The Orbit Bundle Picture of Cotangent Bundle Reduction

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Abstract

Cotangent bundle reduction theory is a basic and well developed subject in which one performs symplectic reduction on cotangent bundles. One starts with a (free and proper) action of a Lie group G on a configuration manifold Q, considers its natural cotangent lift to T^*Q and then one seeks realizations of the corresponding symplectic or Poisson reduced space. We further develop this theory by explicitly identifying the symplectic leaves of the Poisson manifold T^*Q/G, decomposed as a Whitney sum bundle, $T^*(Q/G) \bigoplus \widetilde{\mathfrak{g}}^*$ over Q/G. The splitting arises naturally from a choice of connection on the G-principal bundle $Q \to Q/G$. The symplectic leaves are computed and a formula for the reduced symplectic form is found.

1 Introduction and Overview

A Brief History of Reduction Theory. Reduction theory for mechanical systems with symmetry has its origins in the classical work of Euler and Lagrange in the late 1700s and that of Hamilton, Jacobi, Routh and Poincaré in the period 1830-1910. The immediate goal of reduction theory is to use conservation laws and the associated symmetries to reduce the number of dimensions required to describe a mechanical system. For example, using rotational invariance and conservation of angular momentum, the classical central force problem for a particle moving in $\mathbb{R}^3$ (a three degree of freedom mechanical system) can be reduced to a single second order ordinary differential equation in the radial variable, describing a new, *reduced* mechanical system with just one degree of freedom.

By 1830, variational principles, such as Hamilton's principle and the principle of least action, as well as canonical Poisson brackets were fairly well understood and there were shades of symplectic geometry already in the work of Lagrange. Several classical examples of reduction were understood in that era, such as the elimination of cyclic variables, which we would call today reduction by Abelian groups, which was primarily due to Routh, as well as Jacobi's elimination of the node for interacting particles in $\mathbb{R}^3$ with SO(3) symmetry.

Lie, by 1890, deepened the mathematical understanding of symplectic and Poisson geometry and their link with symmetry. Between 1901 and 1910, Poincaré discovered how to generalize the Euler equations for rigid body mechanics and fluids to general Lie algebras.

Interestingly, these methods, with some exceptions, remained nearly dormant since the time of Poincaré, for over half a century. Meanwhile, Cartan and others developed the needed tools of differential forms and analysis on manifolds, setting the stage for the modern era of reduction theory. Naturally, much attention was also going to exciting developments in relativity theory and quantum mechanics. This modern era began with the fundamental papers of Arnold [1966] and Smale [1970]. Arnold focussed on systems on Lie algebras and their duals, as in the works of Lie and Poincaré, while Smale focussed on the Abelian case giving, in effect, a modern version of Routh reduction.

With hindsight, we now know that the description of many physical systems such as rigid bodies and fluids requires *noncanonical Poisson brackets* and *constrained variational principles* of the sort studied by Lie and Poincaré. One example of noncanonical Poisson brackets on $\mathfrak{g}^*$, the dual of a Lie algebra $\mathfrak{g}$, are called, following Marsden and Weinstein [1983], ***Lie-Poisson brackets***. These structures were known to Lie around 1890, although Lie apparently did not recognize their importance in mechanics. The symplectic leaves in these structures, namely the coadjoint orbit symplectic structures, although implicit in Lie's work, were discovered by Kirillov, Kostant, and Souriau in the 1960's.

To synthesize the Lie algebra reduction methods of Arnold [1966] with the techniques of Smale [1970] on the reduction of cotangent bundles by Abelian groups, Marsden and Weinstein [1974] developed reduction theory in the general context of symplectic manifolds and equivariant momentum maps; related results, but with a different motivation and construction (not stressing equivariant momentum maps) were found by Meyer [1973].

The Symplectic Reduced Space. The construction of the symplectic reduced space is now standard: let (P, Ω) be a symplectic manifold and let a Lie group G act freely and properly on P by symplectic maps. The free and proper assumption is needed if one wishes to avoid singularities in the reduction procedure. Assume that this action has an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. Then the ***symplectic reduced space*** $\mathbf{J}^{-1}(\mu)/G_\mu = P_\mu$ is a symplectic manifold in a natural way; the induced symplectic form Ω_μ is determined uniquely by $\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$ where $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is the projection and $i_\mu : \mathbf{J}^{-1}(\mu) \to P$ is the inclusion. If the momentum map is not equivariant, Souriau [1970] discovered how to centrally extend the group (or algebra) to make it equivariant.

Coadjoint orbits were shown to be symplectic reduced spaces by Marsden and Weinstein [1974]: in the reduction construction, one chooses $P = T^*G$, with G acting by (say left) translation and the corresponding space P_μ is identified with the coadjoint orbit $\mathcal{O}_\mu$ through μ together with its coadjoint orbit symplectic structure. Likewise, the Lie-Poisson bracket on $\mathfrak{g}^*$ is inherited from the canonical Poisson structure on T^*G by *Poisson reduction*, that is, by simply identifying $\mathfrak{g}^*$ with the quotient $(T^*G)/G$. It is not clear who first *explicitly* observed this, but it is *implicit* in many works such as Lie [1890], Kirillov [1962, 1976], Guillemin and Sternberg [1980], and Marsden and Weinstein [1982, 1983], but is *explicit* in Marsden, Weinstein, Ratiu, Schmid, and Spencer [1982] and in Holmes and Marsden [1983].

Kazhdan, Kostant and Sternberg [1978] showed that P_μ is symplectically diffeomorphic to an orbit reduced space $P_\mu \cong J^{-1}(\mathcal{O}_\mu)/G$ and from this it follows that P_μ are the symplectic leaves in P/G. This paper was also one of the first to notice deep links between reduction and integrable systems, a subject continued by, for example, Bobenko, Reyman and Semenov-Tian-Shansky [1989] in their spectacular group theoretic explanation of the integrability of the Kowalewski top.

The way in which the *Poisson* structure on P_μ is related to that on P/G was clarified in a generalization of *Poisson reduction* due to Marsden and Ratiu [1986], a technique that has also proven useful in integrable systems (see, e.g., Pedroni [1995] and Vanhaecke [1996]).

Lagrangian Reduction. Routh reduction for Lagrangian systems is classically associated with systems having cyclic variables (this is almost synonymous with having an Abelian symmetry group); modern accounts can be found in Arnold [1988] and in Marsden and Ratiu [1994], §8.9. A key feature of Routh reduction is that when one drops the Euler-Lagrange equations to the quotient space associated with the symmetry, and when the momentum map is constrained to a specified value (i.e., when the cyclic variables and their velocities are eliminated using the given value of the momentum), then the resulting equations are in Euler-Lagrange form *not* with respect to the Lagrangian itself, but with respect to the *Routhian*. In his classical work, Routh [1877] applied these ideas to stability theory, a precursor to the energy-momentum method for stability (Simo, Lewis, and Marsden [1991]; see Marsden [1992] for an exposition and references). Of course, Routh's stability method is still widely used in mechanics.

Another key ingredient in Lagrangian reduction is the classical work of Poincaré [1901] in which the *Euler-Poincaré equations* were introduced. Poincaré realized that both the equations of fluid mechanics and the rigid body and heavy top equations could all be described in Lie algebraic terms in a beautiful way.

Tangent and Cotangent Bundle Reduction. The simplest case of cotangent bundle reduction is reduction at zero in which case one chooses $P = T^*Q$ and then the symplectic reduced space formed at $\mu = 0$ is given by $P_0 = T^*(Q/G)$, the latter with the canonical symplectic form. Another basic case is when G is Abelian. Here, $(T^*Q)_\mu \cong T^*(Q/G)$ but the latter has a symplectic structure modified by magnetic terms; that is, by the curvature of the mechanical connection.

The Abelian version of cotangent bundle reduction was developed by Smale [1970] and Satzer [1975] and was generalized to the nonabelian case in Abraham and Marsden [1978]. Kummer [1981] introduced the interpretations of these results in terms of a connection, now called the *mechanical connection*. The geometry of this situation was used to great effect in, for example, Guichardet [1984], Iwai [1987] and Montgomery [1984, 1990, 1991a].

Tangent and cotangent bundle reduction evolved into what we now term as the "bundle picture" or the "gauge theory of mechanics". This picture was first developed by Montgomery, Marsden and Ratiu [1984] and Montgomery [1984, 1986]. That work was motivated and influenced by the work of Sternberg [1977] and Weinstein [1978] on a *Yang-Mills construction* that is in turn motivated by Wong's equations, that is, the equations for a particle moving in a Yang-Mills field. The main result of the bundle picture gives a structure to the quotient spaces $(T^*Q)/G$ and $(TQ)/G$ when G acts by the cotangent and tangent lifted actions.

Semidirect Product Reduction. In the simplest case of a semidirect product, one has a Lie group G that acts on a vector space V (and hence on its dual V^*) and then one forms the semidirect product $S = G \circledS V$, generalizing the semidirect product structure of the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \circledS \mathbb{R}^3$.

