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NONCANONICAL HAMILTONIAN FIELD THEORY AND REDUCED MHD

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ABSTRACT. Aspects of noncanonical Hamiltonian field theory are reviewed. Many systems are Hamiltonian in the sense of possessing Poisson bracket structures, yet the equations are not in canonical form. A particular system of this type is considered, namely reduced magnetohydrodynamics (RMHD) which was derived for tokamak modelling. The notion of a Lie-Poisson bracket is reviewed; these are special Poisson brackets associated to Lie groups. The RMHD equations are shown to be Hamiltonian for brackets closely related to the Poisson bracket of a semi-direct product group. The process by which this bracket may be derived from a canonical Lagrangian description by reduction is described.

1. INTRODUCTION. The basic idea underlying noncanonical Hamiltonian field theory is that systems which are not Hamiltonian in the traditional sense can be made so by generalizing the Poisson bracket. In fact, Poisson brackets for most of the major non-dissipative plasma systems have now been obtained. Four of the most basic systems are as follows, in chronological order:

1. Ideal MHD - Morrison and Greene [1980].
2. Maxwell-Vlasov equations - Morrison [1980] and Marsden and Weinstein [1982].
3. Multifluid Plasmas - Spencer and Kaufman [1982].
4. BBGKY hierarchy - Marsden, Morrison and Weinstein (in these proceedings).

For additional historical information and other systems, see Sudarshan and Mukunda [1983] and the reviews of Morrison [1982], Marsden et al., [1983] and the lectures of Holm, Ratiu and Weinstein in these proceedings. The purpose of this paper is to discuss some of the basic ideas and apply them to reduced magnetohydrodynamics (RHMD).

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We now describe some of the uses for Poisson structures that are now surfacing.

1. Categorizing fields. To specify a Hamiltonian field theory, a Hamiltonian and a Poisson bracket are chosen. The structure of the bracket can shed light on the theory, so a categorization by the bracket form is natural.
2. Casimirs. To each bracket there are functions that Poisson commute with every function; these are called Casimirs (see Sudarshan and Mukunda [1983], Littlejohn [1982] and Weinstein's lecture). Casimirs are invariants for any Hamiltonian system when a given bracket is used.
3. Stability. Casimirs are useful in testing for linear and nonlinear stability by a method going back to Arnold in the mid 1960's. See the lectures of Holm and Weinstein, Holm et. al. [1983], [1984] Abarbanel et. al [1984] and Hazeltine, Holm and Morrison [1984] for further information.
4. Quantization. Dashen and Sharp [1968] use noncanonical brackets for quantum observables in the context of current algebras. Goldin's lecture in these proceedings indicate how Poisson structures may be useful in quantization. The quantum approach also can be used to derive classical brackets, as in Dzyaloshinskii and Volovick [1980].
5. Chaos. As in Holmes and Marsden [1983], noncanonical Poisson structures can be used to prove the existence of chaos in perturbations of integrable systems.
6. Limits, Averaging and Perturbations. As in Littlejohn [1979] and Kaufman's lecture in these proceedings, Poisson structures can play a role in understanding the processes by which one passes to averaged systems or limiting systems and to what degree these more idealized models are good approximations to a more encompasing model. A general framework in which these processes are hoped to be understood is given in Montgomery, Marsden and Ratiu's paper in these proceedings.
7. Numerical Schemes. It is hoped that a deeper understanding of Hamiltonian structures will enable one to design algorithms with superior accuracy. For example it is known that algorithms which are energy preserving have better stability properties (see Lewis [1970], Chorin et. al. [1978] and references therein). Also, the successful vorticity algorithms of Chorin-Hald-Beale-Majda are known to be Hamiltonian (see Marsden and Weinstein [1983]). See Holm, Kuperschmidt and Levermore [1984] for some related results.

A tokamak uses a toroidal magnetic field configuration to confine hot plasma (see, for example, Chen [1974]). The physics of a tokamak is complicated and encompasses a wide range of scales. Kinetic and fluid models are typically used. In particular, RMHD is a simple fluid model that is obtained by approximating three dimensional incomressible MHD with the goal of highlighting the dominant physics (Strauss [1976, 1977]). RMHD is a member of a family of such fluid models that strive to explain major tokamak features and yet remain tractable (see Rosenbluth et. al. [1976], Hasegawa and Mima [1977], Hazeltine et. al. [1983] and Hazeltine et. al. [1984]). RMHD has achieved notable success (see Carreras et. al. [1979]). The reader will notice that RMHD is a generalization of the two dimensional Euler equations; perhaps the techniques discussed in the lectures of Zabusky and Beale can be adapted to RMHD.

The paper is organized as follows. In §2 we review some features of canonical and noncanonical Hamiltonian field theory. RMHD and its noncanonical brackets are presented in §3. In §4 the theory of Lie-Poisson brackets is reviewed and the brackets for RMHD are shown to consist of two pieces, one of which is a Lie Poisson bracket for a semi-direct product group. This group is related to the helical lagrangian paths followed by fluid particles in an idealized limit. The methods by which these brackets are obtained from the Lagrangian description by reduction and from ideal MHD by a limiting procedure are outlined in §5.

