Response to Comments on "Alfvén 'resonance' reconsidered: Exact equations for wave propagation across a cold inhomogeneous plasma" [Phys. Plasmas 1, 3523 (1994)]

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We agree with both Goedbloed and Lifschitz (whom we will refer to as GL) and Ruderman, Goossens and Zhelyazkov (RGZ) that the apparent violation of quasi-neutrality in Ref. 1 is not a true problem and rescind this particular argument against MHD. However, the rest of the discussion in our paper is independent of this argument regarding quasi-neutrality, and as discussed below, the conclusions in Ref. 1 about the validity of MHD, about compressibility versus incompressibility, and about Alfvén resonance remain valid. © *1995 American Institute of Physics.*

Validity of the ideal MHD approximation. We agree with GL that MHD is 'a beautiful and classical theory, free of internal contradictions,' and do not dispute the rigor of the mathematics underlying spectral theory. The issue we raise is whether MHD provides a valid description of a real plasma in the physically sensible cold-plasma regime (i.e., $\beta<m_e/m_i$ in order to have $v_A>v_{Te}$). This is a serious question because it is possible to have a beautiful, self-consistent, mathematically rigorous theory that does not correspond to reality. As a simple example, one could construct a new theory identical to ideal MHD except that one arbitrarily sets Ampère's law to be $\nabla\times\mathbf{B}=2\mu_0\mathbf{J}$. Because this new 'theory' is inconsistent with the correct Ampère's law, this new 'theory' is obviously an invalid description of reality, but is nevertheless beautiful, classical, free of internal contradictions and has an associated spectral theory as rigorous as that of regular MHD. Thus freedom from internal contradictions and mathematical rigor are necessary, but not sufficient conditions for a physical theory. The essential question is *which* of the infinity of mathematically rigorous, self-consistent theories is the one that corresponds to physical reality.

The Maxwell–Lorentz equations—the basis of plasma physics—cannot be derived by formal mathematical argument because these equations model the results of experimental observation, i.e., they model physical reality. For example, Faraday *discovered* his 'law' by measuring the voltage appearing on the terminals of a coil linking a changing magnetic flux; he could not have derived his law by mathematical manipulation of previously known formulae.

Even though ideal MHD is mathematically self-consistent, it is sometimes not a consistent approximation of the Maxwell–Lorentz equations, i.e., of physical reality. The specific issue here is the consistency between the approximations used for obtaining the pre-Maxwell Ampère's law and for obtaining the ideal MHD Ohm's law.

Ampère's law: As implied in the comments by GL and RGZ, restricting discussion to phenomena with phase velocity small compared to the speed of light is formally equivalent to letting $c^2\rightarrow\infty$. Since the displacement current scales as $1/c^2$ it is dropped from the full Ampère's law, yielding the pre-Maxwell Ampère's law.

Ohm's law: In the cold plasma approximation, the electron equation of motion is

$$m_e\frac{d\mathbf{u}_e}{dt}=-e[\mathbf{E}+\mathbf{u}_e\times\mathbf{B}]. \tag{1}$$

Since $\omega<<\omega_{ce}$, the perpendicular acceleration $m_e d\mathbf{u}_{e\perp}/dt$ is negligible compared to the magnetic force term and Eq. (1) may be approximated as

$$m_e\frac{d\mathbf{u}_{e\parallel}}{dt}=-e[\mathbf{E}+\mathbf{u}_e\times\mathbf{B}]$$

or using $\mathbf{u}_e=\mathbf{u}_i-\mathbf{J}/ne\simeq\mathbf{U}-\mathbf{J}/ne$,

$$m_e\frac{d\mathbf{u}_{e\parallel}}{dt}=-e\left[\mathbf{E}+\left(\mathbf{U}-\frac{1}{ne}\mathbf{J}\right)\times\mathbf{B}\right].$$

The $\mathbf{J}/ne$ term is the Hall term and may also be dropped since $\omega<<\omega_{ci}$ is being assumed. Thus, the linearized Ohm's law is

$$\frac{c^2}{\omega_{pe}^2}\frac{\partial(\mu_0\tilde{\mathbf{J}}_\parallel)}{\partial t}=\tilde{\mathbf{E}}+\tilde{\mathbf{U}}\times\mathbf{B}. \tag{2}$$

Ideal MHD omits the left hand side of this equation by invoking the argument that $\omega_{pe}^2=ne^2/m_e\epsilon_0\rightarrow\infty$ because the electron mass is small.