Consider the isotropy group G_{a_0} for some $a_0 \in V^*$. The ***semidirect product reduction theorem*** states that each of the *symplectic reduced spaces for the action of* G_{a_0} *on* T^*G *is symplectically diffeomorphic to a coadjoint orbit in* $(\mathfrak{g} \circledS V)^*$*, the dual of the Lie algebra of the semi-direct product*. This semidirect product theory was developed by Guillemin and Sternberg [1980], Ratiu [1980a, 1981, 1982], and Marsden, Ratiu and Weinstein [1984a,b].

This construction is used in applications where one has "advected quantities" (such as the direction of gravity in the heavy top, density in compressible flow and the magnetic field in MHD). Its Lagrangian counterpart was developed in Holm, Marsden, and Ratiu [1998] along with applications to continuum mechanics. Cendra, Holm, Hoyle, and Marsden [1998] applied this idea to the Maxwell-Vlasov equations of plasma physics. Cendra, Holm, Marsden, and Ratiu [1998] showed how Lagrangian semidirect product theory it fits into the general framework of Lagrangian reduction.

Reduction by Stages and Group Extensions. The semidirect product reduction theorem can be viewed using reduction by stages: one reduces T^*S by the action of the semidirect product group $S = G \circledS V$ in two stages, first by the action of V at a point a_0 and followed by the action of G_{a_0}. Semidirect product reduction by stages for actions of semidirect products on general symplectic manifolds was developed and applied to underwater vehicle dynamics in Leonard and Marsden [1997]. Motivated partly by semidirect product reduction, Marsden, Misiolek, Perlmutter, and Ratiu [1998, 2000] gave a significant generalization of semidirect product theory in which one has a group M with a normal subgroup $N \subset M$ (so M is a group extension of N) and M acts on a symplectic manifold P. One wants to reduce P in two stages, first by N and then by M/N. On the Poisson level this is easy: $P/M \cong (P/N)/(M/N)$ but on the symplectic level it is quite subtle.

Cotangent bundle reduction by stages is especially interesting for group extensions. An example of such a group, besides semidirect products, is the Bott-Virasoro group, where the Gelfand-Fuchs cocycle may be interpreted as the curvature of a mechanical connection.

Lagrange-Poincaré and Lagrange-Routh Reduction. Marsden and Scheurle [1993a,b] showed how to generalize the Routh theory to the nonabelian case as well as showing how to get the Euler-Poincaré equations for matrix groups by the important technique of *reducing variational principles*. This approach was motivated by Cendra and Marsden [1987] and Cendra, Ibort, and Marsden [1987]. The work of Bloch, Krishnaprasad, Marsden, and Ratiu [1996] generalized the Euler–Poincaré variational structure to general Lie groups and Cendra, Marsden, and Ratiu [2000] carried out a Lagrangian reduction theory that extends the Euler–Poincaré case to arbitrary configuration manifolds.

This work was in the context of the Lagrangian analogue of Poisson reduction in the sense that no momentum map constraint is imposed.

One of the things that makes the Lagrangian side of the reduction story interesting is the lack of a general category that is the Lagrangian analogue of Poisson manifolds. Such a category, that of *Lagrange-Poincaré bundles* is developed in Cendra, Marsden, and Ratiu [2000], with the tangent bundle of a configuration manifold and a Lie algebra as its most basic examples. That work also develops the Lagrangian analogue of reduction for central extensions and, as in the case of symplectic reduction by stages mentioned above, cocycles and curvatures enter in this context in a natural way.

The Lagrangian analogue of the bundle picture is the bundle $(TQ)/G$, a vector bundle over Q/G; this bundle was studied in Cendra, Marsden, and Ratiu [2000]. In particular, the equations and variational principles are developed on this space. For $Q = G$ this reduces to Euler-Poincaré reduction. A G-invariant Lagrangian L on TQ induces a Lagrangian l on $(TQ)/G$. The resulting equations inherited on this space, given explicitly later, are the *Lagrange–Poincaré equations* (or the *reduced Euler-Lagrange equations*).

The above results develop what we might call ***Lagrange-Poincaré reduction***. On the other hand, in the nonabelian Routh reduction theory of Marsden and Scheurle [1993a] one imposes, as in symplectic reduction, a momentum map constraint. This was put into the bundle context by Jalnapurkar and Marsden [2000] and Marsden, Ratiu and Scheurle [2000]. We may call this ***Lagrange-Routh reduction***.

Applications of Reduction Theory. Reduction theory for mechanical systems with symmetry has proven to be a powerful tool enabling advances in stability theory (from the Arnold method to the energy-momentum method) as well as in bifurcation theory of mechanical systems, geometric phases via reconstruction—the inverse of reduction—as well as uses in control theory from stabilization results to a deeper understanding of locomotion. Methods of Lagrangian reduction have proven very useful in, for example, optimal control problems. It was used in Koon and Marsden [1997] to extend the falling cat theorem of Montgomery [1990] to the case of nonholonomic systems. For a general introduction to some of these ideas and for further references, see, for example, Marsden and Ratiu [1994], Leonard and Marsden [1997] and Marsden and Ostrowski [1998].

Singular Reduction. Singular reduction starts with the observation of Smale [1970] that $z \in P$ is a regular point of $\mathbf{J}$ iff z has no continuous isotropy. Motivated by this, Arms, Marsden, and Moncrief [1981] showed that the level sets $\mathbf{J}^{-1}(0)$ of an equivariant momentum map $\mathbf{J}$ have quadratic singularities at points with continuous symmetry. While such a result is easy for compact group actions on finite dimensional manifolds, the main examples of Arms, Marsden, and Moncrief [1981, 1982] were *infinite dimensional*—both the phase space and the group. We will not be concerned with singular reduction in this paper; we refer to Ortega and Ratiu [2000] for further references and discussion.

There are of course many other aspects of reduction theory and associated techniques that we do not attempt to review here, including resonant systems, nonholonomic mechanics, the method of invariants, etc.

The Main Result. The main new result of this paper is Theorem 4.3. This gives an expression for the reduced symplectic form on the symplectic leaves of $(T^*Q)/G$, each of which is determined to be a fiber products of the form $T^*(Q/G) \times_{Q/G} \widetilde{\mathcal{O}}$, where $\widetilde{\mathcal{O}}$ is the associated coadjoint orbit bundle. The symplectic structure restricted to the orbit bundle involves the curvature of the connection, the orbit symplectic form, and interaction terms that pair tangent vectors to the orbit with the vertical projections of tangent vectors to the configuration space. Our result may be viewed as a symplectic version of the global Poisson bracket formula on reduced cotangent bundles due to Montgomery, Marsden and Ratiu [1984] and Montgomery [1986]; see also Bloam [1999] and Zaalani [1999] for related results.

2 The Symplectic Leaves

Throughout the paper, we let a Lie group G act freely and properly on a manifold Q so that the natural quotient map $\pi : Q \to Q/G$ defines a principal bundle. Let $\mathcal{A}$ be a principal connection this bundle and let $\tilde{\mathfrak{g}}$ denote the associated bundle to the Lie algebra $\mathfrak{g}$, namely $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$, which we regard as a vector bundle over Q/G. We recall the following natural bundle isomorphisms (see Cendra, Holm, Marsden and Ratiu [1998]):

Lemma 2.1. *There are bundle isomorphisms*

$$\alpha_{\mathcal{A}} : TQ/G \to T(Q/G) \oplus \widetilde{\mathfrak{g}} \quad \textit{and} \quad (\alpha_{\mathcal{A}}^{-1})^* : T^*Q/G \to T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^* \tag{2.1}$$

Proof. Given $v_q \in T_qQ$, denote its equivalence class in $T(Q/G)$ by $[v_q]$. We claim that the mapping $\alpha_{\mathcal{A}} : [v_q] \mapsto T_q\pi(v_q) \oplus [q, \mathcal{A}(q)(v_q)]$ is well defined and induces the desired isomorphism of TQ/G with $T(Q/G) \oplus \widetilde{\mathfrak{g}}$. To see this, and to help clarify notations in the sequel, consider another representative of the orbit $[v_q]$, given by $g \cdot v_q$ where we use concatenated notation for the tangent lifted action of G on TQ. We have $T_{g\cdot q}\pi(g \cdot v_q) = T_q\pi(v_q)$ and

$$\begin{aligned}[g \cdot q, \mathcal{A}(g \cdot q)(g \cdot v_q)] &= [g \cdot q, (\phi_g^* \mathcal{A})] = [g \cdot q, \operatorname{Ad}_g \mathcal{A}(q)(v_q)] \\ &= [q, \mathcal{A}(q)(v_q)].\end{aligned}$$

The inverse of this map is given by $v_{[q]} \oplus [q,\xi] \mapsto [\mathrm{hor}_q\, v_{[q]} + \xi_Q(q)]$ as is readily verified. We therefore have a vector bundle isomorphism.