2. HAMILTONIAN DESCRIPTION OF CLASSICAL FIELDS. As in classical texts such as Wentzel [1949] and Goldstein [1980], a system of evolution equations (partial differential or integral equations for example) is said to be in canonical Hamiltonian form if they can be written in the form

$$\frac{\partial \eta^k}{\partial t} = \frac{\delta H}{\delta \pi_k}, \quad \frac{\partial \pi_k}{\partial t} = \frac{-\delta H}{\delta \eta^k}, \quad k = 1,2, \ldots, N \tag{2.1}$$

where $\eta^k(x,t)$ are the basic field variables and $\pi_k(x,t)$ are their conjugate momenta, x belonging to a region V of three space. Here H is a functional of the fields η and π, the dependence being denoted $H[\eta,\pi]$. We recall that the functional derivatives are defined by

$$\lim_{\varepsilon \to 0} \frac{H(\eta,\pi + \varepsilon\bar{\pi}) - H(\eta,\pi)}{\varepsilon} = \int_V \frac{\delta H}{\delta \pi_k} \bar{\pi}_k \, d^3x \tag{2.2}$$

(sum on k), with a similar definition for $\delta H/\delta\eta^k$. The reader should consult one of the aforementioned texts for basic examples of this formalism such as the Klein-Gordon field. This theory from the point of view of symplectic geometry, along with additional examples, is found in Chernoff and Marsden [1974] and Abraham and Marsden [1978, Section 5.5].

Poisson brackets are defined for functionals F and G of the fields η,π by

$$\{F,G\} = \int_V \left(\frac{\delta F}{\delta\eta^k} \frac{\delta G}{\delta\pi_k} - \frac{\delta G}{\delta\eta^k} \frac{\delta F}{\delta\pi_k} \right) d^3x \tag{2.4}$$

(sum on k); note that $\{F,G\}$ is a real valued function of (η,π). It is readily verified that the evolution equations (2.1) are equivalent to

$$\dot{F} = \{F,H\} \tag{2.5}$$

The bracket (2.4) assigns the new functional $\{F,G\}$ to two given ones F and G, and has the following basic properties:

(i) $\{F,G\}$ is linear in F and G (bilinearity)

(ii) $\{F,G\} = -\{G,F\}$ (antisymmetry)

(iii) $\{E, \{F,G\}\} + \{F, \{G,E\}\} + \{G, \{E,F\}\} = 0$ (Jacobi's identity)

(iv) $\{EF,G\} = E\{F,G\} + \{E,G\}F$ (derivation).

(i), (ii) and (iii) define a Lie algebra. A bracket on functionals defined on a phase space P (the space of (η,π) above being an example) satisfying (i)-(iv) is called a Poisson structure. (See Weinstein's lecture in these proceedings).

The four basic plasma physics examples listed in the introduction have equations that can be written in Hamiltonian form (2.5) for a suitable Poisson structure $\{F,G\}$; however, this Poisson structure does not have the canonical form (2.4) and correspondingly, the evolution equations do not have the canonical form (2.1). These examples clearly demonstrate the need for taking the wider view of non-canonical Hamiltonian field theory -- one demands only a Poisson structure and a Hamiltonian functional such that the equations of motion have the form (2.5). If the basic fields of the theory are denoted $\psi^i(x,t)$, $i - 1, \ldots, n$, then the Poisson structure is often of the form

$$\{F,G\} = \int_V \frac{\delta F}{\delta\psi^i} O^{ij} \frac{\delta G}{\delta\psi^j} d^3x \tag{2.6}$$

where O^{ij} is a matrix operator of $\psi = (\psi^i)$. Properties (i) and (iv) are automatic from the form (2.6), and (ii) holds if $O^{ij} = -O^{ji}$. On the other hand, Jacobi's identity is a relatively complicated condition on O^{ij} that requires ingenuity or a deeper insight into how bracket structures arise. Of course (2.6) includes (2.4) as a special case. A common class of Poisson structures have the form (2.6) where

$$O^{ij} = \psi^k c_k^{ij}$$

where c_k^{ij} are structure operators for a Lie algebra. For these, Jacobi's identity follows from Jacobi's identity for c_k^{ij}. These Lie-Poisson structures are examples of Poisson structures and will be considered in §4.

There are three ways to obtain Poisson structures for a given system. First of all, one can proceed by inspection and analogy with known brackets. The verification of Jacobi's identity can be done directly or with the assistance of Lie-Poisson structures. Second, one can introduce potentials (i.e. Clebsch variables) and induce a bracket on functionals of the physical fields by means of canonical brackets on functionals of the potentials and their conjugate momenta. See, for example, Morrison [1982], Holm and Kupershmidt [1983] and Marsden and Weinstein [1983] for accounts of this method. Thirdly, and perhaps most fundamentally, one can first write the theory in terms of a Lagrangian (or material) representation for the matter fields with the basic fields being the particle displacement field η^k and its conjugate momentum π_k. The canonical bracket (2.4) then induces a non-canonical bracket on the Eulerian (or spatial) fields by means of the map taking the Lagrangian to the Eulerian description. This procedure is a special case of reduction and was the method Marsden and Weinstein [1982] used to obtain the Maxwell-Vlasov bracket and which Spencer [1982] used to obtain the multifluid plasma bracket. Marsden, Ratiu and Weinstein [1983] used this method for several other basic systems as well and its basic features are described in Ratiu's lecture in this volume. See the article of Kaufman and Dewar in these proceedings for a related approach.

