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REDUCTION AND HAMILTONIAN STRUCTURES ON DUALS OF SEMIDIRECT PRODUCT LIE ALGEBRAS

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ABSTRACT. With the heavy top and compressible flow as guiding examples, this paper discusses the Hamiltonian structure of systems on duals of semidirect product Lie algebras by reduction from Lagrangian to Eulerian coordinates. Special emphasis is placed on the left-right duality which brings out the dual role of the spatial and body (i.e. Eulerian and convective) descriptions. For example, the heavy top in spatial coordinates has a Lie-Poisson structure on the dual of a semidirect product Lie algebra in which the moment of inertia is a dynamic variable. For compressible fluids in the convective picture, the metric tensor similarly becomes a dynamic variable. Relationships to the existing literature are given.

1. INTRODUCTION. There are natural brackets $\{f,g\}$ defined for $f,g:\mathfrak{g}^* \to \mathbb{R}$ where $\mathfrak{g}^*$ is the dual of a Lie algebra (finite or infinite dimensional); these were discovered by Lie in 1887 and are now called Lie-Poisson brackets. These brackets arise by reduction of canonical Poisson brackets on T^*G, the cotangent bundle of the corresponding group, by left or right invariance (giving structures differing in sign) and are compatible with the Kirillov-Kostant symplectic structures on coadjoint orbits (Marsden and Weinstein [1974]). We review some features of this theory in §4.

Lie-Poisson structures in mechanics have a complex history due, in part, to lack of communication and ignorance of Lie's original discovery. We are concerned here with the line of investigation initiated by Arnold [1966], [1969] in which he gave a clear presentation of the reduction from material (i.e. Lagrangian) coordinates to spatial (i.e. Eulerian) and body (i.e. convective) coordinates for incompressible fluids and the rigid body. Arnold used symplectic structures on coadjoint orbits but did not use the Lie-Poisson bracket. In spite of this, Kuznetsov and Mikhailov [1980], for example, attribute it to him, we think quite appropriately.

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Lie-Poisson structures for semi-direct products have the following history. They were noted for the heavy top in Vinogradov and Kupershmidt [1977]. They appear, using a quantum mechanical motivation, in Dzyaloshinskii and Volovick [1980] (see also Dashen and Sharp [1968], Goldin [1971] and Goldin, Menikoff and Sharp [1980]). For our development, the papers of Guillemin and Sternberg [1980] and Ratiu [1980, 1981, 1982] are crucial. They started developing the abstract setting in which Lie-Poisson structures associated with semi-direct products appear. Simultaneously, Morrison and Greene [1980] and Morrison [1980] published brackets for MHD and the Maxwell-Vlasov equation. It was well-known to workers in the area that the bracket for ideal compressible flow was the Lie-Poisson bracket for the semi-direct product of the diffeomorphism group with functions. Marsden and Weinstein [1982] were the first to put the bracket structures for the Maxwell-Vlasov equation back in the spirit of Arnold and in doing so, corrected one term in the bracket--a correction necessary to ensure Jacobi's identity. The Morrison-Greene bracket for MHD was derived using Clebsch variables and was observed to be a Lie-Poisson bracket for a semidirect product by Holm and Kupershmidt [1983a].

In Ratiu [1980,1981,1982] and Guillemin and Sternberg [1980] a general scheme began emerging in which semi-direct products arose by reduction from T^*G by a subgroup. For example, a special case of their result shows that when $T^*SO(3)$ is reduced by an S^1 subgroup, corresponding to invariance under rotations about the direction of gravity for the heavy top, one automatically gets the Lie-Poisson structure on the dual of the Euclidean group. Some improvements in this theory were given by Ratiu and Van Moerbeke [1982] and Holmes and Marsden [1983]. The sharpest results however, were given by Marsden, Ratiu and Weinstein [1983], who also incorporated the aforementioned fluid and plasma examples into the same scheme.