We next compute $(\alpha_{\mathcal{A}}^{-1})^* : T^*Q/G \to T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^*$, the dual of the inverse map:

$$\begin{aligned}\left\langle (\alpha_{\mathcal{A}}^{-1})^*([\alpha_q], (u_{[q]}, [q,\xi])) \right\rangle &= \left\langle [\alpha_q], [\mathrm{hor}_q \cdot u_{[q]} + \xi_Q(q)] \right\rangle \\ &= \left\langle \alpha_q, \mathrm{hor}_q \cdot u_{[q]} \right\rangle + \langle \alpha_q, \xi_Q(q) \rangle \\ &= \left\langle \mathrm{hor}_q^* \,\alpha_q, u_{[q]} \right\rangle + \langle \mathbf{J}(\alpha_q), \xi \rangle \,,\end{aligned}$$

where $\mathrm{hor}_q^* : T_q^*Q \to T_{[q]}^*(Q/G)$ is dual to the horizontal lift map $\mathrm{hor}_q : T_{[q]}(Q/G) \to T_qQ$ so that we conclude $(\alpha_{\mathcal{A}}^{-1})^*([\alpha_q]) = (\mathrm{hor}_q^*\,\alpha_q, [q, \mathbf{J}(\alpha_q)])$. ■

This bundle isomorphism can be recast as follows (see Cushman and Śniatycki [2000]). Consider the maps

$$\Delta : T^*Q \to T^*Q/G \to \widetilde{\mathfrak{g}}^*; \quad \alpha_q \mapsto [\alpha_q] \mapsto [q, \mathbf{J}(\alpha_q)]$$

and

$$\Gamma : T^*Q \to T^*Q/G \to T^*(Q/G); \quad \alpha_q \mapsto [\alpha_q] \mapsto \mathrm{hor}_q^*\,\alpha_q.$$

Notice that the map $\alpha_q \mapsto \mathrm{hor}_q^*\,\alpha_q$ drops to T^*Q/G, since we have, for all $V_{[q]} \in T_{[q]}(Q/G)$,

$$\begin{aligned}\left\langle \mathrm{hor}_{g\cdot q}^*(g\cdot\alpha_q), V_{[q]} \right\rangle &= \left\langle g\cdot\alpha_q, \mathrm{hor}_{g\cdot q} V_{[q]} \right\rangle = \left\langle \alpha_q, g^{-1}\cdot \mathrm{hor}_{g\cdot q} V_{[q]} \right\rangle \\ &= \left\langle \alpha_q, g^{-1}\cdot (g\cdot \mathrm{hor}_q V_{[q]} \right\rangle = \left\langle \alpha_q, \mathrm{hor}_q V_{[q]} \right\rangle\end{aligned}$$

where we use the fact that $g\cdot \mathrm{hor}_q = \mathrm{hor}_{g\cdot q}$.

We can write $(\alpha_{\mathcal{A}}^{-1})^* = \Gamma \oplus \Delta$. A partial inverse to the projection Δ is given in the next proposition,

Proposition 2.2. *Consider the map,*

$$\sigma : Q \times \mathfrak{g}^* \to T^*Q/G \tag{2.2}$$

given by $(q,\nu) \mapsto \mathcal{A}(q)^*\nu$. *This map is equivariant with respect to the diagonal action of* G *on* $Q \times \mathfrak{g}^*$ *and the cotangent lifted action of* G *on* T^*Q*, and so uniquely defines a map on the quotient,*

$$\widetilde{\sigma} : \widetilde{\mathfrak{g}}^* \to T^*Q/G \tag{2.3}$$

This is a fiber preserving bundle map which is injective on each fiber and satisfies $\Delta \circ \widetilde{\sigma} = id|_{\widetilde{\mathfrak{g}}^*}$

Proof. Under the map σ, $g \cdot (q,\nu) = (g\cdot q, \mathrm{Ad}^*_{g^{-1}}\nu) \mapsto \mathcal{A}(g\cdot q)^*(\mathrm{Ad}^*_{g^{-1}}\nu)$. However, for all $v \in T_{g\cdot q}Q$,

$$\begin{aligned}\left\langle \mathcal{A}(g\cdot q)^*(\mathrm{Ad}^*_{g^{-1}}\nu, v\right\rangle &= \left\langle \mathrm{Ad}^*_{g^{-1}}\nu, \mathcal{A}(g\cdot q)v\right\rangle = \langle \nu, \mathrm{Ad}_{g^{-1}}\mathcal{A}(g\cdot q)v\rangle \\ &= \langle \nu, (\psi_{g^{-1}})^*\mathcal{A}(g\cdot q)v\rangle = \langle \nu, \mathcal{A}(q)(T\psi_{g^{-1}}v)\rangle \\ &= \langle g\cdot \mathcal{A}(q)^*\nu, v\rangle ,\end{aligned}$$

from which we conclude equivariance of σ. Also, for each $[q,\nu] \in \widetilde{\mathfrak{g}}^*$,

$$\Delta(\widetilde{\sigma}([q,\nu])) = \Delta([\mathcal{A}(q)^*\nu]) = [q, \mathbf{J}(\mathcal{A}(q)^*\nu)] = [q,\nu],$$

since, for all $\xi \in \mathfrak{g}$, $\langle \mathbf{J}(\mathcal{A}(q)^*\nu), \xi\rangle = \langle \nu, \mathcal{A}(q)(\xi_Q(q))\rangle = \langle \nu, \xi\rangle$. ■

We next determine the image, under $(\alpha_{\mathcal{A}}^{-1})^*$ of the symplectic leaves of the Poisson manifold T^*Q/G, which we know from the symplectic correspondence theorem (see Weinstein[1983], Blaom[1998]) are given by $\mathbf{J}^{-1}(\mathcal{O})/G$ for each coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$.

Theorem 2.3. *We have $(\alpha_{\mathcal{A}}^{-1})^*(\mathbf{J}^{-1}(\mathcal{O})/G) = T^*(Q/G) \times_{Q/G} \widetilde{\mathcal{O}}$, where $\widetilde{\mathcal{O}} = (Q\times \mathcal{O})/G$ is the associated bundle using the coadjoint action of G on $\mathcal{O}$.*

Proof. From the definition of the bundle isomorphism $\alpha_{\mathcal{A}}$,

$$\begin{aligned}(\alpha_{\mathcal{A}}^{-1})^*(\mathbf{J}^{-1}(\mathcal{O})/G) &= \{(\Gamma(\alpha_q), \Delta(\alpha_q)) \mid \mathbf{J}(\alpha_q) \in \mathcal{O}\} \\ &= \{(\mathrm{hor}^*_q \alpha_q, [q, \mathbf{J}(\alpha_q)]) \mid \mathbf{J}(\alpha_q) \in \mathcal{O}\}\end{aligned}$$

First we characterize the sets $T^*_qQ \cap \mathbf{J}^{-1}(\mathcal{O})$, using the connection $\mathcal{A}$. Denote by $\mathbf{J}_q$, the restriction of $\mathbf{J}$ to T^*_qQ, and let $\sigma_q : \mathfrak{g} \mapsto T_qQ$, be the injective infinitesimal generator map. Using the fact that $\mathbf{J}_q = \sigma_q^*$, we have for all $\xi \in \mathfrak{g}$

$$\langle \mathbf{J}(\alpha_q + \mathcal{A}_\mu(q)), \xi\rangle = \langle \alpha_q, \xi_Q(q)\rangle + \langle \mu, \xi\rangle = \langle \mu, \xi\rangle \tag{2.4}$$

where the second equality holds since $\alpha_q \in V^0$, the annihilator of the vertical sub-bundle of TQ. We conclude that

$$T^*_qQ \cap \mathbf{J}^{-1}(\mathcal{O}) = \{V^0_q + \mathcal{A}_\mu(q) \mid \mu \in \mathcal{O}\}.$$

Recall that since σ_q^* is surjective, $\tau \circ \mathbf{J}^{-1}(\mathcal{O}) = Q$, where $\tau_{T^*Q} : T^*Q \to Q$ is the cotangent bundle projection. Now apply hor^*_q to each fiber over Q in $\mathbf{J}^{-1}(\mathcal{O})$. That is, for each $q \in Q$, we consider

$$\mathrm{hor}^*_q(\mathbf{J}^{-1}(\mathcal{O}) \cap T^*_qQ) \tag{2.5}$$

First, note that for all $X_{[q]} \in T_{[q]}(Q/G)$,

$$\left\langle \mathrm{hor}^*_q(\mathcal{A}_\mu(q)), X_{[q]}\right\rangle = \left\langle \mu, \mathcal{A}(q)(\mathrm{hor}_q(X_{[q]})\right\rangle = 0 \tag{2.6}$$

so that $\mathrm{hor}_q^*(\mathcal{A}_\mu(q)) = 0$. Furthermore, since hor_q is injective, $\mathrm{hor}_q^* : T_q^*Q \to T_{[q]}^*(Q/G)$ is surjective with $\ker \mathrm{hor}_q^* = H^0$, where H^0 denotes the annihilator of the horizontal subbundle of TQ. Thus, as a linear map, $\mathrm{hor}_q^* : V^0 \to T_{[q]}^*(Q/G)$ is an isomorphism. Consider the set of pairs, $\{(\Gamma(\alpha_q), \Delta(\alpha_q)) \mid \mathbf{J}(\alpha_q) \in \mathcal{O}\}$. Each α_q can be uniquely expressed as $\beta_q + \mathcal{A}_\mu(q)$ for some $\beta_q \in V^0$ and $\mu \in \mathcal{O}$. For a fixed μ, let β_q range over V_q^0. This generates the set $T_{[q]}^*(Q/G) \times [q, \mu]$ since $\mathbf{J}$ vanishes on V^0. The result now follows by varying $\mu \in \mathcal{O}$. ■

3 Orbit Reduction

Let us recall the characterizing property of the reduced symplectic forms in the orbit reduction setting of Kazhdan, Kostant and Sternberg [1978].