3. REDUCED MHD AND ITS BRACKET. As noted in the introduction, the RMHD equations are obtained by approximating the ideal incompressible MHD equations with the goal of describing the dominant tokamak physics. The approximation is tailored to the tokamak toroidal geometry and is discussed in the original papers of Strauss [1976,7]; see also Morrison and Hazeltine [1983].

The tokamak geometry is sometimes described by <u>toroidal coordinates</u>: (r,θ) represent polar coordinates in a plane perpendicular to the major toroidal axis; this plane is called the <u>poloidal plane</u>. The angular coordinate along the major axis of the torus is denoted ζ and is called the <u>toroidal angle</u>. Thus, θ and ζ are 2π-periodic while $0 \leq r \leq a$, where $r = a$ represents the torus boundary.

The RMHD fields are obtained by considering the components of the three dimensional velocity field v and magnetic field B in the poloidal plane.

The divergence free assumption on v (to lowest order) and the equation $\nabla\cdot B = 0$ for B leads one to consider corresponding potentials for their poloidal projections, namely

(i) a <u>scalar vorticity</u> $U(r,\theta,\zeta,t)$ (so $\nabla \times (U\hat{\zeta})$ is the poloidal velocity, where $\hat{\zeta}$ is a unit vector in the ζ direction)

and (ii) a <u>poloidal flux function</u> (or magnetic potential) $\psi(r,\theta,\zeta,t)$ (so $\nabla \times \psi\hat{\zeta}$ is the poloidal magnetic field).

The toroidal components in the RMHD approximation to leading order are regarded as constant.

The <u>RMHD equations</u> in what is called the low β limit (i.e. neglecting pressure effects) are

$$\frac{\partial U}{\partial t} = [U,\phi] + [\psi,J] - \frac{\partial J}{\partial \zeta} \tag{3.1a}$$

$$\frac{\partial \psi}{\partial t} = [\psi,\phi] - \frac{\partial \phi}{\partial \zeta} \tag{3.1b}$$

where $[f,g] = \frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial r}\right)$ is the canonical Poisson bracket in the poloidal plane and where

$$\nabla_\perp^2 \phi = U, \quad \text{so } \phi \text{ is the velocity stream function}$$

and

$$\nabla_\perp^2 \psi = J, \quad \text{the toroidal current} \quad .$$

Here, $\nabla_\perp = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial \theta}$ is the poloidal gradient operator and $\hat{r}$ and $\hat{\theta}$ are unit vectors along the r and θ coordinate axes. We recall that the MHD current is $J = \nabla \times B$ so for B in the poloidal plane, J points in the toroidal direction.

The equations (3.1) are to be supplemented with appropriate boundary conditions on ϕ and ψ at the boundary $r = a$.

We now describe the sense in which equations (3.1) are Hamiltonian. There is a conserved Hamiltonian, which is just the kinetic energy of the fluid plus the magnetic field energy:

$$H = \frac{1}{2}\int_V (|\nabla_\perp\phi|^2 + |\nabla_\perp\psi|^2)d^3x \tag{3.2}$$

where V is the torus, $0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le \zeta \le 2\pi$. There are additional constants of the motion that are important (for the stability analysis for example) which won't be discussed here; see Morrison and Hazeltine [1983].

Poisson brackets for the RMHD equations (3.1) are as follows; let F and G be functionals of U and ψ and set

$$\{F,G\} = \int_V \left\{U\left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}\right] + \psi\left(\left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta U}\right] + \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta \psi}\right]\right) + \left(\frac{\delta F}{\delta \psi}\frac{\partial}{\partial \zeta}\frac{\delta G}{\delta U} - \frac{\delta G}{\delta \psi}\frac{\partial}{\partial \zeta}\frac{\delta F}{\delta U}\right)\right\} d^3x \ . \tag{3.3}$$

The bracket is due to Morrison and Hazeltine [1983]. Using the fact that $\frac{\delta H}{\delta U} = -\phi$ and $\frac{\delta H}{\delta \psi} = -J$, it is easy to show that the equations (3.1) are equivalent to the Hamiltonian form

$$\dot{F} = \{F,H\} \ . \tag{3.4}$$

The only property of the bracket (3.3) which is not obvious is Jacobi's identity. It is verified directly in Morrison and Hazeltine [1983]. In the next section we shall verify that the first two terms of (3.3) are a Lie-Poisson bracket for a semi-direct product; this will give another proof of the Jacobi identity. In the final section we shall discuss the derivation of (3.3) by reduction and approximation (the method of Clebsch potentials is discussed in Morrison and Hazeltine [1983]).

