, we first eliminate the translation subgroup so that $G = SO(3)$ remains. With the ***Jacobi coordinates*** r and s defined as in Figure 2.1, we have

$$\alpha = \frac{\dot{r} \cdot n}{\|s\|}\hat{r} + \frac{\dot{s} \cdot n}{\|r\|}\hat{s} + \frac{1}{\gamma}\left\{ \frac{1}{2}\|r\|^2 s \cdot \dot{r} - \frac{2}{3}\|s\|^2 r \cdot \dot{s} \right\} n$$

where

$$\hat{r} = \frac{r}{\|r\|}, \quad \gamma = \frac{1}{2}\|r\|^2 + \frac{2}{3}\|s\|^2 \quad \text{and} \quad n = \frac{r \times s}{\|r\|\|s\|},$$

provided r and s are perpendicular. (The formula in the general case is a little more complicated.)

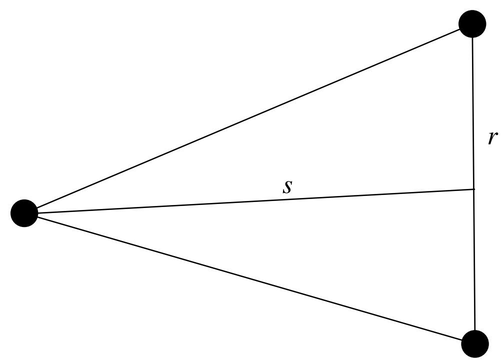


Figure 2.1: Jacobi coordinates

Cotangent Bundle Reduction The mechanical connection plays an important role in the reduction process. Here we form the ***reduced space*** (again, staying away from singularities)

$$P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$$

where $G_\mu = \{g \in G \mid g \cdot \mu = \mu\}$ is the isotropy subgroup for the coadjoint action of G on $\mathfrak{g}^*$, at μ. By a general theorem of Marsden and Weinstein [1974] (see also Meyer [1973]), P_μ inherits a symplectic structure. Also, any G invariant Hamiltonian system on T^*Q drops to one on P_μ.

After reduction, one would like to know how to *reconstruct* the dynamics back on the original phase space. It is here that geometric phases and holonomy come in. We shall remark on this below, but for the general theory, the reader should consult Marsden, Montgomery and Ratiu [1990].

In many examples, such as the double spherical pendulum and the ozone molecule, we work not in the general context of symplectic manifolds, but in the cotangent bundle context. Thus, in this case, it is good to know to what extent P_μ is a cotangent bundle. The cotangent bundle reduction theorem answers this.

There are two versions of the cotangent bundle reduction theorem. The first (see Abraham and Marsden [1978]) states that there is a symplectic embedding

$$(T^*Q)_\mu \to T^*(Q/G_\mu),$$

where $(T^*Q)_\mu$ embeds as a symplectic subbundle. However, $T^*(Q/G_\mu)$ does not carry the canonical symplectic structure, but rather it carries the symplectic two form

$$\Omega_\mu = \Omega_{\text{canonical}} + \beta_\mu$$

where β_μ is the two form on Q/G_μ induced by $\mathbf{d}\alpha_\mu$. Here it is important to note that α_μ does not give a well defined one form on Q/G_μ, but $\mathbf{d}\alpha_\mu$, its exterior derivative, does.

The key step in proving the above is to choose a point $\alpha_q \in \mathbf{J}^{-1}(\mu)$ and shift it by α_μ; $\alpha_q \mapsto \alpha_q - \alpha_\mu(q)$, which produces a point in $\mathbf{J}^{-1}(0)$ since $\alpha_\mu(q) \in \mathbf{J}^{-1}(\mu)$. This is the map that readily drops to the quotient and produces the desired embedding.

For example, for the ordinary rigid body, $Q = G = SO(3), G_\mu = S^1$ (rotations about the axis μ) and $(T^*Q)_\mu$ embeds in $T^*(SO(3)/S^1) = T^*(S^2)$ as the zero section. In this case then, one recovers the body angular momentum sphere and the dynamics on it. In general, for $Q = G, (T^*Q)_\mu$ is a coadjoint orbit, realized in the zero section of $T^*(G/G_\mu) = T^*(Q_\mu)$. In this case, the reduced symplectic form is "all magnetic", since $\Omega_{\text{canonical}}$ vanishes on the zero section.

When G is abelian, $G = G_\mu$ and we have equality

$$(T^*Q)_\mu = T^*(Q/G),$$

but the magnetic term is still present in general.

For example, for the double spherical pendulum, $(T^*q)_\mu = T^*((S^2 \times S^2)/S^1)$ is 6-dimensional (but it has two singularities!), and the magnetic term is the 2-form associated with the 3×3 matrix S described earlier — it is 3×3 since β_μ is a 2-form on configuration space. For the associated Poisson brackets, one adds the term

$$\beta_{ij} \frac{\partial F}{\partial p_i} \frac{\partial K}{\partial p_j}$$

to the canonical bracket of F and K, as was also mentioned earlier.