Theorem 3.1. *Let μ be a regular value of an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ of a left symplectic action of G on the symplectic manifold (P, Ω) and assume that the symplectic reduced space P_μ is a manifold with π_μ a submersion. Let $\mathcal{O}$ be the coadjoint orbit through μ in $\mathfrak{g}_+^*$. Then*

1. *$\mathbf{J}$ is transversal to $\mathcal{O}$ so $\mathbf{J}^{-1}(\mathcal{O})$ is a manifold*
2. *$\mathbf{J}^{-1}(\mathcal{O})/G$ has a unique differentiable structure such that the canonical projection $\pi_\mathcal{O} : \mathbf{J}^{-1}(\mathcal{O}) \to \mathbf{J}^{-1}(\mathcal{O})/G$ is a surjective submersion*
3. *there is a unique symplectic structure $\Omega_\mathcal{O}$ on $\mathbf{J}^{-1}(\mathcal{O})/G$ such that*

$$\iota_\mathcal{O}^* \Omega = \pi_\mathcal{O}^* \Omega_\mathcal{O} + \mathbf{J}_\mathcal{O}^* \omega_\mathcal{O}^+, \tag{3.1}$$

where $\iota_\mathcal{O} : \mathbf{J}^{-1}(\mathcal{O}) \to P$ is the inclusion, $\mathbf{J}_\mathcal{O} = \mathbf{J}|\mathbf{J}^{-1}(\mathcal{O})$, and $\omega_\mathcal{O}^+$ is the "+" orbit symplectic structure on $\mathcal{O}$.

By considering a momentum shift we can realize a bundle isomorphism between $\mathbf{J}^{-1}(\mathcal{O})$ and the space $V^0 \times \mathcal{O}$. Since it will be shown that this isomorphism is G equivariant, it determines a unique diffeomorphism between $V^0 \times \mathcal{O}/G$ and $\mathbf{J}^{-1}(\mathcal{O})/G$. We will characterize the symplectic form on the former by pulling back the "characterizing" symplectic form on $\mathbf{J}^{-1}(\mathcal{O})$. Furthermore, it will be shown that $V^0 \times \mathcal{O}/G$ is diffeomorphic to $T^*(Q/G) \oplus \widetilde{\mathcal{O}}$, so that the reduced symplectic form can be expressed on this space as well. Since $\mathbf{J}^{-1}(\mathcal{O}) \subset T^*Q$, the reduced symplectic form is determined by the restriction of the canonical symplectic form in T^*Q to $\mathbf{J}^{-1}(\mathcal{O})$, which in turn is determined by the restriction of the canonical one-form to $\mathbf{J}^{-1}(\mathcal{O})$.

Lemma 3.2. *There is a G-equivariant bundle isomorphism,*

$$\chi : V^0 \times \mathcal{O} \to \mathbf{J}^{-1}(\mathcal{O}) \tag{3.2}$$

that uniquely determines a diffeomorphism, $\bar{\chi}$ on the quotient spaces so that the following diagram commutes

$$\begin{array}{ccc} V^0 \times \mathcal{O} & \xrightarrow{\ \chi\ } & \mathbf{J}^{-1}(\mathcal{O}) \\ \Big\downarrow{\scriptstyle \widetilde{\pi}_G} & & \Big\downarrow{\scriptstyle \pi_{\mathcal{O}}} \\ (V^0 \times \mathcal{O})/G & \xrightarrow[\ \bar{\chi}\ ]{} & \mathbf{J}^{-1}(\mathcal{O})/G \end{array}$$

Proof. The map χ is given by

$$\chi(\alpha_q, \nu) = \alpha_q + \mathcal{A}(q)^*\nu \tag{3.3}$$

This map takes values in $\mathbf{J}^{-1}(\mathcal{O})$, as is seen from the proof of the previous theorem. From the characterization of the fibers of the bundle $\mathbf{J}^{-1}(\mathcal{O}) \to Q$, it follows that this map is onto. Since it is a momentum shift, it is clearly invertible with inverse

$$\alpha_q \mapsto \alpha_q - \mathcal{A}(q)^*\mathbf{J}(\alpha_q) \tag{3.4}$$

We check G-equivariance as follows:

$$\begin{aligned} \chi(g \cdot (\alpha_q, \nu)) &= \chi(g \cdot \alpha_q, g \cdot \nu) = g \cdot \alpha_q + \mathcal{A}(g \cdot q)^*(g \cdot \nu) \\ &= g \cdot \alpha_q + g \cdot (\mathcal{A}(q)^*\nu) = g \cdot (\chi(\alpha_q, \nu)) \end{aligned}$$

where the third equality uses the invariance properties of the connection. ■

Because χ is G-equivariant, the pull back by χ of the G-invariant form on $\mathbf{J}^{-1}(\mathcal{O})$, $\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}}$, given by $\chi^*(\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}})$, is a G-invariant form on $V^0 \times \mathcal{O}$. In fact, the form drops to the quotient by the diagonal G action since

$$\chi^*(\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}}) = \widetilde{\pi}_G^*\bar{\chi}^*\Omega_{\mathcal{O}}, \tag{3.5}$$

where $\widetilde{\pi}_G : V^0 \times \mathcal{O} \to V^0 \times \mathcal{O}/G$ denotes the projection. This follows since the diagram in the preceding theorem is commutative.

3.1 The Two-form on $V^0 \times \mathcal{O}$

We proceed to characterize the structure of the two-form $\chi^*(\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}})$ on $V^0 \times \mathcal{O}$. By construction,

$$\chi^*(\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}}) = \chi^*\iota_{\mathcal{O}}^*\Omega - \chi^*\mathbf{J}_{\mathcal{O}}^*\omega_{\mathcal{O}}^+ \tag{3.6}$$

The Second Term. We claim that

$$\chi^* \mathbf{J}_{\mathcal{O}}^* \omega_{\mathcal{O}}^+ = \pi_2^* \omega_{\mathcal{O}}^+, \tag{3.7}$$

where $\pi_2 : V^0 \times \mathcal{O} \to \mathcal{O}$ is projection on the second factor. This follows since for all $\xi \in \mathfrak{g}$,

$$\begin{aligned} \langle \mathbf{J}_{\mathcal{O}} \circ \chi(\alpha_q, \nu), \xi \rangle &= \langle \alpha_q + \mathcal{A}(q)^* \nu, \xi_Q(q) \rangle = \langle \alpha_q, \xi_Q(q) \rangle + \langle \mathcal{A}(q)^* \nu, \xi_Q(q) \rangle \\ &= 0 + \langle \nu, \xi \rangle, \end{aligned}$$

so that $\mathbf{J}_{\mathcal{O}} \circ \chi = \pi_2$.

The First Term. The first term is a little more complicated and splits into a sum of terms. We begin by considering the pull back by χ of the restriction to $\mathbf{J}^{-1}(\mathcal{O})$ of the canonical one-form, and then we compute the exterior derivative of this one-form.

Lemma 3.3. *We have*

$$\chi^* \iota_{\mathcal{O}}^* \Theta = \pi_1^* \iota_{V^0}^* \Theta + \varpi, \tag{3.8}$$

where $\iota_{V^0} : V^0 \to T^*Q$ *is inclusion,* Θ *is the canonical one-form on* T^*Q*,* $\pi_1 : V^0 \times \mathcal{O} \to V^0$ *is projection on the first factor, and* $\varpi \in \Omega^1(V^0 \times \mathcal{O})$ *is given by*

$$\varpi(\alpha_q, \nu)(X_{\alpha_q}, X_\nu^{\xi'}) = \left\langle \nu, \mathcal{A}(q)(T_{\alpha_q} \tau_{T^*Q}(X_{\alpha_q})) \right\rangle \tag{3.9}$$

for $(X_{\alpha_q}, X_\nu^{\xi'}) \in T_{(\alpha_q, \nu)}(V^0 \times \mathcal{O})$*, where* $X_\nu^{\xi'} \in T_\nu \mathcal{O}$ *denotes the infinitesimal generator for the left action of* G *on* $\mathcal{O}$*,* $X_\nu^{\xi'} = -\operatorname{ad}_{\xi'}^* \nu$*.*

Proof. Let $t \mapsto (\alpha_q(t), \operatorname{Ad}_{\exp -t\xi'}^* \nu)$ be a curve in $V^0 \times \mathcal{O}$, through the point (α_q, ν) such that $\frac{d}{dt}\big|_{t=0} \alpha_q(t) = X_{\alpha_q} \in T_{\alpha_q} V^0$. Since $\frac{d}{dt}\big|_{t=0} \operatorname{Ad}_{\exp -t\xi'}^* \nu = X_\nu^{\xi'}$, we get

$$\chi^* \iota_{\mathcal{O}}^* \Theta(\alpha_q, \nu)(X_{\alpha_q}, X_\nu^{\xi'}) = \iota_{\mathcal{O}}^* \Theta(\alpha_q + \mathcal{A}(q)^* \nu)(T_{(\alpha_q, \nu)} \chi(X_{\alpha_q}, X_\nu^{\xi'}) \tag{3.10}$$