The essential problem is that in deriving the pre-Maxwell Ampère's law we have *formally assumed* that $c^2\rightarrow\infty$, so even though the electron mass is small, it is inconsistent to neglect c^2/ω_{pe}^2. Dropping the electron inertia term changes the cold Maxwell–Lorentz system from being a fourth-order system to being a second-order system. As discussed in Ref. 1, the fourth order system has two modes, the compressional mode which involves $B_\parallel$ and $\mathbf{J}_\perp$ and the shear mode which involves $\mathbf{B}_\perp$ and $J_\parallel$. Dropping the electron inertia does not affect the compressional mode (except in the vicinity of the $\omega=k_z v_A$ layer), and indeed the cold limit of the mode given in Eq. (1) of GL is the compressional mode. To see this, consider the zero-pressure, $B_y=0$ form of Eq. (1) of GL

$$\frac{d}{dx}\left[\frac{(\mu_0\rho\omega^2-k_z^2B^2)B^2}{\mu_0\rho\,\omega^2-(k_y^2+k_z^2)B^2}\frac{dU_x}{dx}\right]+(\mu_0\rho\omega^2-k_z^2B^2)U_x=0;$$

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here we have used $U_x=-i\omega X$. This equation is identical to Eq. (32) of Ref. 1 and in Sec. VI A of Ref. 1 it was shown that this equation is mathematically equivalent to the compressional wave equation

$$\nabla_\perp\cdot\left(\frac{1}{S-n_z^2}\nabla_\perp\tilde{B}_z\right)+\tilde{B}_z=0$$

expressed in dimensionless variables.

Incompressibility and cold plasma assumptions. MHD analyses that invoke incompressibility typically do so without substantive justification. The usual 'justification' is to say one is taking 'the limit $\gamma\rightarrow\infty$', but this is a mathematical formalism just like letting $c\rightarrow\infty$. Both γ and c do not really change; what one is really doing is modeling some physically attainable regime which behaves as if γ and/or c became infinite. Thus it is necessary to specify exactly which physically attainable regime behaves *as if* $\gamma\rightarrow\infty$. RGZ have shown that this regime is where $c_s^2>>v_A^2$; i.e., this is the regime where incompressibility is a valid approximation for the MHD equations. RGZ do this by showing that the compressible warm plasma equation corresponds to the 'incompressible' Eq. (27) of Ref. 1 *only* when $c_s^2>>v_A^2$ For $\gamma\mu_0P>>B^2$ one can approximate Eq. (2) of GL as $D\simeq(\rho\omega^2-k^2\gamma P)(\rho\omega^2-F^2)$ and so it is clear that the more general Eq. (1) of GL also becomes similar to Eq. (27) of Ref. 1 when $c_s^2>>v_A^2$ and $\omega^2<<k^2c_s^2$. Explicitly stating the criterion as $c_s^2>>v_A^2$ (rather than the obscure mathematical maneuver of letting $\gamma\rightarrow\infty$) is of great physical relevance because $\beta=\mu_0n\kappa T/B^2=c_s^2/v_A^2$. Thus, the often-used 'incompressible MHD' approximation could *only* be valid for plasmas having $\beta>>1$. Ultra-high β means the plasma is nearly un-magnetized — no magnetic fusion device is in this regime, nor are most of the plasmas for which ideal MHD is typically used. Since $\omega/k_\parallel\sim v_A<<c_s$ one would expect the equation of state to be isothermal, not adiabatic. Furthermore, the substantial parallel electron pressure would have to be included in Eq. (2) and would balance the parallel electric field so that it would still be incorrect to assume, as done in ideal MHD, that the parallel electric field vanishes.

Thus, incompressibility cannot be invoked on a given plasma by imposing the formal limit $\gamma\rightarrow\infty$, but is rather a consequence of the plasma having $\beta>>1$. The common practice of prescribing incompressibility for $\beta<1$ plasmas, the usual plasmas to which ideal MHD is applied, constitutes overdetermining the system of equations (e.g., see Ref. 2 where incompressibility is imposed on a $\beta=0.5$ plasma).