In the present paper we take the point of view of Poisson manifolds and shall be as concrete as possible, using the heavy top and compressible flow as detailed motivating examples for the general theory. In Marsden, Ratiu and Weinstein [1983] we studied the role of symplectic reduction and determined the symplectic leaves of the reduced spaces for T^*G divided by an isotropy subgroup of a representation of G on a vector space V. These were shown to be symplectically diffeomorphic to the coadjoint orbits in the dual of the semi-direct product $\mathfrak{g} \ltimes V$. This provided a satisfactory explanation of why semi-direct products occur in so many examples. Indeed, T^*G represents the basic Lagrangian phase space and reduction by the subgroup of symmetries represents the passage from Lagrangian to Eulerian or convective coordinates. In addition to the Poisson point of view, the new results in the present paper are:

a) A demonstration is given (in §4.4) that a generalization of the Poisson map of Holm, Kupershmidt and Levermore [1983] can be directly constructed from the setting of Marsden Ratiu and Weinstein [1983]. The basic idea is that by reducing $T^*(G \times V)$ by V one can pass from a Poisson map of $T^*(G \times V) \to (\mathfrak{g} \ltimes V)^*$ to one from $T^*G \times V^* \to (\mathfrak{g} \ltimes V)^*$. In addition, we give both the right and left reductions--they are not trivially related. The generalization to allow V to be a non abelian group is not hard; for example, it is covered by Montgomery, Marsden and Ratiu's contribution to these proceedings, dealing with semi-direct product bundles.

b) A derivation is given (in §5.2) of the Hamiltonian structure for the heavy top in spatial coordinates (it is usually given in body coordinates). Here the moment of inertia tensor is a dynamic variable; cf. Guillemin and Sternberg [1980].

c) A derivation is given (in §5.4) of the Hamiltonian structure for the equations of compressible flow in convective ("body") coordinates (it is usually given in Eulerian ("spatial") coordinates.) Here the metric tensor is a dynamic variable. In a future paper this formulation will be connected with the results of Simo and Marsden [1983] on the Doyle-Ericksen formula for the stress tensor ($\sigma = 2\rho\partial e/\partial g$, where e is the internal energy and g is the metric tensor), which is closely related to the covariant Hamiltonian formulation of elasticity (see also Marsden and Hughes [1983]).

The left-right duality which is emerging as a basic, yet usually overlooked, ingredient in the Lagrangian to Eulerian map is summarized as follows:

$$\begin{tikzcd} & \begin{pmatrix}\text{Lagrangian}\\ \text{representation}\end{pmatrix} \arrow[dl, "\text{Left translations}"'] \arrow[dr, "\text{Right translations}"] & \\ \begin{pmatrix}\text{body or convective}\\ \text{representation}\end{pmatrix} & & \begin{pmatrix}\text{spatial or Eulerian}\\ \text{represenation}\end{pmatrix} \end{tikzcd}$$

If the basic Lagrangian space one starts with is T^*G, as is appropriate for

i) the free rigid body ($G = SO(3)$)

ii) incompressible flow (G = volume preserving diffeomorphism)

or iii) the Poisson-Vlasov equation (G = canonical transformation), the picture specializes to

$$
\begin{tikzcd}
 & T^*G \arrow[dl, "\text{left translation implemented by } J_R"'] \arrow[dr, "\text{right translation implemented by } J_L"] & \\
\mathfrak{g}^*_- & & \mathfrak{g}^*_+
\end{tikzcd}
$$

where $\mathfrak{g}^*_\pm$ is $\mathfrak{g}^*$ with + or - Lie-Poisson structures (this is reviewed briefly in §4.2 below) and J_R and J_L are the momentum maps associated to the right and left actions of G respectively.

If the basic Lagrangian space one starts with is the cotangent bundle $T^*(G \times V)$ i.e. the basic configuration variables are $G \times V$, then we get a more detailed picture:

$$
\begin{tikzcd}
T^*G \times V^* \arrow[d, "\tilde{J}_R"'] & T^*(G \times V) \arrow[l, "P_L"'] \arrow[r, "P_R"] \arrow[dl, "J_R"] \arrow[dr, "J_L"'] & T^*G \times V^* \arrow[d, "\tilde{J}_L"] \\
(\mathfrak{g} \times V)^*_- & & (\mathfrak{g} \times V)^*_+ \\
\text{(convective)} & & \text{(spatial)}
\end{tikzcd}
$$

Here, J_R, $\tilde{J}_R$, J_L and $\tilde{J}_L$ are momentum maps for the left and right actions of $G \ltimes V$ on $T^*(G \times V)$ and $T^*G \times V^*$. These maps include, as special cases, Poisson maps found by Guillemin and Sternberg, Ratiu, Kupershmidt, and Holm, Kupershmidt and Levermore. The maps P_L and P_R are Poisson maps implementing the reduction by V; while P_L just projects out V, P_R involves a fiber translation by a differential (such maps play an important role in Guillemin and Sternberg [1980] and in Marsden and Weinstein [1982]). This asymmetry between left and right occurs because we chose G to act on V on the left.