4. LIE-POISSON BRACKETS AND SEMI-DIRECT PRODUCTS.

A key feature of the first two terms of (3.3) is the linear dependence on U and ψ. Brackets of this type are called Lie-Poisson and the associated phase space is the dual of a Lie algebra. We shall describe this construction in this section and shall show that the first two terms of (3.3) are Lie-Poisson brackets on the dual of a semi-direct product Lie algebra. The last term of (3.3) will be discussed in the final subsection.

A. Lie Poisson Brackets. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We recall (see Abraham and Marsden [1978, Sect. 4.1] for background) that $\mathfrak{g}$ is the tangent space to G at the identity and that for $\xi,\eta \in \mathfrak{g}$, the Lie bracket of ξ and η is given by the formula

$$[\xi,\eta] = \frac{d}{ds}\frac{d}{dr} g(s)\ h(r)\ g(s)^{-1}\Big|_{r=s=0} \tag{4.1}$$

where $g(s)$ and $h(r)$ are arbitrary smooth curves in G such that

$$g(0) = e, \quad h(0) = e, \quad g'(0) = \xi \quad \text{and} \quad h'(0) = \eta \ .$$

Let $\mathfrak{g}^*$ be the dual space of linear functionals on $\mathfrak{g}$ with the pairing between elements $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$ being denoted $\langle\mu,\xi\rangle$. In the infinite dimensional case we choose $\mathfrak{g}^*$ together with a pairing satisfying: $\langle\mu,\xi\rangle = 0$ for all μ implies $\xi = 0$ (a non-degeneracy condition) in a way appropriate for the problem at hand.

For $F:\mathfrak{g}^* \to \mathbb{R}$, we define $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$ by

$$\frac{d}{d\varepsilon}F(\mu + \varepsilon\bar{\mu})\Big|_{\varepsilon=0} = \left\langle \bar{\mu}, \frac{\delta F}{\delta \mu} \right\rangle \tag{4.2}$$

which is consistent with (2.2) if $\langle\ ,\ \rangle$ is taken to be the L^2-pairing. The Lie-Poisson bracket is defined by

$$\{F,G\}_{\pm} = \pm\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle. \tag{4.3}$$

There are two choices, $+$ or $-$. For this paper, we shall use the $+$ bracket, but the $-$ bracket is also used. To understand the $\pm$ distinction we need to recall how (4.3) is derived.

If F and G are real valued functions on $\mathfrak{g}^*$, we can extend them by left translation to functions F_L and G_L on T^*G, so F_L restricted to $T^*_eG = \mathfrak{g}^*$ is F. But T^*G carries canonical Poisson brackets $\{\cdot,\cdot\}_{T^*G}$ and we have

$$\{F_L, G_L\}_{T^*G} \quad \text{restricted to} \quad T^*_eG \quad \text{is} \quad \{F,G\}_- .$$

Similarly, extending by right invariance,

$$\{F_R, G_R\}_{T^*G} \quad \text{restricted to} \quad T^*_eG \quad \text{is} \quad \{F,G\}_+$$

(see Marsden, Weinstein et. al. [1983] for details of the proof). Thus, the $\pm$ Lie-Poisson brackets are naturally obtained from canonical brackets on T^*G. The process just described of getting brackets on $\mathfrak{g}^*$ from those on T^*G is a special case of a more general procedure called reduction (Marsden and Weinstein [1974]). Thus, whether one uses the $\pm$ bracket depends on whether the system under investigation corresponds to a left $(-)$ or right $(+)$ invariant system on T^*G. In fact the space T^*G often corresponds to material, or Lagrangian coordinates. The above picture relating T^*G and $\mathfrak{g}^*$ has its origins in the fundamental work of Arnold [1966]; see Ratiu's lecture in these proceedings for further information.

The Lie-Poisson brackets (4.3) make $\mathfrak{g}^*$ into a Poisson manifold. The properties (i)-(iv) of §2 can all be verified directly. For example, Jacobi's identity follows from symmetry of the second variations and from Jacobi's identity for the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$. Alternatively one can simply observe that T^*G, being a canonical manifold (cotangent space), is a Poisson manifold and that the Poisson bracket properties are inherited on $\mathfrak{g}^*$ from T^*G by the reduction procedure described above.

B. The Lie-Poisson bracket for the group of canonical transformations. The first term of (3.3) conforms to the Lie-Poisson format (4.3). A bracket of this type occurs for the Vlasov Poisson equation (see Morrison [1980] and Marsden and Weinstein [1982]) and for the two dimensional incompressible Euler equations (see Morrison [1982] and Marsden and Weinstein [1983]. We note that Arnold [1966] discussed the Hamiltonian formuation of Euler's equations, but did not explicitly give this bracket.)