Does $\tilde{B}_x$ have a logarithmic singularity? In the previous version of their comment RGZ asserted:

'there is a fundamental difference between dissipative MHD and the mathematical model of a cold plasma. In contrast to viscosity and/or resistivity in MHD *the effect of electron inertia does not remove the Alfvén singularity.* '

In support of this assertion, RGZ claimed to show that $\tilde{B}_x$ should have a logarithmic singularity at the Alfvén layer where $\omega=k_zv_A$. RGZ used Eq. (75) of Ref. 1 to give the relation

$$\tilde{B}_x=\frac{v_A^2}{\omega^2-k_z^2v_A^2}\left(ik_z\frac{d\tilde{B}_z}{dx}+\frac{\omega k_y}{v_A^2}\tilde{E}_z\right)\tag{3}$$

and then correctly stated that a logarithmically singular $\tilde{B}_x$ would be obtained using the solutions given by Eqs. (142) and (143) of Ref. 1.

In our response to this earlier criticism by RGZ we showed that there is still no Alfvén resonance because of a subtle error in Eq. (141) of Ref. 1. If Eqs. (136)–(140) are used to transform Eq. (135) and *all* terms are kept, one finds

$$\xi^2g''_\pm+\xi g'_\pm-(4\pm\xi^2+\xi^4/4)g_\pm=0\tag{4}$$

where, unlike Eq. (141) of Ref. 1, the seemingly unimportant $\xi^4/4$ term is now retained. The $Y_2(\xi)$ Bessel function is an exact solution for $g_-(\xi)$ of Eq. (141) of Ref. 1, but is an incorrect solution of Eq. (4). By explicitly substituting the series expansion $Y_2(\xi)\ =\xi^{-2}+1/4-(\xi^2/16)\ln(\xi/2)\ +\ldots$ into Eq. (4), it is seen that the previously dropped term gives a contribution $\sim\xi^4Y_2(\xi)\sim\xi^2$ which is of the same order as terms resulting from derivatives of the logarithmic term in $Y_2(\xi)$. Thus, we conclude that $Y_2(\xi)$ is not a correct solution of Eq. (4) and, as shown below, the correct, exact solution of Eq. (4) is $g_\pm(\xi)=\xi^{-2}\exp(\mp\xi^2/4)$.

In hindsight, transforming Eq. (135) to Eq. (141) in Ref. 1 turns out to have been unnecessary because Eq. (135) may be solved exactly, yielding the two independent solutions

$$Q_\pm^{(1)}(x)=e^{\mp n_yx}\tag{5}$$

and

$$Q_\pm^{(2)}(x)=e^{\mp n_yx}-e^{\pm n_yx}\pm2n_yxe^{\pm n_yx};\tag{6}$$

for small x note that $Q_\pm^{(1)}(x)\simeq1\mp n_yx$ and $Q_\pm^{(2)}(x)\simeq2(n_yx)^2\pm2(n_yx)^3/3$. Hence, the exact general solution of Eq. (135) of Ref. 1 is

$$\begin{aligned}Q_-(x)&=aQ_-^{(1)}(x)+bQ_-^{(2)}(x),\\Q_+(x)&=\kappa Q_+^{(1)}(x)+\lambda Q_+^{(2)}(x),\end{aligned}\tag{7}$$

where a,b,κ, and λ are constants chosen to match the WKB solutions at large x.

That $\tilde{B}_x$ is not singular at $x=0$ can be seen using Eq. (75) of Ref. 1 to express $\tilde{B}_x$ in normalized form as

$$\tilde{B}_x=\frac{1}{S-n_z^2}\left(in_z\frac{d\tilde{B}_z}{dx}+n_yS\tilde{E}_z\right).\tag{8}$$

In the limit $S\rightarrow n_z^2$ we may write $S-n_z^2=xS'$ so that Eq. (8) becomes

$$\lim_{S\rightarrow n_z^2}\tilde{B}_x=\frac{1}{xS'}\left(in_z\frac{d\tilde{B}_z}{dx}+n_yn_z^2\tilde{E}_z\right).\tag{9}$$

Multiplying Eq. (132) of Ref. 1 by in_z and expanding gives

$$in_z\frac{d\tilde{B}_z}{dx}+n_yn_z^2\tilde{E}_z=in_zx\left(\frac{d^2\tilde{B}_z}{dx^2}-n_y^2\tilde{B}_z\right)\tag{10}$$

so

$$\lim_{x\rightarrow0}\tilde{B}_x=\frac{in_z}{S'}\left(\frac{d^2\tilde{B}_z}{dx^2}-n_y^2\tilde{B}_z\right).\tag{11}$$

Since the exact $Q_\pm$ given in Eqs. (7) are finite and regular at $x=0$, then $\tilde{B}_x=(Q_+-Q_-)/2i$ is also finite and regular at

$x\simeq 0$ and so there is indeed no Alfvén singularity in a cold plasma when finite electron inertia is taken into account.