The plan of the paper is as follows. In sections 2 and 3, concrete and detailed expositions of the Hamiltonian structure for the heavy top and compressible flow are given. Here things are done more or less by bare hands both to motivate and show the power of the abstract theory, presented in §4. In §5 we return to the examples to show how the theory works. There are other examples as well. See Marsden, Ratiu and Weinstein [1983] for MHD, multifluid plasmas and the Maxwell-Vlasov equations, Montgomery, Marsden and Ratiu [1984, and this volume] for the Yang-Mills-Vlasov equations, Marsden and Weinstein [1983] for incompressible flow, and Abarbanel, Holm, Marsden and Ratiu [1984] for stratified flow.

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2. THE HEAVY TOP.

2.1 Configuration Space. A top is, by definition a rigid body moving about a fixed point in three dimensional space. A reference configuration $\mathcal{B}$ of the body is the closure of an open set in $\mathbb{R}^3$ with piecewise smooth boundary. Points in $\mathcal{B}$ denoted $\underline{X} = (X^1, X^2, X^3) \in \mathcal{B}$ are called material points and X^i, $i = 1,2,3$ material coordinates. A configuration of $\mathcal{B}$ is a mapping $\eta:\mathcal{B} \to \mathbb{R}^3$ that has certain smoothness properties, is orientation preserving, and is invertible on its image. The points of the target space of η are called spatial points and are denoted by lower case letters $\underline{x} = (x^1,x^2,x^3)$ $\mathbb{R}^3$; x^i, $i = 1,2,3,$ are called spatial coordinates. A motion of $\mathcal{B}$ is a time dependent family of configurations, written $\underline{x} = \eta(\underline{X},t) = \eta_t(\underline{X})$ or simply $\underline{x}(\underline{X},t)$ or $\underline{x}_t(\underline{X})$. Spatial quantities are maps whose doman is $\mathbb{R}^3$, i.e. they are functions of $\underline{x}$. They are lower case letters such as z (if scalar valued) or $\underline{z}$ (if vector valued). By composition with η_t, spatial quantities become functions of the material points $\underline{X}$.

Dually, one can consider material quantities such as scalar maps $Z:\mathcal{B} \to \mathbb{R}$ or vector maps $\underline{Z}:\mathcal{B} \to \mathbb{R}^3$. Then we can form spatial quantities by composition: $z_t = Z_t \circ \eta_t^{-1}$ and $\underline{z}_t = \underline{Z}_t \circ \eta_t^{-1}$.

In addition to the material and spatial coordinates, there is a third set, the convected or body coordinates. These are the coordinates associated with a moving basis. Although these are defined in general (Marsden and Hughes [1983] p. 41) we shall first consider them in the context of a rigid body.

In §4,5 we shall see the following picture emerge of which the present discussion is a special case:

$$
\begin{tikzcd}
 & \begin{array}{c}\text{(Lagrangian or}\\ \text{material coordinates)}\\ T^*G\end{array} \arrow[dl, "\text{left translation}"'] \arrow[dr, "\text{right translation}"] & \\
\begin{array}{c}\mathfrak{g}^*\\ \text{(body or convected}\\ \text{coordinates)}\end{array} & & \begin{array}{c}\mathfrak{g}^*\\ \text{(Eulerian or spatial}\\ \text{coordinates)}\end{array}
\end{tikzcd}
$$

Rigidity of the top means that the distances between points of the body are fixed as the body moves. This says that if the configuration $\underline{x}(\underset{\sim}{X},t)$ represents the position of a particle that was at $\underline{X}$ when $t = 0$, we have

$$\underline{x}(\underline{X},t) = A(t)\underline{X} \quad \text{i.e.} \quad x^i = A^i_j(t)X^j \ , \quad i,j = 1,2,3, \text{ sum on } j \tag{2.1}$$

where $A(t) = (A^i{}_j(t))$ is an orthogonal matrix. Since the motion is assumed to be at least continuous and $A(0)$ is the identity matrix, it follows that $\det(A(t)) = 1$ and thus $A(t) \in SO(3)$, the proper orthogonal group. Thus, <u>the configuration space of the heavy top may be identified with SO(3). Consequently the phase space of the top is the cotangent bundle</u> $T^*(SO(3))$, <u>which will be described in 2.4.</u>