The other way of looking at the cotangent bundle reduction theorem is to view $(T^*Q)_\mu$ as a coadjoint orbit bundle over $T^*(Q/G)$, as in Montgomery, Marsden and Ratiu [1984] and Montgomery [1986]. For example, for the ozone molecule one finds that $(T^*Q)_\mu$ is an S^2-bundle over $T^*(Q/G)$ where ***shape space*** Q/G is parametrized by $\|r\|, \|s\|$ and $r \cdot s$.

Lagrangian Reduction The above constructions emphasize the Hamiltonian point of view. On the Lagrangian side we proceed as follows (following Marsden and Scheurle [1993]).

Define the ***Routhian*** by

$$R^\mu = L - \alpha_\mu$$

where

$$L(q, v) = \frac{1}{2}\|v\|^2 - V(q)$$

is the Lagrangian. A straightforward calculation shows that in terms of the Routhian, the standard variational principle

$$\delta \int L dt = 0$$

can be written

$$\delta \int R^\mu dt = \int \beta_\mu(\dot{q}, \delta q)$$

for solutions with $\mathbf{J} = \mu$. Here β_μ is the magnetic term as defined in §**1**. The following algebraic identity for R^μ is useful:

$$R^\mu(q, v) = \frac{1}{2}\|\mathrm{hor}(q, v)\|^2 - V_\mu(q)$$

where

$$\mathrm{hor}(q, v) = v - (\alpha(q, v))_Q$$

is the ***horizontal projection*** for the connection α and where

$$\begin{aligned} V_\mu(q) &= V(q) + \frac{1}{2}\langle \mu, \mathbb{I}(q)^{-1}\mu\rangle \\ &= V(q) + \frac{1}{2}\mathbb{I}^{ab}\mu_a\mu_b \end{aligned}$$

is the ***amended potential***.

Because of this identity for R^μ, and because β_μ naturally drops to Q/G_μ, we see that our variational principle in terms of R^μ drops to Q/G_μ. A key point is that because R^μ depends on v only through its horizontal part, we can relax the fixed endpoint boundary conditions to the condition that the endpoints lie on G_μ-orbits.

Thus, if one adopts the variational principle in the sense of Lagrange and d'Alembert, then Lagrangian reduction is a natural counterpart to the cotangent bundle reduction theorem. For G abelian, this corresponds to the classical Routh procedure.

One caution is noteworthy, however. In general, our reduced variational principle will be degenerate. This degeneracy occurs precisely when G is *not* abelian and will introduce additional constraints in the sense of Dirac. Interestingly, these constraints correspond exactly to the restriction of the bundle $T(Q/G_\mu)$ to its symplectic subbundle $(T^*Q)_\mu$ (where we identify vectors and covectors by the Legendre transformation).

As an extreme case, consider again the rigid body. Here the Routhian is independent of v altogether, and the variational principle becomes

$$\delta \int V_\mu(q)dt = \int \beta_\mu(\dot{q}, \delta q)$$

which are the *first-order* Euler equations on $Q/G_\mu = S^2$.

It is worthwhile to note that there is also a theory of Lagrangian reduction that does not set the momentum map equal to a constant. In this respect, the theory is the Lagrangian analogue of Poisson reduction on the Hamiltonian side. In particular, when one reduces a Lagrangian system on TG for a Lie group G, one gets the Euler-Poincaré equations for a Lagrangian on a Lie algebra, which are related to the Lie-Poisson equations on the dual of the Lie algebra by a Legendre transformation. One of the most interesting aspects of this is the way that the Euler-Poincaré equations couple to the equations for internal variables through curvature terms. We refer to Marsden and Scheurle [1993a] for details.

Relative Equilibria and Stability In the general setting, ***relative equilibria*** are dynamic solutions that are also one parameter group orbits. In our context of cotangent bundles, one shows that relative equilibria with $\mathbf{J} = \mu$ are critical points of V_μ.

For example, for the double spherical pendulum,

$$\begin{aligned} V_\mu(q_1, q_2) &= m_1 g q_1 \cdot k + m_2 g (q_1 + q_2) \cdot k \\ &\quad + \frac{1}{2} \frac{\mu^2}{m_1 \|q_1^\perp\|^2 + m_2 \|(q_1 + q_2)^\perp\|^2}. \end{aligned}$$

A study of the critical points of V_μ leads to the cowboy and straight out solutions mentioned earlier. One has, in terms of (r_1, r_2, φ),

$$\begin{aligned} V_\mu &= -m_1 g_1 \sqrt{l_1^2 - r_1^2} - m_2 g \left(\sqrt{l_1^2 - r_1^2} + \sqrt{l_2^2 - r_2^2} \right) \\ &\quad + \frac{1}{2} \frac{\mu^2}{(m_1 + m_2) r_1^2 + m_2 r_2^2 + 2 m_2 r_1 r_2 \cos\varphi}. \end{aligned}$$

To study stability, one computes $\delta^2 V_\mu$; it is a little complicated, but due to the discrete symmetries, as mentioned before, it has the form

$$\delta^2 V_\mu = \begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

and leads to our earlier assertions about the signature.