The above response to RGZ's original criticism shows that $\tilde{B}_x$ does not have a logarithmic singularity and RGZ appear to have accepted this response as valid. However, in the revised version of their comment RGZ again state 'there is a fundamental difference between dissipative MHD and the mathematical model of a cold collisionless plasma', but now introduce a new and completely different issue, namely concern about matching in the vicinity of $x=0$.

RGZ now claim that it is impossible to match Eqs. (7) to $\tilde{E}_z$ and $\tilde{B}_z$ outside the $x\simeq 0$ neighborhood because the Eq. (7) solutions are exponential in character. RGZ base this claim on the assertion that 'far away from the resonant point $\tilde{B}_z$ tends to a constant value and $\tilde{E}_z$ tends to zero.' This assertion is incorrect; $\tilde{E}_z$ and $\tilde{B}_z$ are determined by Eqs. (127) and (128) of Ref. 1 together with specified boundary conditions, and as seen from Figs. 2 and 3 of Ref. 1, $\tilde{E}_z$ does not tend to zero away from $x=0$. Equations (7) are the solutions in the vicinity of $x=0$ and give the information required to match the exterior solutions to the left of $x=0$ to the exterior solutions on the right. Given $\tilde{E}_z$, $\tilde{B}_z$, $\tilde{E}_z'$ and $\tilde{B}_z'$ at $x=-\delta$ determines the constants a,b,κ, and λ; these constants can then be used together with Eqs. (7) to calculate $\tilde{E}_z$, $\tilde{B}_z$, $\tilde{E}_z'$ and $\tilde{B}_z'$ at $x=+\delta$ and so match to the exterior solutions to the right of $x=0$. The numerical integrations given in Figs. 2 and 3 of Ref. 1 would not have succeeded if there were a matching problem at the $x=0$ layer. In fact the transition is so smooth that it is hard to see where the Alfvén layer is; if RGZ were correct there would be an abrupt jump at the Alfvén layer (the location of the Alfvén layer for Fig. 2 is at 0.05 m, i.e., where $n=7.2\times 10^{18}\ \mathrm{m}^{-3}$, while the location of the Alfvén layer for Fig. 3 is at 0.93 m).

RGZ then go on to say that it does not make sense to consider a driven problem in a dissipationless plasma. This is not true as can be seen from the everyday example of a radio transmitter connected to a dissipationless transmission line. What counts are the boundary conditions at the ends. If the net energy flux into the transmission line equals the net flux out, then there will be no accumulation of energy in the line. This can be achieved by having (i) a non-driven standing wave, (ii) a matched load, or (iii) an unmatched load with a matching impedance at the source to absorb the reflected wave. The argument by RGZ has nothing to do with Alfvén resonance and, if true, would mean that all studies of waves in dissipationless plasmas would be incorrect. Equations (127) and (128) of Ref. 1 constitute a fourth order system with no singularities and have a unique solution if four boundary conditions are specified.

Comparison with RGZ's matched asymptotic expansions. RGZ have written Eqs. (127) and (128) of Ref. 1 in dimensioned form and have assumed that $k_zL\sim 1$ while $k_yL<<1$. Since the inhomogeneity is in the x direction, there is no reason why the z and y wavelengths should be determined by the scale length in the x direction.

RGZ assume as a boundary condition that $E_z\rightarrow 0$ away from the $\omega^2=k_z^2v_A^2$ layer. This constitutes prejudging the outcome, since if the fast B_z mode converts into a slow E_z mode at the $\omega^2=k_z^2v_A^2$ layer, then $\tilde{E}_z$ will certainly not vanish on the low density side of the layer.