Now we are ready to define convected, or body coordinates. Let $\underset{\sim}{E}_1$, $\underset{\sim}{E}_2$, $\underset{\sim}{E}_3$ be an orthonormal basis relative to which material coordinates $\underline{X} = (X^1, X^2, X^3)$ are defined and $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ be an orthonormal basis associated to spatial coordinates. Let the time dependent basis $\underset{\sim}{\xi}_1$, $\underset{\sim}{\xi}_2$, $\underset{\sim}{\xi}_3$ be defined by

$$\underset{\sim}{\xi}_i = A(t)\underset{\sim}{E}_i$$

so $\underset{\sim}{\xi}_i$ move attached to the body. The <u>body coordinates</u> of a vector in $\mathbb{R}^3$ are its components relative to $\underset{\sim}{\xi}_i$. For $\underline{v} \in \mathbb{R}^3$, its spatial coordinates v^i are related to its body coordinates ν^j by

$$v^i = A^i{}_j \nu^j$$

where $A^i{}_j$ is the matrix of A relative to $\underset{\sim}{E}_i$ and $\underset{\sim}{e}_i$. Of course the components of a vector $\underline{V}$ relative to $\underset{\sim}{E}_i$ are the same as the components of $A\underline{V}$ relative to $\underset{\sim}{\xi}_i$. In particular, the body coordinates of $\underline{x}$ are X^i.

2.2 <u>Euler Angles</u> are the traditional way to express the relationship between space and body coordinates, i.e. to parametrize SO(3). In what follows we shall adopt the conventions of Arnold [1978] and Goldstein [1980] which are different from those of Whittaker [1917].

One can pass from the spatial basis $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ to the body basis $(\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2, \underset{\sim}{\xi}_3)$, by means of three consecutive counterclockwise rotations performed in a specific order: first rotate by the angle ϕ around $\underset{\sim}{e}_3$ and denote the new position of $\underset{\sim}{e}_1$ by ON (line of nodes), then rotate by the angle θ around ON, and finally rotate by the angle ψ around $\underset{\sim}{\xi}_3$ (see Fig. 1). Consequently $0 \le \phi, \psi < 2\pi$ and $0 \le \theta < \pi$. Note that there is a bijective map between the (ϕ, ψ, θ) variables and SO(3). However, this

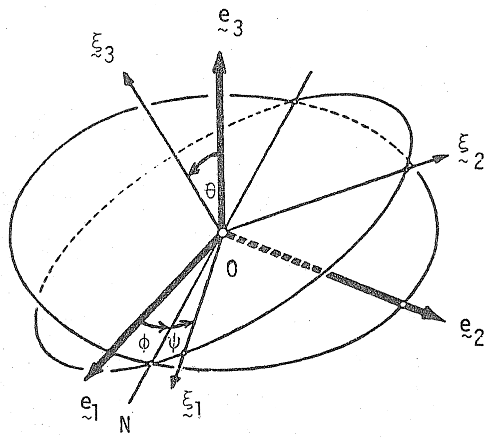

Figure 1

bijective map does not define a chart, since its differential vanishes, for example, at $\phi = \psi = \theta = 0$. The differential is non-zero for $0 < \phi < 2\pi$, $0 < \psi < 2\pi$, $0 < \theta < \pi$ and on this domain, the Euler angles do form a chart. Explicitly this is given by $(\phi,\psi,\theta) \mapsto A$, where A is uniquely determined by $\underline{x} = A\underline{X}$ and has the matrix relative to $\underset{\approx}{E}_i$ and $\underset{\approx}{e}_i$ given by

$$A = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \tag{2.2}$$

With the aid of the chart given by Euler angles we induce a natural chart $(\phi,\psi,\theta,\ \dot\phi,\dot\psi,\dot\theta)$ on the tangent bundle $T(SO(3))$ of the proper rotation group $SO(3)$. Then using a Legendre transformation given by a certain metric on $SO(3)$ uniquely determined by the mass distribution of the top, we will define a mapping to the natural chart $(\phi,\psi,\theta,\ p_\phi,\ p_\psi,\ p_\theta)$ on the cotangent bundle $T^*(SO(3))$ which is the canonical phase space. This will be done in §2.4.