The nonabelian case is more complicated but still is quite interesting and structured.

One splits the space of variations of (a concrete realization of) Q/G_μ into variations in G/G_μ and variations in Q/G. With the appropriate splitting, one gets the block diagonal structure

$$\delta^2 V_\mu = \begin{bmatrix} \text{Arnold form} & 0 \\ & \\ 0 & \text{Smale form} \end{bmatrix}$$

where the Arnold form means $\delta^2 V_\mu$ computed on the tangent space to the coadjoint orbit $O_\mu \cong G/G_\mu$, and the Smale form means $\delta^2 V_\mu$ computed on Q/G. This method turns out to be a powerful one when applied to specific systems such as spinning satellites with flexible appendages. These results are part of the ***energy-momentum method*** of Simo, Posbergh and Marsden [1990] and Simo, Lewis and Marsden [1991].

Perhaps even more interesting is the structure of the linearized *dynamics* near a relative equilibrium. That is, both the ***augmented Hamiltonian*** $H_\xi = H - \langle J, \xi \rangle$ *and* the symplectic structure can be *simultaneously* brought into the following normal form:

$$\delta^2 H_\xi = \begin{bmatrix} \text{Arnold form} & 0 & 0 \\ & & \\ 0 & \text{Smale form} & 0 \\ & & \\ 0 & 0 & \text{Kinetic Energy} > 0 \end{bmatrix}$$

and

$$\text{Symplectic Form} = \begin{bmatrix} \text{coadjoint orbit form} & * & 0 \\ -* & \text{magnetic (coriolis)} & I \\ 0 & -I & 0 \end{bmatrix}$$

where the columns represent the ***coadjoint orbit variables*** (G/G_μ), the ***shape variables*** (Q/G) and the ***shape momenta*** respectively. The term $*$ is an ***interaction term*** between the group variables and the shape variables. The magnetic term is the curvature of the μ-component of the mechanical connection, as we described earlier.

For $G = SO(3)$, this form captures all the essential features in a well-organized way: centrifugal forces in V_μ, coriolis forces in the magnetic term and the interaction between internal and rotational modes. In fact in this case, the splitting of variables solves an important problem in mechanics: *how to efficiently separate rotational and internal modes near a relative equilibrium.*

Now suppose that we have a compact discrete group Σ acting by isometries on Q, and preserving the potential. This action lifts to the cotangent bundle. The resulting fixed point space is the cotangent bundle of the fixed point space Q_Σ. This fixed point space represents the Σ-symmetric configurations.

This action also gives an action on the quotient space, or shape space Q/G. We can split the tangent space to Q/G at a configuration corresponding to the relative equilibrium according to this discrete symmetry. Here, one must check that the amended potential is invariant under Σ. In general, this need not be the case, since the discrete group need not leave the value of the momentum μ invariant. However, there are two important cases for which this is verified. The first is for $SO(3)$ with $\mathbb{Z}_2$ acting by conjugation, where it maps μ to its negative, so in this case, from the formula $V_\mu(q) = V(q) - \mu \mathbb{I}(q)^{-1}\mu$ we see that indeed V_μ is invariant. The second case, which is relevant for the water molecule, is when Σ acts trivially on G. Then Σ leaves μ invariant, and so V_μ is again invariant. Under one of these assumptions, one finds that the Smale form block diagonalizes, which we refer to as the ***subblocking property***. The blocks in the Smale form are the Σ-symmetric variations, and their complement chosen to be as the annihilator of the symmetric dual variations. This can also be applied to the symplectic form, showing that it subblocks as well.

For the ozone (or water) molecule in a symmetric configuration $(r \perp s)$ one finds

$$\delta^2 V_\mu = \begin{bmatrix} \alpha & \beta & 0 & 0 & 0 \\ \beta & \delta & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & b & d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

where $\begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix}$ is the 2×2 Arnold block, corresponding to the ***rigid variations*** and the ***internal variations*** split into ***symmetric variations*** with the block

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

and the scalar e corresponding to the non-symmetric internal variations.

In this case, the linearized equations take on the general form $M\ddot{q} + S\dot{q} + \Lambda q = 0$ before, except that now these equations for the *internal variables* get *coupled* to the rigid body equations for the *rigid variables*.