RGZ calculate the width of a 'dispersion' layer by assuming that the dominant terms in their Eq. (4) are the first and third, in which case this equation becomes

$$\frac{d}{d\bar{x}}\left(\frac{1}{\omega^2-v_A^2k_z^2}\frac{d\tilde{E}_z}{d\bar{x}}\right)-\frac{1}{\omega^2l_e^2}\tilde{E}_z=0. \tag{12}$$

Near the Alfvén layer, they assume that $s=x-x_A$, and write $\omega^2-v_A^2k_z^2=s\Delta$ where Δ is defined by their Eq. (8). They then make two critical assumptions: (i) that there is a unique s where the two terms in the above equation balance, and (ii) this critical s can be estimated by assuming that derivatives $d/d\bar{x}$ can be replaced by $1/s$ so that the left hand term becomes $E_z/s^3\Delta$. With these assumptions they determine their δ_A, which is supposed to be the special value of s at which the two terms balance. This argument by RGZ is incorrect for several reasons. First of all, Eq. (12) is a differential equation which can be solved for all values of x, so there is no special value of s where the two terms balance; by definition, a solution to Eq. (12) balances the two terms everywhere. Second, it is improper to assume that derivatives can be replaced by $1/s$ in regions near the layer. To see this, note that Eq. (12) can be written as

$$\frac{d}{ds}\left(\frac{1}{s}\frac{dE_z}{ds}\right)-\frac{\Delta}{\omega^2l_e^2}E_z=0 \tag{13}$$

which for small s has two approximate solutions $E_z=1+\Delta s^3/3\omega^2l_e^2$ and $E_z=s^2+\Delta s^5/15\omega^2l_e^2$. It is incorrect to assume that $(s^{-1}E_z')'\sim E_z/s^3$ for either of these solutions. The condition $s\sim(\omega^2l_e^2/|\Delta|)^{1/3}$ is simply the upper value of s for which the leading term in the Frobenius solutions dominates higher order terms, as can be seen by comparing the magnitudes of the first and second terms in the two respective solutions. For large s, Eq. (13) becomes

$$\frac{d^2E_z}{ds^2}-\frac{s\Delta}{\omega^2l_e^2}E_z\simeq 0 \tag{14}$$

which for negative s (low density side of the $\omega^2=k_z^2v_A^2$ layer) is just the slow wave equation which has solutions $E_z\sim\exp(\pm i\int k_xdx)$ where $k_x^2=-s\Delta/\omega^2l_e^2$. On changing back to dimensionless variables, it is seen that this k_x^2 is just the E wave given by Eq. (88) of Ref. 1. Thus, it is incorrect to assume that $E_z\rightarrow 0$ on the low density side of the Alfvén layer, because on this side Eq. (14) describes a propagating non-MHD wave which has finite E_z.

The inertial electron Alfvén cone: A counter-example to the predictions of ideal MHD. To emphasize the fact that ideal MHD gives an incomplete and sometimes incorrect description, consider the inertial electron shear Alfvén resonance cone. This non-trivial phenomenon cannot be described by ideal MHD, yet occurs in the $\omega<<\omega_{ci}$ cold plasma regime where ideal MHD supposedly provides a complete description of plasma behavior.

This cone was first derived by Borg *et al.*[3] using spatial Fourier analysis (and has been discussed more recently by Morales *et al.*[4]). The Fourier analysis derivation is quite complicated so that it is not obvious that the cone is an exact solution. We give here a brief alternate derivation that avoids spatial Fourier analysis and shows directly that the cone is an exact *non-MHD* shear wave solution of the Maxwell–Lorentz equations in a $\omega << \omega_{ci}$ cold, uniform plasma.

We use cylindrical coordinates (r,θ,z), assume that the only finite fields are $\tilde{E}_r, \tilde{E}_z, \tilde{J}_r, \tilde{J}_z, \tilde{B}_\theta$ (corresponding to shear wave polarization), and consider solutions independent of θ. The relevant equations are the parallel component of Ampère's law

$$\frac{1}{r}\frac{\partial(r\tilde{B}_\theta)}{\partial r} = \mu_0 \tilde{J}_z, \tag{15}$$

the perpendicular component of Ampère's law

$$-\frac{\partial \tilde{B}_\theta}{\partial z} = \mu_0 \tilde{J}_r, \tag{16}$$

the azimuthal component of Faraday's law

$$\frac{\partial \tilde{E}_r}{\partial z} - \frac{\partial \tilde{E}_z}{\partial r} = -\frac{\partial \tilde{B}_\theta}{\partial t}, \tag{17}$$

the parallel component of Eq. (2),

$$\frac{\partial \tilde{J}_z}{\partial t} = \frac{\omega_{pe}^2}{c^2 \mu_0}\tilde{E}_z \tag{18}$$

and the polarization current

$$\tilde{J}_r = \frac{1}{\mu_0 v_A^2}\frac{\partial \tilde{E}_r}{\partial t}. \tag{19}$$