If G is the group of canonical transformations of $\mathbb{R}^2$ then the Lie algebra $\mathfrak{g}$ of G consists of the Hamiltonian vector fields. (See Ebin and Marsden [1970] for the function space topologies used to make these assertions precise). If we identify Hamiltonian vector fields with their generating functions (a constant is dropped in making this identification) then the Lie algebra $\mathfrak{g}$ is identified with functions and the Lie bracket is the Poisson bracket (see Marsden and Weinstein [1982]; here we use the standard left Lie bracket, while they used the right Lie bracket). The dual of $\mathfrak{g}^*$ is identified with functions on $\mathbb{R}^2$ (or more properly densities on $\mathbb{R}^2$) and the pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$ is the usual L^2 pairing. Thus, we conclude that the first term of (3.3) is the (+) Lie-Poisson bracket for the group of canonical transforamtions on $\mathbb{R}^2$. How this term arises from a Lagrangian description is discussed in §5.

C. Semi-direct products. We now want to show that the first two terms of (3.3) taken together still define a Lie-Poisson bracket. This involves the notion of a semi-direct product, so we review the abstract construction first.

Let G be a group and V a vector space. Let ρ be a representation of G on V, so ρ is a homomorphism from G to the group of invertible linear transformations of V. We write $\rho_g(v)$ for $\rho(g)(v)$ for notational convenience. The semi-direct product $G \ltimes V$ is, as a set, $G \times V$, and has group multiplication given by

$$(g,u_1)\cdot(g_2,u_2) = (g_1g_2,\ u_1 + \rho_{g_1}(u_2)) \ . \tag{4.4}$$

One easily checks that $G \ltimes V$ is a group. Using formula (4.1) and $(g,u)^{-1} = (g^{-1},-\rho_{g^{-1}})$, one can readily prove that the Lie bracket for $G \ltimes V$ is given by

$$[(\xi_1,v_1),\ (\xi_2,v_2)] = ([\xi_1,\xi_2],\ \rho'_{\xi_1}(v_2) - \rho'_{\xi_2}(v_1)) \tag{4.5}$$

where $\rho'_\xi : V \to V$ is defined by

$$\rho'_\xi(v) = \frac{d}{d\varepsilon}\rho_{g(\varepsilon)}(v)\Big|_{\varepsilon=0}$$

where $g(\varepsilon)$ is any curve in G satisfying $g(0) = e$ and $g'(0) = \xi$. This Lie algebra is denoted $\mathfrak{g} \ltimes V$.

For example, the Euclidean group $E(3)$ of rigid motions of $\mathbb{R}^3$ is the semi-direct product of the rotation group $O(3)$ and the translation group $\mathbb{R}^3$; $E(3) = O(3) \ltimes \mathbb{R}^3$ where $O(3)$ acts on $\mathbb{R}^3$ by matrix multiplication. This of course is well-known (see, for example, Talman [1968] or Sudarshan and Mukunda [1983, p. 251ff]).

If we identify $(\mathfrak{g} \times V)^*$ with $\mathfrak{g}^* \times V^*$, combining (4.3) and (4.5), we see that the Lie Poisson brackets on $(\mathfrak{g} \ltimes V)^*$ are given by

$$\{F,G\}_\pm = \pm\left(\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle + \left\langle a, \rho'_{\frac{\delta F}{\delta \mu}}\left(\frac{\delta G}{\delta a}\right) - \rho'_{\frac{\delta G}{\delta \mu}}\left(\frac{\delta F}{\delta a}\right)\right\rangle\right) \tag{4.6}$$

where $(\mu,a) \in \mathfrak{g}^* \times V^*$ and $\langle\ ,\ \rangle$ denotes the pairing on the appropriate space.

Now let $G = \mathrm{Sym}(\mathbb{R}^2)$ be the group of canonical transformation η of $\mathbb{R}^2$ or of a region in $\mathbb{R}^2$ and let $V = F(\mathbb{R}^2)$ be the space of functions k on $\mathbb{R}^2$ and let G act on V by Lie transport: $\rho_\eta(k) = k\circ\eta^{-1}$. (The inverse is to make it a (left) representation). The induced action of $\mathfrak{g}$ on V is by Lie differentiation of vector fields or in terms of functions, by Poisson brackets:

$$\rho'_f(k) = \{f,k\} \tag{4.7}$$

where $\{\ ,\ \}$ is the standard Poisson bracket in $\mathbb{R}^2$ (the same as $[\ ,\ ]$ used in 3.1). Substituting (4.7) in (4.6) with $a = \psi$, $\mu = U$ and using the + Lie-Poisson structure, (4.6) reduces to the first two terms in (3.3). In summary, we have proved that <u>the first two terms of (3.3) are the Lie-Poisson bracket associated with the semi-direct product of canonical transformations and functions</u>, $\mathrm{Sym}(\mathbb{R}^2) \ltimes F(\mathbb{R}^2)$.

D. <u>Helical Symmetry</u>. Finally, we consider the last term of (3.3). First of all, this term is in almost canonical form and Jacobi's identity for it is readily checked. Combined with the result of part C, this verifies that indeed (3.3) defines a Poisson structure.