Finally, a word about singular states. For the double spherical pendulum such a state is the straight down state. The reduced linearized equations become singular as $\mu \to 0$. One can regularize them by rescaling the variables and defining a new Lagrangian for the linearized equations by

$$\begin{aligned} & L_2(\delta r_1, \delta r_2, \delta\varphi, \delta\dot{r}_1, \delta\dot{r}_2, \delta\dot{\varphi}) \\ &= \lim_{\mu \to 0} \frac{1}{\mu} L_2^\mu(\sqrt{\mu}\delta r_1, \sqrt{\mu}\delta r_2, \delta\varphi, \sqrt{\mu}\delta\dot{r}_1, \sqrt{\mu}\delta\dot{r}_2, \delta\dot{\varphi}), \end{aligned}$$

where L_2^μ is the Lagrangian for the linearized equations at a relative equilibrium with $\mu \neq 0$. This regularization shows interesting bifurcation behavior in the straight down state, as in Dellnitz, Marsden, Melbourne and Scheurle [1992]. In particular, both *splitting* (as in Figure 1.5) and *passing* generically occur.

3 Geometric Phases and Control

To get the idea of geometric phases, we consider some simple examples. Already the rigid body example in §**1** is one basic example, and we will return to this type of example in the context of control, at the end of the section.

Three Basic Examples of Phases The first example consists of two planar rigid bodies connected at their centers of mass by a pin joint. Imagine a torque can be exerted at the joint so that the two bodies can rotate relative to one another, but that the total angular momentum is zero. See Figure 3.1.

If we let I_1 and I_2 be the moments of inertia of the two bodies and θ_1 and θ_2 the angles they make with respect to an inertial frame, then conservation of angular momentum gives

$$I_1\dot{\theta}_1 + I_2\dot{\theta}_2 = 0.$$

Letting $\psi = \theta_2 - \theta_1$ be the angle between the bodies, we get

$$(I_1 + I_2)d\theta_1 + I_2 d\psi = 0$$

and so if ψ goes through an angle $2\pi k$ (*i.e.,* if the joint motor causes k revolutions) then

$$\Delta\theta_1 = -\frac{I_2}{I_1 + I_2} \cdot 2\pi k$$

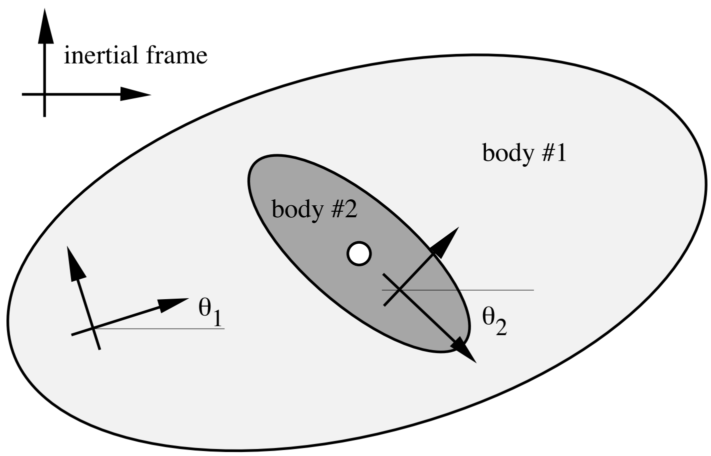


Figure 3.1: Two planar rigid bodies coupled at their centers of mass by a pin joint

gives the corresponding phase shift on the "large" body. This example shows how one can reorient body #1 by using a motor to spin body #2. Of course one would like to understand similar phenomena for a rigid body in 3-space with various internal variables, such as rotors.

A second example illustrates the role of slowly varying system parameters. This example, due to Hannay and Berry, considers a bead free to slide on a closed loop of wire in the plane. Imagine that the bead travels around the loop for time T. Now we repeat this situation except while the bead is in motion, we *slowly* (adiabatically) rotate the hoop (we move the whole system) through 360° and compare the two resulting positions. One finds a shift in position by a distance $4\pi A/L$ where A is the area enclosed by the hoop and where L is the length of the hoop. This quantity is purely geometric and again can be seen as a holonomy (but there are some subtleties; see Marsden, Montgomery and Ratiu [1990]).

A third example is the familiar Foucault pendulum. It is well known that the angular shift in the plane of the pendulum is given by $2\pi \cos\alpha$ where α is the colatitude. One can check that this equals the holonomy for parallel translation of an orthonormal frame around this corresponding line of latitude; see Figure 3.2 for the standard illustration of this holonomy. Establishing this geometric link using fundamental principles requires some work (Montgomery [1988], Marsden, Montgomery and Ratiu [1990]).

Generalities on Geometric Phases and Control The general procedure is to first consider the case in which no external parameters are varied. Here, one considers the reduction bundle

$$\mathbf{J}^{-1}(\mu) \to P_\mu$$

with group G_μ and uses the mechanical connection to induce a connection on this bundle. (As we mentioned before, in the special case $G = SO(3) = Q$, one gets the canonical one form.) The holonomy of this connection gives the geometric phase.