These equations describe the shear wave and in particular, using Eq. (2) we see that

$$\tilde{U}_\theta = -\tilde{E}_r/B = \hat{\theta}\cdot \int dt\, \frac{\tilde{\mathbf{J}}\times\mathbf{B}}{\rho} \tag{20}$$

which would correspond to the Y defined by GL (assuming cold plasma and $B_y=0$). The shear wave equation is obtained as follows: $\tilde{J}_r$ is eliminated to obtain

$$\frac{1}{v_A^2}\frac{\partial \tilde{E}_r}{\partial t} = -\frac{\partial \tilde{B}_\theta}{\partial z}. \tag{21}$$

Equation (18) and the time derivative of Eq. (15) give

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{B}_\theta}{\partial t}\right) = \frac{\omega_{pe}^2}{c^2}\tilde{E}_z. \tag{22}$$

Substitution for $\partial \tilde{E}_r/\partial t$ and $\tilde{E}_z$ in the time derivative of Eq. (17) gives the cold plasma shear wave partial differential equation

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial r}\left(\frac{c^2}{\omega_{pe}^2 r}\frac{\partial(r\tilde{B}_\theta)}{\partial r}\right) - \tilde{B}_\theta\right] + \frac{\partial}{\partial z}\left(v_A^2\frac{\partial \tilde{B}_\theta}{\partial z}\right) = 0. \tag{23}$$

The slab dispersion relation associated with this equation is Eq. (159) of Ref. 1. If, as in ideal MHD, c^2/ω_{pe}^2 is arbitrarily set to zero, then the slab dispersion relation associated with Eq. (23) corresponds to Eq. (7) of GL and the cold plasma shear Alfvén wave becomes ill-defined because it is missing the electron inertia term. [In a warm plasma (i.e., $\beta > m_e/m_i$) the electron inertial term does not get smaller; it is neglected because the parallel electron pressure is larger. Equation (7) of GL results from omitting *both* electron inertia and parallel pressure and so artificially forces the parallel electric field to vanish in all cases.]

If we consider a perturbation with time dependence $\exp(-i\omega t)$ and define $\rho = \omega_{pe} r/c$ and $\zeta = \omega z/v_A$, Eq. (23) becomes

$$\frac{\partial^2 \tilde{Q}}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial \tilde{Q}}{\partial \rho} - \tilde{Q} = \frac{\partial^2 \tilde{Q}}{\partial \zeta^2}. \tag{24}$$

where $\tilde{Q} = \rho \tilde{B}_\theta$ is proportional to the field-aligned current. We define the auxiliary coordinate η,

$$\eta^2 = \rho^2 - \zeta^2 \tag{25}$$

and express the $\tilde{Q}$ derivatives in terms of η derivatives, e.g. $\partial^2\tilde{Q}/\partial\rho^2 = (1/\eta - \rho^2/\eta^3)\partial\tilde{Q}/\partial\rho + (\rho^2/\eta^2)\partial^2\tilde{Q}/\partial\eta^2$. Substituting for $\partial^2\tilde{Q}/\partial\rho^2, \partial\tilde{Q}/\partial\rho$, and $\partial^2\tilde{Q}/\partial\zeta^2$ in Eq. (24) and using Eq. (25) gives

$$\frac{\partial^2 \tilde{Q}}{\partial \eta^2} - Q = 0 \tag{26}$$

which has solutions $Q = e^{\pm\eta}$. Thus, solutions of Eq. (23) are

$$\tilde{B}_\theta = \exp(\pm i\sqrt{\zeta^2 - \rho^2})/\rho. \tag{27}$$

Using Eq. (22) gives the parallel electric field,

$$\tilde{E}_z = = \pm\omega\frac{\exp(\pm i\sqrt{\zeta^2-\rho^2})}{\sqrt{\zeta^2-\rho^2}} \tag{28}$$

which is just Borg *et al.*'s resonance cone. In un-normalized variables, Eq. (28) is

$$\tilde{E}_z = \pm\omega\frac{\exp(\pm i\sqrt{(\omega z/v_A)^2 - (\omega_{pe} r/c)^2})}{\sqrt{(\omega z/v_A)^2 - (\omega_{pe} r/c)^2}}. \tag{29}$$