Now,

$$\begin{aligned} T_{(\alpha_q, \nu)} \chi(X_{\alpha_q}, X_\nu^{\xi'}) &= \left. \frac{d}{dt} \right|_{t=0} (\chi(\alpha_q(t), \operatorname{Ad}_{\exp -t\xi'}^* \nu)) \\ &= \left. \frac{d}{dt} \right|_{t=0} (\alpha_q(t) + \mathcal{A}(q)^* (\operatorname{Ad}_{\exp -t\xi'}^* \nu)) \\ &= X_{\alpha_q} - \mathcal{A}(q)^* (\operatorname{ad}_{\xi'}^* \nu) \end{aligned}$$

where we use the fact that the curve $t \mapsto \mathcal{A}(q)^*(\mathrm{Ad}^*_{\exp -t\xi'} \nu)$ lies in the single fiber, $T^*_q Q$ for all t. Thus, the right hand side in (3.10) becomes

$$\begin{aligned}
\iota^*_{\mathcal{O}}\Theta(\alpha_q + \mathcal{A}(q)^*\nu)(T_{(\alpha_q,\nu)}\chi(X_{\alpha_q}, X^{\xi'}_\nu)) &= \Theta(\alpha_q + \mathcal{A}(q)^*\nu)(T_{(\alpha_q,\nu)}\chi(X_{\alpha_q}, X^{\xi'}_\nu)) \\
&= \Theta(\alpha_q + \mathcal{A}(q)^*\nu)(X_{\alpha_q} - \mathcal{A}(q)^*(\mathrm{ad}^*_{\xi'} \nu)) \\
&= \left\langle \alpha_q + \mathcal{A}(q)^*\nu, T\tau_{T^*Q} \cdot X_{\alpha_q} \right\rangle \\
&= \left\langle \alpha_q, T\tau_{T^*Q} \cdot X_{\alpha_q} \right\rangle + \left\langle \nu, \mathcal{A}(q)(T\tau_{T^*Q} \cdot X_{\alpha_q}) \right\rangle \\
&= \pi_1^*\iota^*_{V^0}\Theta + \varpi
\end{aligned}$$

The third equality holds because, for all t,

$$\tau_{T^*Q}(\alpha_q(t) + \mathcal{A}(q)^*(\mathrm{Ad}^*_{\exp -t\xi'} \nu)) = \tau_{T^*Q}(\alpha_q(t)). \quad \blacksquare$$

Computing the exterior derivative,

$$\chi^*\iota^*_{\mathcal{O}}(-\mathbf{d}\Theta) = -\mathbf{d}(\pi_1^*\iota^*_{V^0}\Theta + \varpi) = \pi_1^*\iota^*_{V^0}\Omega - \mathbf{d}\varpi,$$

so that

$$\chi^*(\pi^*_{\mathcal{O}}\Omega_{\mathcal{O}}) = \chi^*\iota^*_{\mathcal{O}}\Omega - \chi^*\mathbf{J}^*_{\mathcal{O}}\omega^+_{\mathcal{O}} = \pi_1^*\iota^*_{V^0}\Omega - \mathbf{d}\varpi - \pi_2^*\omega^+_{\mathcal{O}}.$$

The form ϖ is $\varpi = (\tau_{T^*Q} \times \mathrm{id})^*\alpha$, the pull back to $V^0 \times \mathcal{O}$ of the one-form α on $Q \times \mathcal{O}$ defined by

$$\alpha(q,\nu)(X_q, X_\nu) = \langle \nu, \mathcal{A}(q)(X_q) \rangle . \tag{3.11}$$

We are implicitly restricting the domain of τ_{T^*Q} to the sub-bundle V^0.

3.2 Computation of $\mathbf{d}\alpha$

The philosophy of the computation will be to make use of the connection to decompose tangent vectors to Q in terms of their horizontal and vertical parts. Of course we expect the curvature of the connection to appear in the resulting formula. However the presence of the pairing with ν, which varies over the coadjoint orbit $\mathcal{O}$ must be dealt with carefully.

We begin with an elementary but useful fact concerning the Jacobi-Lie bracket of vector fields on the cartesian product of two manifolds.

Lemma 3.4. *Let M and N be two smooth manifolds of dimension m and n respectively and consider their Cartesian product $M \times N$. Suppose we have two vector fields (X^M, X^N) and (Y^M, Y^N) on $M \times N$, each with the property that the tangent vector to M is independent of N, and that the tangent vector to N is independent of M. Then, the Jacobi Lie bracket of these two vector fields is also of this type. In fact, we have,*

$$[(X^M, X^N), (Y^M, Y^N)] = ([X^M, Y^M]_M, [X^N, Y^N]_N) \tag{3.12}$$

This is readily proved using the local coordinate expression of the bracket.

To determine $\mathbf{d}\alpha \in \Omega^2(Q \times \mathcal{O})$, it suffices, by bilinearity and skew symmetry, to compute its value on pairs of tangent vectors to Q of the type

- $\mathrm{hor}_q, \mathrm{hor}_q$
- $\mathrm{hor}_q, \mathrm{ver_q}$
- $\mathrm{ver_q}, \mathrm{ver_q}$

To carry this out, we will extend each tangent vector to be horizontal or vertical in an entire neighborhood of the point in question and use the fact that $\mathbf{d}\alpha$ is a tensor.

Case 1. $X_q, Y_q \in Hor_q Q$**.** We consider $(X_q, X_\nu^{\xi'}), ((Y_q, Y_\nu^{\eta'}) \in T_{(q,\nu)}(Q \times \mathcal{O})$. Extend X_q to the horizontal vector field $\widetilde{X}_Q \in \mathrm{Hor}\, Q$ and similarly extend Y_q to $\widetilde{Y}_Q$. We extend the second components of each tangent vector in the obvious way to be infinitesimal generators of the given Lie algebra element. That is we extend $X_\nu^{\xi'}$ to $\xi'_{\mathcal{O}}$ and similarly for $Y_\nu^{\eta'}$. Denote by $\widetilde{X}$, the extended vector field on a neighborhood of $Q \times \mathcal{O}$ given by $(\widetilde{X}_Q, \xi'_{\mathcal{O}})$, and similarly for $\widetilde{Y}$. We then have,

$$\begin{aligned} &\mathbf{d}\alpha(q,\nu)((X_q, X_\nu^{\xi'}), (Y_q, Y_\nu^{\eta'})) \\ &\quad = (X_q, X_\nu^{\xi'}) \cdot \alpha(\widetilde{Y}_Q, \eta'_{\mathcal{O}}) - (Y_q, Y_\nu^{\eta'}) \cdot \alpha(\widetilde{X}_Q, \xi'_{\mathcal{O}}) - \alpha([\widetilde{X}, \widetilde{Y}])(q,\nu) \end{aligned} \tag{3.13}$$

Notice that the first term vanishes since, if we take a curve $t \mapsto (q(t), \nu(t))$ through the point (q, ν) such that $(\dot{q}(0), \dot{\nu}(0)) = (X_q, X_\nu^{\xi'})$, we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \alpha(q(t),\nu(t))(\widetilde{Y}_Q(q(t)), \eta'_{\mathcal{O}}(\nu(t))) &= \left.\frac{d}{dt}\right|_{t=0} \left\langle \nu(t), \mathcal{A}(q(t)) \cdot \widetilde{Y}_Q(q(t)) \right\rangle \\ &= 0 \end{aligned} \tag{3.14}$$

since for all t, $\widetilde{Y}_Q(q(t)) \in \mathrm{Hor}_{q(t)}\, Q$. Similarly, the second term vanishes. Now, by Lemma 3.4, we have $[\widetilde{X}, \widetilde{Y}]_{Q \times \mathcal{O}} = ([\widetilde{X}_Q, \widetilde{Y}_Q]_Q, [\xi'_{\mathcal{O}}, \eta'_{\mathcal{O}}]_{\mathcal{O}})$ leaving

$$\mathbf{d}\alpha(q,\nu)((X_q, X_\nu^{\xi'}), (Y_q, Y_\nu^{\eta'})) = -\left\langle \nu, \mathcal{A}(q)([\widetilde{X}_Q, \widetilde{Y}_Q](q)) \right\rangle \tag{3.15}$$

However, since $\widetilde{X}_Q$ and $\widetilde{Y}_Q$ are horizontal vector fields, it follows that

$$\mathcal{A}([\widetilde{X}_Q, \widetilde{Y}_Q]) = -\,\mathrm{Curv}_{\mathcal{A}}(\widetilde{X}_Q, \widetilde{Y}_Q) \tag{3.16}$$

so that

$$\mathbf{d}\alpha(q,\nu)((X_q, X_\nu^{\xi'}), (Y_q, Y_\nu^{\eta'})) = \langle \nu, \mathrm{Curv}_{\mathcal{A}}(X_q, Y_q) \rangle\,. \tag{3.17}$$