If we confine ourselves to solutions with helical symmetry, then the last term of (3.3) can be transformed away and the entire bracket then becomes Lie-Poisson. This proceeds as follows: fix a number q_0 and consider the additive group $\mathbb{R}$ acting on (r,θ,ζ) space by

$$H_s^{q_0}(r,\theta,\zeta) = (r,\theta + sq_0^{-1}, \zeta + s). \tag{4.8}$$

This is the group of helical transformations with pitch q_0. If ψ_h is invariant under H_s, it has the form

$$\psi(r,\theta,\zeta,t) = \hat{\psi}(r,\theta - q_0^{-1}\zeta,t) \tag{4.9}$$

as is easily checked, and similarly for U. One can check that for helically symmetric functions, the transformation $\psi \to \psi_h$, $U \to U_h$ given by

$$\begin{aligned} \psi_h(r,\theta,t) &= \hat{\psi}(r,\theta,t) + \frac{r^2}{2q_0} \\ U_h(r,\theta,t) &= \hat{U}(r,\theta,t) \end{aligned} \tag{4.10}$$

transforms the bracket (3.3) to

$$\{F,G\} = \int_U U_h\left[\frac{\delta F}{\delta U_h}, \frac{\delta G}{\delta U_h}\right] + \psi_h\left(\left[\frac{\delta F}{\delta \psi_h}, \frac{\delta G}{\delta U_h}\right] - \left[\frac{\delta G}{\delta \psi_h}, \frac{\delta F}{\delta U_h}\right]\right) d^2x \tag{4.11}$$

Thus, in the single helicity case, the bracket (3.3) transforms via (4.9) to the Lie-Poisson bracket (4.11). (See Morrison and Hazeltine [1983]).

One can also transform away the third term in (3.3) by using Lie transforms. One attemps to solve the equation

$$\frac{\partial \psi_h}{\partial \zeta} + [\psi_h, \psi] = 0 \tag{4.12}$$

for ψ_h given ψ. In general this is impossible because ζ must be a periodic variable. However, if it were possible, one sees that formally, this transforms away the third term of (3.3) (see the Appendix for this calculation). Following the dictates of Lie transform theory, we get a good approximation to (4.12) by averaging (see Guckenheimer and Holmes [1983, Chapter 4]). Since the helicity condition (4.9) gives the solution (4.10) to (4.12), it is natural to average ψ first with respect to H_s^q:

$$\psi_{av}(r,\theta,\zeta) = \int_0^{2\pi} \psi \circ H_s^{q_0}(r,\theta,\zeta)ds \tag{4.13}$$

where t is suppressed. Then ψ_{av} is helically invariant. Now let $\psi_h = \psi_{av} + r^2/2q_0$, $U_h = U_{av}$. The map

$$(\psi,U) \to (\psi_h,U_h) \tag{4.14}$$

transforms the bracket (3.3) into (4.11), which is shown just as in the appendix.

In fact, one can verify that (4.14) is a momentum map for the action of the semi-direct product $\mathrm{Sym}(\mathbb{R}^2) \times F(\mathbb{R}^2)$ on the space of $U(r,\theta,\zeta)$ and $\psi(r,\theta,\zeta)$ with the bracket (3.3) given as follows. (See Ratiu's lecture, Abraham and Marsden [1978] and Marsden, Weinstein, et. al. [1983] for the basic definitions and properties of momentum maps). If $\eta \in \mathrm{Sym}(\mathbb{R}^2)$ and $f \in F(\mathbb{R}^2)$, let them act on (ψ, U) by

$$(\psi,U) \mapsto (\psi_{av}, U_{av})$$

where

$$\psi_{av}(r,\theta,\zeta) = \psi(\eta(r,\theta + f(r,\theta)q_0^{-1}),\zeta + f(r,\theta))$$

and

$$U_{av}(r,\theta,\zeta) = U(\eta(r,\theta),\zeta) \quad .$$

This remark is consistent with the fact that momentum maps are always Poisson maps and the fact that Lie transforms, averaging and reduction are closely related.

5. REMARKS AND CONJECTURES ON LAGRANGIAN COORDINATES, REDUCTIONS AND APPROXIMATIONS. The preceeding discussions still leave open the question of how to derive brackets like (3.3) or (4.11). For the single helicity case, the derivation of (4.11) from canonical Lagrangian coordinates proceeds as follows. We assume that individual fluid particles follow trajectories that commute with the helical group and that the magnetic field is Lie transported. Thus, the particles move by means of volume preserving diffeomorphisms $\phi:\mathbb{R}^3 \to \mathbb{R}^3$ such that $H_s^{q_0}\circ\phi = \phi\circ H_s^{q_0}$ for each s. Call the group of such ϕ's, $\mathcal{H}$. Now we add a constraint that is consistent with the RMHD approximation namely that the toroidal speed of the particles is fixed; thus the configuration space for the fluid is $\mathcal{H}/S^1$ where S^1 is the group of $H_s^{q_0}$. But $\mathcal{H}/S^1$ is the group of transformations of the helices (orbits of the action (4.8)), which is isomorphic to $\mathrm{Sym}(\mathbb{R}^2)$. $\mathcal{H}/S^1$ then is the basic configuration space for a single helicity fluid.