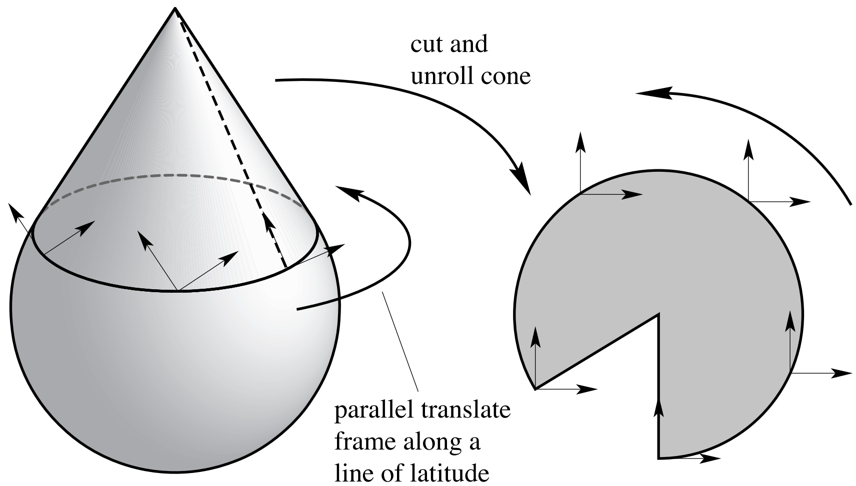


Figure 3.2: Holonomy for the Foucault pendulum.

This construction presupposes the phase corresponds to a closed loop in P_μ. However, there is evidence that phases can be well defined and meaningful even when the dynamics on P_μ is chaotic. This is an interesting topic for further research.

For moving systems, one adds to the mechanical connection a “Cartan term” encoding the coriolis, centrifugal and Euler forces due to the movement. Then averaging produces a new connection, the ***Cartan-Hannay-Berry connection*** whose holonomy gives the geometric phase.

Let us return to the non-adiabatic situation, but add the complication of controlling the internal variables. Perhaps one wants to manipulate the internal variables having certain attitude objectives in mind.

Suppose one wants to control the internal variables on Q/G with a holonomy on Q prescribed. Here we are thinking of $Q \to Q/G$ as a principle bundle and our mechanical connection is defined by declaring horizontal to be orthogonal to the G-orbits, as in Figure 3.3.

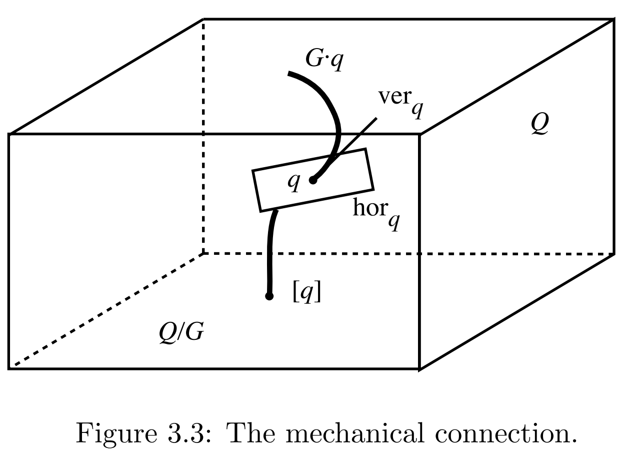


Figure 3.3: The mechanical connection.

The orthogonality condition means that horizontal curves in Q have zero angular momentum.

This bundle is a possible arena for the dynamics of "colored" particles. That is, it makes sense to discuss the motion of particles in Q/G in the presence of the gauge field α, the mechanical connection. These general equations, called ***Wong's equations*** (see Montgomery [1984] and references therein) have the Hamiltonian, interestingly, given by the kinetic part of the Routhian; namely

$$H_{\text{Wong}} = \frac{1}{2}\|\text{hor}(q, v)\|^2,$$

but regarded as a function on $T(Q/G)$. These equations reduce to the Lorentz force law for a particle in a magnetic field when $G = S^1$.

We recall that for constant magnetic fields, charged particles move in circles. Many control laws (such as those for car and satellite parking) involve repeated circular excursions in the internal variables as well, and this is undoubtedly not an accident.

Indeed, a remarkable result of Wilczeck, Shapere and Montgomery (see Montgomery [1990]) shows that the optimal (using the energy cost function) trajectory in Q/G achieving a given holonomy is indeed the path of a particle moving in the connection field of the mechanical connection. This link between optimal control and Yang-Mills particles is one of the surprising results joining these two apparently unrelated areas of research.

Stabilization of a rigid body with internal rotors We consider a rigid body with internal rotors as in Figure 3.4. For concreteness, suppose there are 3 rotors and that they are driven by rotor torques.