Thus $\tilde{E}_z$ is divergent on the conical surface

$$r = \frac{\omega}{\omega_{gm}} z \tag{30}$$

where $\omega_{gm} = v_A\omega_{pe}/c = \sqrt{\omega_{ci}\omega_{ce}}$. The speed of light does not appear in the cone angle indicating that the resonance cone is a slow wave phenomenon. Warm plasma effects neglected in this derivation would keep the cone finite (this has been thoroughly discussed in the context of high frequency resonance cones, e.g., see Ref. 5). The cold plasma shear Alfvén resonance cone has been observed experimentally both by Ono[6] and by Gekelman *et al.*[7]

Equation (29) is an experimentally confirmed, exact solution to the $\omega << \omega_{ci}$ cold plasma Maxwell-Lorentz system and yet contradicts the predictions of ideal MHD. Ideal MHD is in error here because of the improper assumption that ω_{pe}/c is infinite.

In their revised comments RGZ attempt to dismiss the resonance cone, a macroscopic plasma phenomenon, as being outside the framework of MHD applicability and make the interesting assertion that ideal MHD is only valid for scale lengths $L >> c/\omega_{pi}$ (if true, this assertion would mean that the Tokapole II plasma where the Alfvén 'resonance' was claimed to have been observed[8] is not describable by

MHD since the Tokapole II minor radius is $a=7-10$ cm and $c/\omega_{pi} \geqslant 7$ cm for the Tokapole $n \leqslant 10^{19}$ m^{-3} hydrogen plasma). The condition $L >> c/\omega_{pi}$ in fact, is not one of the standard assumptions made in deriving ideal MHD, but is rather a condition relating to *non-linear* Hall magnetic diffusion [cf. discussion after Eq. (9) of Ref. 9]. For linear problems, such as discussed in Ref. 1, the ion skin depth c/ω_{pi} is of no relevance. The fact that the resonance cone is 'outside of the framework of applicability of MHD' merely restates the fact that MHD does not properly describe the behavior of shear waves in a cold, collisionless plasma. Shear Alfvén wave resonance cones involve very large $\tilde{E}_z$ (infinite in the cold plasma approximation) and so cannot even be approximated by MHD because MHD is based on the incorrect assumption that $\tilde{E}_z=0$.

Conclusions. In the $\beta << 1$ cold plasma limit, Eq. (1) presented by GL may be identified as the compressional mode because in this limit it becomes equivalent to Eq. (116) of Ref. 1. A consequence of RGZ's analysis is that $\beta >> 1$ is the only regime where the 'incompressibility' assumption could conceivably be applied to the plasma as a whole, and in this limit one might expect the equations derived by Uberoi[10] and by Tataronis and Grossmann[2] could apply to a real plasma. However, the $\beta >> 1$ ideal MHD regime is not self-consistent because, if a collisionless plasma has $\beta >> 1$, the parallel electron pressure term (neglected in ideal MHD) will act instead of electron inertia to balance the parallel electric field (which is assumed to vanish in ideal MHD). Furthermore, the usual plasmas where incompressibility is invoked have β less than unity, and so assuming incompressibility for these typical plasmas overdetermines the system of equations.

The non-existence of the Alfvén resonance in a cold plasma has been shown more precisely by correcting a subtle error found in response to criticism by RGZ.

The omission of electron inertia in cold ideal MHD means that the formal limit $c^2 \rightarrow \infty$ is not consistently applied in deriving ideal MHD — thus ideal MHD is not a consistent approximation of the Maxwell-Lorentz equations. The experimentally observed shear Alfvén resonance cone has been shown to be an exact solution of the cold plasma Maxwell-Lorentz equations and provides a substantive contradiction to cold plasma ideal MHD.

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