Case 2. $X_q \in Hor_q Q, Y_q \in Ver_q Q$**.** Using the same notation for vector fields as in the previous case, we let $\widetilde{X}_Q$ denote the horizontal vector field extending X_q. Let $\eta = \mathcal{A}(q)(Y_q)$. Since Y_q is vertical we have $\eta_Q(q) = Y_q$. Then η_Q is a vertical extension of Y_q. With these extensions, we have

$$\begin{aligned}&\mathbf{d}\alpha(q,\nu)(X_q, X_\nu^{\xi'}), (Y_q, Y_\nu^{\eta'}) \\ &\quad = (X_q, X_\nu^{\xi'}) \cdot \alpha(\eta_Q, \eta'_{\mathcal{O}}) - (Y_q, Y_\nu^{\eta'}) \cdot \alpha((\widetilde{X}_Q, \xi'_{\mathcal{O}})) - \alpha(q,\nu)([\widetilde{X}, \widetilde{Y}])\end{aligned} \tag{3.18}$$

Consider the first term. Let $t \mapsto (q(t), \nu(t))$ be a curve through (q,ν) with $(\dot{q}(0), \dot{\nu}(0)) = (X_q, -\operatorname{ad}^*_{\xi'} \nu)$. Then

$$\begin{aligned}(X_q, X_\nu^{\xi'}) \cdot \alpha(\eta_Q, \eta'_{\mathcal{O}}) &= \left.\frac{d}{dt}\right|_{t=0} \alpha(q(t), \nu(t))(\eta_Q(q(t)), \eta'_{\mathcal{O}}(\nu(t))) \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle \nu(t), \mathcal{A}(q(t))(\eta_Q(q(t)) \rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle \nu(t), \eta \rangle = \left\langle -\operatorname{ad}^*_{\xi'} \nu, \eta \right\rangle\end{aligned}$$

The second term vanishes since $\alpha(\widetilde{X}_Q, \xi'_{\mathcal{O}}) = 0$ for $\widetilde{X}_Q \in \operatorname{Hor} Q$. Recall that for $\widetilde{X}_Q$ a horizontal vector field, we have, for all $\eta \in \mathfrak{g}$,

$$[\widetilde{X}_Q, \eta_Q] \in \operatorname{Hor} Q \tag{3.19}$$

This fact, together with Lemma 3.4, gives

$$\alpha(q,\nu)([\widetilde{X}, \widetilde{Y}]) = \left\langle \nu, \mathcal{A}(q)([\widetilde{X}_Q, \eta_Q]) \right\rangle = 0, \tag{3.20}$$

so that

$$\mathbf{d}\alpha(q,\nu)(X_q, X_\nu^{\xi'}), (Y_q, Y_\nu^{\eta'}) = \left\langle -\operatorname{ad}^*_{\xi'} \nu, \eta \right\rangle. \tag{3.21}$$

Case 3. $X_q, Y_q \in Ver_q Q$**.** Let $\xi = \mathcal{A}(q)(X_q)$ and $\eta = \mathcal{A}(q)(Y_q)$ We choose extensions to be vertical globally. Thus, $\widetilde{X}_Q = \xi_Q$ and $\widetilde{Y}_Q = \eta_Q$. Then we compute each term in the expression for $\mathbf{d}\alpha$.

The first term will again be

$$\left\langle -\operatorname{ad}^*_{\xi'} \nu, \eta \right\rangle \tag{3.22}$$

since $\alpha(q,\nu)(\eta_Q, \eta'_{\mathcal{O}}) = \langle \nu, \eta \rangle$ i.e. $\iota_{(\eta_Q, \eta'_{\mathcal{O}})} \alpha : Q \times \mathcal{O} \to \mathbb{R}$ is independent of Q.

The second term is computed similarly to be $\left\langle \operatorname{ad}^*_{\eta'} \nu, \xi \right\rangle$. For the last term, recall that for left actions, $G \times Q \to Q$, we have

$$[\xi_Q, \eta_Q] = -[\xi, \eta]_Q \tag{3.23}$$

so that

$$\alpha(q,\nu)([\widetilde{X},\widetilde{Y}]) = \langle \nu, \mathcal{A}(q)([\xi_Q,\eta_Q])\rangle = -\langle \nu, \mathcal{A}(q)([\xi,\eta]_Q)(q)\rangle = -\langle \nu, [\xi,\eta]\rangle .$$

Therefore,

$$\mathbf{d}\alpha(q,\nu)(X_q, X_\nu^{\xi'}),(Y_q,Y_\nu^{\eta'}) = \langle \nu,[\eta,\xi']\rangle + \langle \nu,[\eta',\xi]\rangle + \langle \nu,[\xi,\eta]\rangle . \tag{3.24}$$

We now collect these results to obtain a formula for the two form relative to a decomposition of the tangent vectors to Q into their horizontal and vertical projections.

Theorem 3.5. *Let* $(X_q, X_\nu^{\xi'}),(Y_q,Y_\nu^{\eta'}) \in T_{(q,\nu)}(Q\times\mathcal{O})$. *Let*

$$\xi = \mathcal{A}(q)(X_q) \quad \textit{and} \quad \eta = \mathcal{A}(q)(Y_q)$$

so that

$$X_q = \xi_Q(q) + \operatorname{Hor}_q X_q, \quad Y_q = \eta_Q(q) + \operatorname{Hor}_q Y_q \tag{3.25}$$

where Hor_q *denotes the horizontal projection onto the horizontal distribution. We then have*

$$\begin{aligned}&\mathbf{d}\alpha(q,\nu)((X_q, X_\nu^{\xi'}),(Y_q,Y_\nu^{\eta'}))\\ &\quad = \langle \nu,[\eta',\xi]\rangle + \langle \nu,[\eta,\xi']\rangle + \langle \nu,[\xi,\eta]\rangle + \langle \nu, \operatorname{Curv}_{\mathcal{A}}(q)(X_q,Y_q)\rangle\end{aligned} \tag{3.26}$$

Proof. The proof is a straightforward computation:

$$\begin{aligned}
&\mathbf{d}\alpha(q,\nu)((X_q, X_\nu^{\xi'}),(Y_q,Y_\nu^{\eta'}))\\
&\quad = \mathbf{d}\alpha(q,\nu)((\xi_Q(q)+\operatorname{Hor}_q X_q, X_\nu^{\xi'}),(\eta_Q(q)+\operatorname{Hor}_q Y_q, Y_\nu^{\eta'}))\\
&\quad = \mathbf{d}\alpha(q,\nu)((\xi_Q(q),\tfrac{1}{2}X_\nu^{\xi'}) + (\operatorname{Hor}_q X_q, \tfrac{1}{2}X_\nu^{\xi'}),(\eta_Q(q),\tfrac{1}{2}Y_\nu^{\eta'})\\
&\qquad + (\operatorname{Hor}_q Y_q, \tfrac{1}{2}Y_\nu^{\eta'}))\\
&\quad = \mathbf{d}\alpha(q,\nu)((\xi_Q(q),\tfrac{1}{2}X_\nu^{\xi'}),(\eta_Q(q),\tfrac{1}{2}Y_\nu^{\eta'})) + \mathbf{d}\alpha(q,\nu)((\xi_Q(q),\tfrac{1}{2}X_\nu^{\xi'}),\\
&\qquad (\operatorname{Hor}_q Y_q, \tfrac{1}{2}Y_\nu^{\eta'})) + \mathbf{d}\alpha(q,\nu)((\operatorname{Hor}_q X_q,\tfrac{1}{2}X_\nu^{\xi'}),(\eta_Q(q),\tfrac{1}{2}Y_\nu^{\eta'}))\\
&\qquad + \mathbf{d}\alpha(q,\nu)((\operatorname{Hor}_q X_q,\tfrac{1}{2}X_\nu^{\xi'}),(\operatorname{Hor}_q Y_q,\tfrac{1}{2}Y_\nu^{\eta'}))\\
&\quad = \left\langle \nu,[\tfrac{1}{2}\eta',\xi]\right\rangle + \langle \nu,[\xi,\eta]\rangle + \langle \nu,[\xi,\eta]\rangle + \left\langle \nu,[\eta,\tfrac{1}{2}\xi']\right\rangle\\
&\qquad - \left\langle \operatorname{ad}^*_{\frac{1}{2}\eta'}\nu,\xi\right\rangle + \left\langle -\operatorname{ad}^*_{\frac{1}{2}\xi'}\nu,\eta\right\rangle + \langle \nu, \operatorname{Curv}_{\mathcal{A}}(X_q,Y_q)\rangle\\
&\quad = \langle \nu,[\eta',\xi]\rangle + \langle \nu,[\eta,\xi']\rangle + \langle \nu,[\xi,\eta]\rangle\\
&\qquad + \langle \nu, \operatorname{Curv}_{\mathcal{A}}(q)(X_q,Y_q)\rangle
\end{aligned}$$

where we have used the relations determined in the previous section. ■

4 The reduced form

Recall that

$$\begin{aligned}\chi^*(\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}}) &= \chi^*\iota_{\mathcal{O}}^*\Omega - \chi^*\mathbf{J}_{\mathcal{O}}^*\omega_{\mathcal{O}}^+ \\ &= \pi_1^*\iota_{V^0}^*\Omega - \mathbf{d}\varpi - \pi_2^*\omega_{\mathcal{O}}^+ \\ &= \pi_1^*\iota_{V^0}^*\Omega - (\tau_{T^*Q}\times \mathrm{id})^*\mathbf{d}\alpha - \pi_2^*\omega_{\mathcal{O}}^+ \end{aligned} \tag{4.1}$$