Let us first set up the notation. Let

- $\mathbb{I}_{\text{body}}$ = inertia tensor of carrier
- $\mathbb{I}_{\text{rotor}}$ = diagonal matrix of rotor inertias
- $\mathbb{I}_{\text{lock}}$ = locked inertia tensor
- Ω = carrier body angular velocity
- Ω_r = vector of rotor angular velocities
- $m = \mathbb{I}_{\text{lock}}\Omega + \mathbb{I}_{\text{rotor}}\Omega_r$ = momentum conjugate to Ω
- $l = \mathbb{I}_{\text{rotor}}(\Omega + \Omega_r)$ = momentum conjugate to Ω_r

The equations of motion are

$$\begin{aligned} \dot{m} &= m \times \Omega \\ \dot{l} &= u, \end{aligned}$$

where u is the vector of rotor torques.

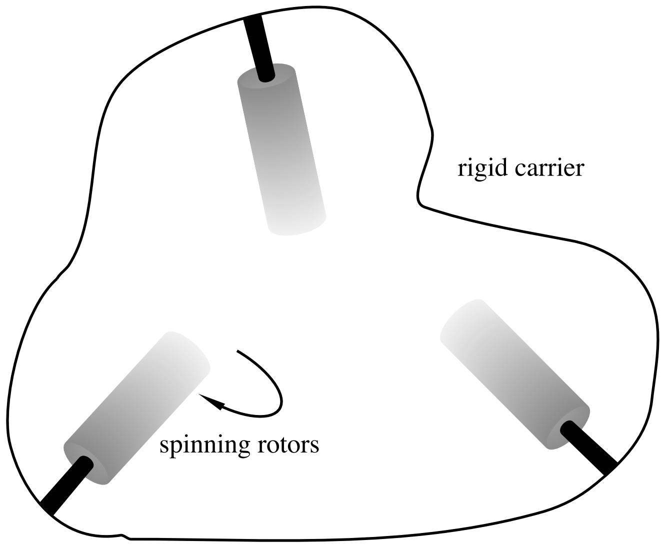


Figure 3.4: A rigid body with rotors.

Following Bloch, Krishnaprasad, Marsden and Sanchez [1992], one can ask: can we manipulate the rotors with a control law so that an otherwise unstable motion stabilizes?

To answer this question, let us specialize further so we consider motion about the middle axis and place a rotor along the third axis, as in Figure 3.5.

Let us also specialize the notation. Let

- $I_1 > I_2 > I_3$ be the rigid body moments of inertia
- $J_1 = J_2$ and J_3 be the rotor moments of inertia
- Ω be the body angular velocity of the rigid body (the carrier)
- α be the rotor angle relative to the carrier
- $m_1 = \lambda_1 \Omega_1$ where $\lambda_1 = J_1 + I_1$
- $m_2 = \lambda_2 \Omega_2$ where $\lambda_2 = J_2 + I_2$
- $m_3 = \lambda_3 \Omega_3 + J_3 \dot{\alpha}$ where $\lambda_3 = J_3 + I_3$
- $l_3 = J_3(\Omega_3 + \dot{\alpha})$

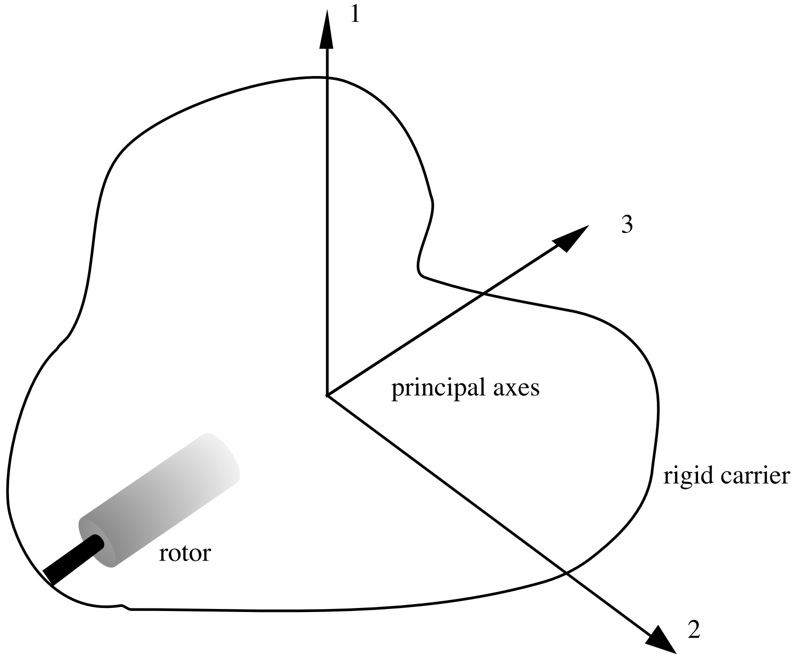


Figure 3.5: Rigid body with one rotor along the third axis.