Thus, the phase space is $T^*\mathrm{Sym}(\mathbb{R}^2)$. Now the magnetic potential is Lie transported by the helical action of $\mathrm{Sym}(\mathbb{R}^2)$ as in §4D. Thus, one can reduce $T^*\mathrm{Sym}(\mathbb{R}^2)$ as described in Ratiu's lecture to obtain a Lie-Poisson structure for the semi-direct product of $\mathrm{sym}(\mathbb{R}^2)$ and the space on which the magnetic potential lives. This produces exactly the structure (4.11).

To obtain the bracket (3.3) from a canonical Lagrangian picture we proceed as follows. As above, we build the RMHD approximation we have in mind into the Lagrangian configuration space. Choose $q_0 = \infty$ so $H_s^\infty = H_s$ is just translation in the ζ-direction; these H_s form an S^1 group. Now to allow ζ dependence we choose the basic configuration space to be the group C of volume preserving transformations that map S^1 orbits to S^1 orbits. Again the magnetic potential is Lie transported. However, our magnetic field will be assumed to have a dynamic component only in the poloidal plane so it is consistent to choose the subgroup S of C consisting of diffeomorphisms that are the identity in the r,θ variables ("streaming" diffeomorphisms: $(r,\theta,\zeta) \mapsto (r,\theta,\zeta + g(r,\theta))$). Our basic configuration space is then C/S, which is roughly, speaking, the ζ-dependent diffeomorphisms of the (r,θ) plane, and so the phase space is $T^*(C/S)$. However, the magnetic potential is Lie transported, so we need to reduce $T^*(C/S)$ by the further symmetry group corresponding to the magnetic potential and the S^1 invariance. We note that C/S is a bundle over the ζ axis with fiber $\mathrm{Sym}(\mathbb{R}^2)$, the diffeomorphism group of the ζ = constant poloidal planes. We now perform the reduction procedure described in Ratiu's lecture fiberwise. By the formulas in the paper of Montgomery, Marsden and Ratiu in these proceedings, the bracket on the quotient space is the semi-direct Lie-Poisson bracket plus a canonical bracket for the ζ dependence. Finally, the S^1 symmetry quotient inserts a $\partial/\partial\zeta$ in this canonical bracket, to produce the bracket (3.3).

The last step in this construction can be illustrated by the wave equation on the ζ axis. The canonical bracket on the phase space $F(S^1) \times F(S^1)^*$ is

$$\{F,G\} = \int_{S^1} \left(\frac{\delta F}{\delta\phi}\frac{\delta G}{\delta\pi} - \frac{\delta F}{\delta\pi}\frac{\delta G}{\delta\phi}\right) d\zeta \quad .$$

However the bracket on the reduced space with ζ-translations divided out is

$$\{F,G\} = \int_{S^1} \left(\frac{\delta F}{\delta\phi}\frac{\partial}{\partial\zeta}\frac{\delta G}{\delta\pi} - \frac{\delta F}{\delta\phi}\frac{\partial}{\partial\zeta}\frac{\delta F}{\delta\pi}\right) d\zeta$$

i.e. we change the cosymplectic operator as follows:

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \frac{\partial}{\partial\zeta} \\ -\frac{\partial}{\partial\zeta} & 0 \end{pmatrix} \quad .$$

This is proved the same way as the corresponding assertion for Maxwell's equation (see Marsden and Weinstein [1982]).

We just mention that there is another way of getting (3.3) directly from the (incompressible homogeneous version of) the Morrison-Greene bracket for MHD. Namely, one inserts the decomposition $\underline{v} = v_{\parallel}\hat{\zeta} + \hat{\zeta} \times \nabla_{\perp}\phi$ and $B = B_{\parallel}\hat{\zeta} + \hat{\zeta} \times \nabla_{\perp}\psi$ into that bracket. With $\mathrm{div}\, B = 0$, $B_{\parallel} = 1$ and $v_{\parallel} = 1$, the expression (3.3) results. One can also use this procedure to derive more complex brackets with a $v_{\parallel}$ dependence. (These are related to the hamiltonian structure of the Hazeltine equations [1983] which will be the subject of another publication (Hazeltine, Holm and Morrison [1984]).) The rough idea is that if a factor $B_{\parallel}$ is inserted in the last term of (3.3), linearity of the brackets in the field variables is restored. This is consistent with the fact that the Morrison-Greene bracket is Lie-Poisson for a semi-direct product (Holm and Kupershmidt [1983]) and can be derived from canonical brackets in a Lagrangian representation (Marsden, Ratiu and Weinstein [1983]). The procedure of neglecting $\delta/\delta v_{\parallel}$ and $\delta/\delta B_{\parallel}$ terms in this bracket can be viewed as an approximation procedure analogous to the limit $c \to \infty$ which converts the Maxwell-Vlasov to the Poisson-Vlasov bracket.