We have already established the G-invariance of this form. Notice that the first term is independently G-invariant since, if we denote the action of G on $V^0\times\mathcal{O}$ by $\psi^{V^0\times\mathcal{O}}$ and the action of G on T^*Q by ψ, we have

$$\begin{aligned}(\psi_g^{V^0\times\mathcal{O}})^*\pi_1^*\iota_{V^0}^*\Omega &= \pi_1^*\psi_g^*\iota_{V^0}^*\Omega = \pi_1^*\iota_{V^0}^*\psi_g^*\Omega \\ &= \pi_1^*\iota_{V^0}^*\Omega\end{aligned}$$

since $\pi_1\circ\psi_g^{V^0\times\mathcal{O}}(\alpha_q,\nu) = g\cdot\alpha_q = \psi_g\circ\pi_1(\alpha_q,\nu)$. Thus, the sum of the last two terms is G invariant. Furthermore, the G invariance of the last two terms as forms on $V^0\times\mathcal{O}$, is really G invariance of a form on $Q\times\mathcal{O}$ since

$$\begin{aligned}(\tau_{T^*Q}\times\mathrm{id})\circ\psi_{\mathrm{g}}^{\mathrm{V}^0\times\mathcal{O}}(\alpha_{\mathrm{q}},\nu) &= (\tau_{T^*Q}\times\mathrm{id})(\mathrm{g}\cdot\alpha_{\mathrm{q}},\mathrm{g}\cdot\nu) \\ &= (g\cdot q, g\cdot\nu) = \psi_g^{Q\times\mathcal{O}}(q,\nu) \\ &= \psi_g^{Q\times\mathcal{O}}\circ(\tau_{T^*Q}\times\mathrm{id})(\alpha_{\mathrm{q}},\nu)\end{aligned}$$

4.1 The Part that Drops to $\widetilde{\mathcal{O}}$

We begin with the proof of the vanishing of the two-form $\mathbf{d}\alpha+\pi_2^*\omega_{\mathcal{O}}^+$ on vertical vectors.

Proposition 4.1. *The two form, $\mathbf{d}\alpha+\pi_2^*\omega_{\mathcal{O}}^+$ on $Q\times\mathcal{O}$ vanishes on vertical vectors of the bundle $Q\times\mathcal{O}\to\widetilde{\mathcal{O}}$. It therefore uniquely determines a two form on $\widetilde{\mathcal{O}}$.*

Proof. For the two form $\mathbf{d}\alpha+\pi_2^*\omega_{\mathcal{O}}^+$ on $Q\times\mathcal{O}$ to drop to the quotient, $\widetilde{\mathcal{O}}$, we must have both G invariance of the form and also the property that it vanish on the vertical fibers. To see this, fix $\xi\in\mathfrak{g}$ and let $(Y_q, Y_\nu^{\eta'})\in T_{(q,\nu)}(Q\times\mathcal{O})$. As usual, let $\eta = \mathcal{A}(q)(Y_q)$. Since the action of G on $Q\times\mathcal{O}$ is the diagonal action, we have

$$\xi_{Q\times\mathcal{O}}(q,\nu) = \left.\frac{d}{dt}\right|_{t=0}(\exp t\xi\cdot q, \mathrm{Ad}^*_{\exp -t\xi}\nu) = (\xi_Q(q), X_\nu^\xi) \tag{4.2}$$

We then have,

$$\begin{aligned}(\mathbf{d}\alpha + \pi_2^*\omega_{\mathcal{O}}^+)(q,\nu)((\xi_Q(q), X_\nu^\xi), (Y_q, Y_\nu^{\eta'}))&\\ = \mathbf{d}\alpha(q,\nu)((\xi_Q(q), X_\nu^\xi), (Y_q, Y_\nu^{\eta'}))&\\ + \omega_{\mathcal{O}}^+(-\operatorname{ad}_\xi^*\nu, -\operatorname{ad}_{\eta'}^*\nu)&\\ = \langle \nu, [\eta', \xi]\rangle + \langle \nu, [\eta, \xi]\rangle + \langle \nu, [\xi, \eta]\rangle&\\ + \omega_{\mathcal{O}}^+(\operatorname{ad}_\xi^*\nu, \operatorname{ad}_{\eta'}^*\nu)&\\ = \langle \nu, [\eta', \xi]\rangle + \langle \nu, [\xi, \eta']\rangle = 0&\end{aligned}$$

Notice that the curvature term in the formula for $\mathbf{d}\alpha$ vanishes since it is evaluated on a vertical vector $\xi_Q(q)$. ■

4.2 The Part that Drops to $T^*(Q/G)$

We now characterize the first term of

$$\pi_1^*\iota_{V^0}^*\Omega - (\tau_{T^*Q} \times \mathrm{id})^*\mathbf{d}\alpha - \pi_2^*\omega_{\mathcal{O}}^+ \tag{4.3}$$

as the pull back relative to Γ of the canonical form on $T^*(Q/G)$.

Proposition 4.2. *Denote by*

$$\pi_{\mathcal{A}} : V^0 \to T^*(Q/G) \tag{4.4}$$

the map given by $\alpha_q \mapsto \operatorname{hor}_q^*\alpha_q \in T_{[q]}^*(Q/G)$. Note that this is simply the map Γ restricted to V^0. Let Θ denote the canonical one-form on T^*Q and $\Theta_{Q/G}$ the canonical one-form on $T^*(Q/G)$. We then have

$$\pi_{\mathcal{A}}^*\Theta_{Q/G} = \iota_{V^0}^*\Theta \tag{4.5}$$

from which it follows that

$$\pi_{\mathcal{A}}^*\Omega_{Q/G} = \iota_{V^0}^*\Omega \tag{4.6}$$

Proof. Let $X_{\alpha_q} \in T_{\alpha_q}V^0$. We have

$$\begin{aligned}\pi_{\mathcal{A}}^*\Theta_{Q/G}(X_{\alpha_q}) &= \Theta_{Q/G}(\operatorname{hor}_q^*\alpha_q)\\ &= \left\langle \operatorname{hor}_q^*\alpha_q, T\tau_{Q/G} \circ T\pi_{\mathcal{A}} \cdot X_{\alpha_q}\right\rangle\end{aligned}$$

We need to compute the derivative of the composition,

$$\tau_{Q/G} \circ \pi_{\mathcal{A}} : V^0 \to Q/G \tag{4.7}$$

Let $t \mapsto \alpha_q(t) \in V^0$ be a smooth curve through α_q such that $\dot{\alpha}_q(0) = X_{\alpha_q}$. Let $q(t) = \tau_{T^*Q}(\alpha_q(t))$. Then

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \tau_{Q/G} \circ \pi_{\mathcal{A}}(\alpha_q(t)) &= \left.\frac{d}{dt}\right|_{t=0} \tau_{Q/G}(\mathrm{hor}^*_{q(t)}(\alpha_q(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} [q(t)] = T\pi \circ T\tau_{T^*Q} \cdot X_{\alpha_q} \end{aligned}$$

Thus,

$$\left\langle \mathrm{hor}^*_q \alpha_q, T\tau_{Q/G} \circ T\pi_{\mathcal{A}} \cdot X_{\alpha_q} \right\rangle = \left\langle \alpha_q, \mathrm{hor}_q \circ T\pi \circ T\tau_{T^*Q} \cdot X_{\alpha_q} \right\rangle \tag{4.8}$$

On the other hand, we have

$$\begin{aligned} \iota^*_{V^0}\Theta(\alpha_q)(X_{\alpha_q}) &= \left\langle \alpha_q, T\tau_{T^*Q} \cdot X_{\alpha_q} \right\rangle \\ &= \left\langle \alpha_q, \mathrm{Hor}_q\, T\tau_{T^*Q} \cdot X_{\alpha_q} + \mathrm{Ver_q T\tau_{T^*Q} \cdot X_{\alpha_q}} \right\rangle \\ &= \left\langle \alpha_q, \mathrm{Hor}_q\, T\tau_{T^*Q} \cdot X_{\alpha_q} \right\rangle \\ &= \left\langle \alpha_q, \mathrm{hor}_q \circ T\pi \circ T\tau_{T^*Q} \cdot X_{\alpha_q} \right\rangle \end{aligned}$$

where the third equality follows from the fact that α_q annihilates vertical vectors. ■

4.3 A Final Piece of Diagram Chasing

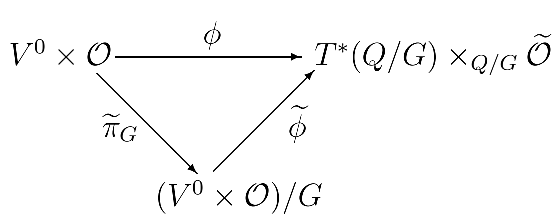
Recall that we have the following maps:

$$V^0 \times \mathcal{O} \quad \overset{\tau_{T^*Q} \times \mathrm{id}}{\longrightarrow} \quad Q \times \mathcal{O} \quad \overset{\pi_\mathrm{G}}{\longrightarrow} \quad \widetilde{\mathcal{O}}.$$

Define the map ϕ as follows:

$$\phi(\alpha_q, \nu) = (\mathrm{hor}^*_q \circ \pi_1, \pi_G \circ (\tau_{T^*Q} \times \mathrm{id})) \tag{4.9}$$

It is easy to see that ϕ is G-invariant, so that we have the following commutative diagram.