Then the equations of motion become

$$\begin{aligned}
\dot{m}_1 &= m_2 m_3 \left(\frac{1}{I_3} - \frac{1}{\lambda_2} \right) - \frac{l_3 m_2}{I_3} \\
\dot{m}_2 &= m_1 m_3 \left(\frac{1}{\lambda_1} - \frac{1}{I_3} \right) + \frac{l_3 m_1}{I_3} \\
\dot{m}_3 &= m_1 m_2 \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) := m_1 m_2 a_3 \\
\dot{l}_3 &= u.
\end{aligned}$$

Now we choose the feedback control law

$$u = k m_1 m_2 a_3$$

where k is a (gain) parameter. This law is chosen because using it still makes the whole system Hamiltonian (in fact, these forces are the curvature of a certain feedback connection!) *and* retains a symmetry, so leads to the conservation law

$$l_3 - k m_3 = p,$$

a constant. We can then perform ordinary reduction, by noting p is a conserved quantity for a (k-dependent) $\mathbb{R}$-action and we can eliminate the rotor angle α and the momentum l_3 by substitution. The equations we get are

$$\begin{aligned}\dot{m}_1 &= m_2\left(\frac{(1-k)m_3 - p}{I_3}\right) - \frac{m_3 m_2}{\lambda_2}\\ \dot{m}_2 &= -m_1\left(\frac{(1-k)m_3 - p}{I_3}\right) + \frac{m_1 m_3}{\lambda_1}\\ \dot{m}_3 &= a_3 m_1 m_2.\end{aligned}$$

A study of this reduction, or a direct calculation leads to

Proposition 3.1. *The preceding equations are Hamiltonian relative to the rigid body bracket*

$$\{F, K\}(m) = -m \cdot (\nabla F \times \nabla K)$$

and the Hamiltonian

$$H = \frac{1}{2}\left(\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda_2} + \frac{[(1-k)m_3 - p]^2}{(1-k)I_3}\right) + \frac{1}{2}\frac{p^2}{J_3(1-k)}.$$

If one prefers, one can do a momentum shift by p and put the gyroscopic effects in the bracket. One can also do the whole procedure on the Lagrangian side, and use the theory of Lagrangian reduction. (The details of how to do this will be given in a forthcoming publication).

For $k = 0$, there are no torques and one recovers the case of a free spinning rotor. For $k = J_3/\lambda_3$ one gets the dual spin case where the rotor is constrained to rotate at constant angular velocity; see Krishnaprasad [1985] and Sanchez [1989].

The stabilization works as follows:

Proposition 3.2. *Consider $p = 0$ and the relative equilibrium $(0, M, 0)$. If*

$$k > 1 - \frac{J_3}{\lambda_2},$$

then $(0, M, 0)$ is stable.

Roughly, the flows on the momentum spheres change as in Figure 3.6.

This proposition is readily proved by the energy-momentum or energy-Casimir method. We look at $H + C$ where $C = \varphi(\|m\|^2)$, pick φ so that

$$\delta(H + C)\mid_{(0,M,0)} = 0$$

and show that $\delta^2(H + C)$ is negative definite if $k > 1 - (J_3/\lambda_2)$ and $\varphi''(M^2) < 0$.

Finally, we observe that this problem has attitude phase formulas similar to those for the free rigid body. We refer the reader to Bloch et al. [1992] for details.

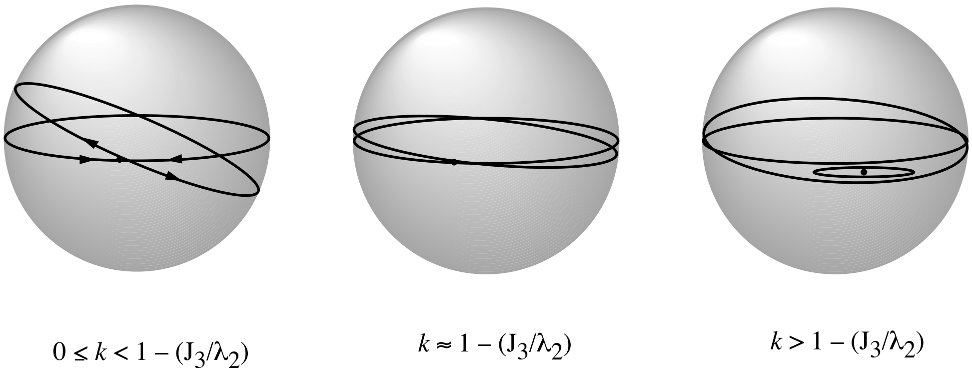


Figure 3.6: Flows on S^2 for various gains.

References

Abraham, R. and J.E. Marsden [1978] *Foundations of Mechanics.* Second Edition, Addison-Wesley Publishing Co., Reading, Mass..

Berry, M. [1985] Classical adiabatic angles and quantal adiabatic phase, *J. Phys. A. Math. Gen.* **18**, 15–27.

Berry, M. and J. Hannay [1988] Classical non-adiabatic angles, *J. Phys. A. Math. Gen.* **21**, 325–333.