We hope that the bundle point of view sketched in the paper of Montgomery, Marsden and Ratiu in these proceedings will shed light on how these processes of averaging, reduction and limits all tie together into a coherent picture.

APPENDIX

If ψ is a given function then the formal solution to (4.12) can be obtained by integrating the characteristic equations where ψ acts as a Hamiltonian and ζ plays the role of time. One obtains $\psi_h(r_0(r,\theta,s), \theta_0(r,\theta,\zeta), 0)$, where we have suppressed the parametric time dependence. Shortly we will implicitly differentiate (4.12) in order to formally transform ψ variational derivatives into derivatives with respect to ψ_n.

Let us suppose that P is some functional of ψ that has the first variation

$$
\begin{aligned}
DP[\psi]\cdot\delta\psi &= \int_V \frac{\delta P}{\delta\psi}\, \delta\psi d^3x \\
&= \int_V \frac{\delta P}{\delta\psi_h}\, \delta\psi_h\, d^3x
\end{aligned}
\tag{A.1}
$$

In the second equality we assume P is a functional of ψ_h through (4.12). If we define the operator $\mathcal{L}$ by, $\mathcal{L}f = \partial f/\partial\zeta + [f,\psi]$, then linearization of (4.12) yields

$$
\delta\psi_h = \mathcal{L}^{-1}[\psi_h, \delta\psi] \tag{A.2}
$$

where we have used $\mathcal{L}^{-1}$ to mean the inverse of $\mathcal{L}$. In practice $\mathcal{L}^{-1}$ is obtained by integrating over characteristics. For our purposes it will be sufficient to pretend that the appropriate analysis has been done and that we can freely invert $\mathcal{L}$ when needed. Inserting (A.2) into (A.1) yields

$$DP\cdot\delta\psi = \int_V \frac{\delta P}{\delta\psi}\,\delta\psi\, d^3x = \int \frac{\delta P}{\delta\psi_h}\,\mathcal{L}^{-1}[\psi_h,\delta\psi]\, d^3x \tag{A.3a}$$

$$= \int_V [\psi_h,\delta\psi](\mathcal{L}^{-1})^+\,\frac{\delta P}{\delta\psi_h}\, d^3x \tag{A.3b}$$

where $(\mathcal{L}^{-1})^+$ is the formal adjoint of $\mathcal{L}^{1}$. Equation (A.3b) can be further transfomred by using the identity $\int_V f[g,h]\,d^3x = \int_V g[h,f]\,d^3x$; we obtain

$$DP\cdot\delta\psi = \int_V \left[(\mathcal{L}^{-1})^+\frac{\delta P}{\delta\psi_h},\ \psi_h\right]\delta\psi\ \ d^3x. \tag{A.4}$$

If Eq. (A.3b) is to hold for all variations $\delta\psi$ then evidently

$$\frac{\delta P}{\delta\psi} = \left[(\mathcal{L}^{-1})^+\frac{\delta P}{\delta\psi_h},\ \psi_h\right]. \tag{A.5}$$

Operating on both sides of (A.5) with $\mathcal{L}$ and using $\mathcal{L}[f,g] = [\mathcal{L}f,g] + [f,\mathcal{L}g]$, which is not difficult to establish, yields

$$\mathcal{L}\ \frac{\delta P}{\delta\psi} = \left[\mathcal{L}(\mathcal{L}^{-1})^+\frac{\delta P}{\delta\psi_h},\ \psi_h\right] \tag{A.6a}$$

$$\frac{\partial}{\partial\zeta}\frac{\delta P}{\delta\psi} + \left[\frac{\delta P}{\delta\psi},\psi\right] = \left[\psi_h,\ \frac{\delta P}{\delta\psi_h}\right]. \tag{A.6b}$$

Equation (A.6b) follows from the fact that an anti-self-adjoint linear operator will have an anti-self-adjoint inverse. From Eq. (A.6b) it follows immediately that the RMHD bracket becomes

$$\{F,G\} = \int_V \left\{U\left[\frac{\delta F}{\delta U},\ \frac{\delta G}{\delta U}\right] + \psi_h\left(\left[\frac{\delta F}{\delta\psi_h},\ \frac{\delta G}{\delta U}\right] + \left[\frac{\delta F}{\delta U},\ \frac{\delta G}{\delta\psi_h}\right]\right)\right\} d^3x\quad . \tag{A.7}$$

Hence we have transformed away the non Lie-Poisson term and the bracket possesses the algebraic interpretation of Section 4C. Moreover, it appears that we have replaced a three-dimensional problem with a two-dimensional problem!

In spite of the rosey picture painted above, there is a catch, which is associated with a periodicity constraint on (4.12). Recall ψ was required to be periodic in θ and ζ; if ψ_h is to be single-valued then it too must be periodic. Flows with periodic Hamiltonian's typically are not periodic -- indeed such would be an exception. So our problem lies in the fact that appropriate ψ_h do not in general exist. There are, however, the special single helicity solutions discussed in Section 4D.

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