It is straightforward to check that the map $\widetilde{\phi}$ is invertible and therefore determines a bundle isomorphism.

Theorem 4.3. *Denote by ω_{red} the reduced symplectic form on the symplectic reduced space $T^*(Q/G) \times_{Q/G} \widetilde{\mathcal{O}}$. We then have the formula*

$$\omega_{\mathrm{red}} = \Omega_{Q/G} - \beta$$

where β is the unique two form on $\widetilde{\mathcal{O}}$ determined by

$$\pi_G^*\beta = \mathbf{d}\alpha + \pi_2^*\omega_{\mathcal{O}}^+$$

and, as in Theorem 3.5,

$$\begin{aligned} &\mathbf{d}\alpha(q,\nu)((X_q, X_\nu^{\xi'}), (Y_q, Y_\nu^{\eta'})) \\ &\quad = \langle \nu, [\eta', \xi] \rangle + \langle \nu, [\eta, \xi'] \rangle + \langle \nu, [\xi, \eta] \rangle + \langle \nu, \mathrm{Curv}_{\mathcal{A}}(q)(X_q, Y_q) \rangle \end{aligned} \tag{4.10}$$

Proof. The two-form ω_{red} is the unique two-form on $T^*(Q/G) \times_{Q/G} \widetilde{\mathcal{O}}$ such that $\widetilde{\phi}^*\omega_{\mathrm{red}} = \bar{\chi}^*\Omega_{\mathcal{O}}$, where $\bar{\chi}^*\Omega_{\mathcal{O}}$ (see equation 3.5) is the unique two-form on $(V^0 \times \mathcal{O})/G$ such that

$$\widetilde{\pi}_G^*\bar{\chi}^*\Omega_{\mathcal{O}} = \pi_1^*\iota_{V^0}^*\Omega - (\tau_{T^*Q} \times \mathrm{id})^*\mathbf{d}\alpha - \pi_2^*\omega_{\mathcal{O}}^+.$$

We then have

$$\widetilde{\pi}_G^*\widetilde{\phi}^*\omega_{\mathrm{red}} = \pi_1^*\iota_{V^0}^*\Omega - (\tau_{T^*Q} \times \mathrm{id})^*\mathbf{d}\alpha - \pi_2^*\omega_{\mathcal{O}}^+$$

However, since $\widetilde{\phi} \circ \widetilde{\pi}_G = \phi$, we have

$$\phi^*\omega_{\mathrm{red}} = \pi_1^*\iota_{V^0}^*\Omega - (\tau_{T^*Q} \times \mathrm{id})^*\mathbf{d}\alpha - \pi_2^*\omega_{\mathcal{O}}^+ \tag{4.11}$$

from which we can read off ω_{red}:

$$\begin{aligned} &\phi^*\omega_{\mathrm{red}}(\alpha_q, \nu)((X_{\alpha_q}, X_\nu), (Y_{\alpha_q}, Y_\nu)) = \\ &\quad \omega_{\mathrm{red}}(\mathrm{hor}_q^* \alpha_q, [q,\nu])\big((T(\mathrm{hor}^* \circ \pi_1)(X_{\alpha_q}, X_\nu), \\ &\quad T(\pi_G \circ \tau_{T^*Q} \times \mathrm{id})(\mathrm{X}_{\alpha_q}, \mathrm{X}_\nu)), (\mathrm{T}(\mathrm{hor}^* \circ \pi_1)(\mathrm{Y}_{\alpha_q}, \mathrm{Y}_\nu), \\ &\quad T(\pi_G \circ \tau_{T^*Q} \times \mathrm{id})(\mathrm{Y}_{\alpha_q}, \mathrm{Y}_\nu))\big) \end{aligned}$$

Note that $T(\mathrm{hor}^* \circ \pi_1)(X_{\alpha_q}, X_\nu) = T\pi_{\mathcal{A}} X_{\alpha_q}$ and

$$T(\pi_G \circ (\tau_{T^*Q} \times \mathrm{id}))(\mathrm{X}_{\alpha_q}, \mathrm{X}_\nu) = \mathrm{T}_{(q,\nu)}\pi_{\mathrm{G}} \cdot (\mathrm{T}\tau_{\mathrm{T^*Q}}\mathrm{X}_{\alpha_q}, \mathrm{X}_\nu)$$

The right hand side of equation (4.11) becomes

$$\begin{aligned} \Omega(\alpha_q)(X_{\alpha_q}, Y_{\alpha_q}) &- (\tau_{T^*Q} \times \mathrm{id})^*\pi_{\mathrm{G}}^*\beta(\mathrm{X}_{\alpha_q}, \mathrm{Y}_{\alpha_q}) \\ &= \Omega_{Q/G}(\pi_{\mathcal{A}}(\alpha_q))(T\pi_{\mathcal{A}} X_{\alpha_q}, T\pi_{\mathcal{A}} Y_{\alpha_q}) \\ &\quad - \beta(([q,\nu])(T_{(q,\nu)}\pi_G \cdot (T\tau_{T^*Q} X_{\alpha_q}, X_\nu), \\ &\quad T_{(q,\nu)}\pi_G \cdot (T\tau_{T^*Q} Y_{\alpha_q}, Y_\nu)), \end{aligned}$$

from which the the claim follows. ■

5 The extreme cases

The obvious extreme cases are $Q = G$ and G Abelian. We first consider the case $Q = G$. Then, Q/G reduces to a point, and the associated bundle is simply the coadjoint orbit through ν_0. $\widetilde{\mathcal{O}} = Q \times \mathcal{O}/G = G \times \mathcal{O}/G \simeq \mathcal{O}$. Consider a tangent vector to the coadjoint orbit through a point ν, given by $-\operatorname{ad}^*_{\xi'} \nu$. Represent this tangent vector with the curve through ν, $t \mapsto \operatorname{Ad}^*_{\exp -t\xi'} \nu$. We must find a lift to $G \times \mathcal{O}$ of such a tangent vector. The projection, $\pi_G : G \times \mathcal{O} \to \mathcal{O}$ is given by $(g, \nu) \mapsto g^{-1}\nu$ since $[g, \nu] = [e, g^{-1}\nu]$. More generally, consider a curve through $(e, \nu) \in G \times \mathcal{O}$ denoted by $t \mapsto (g(t), \nu(t))$. Let $\xi = \dot{g}(0)$. Since $\mathcal{A}(e)(\dot{g}(0)) = \dot{g}(0)$, this is consistent notation. The projection of this curve to $\mathcal{O}$ is given by

$$\pi_G(g(t), \nu(t)) = g(t)^{-1}\nu(t) = \operatorname{Ad}^*_{g(t)} \nu(t) \tag{5.1}$$

and therefore we require

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad}^*_{g(t)} \nu(t) = -\operatorname{ad}^*_{\xi'} \nu,$$

which implies

$$\operatorname{Ad}^*_g \dot{\nu}(o) + \operatorname{ad}^*_{\dot{g}(0)} \nu = -\operatorname{ad}^*_{\xi'} \nu, \tag{5.2}$$

from which it follows that

$$\dot{\nu}(0) = -\operatorname{ad}^*_{\xi'} \nu - \operatorname{ad}^*_{\xi} \nu$$

Equations (4.1) and (4.10) give

$$\begin{aligned}
\omega_{\mathrm{red}}(\nu)(-\operatorname{ad}^*_{\xi'} \nu, \operatorname{ad}^*_{\eta'} \nu) &= \Omega_{Q/G} - \beta(e,\nu)((\xi, X_\nu^{\xi'+\xi}), (\eta, Y_\nu^{\eta'+\eta})) \\
&= -\big(\mathbf{d}\alpha + \pi_2^* \omega_{\mathcal{O}}^+\big)(e,\nu)((\xi, X_\nu^{\xi'+\xi}), (\eta, Y_\nu^{\eta'+\eta})) \\
&= -\big(\langle \nu, [\eta' + \eta] \rangle + \langle \nu, [\eta, \xi' + \xi] \rangle + \langle \nu, [\xi, \eta] \rangle \\
&\quad + \operatorname{Curv}_{\mathcal{A}}(e)(\xi, \eta) + \langle \nu, [\xi' + \xi, \eta' + \eta] \rangle \big) \\
&= -\langle \nu, [\xi', \eta'] \rangle,
\end{aligned}$$

where the last equality follows from the fact that the curvature term vanishes on vertical vectors and an expansion of the Lie algebra brackets.

For G Abelian, the fibers of the $\widetilde{\mathcal{O}}$ bundle collapse and we are left with just $T^*(Q/G)$. The reduced symplectic form, from equations 4.1 and 4.10 is then

$$\omega_{red} = \Omega - \langle \nu, \operatorname{Curv}_{\mathcal{A}} \rangle \tag{5.3}$$

since all brackets vanish.

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