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden and T.S. Ratiu [1994] Dissipation induced instabilities, *Ann. Inst. H. Poincaré* (to appear), and The Euler-Poincaré equations and double bracket dissipation (preprint).

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden and G. Sánchez de Alvarez [1992] Stabilization of rigid body dynamics by internal and external torques, *Automatica* **28**, 745–756.

Dellnitz, M., J.E. Marsden, I. Melbourne and J. Scheurle [1992] Generic bifurcations of pendula, *Int. Series on Num. Math.* ed. by G. Allgower, K. Böhmer and M. Golubitsky, Birkhaüser, Boston, **104**, 111-122.

Dellnitz, M., I. Melbourne and J.E. Marsden [1992] Generic bifurcation of Hamiltonian vector fields with symmetry, *Nonlinearity* **5**, 979–996.

Guichardet, A. [1984] On rotation and vibration motions of molecules, *Ann. Inst. H. Poincaré* **40**, 329–342.

Iwai, T. [1987] A geometric setting for classical molecular dynamics, *Ann. Inst. Henri Poincaré, Phys. Th.* **47**, 199–219.

Krishnaprasad, P.S. [1985] Lie-Poisson structures, dual-spin spacecraft and asymptotic stability, *Nonlinear Analysis, Theory, Methods, and Applications* **9**, 1011–1035.

Kummer, M. [1981] On the construction of the reduced phase space of a Hamiltonian system with symmetry, *Indiana Univ. Math. J.* **30**, 281–291.

Marsden, J.E. [1992] *Lectures on Mechanics.* London Mathematical Society Lecture Notes, **174**, Cambridge University Press.

Marsden, J.E., R. Montgomery and T.S. Ratiu [1990] *Reduction, symmetry, and phases in mechanics.* Memoirs AMS **436**

Marsden, J.E. and J. Scheurle [1993] Lagrangian reduction and the double spherical pendulum, *ZAMP* **44**, 17–43, and the reduced Euler-Lagrange equations, *Fields Institute Communications*, to appear.

Marsden, J.E. and A. Weinstein [1974] Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.* **5**, 121–130.

Meyer, K.R. [1973] Symmetries and integrals in mechanics, in *Dynamical Systems*, M. Peixoto (ed.), Academic Press, 259–273.

Montgomery, R. [1984] Canonical formulations of a particle in a Yang-Mills field, *Lett. Math. Phys.* **8**, 59–67.

Montgomery, R. [1986] The bundle picture in mechanics, *Thesis, UC Berkeley.*

Montgomery, R. [1988] The connection whose holonomy is the classical adiabatic angles of Hannay and Berry and its generalization to the non-integrable case, *Comm. Math. Phys.* **120**, 269–294.

Montgomery, R. [1990] Isoholonomic problems and some applications, *Comm. Math Phys.* **128**, 565–592.

Montgomery, R. [1991] How much does a rigid body rotate? A Berry's phase from the 18^{th} century, *Am. J. Phys.* **59**, 394–398.

Montgomery, R., J.E. Marsden and T.S. Ratiu [1984] Gauged Lie-Poisson structures, *Cont. Math. AMS* **28**, 101–114.

Sánchez de Alvarez, G. [1989] Controllability of Poisson control systems with symmetry, *Cont. Math. AMS* **97**, 399–412.

Shapere, A. and F. Wilczeck [1989] Geometry of self-propulsion at low Reynolds number, *J. Fluid Mech.* **198**, 557–585.

Simo, J.C., D.R. Lewis and J.E. Marsden [1991] Stability of relative equilibria I: The reduced energy momentum method, *Arch. Rat. Mech. Anal.* **115**, 15-59.

Simo, J.C., T.A. Posbergh and J.E. Marsden [1990] Stability of coupled rigid body and geometrically exact rods: block diagonalization and the energy-momentum method, *Physics Reports* **193**, 280–360.

Simo, J.C., T.A. Posbergh and J.E. Marsden [1991] Stability of relative equilibria II: Three dimensional elasticity, *Arch. Rat. Mech. Anal.* **115**, 61–100.

Smale, S. [1970] Topology and Mechanics, *Inv. Math.* **10**, 305–331, **11**, 45–64.

van der Meer, J.C. [1985] *The Hamiltonian Hopf Bifurcation.* Springer Lecture Notes in Mathematics **1160**.

van der Meer, J.C. [1990] Hamiltonian Hopf bifurcation with symmetry, *Nonlinearity* **3**, 1041–1056.

Walsh, G. and S. Sastry [1993] On reorienting linked rigid bodies using internal motions. *IEEE Transactions on Robotics and Automation*, (to appear).

Wilczek, F. and A. Shapere [1989] Geometry of self-propulsion at low Reynold's number, Efficiencies of self-propulsion at low Reynold's number, *J. Fluid Mech.* **19**, 557–585, 587–599.