2.3 The Lie Algebra so(3) and Its Dual. In order to simplify the computations and identify the geometrical structure of the Hamiltonian of the heavy top, a summary of the Lie algebra $\mathfrak{so}(3)$ and its dual are needed.

The proper rotation group $SO(3)$ has as Lie algebra the 3×3 infinitesimal rotation matrices, i.e. the space $so(3)$ of 3×3 skew-symmetric matrices; the bracket operation is the commutator of matrices. The Lie algebra $so(3)$ is identified with $\mathbb{R}^3$ by associating to the vector $\underline{v} = (v^1, v^2, v^3) \in \mathbb{R}^3$, the matrix $\hat{\underline{v}} \in so(3)$ given by

$$\hat{\underline{v}} = \begin{bmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{bmatrix} . \tag{2.3}$$

Then we have the following identities:

$$(\underline{u} \times \underline{v})\hat{} = [\hat{\underline{u}}, \hat{\underline{v}}] \tag{2.4}$$

$$\hat{\underline{u}} \cdot \underline{v} = \underline{u} \times \underline{v} \tag{2.5}$$

$$[\hat{\underline{u}}, \hat{\underline{v}}] \cdot \underline{w} = (\underline{u} \times \underline{v}) \times \underline{w} \tag{2.6}$$

$$\underline{u} \cdot \underline{v} = -\frac{1}{2} \mathrm{Tr}(\underline{\hat{u}\hat{v}}) . \tag{2.7}$$

Moreover if $A \in SO(3)$ and $\underline{v} \in \mathbb{R}^3$, then the adjoint action (conjugation) is given by

$$(A\underline{v})\hat{} = \mathrm{Ad}_A \underline{v}\hat{} := A\hat{\underline{v}}A^{-1} . \tag{2.8}$$

Consequently, since the adjoint action is a Lie algebra homomorphism, for all $A \in SO(3)$, $\underline{u}, \underline{v} \in \mathbb{R}^3$ we recover the vector algebra identity

$$A(\underline{u} \times \underline{v}) = A\underline{u} \times A\underline{v}. \tag{2.9}$$

In what follows we shall identify the dual $so(3)^*$ with $\mathbb{R}^3$ by the inner product, i.e. $\tilde{\underline{m}} \in so(3)^*$ corresponds to $\underline{m} \in \mathbb{R}^3$ by $\tilde{\underline{m}}(\underline{v}) = \underline{m} \cdot \underline{v}$, for all $\underline{v} \in \mathbb{R}^3$. Then the coadjoint action of $SO(3)$ on $so(3)^*$ is represented by the usual action of $SO(3)$ on $\mathbb{R}^3$, i.e.

$$\mathrm{Ad}^*_{A^{-1}} \underline{m} = A\underline{m} \tag{2.10}$$

since $(A^{-1})^T = A$.

2.4 <u>The Hamiltonian</u>. If $\underline{X} \in \mathcal{B}$ is a point of the body, then the trajectory followed by $\underline{X}$ in space is $\underline{x}(t) = A(t)\underline{X}$, where $A(t) \in SO(3)$. The <u>material</u> or <u>Lagrangian velocity</u> $\underline{V}(\underline{X}, t)$ is defined by

$$\underline{V}(\underline{X}, t) = \partial\underline{x}(\underline{X}, t)/\partial t = \dot{A}(t)\underline{X}, \tag{2.11}$$

the spatial or Eulerian velocity $\underline{v}(\underline{x},t)$ by

$$\underline{v}(\underline{x},t) = \partial \underline{x}(\underline{X},t)/\partial t = \underline{V}(\underline{X},t) = \dot{A}(t)A(t)^{-1}\underline{x}, \tag{2.12}$$

and the body or convective velocity $\underline{\mathcal{V}}(\underline{X},t)$ by

$$\begin{aligned}\underline{\mathcal{V}}(\underline{X},t) &= -\partial \underline{X}(\underline{x},t)/\partial t = A(t)^{-1}\dot{A}(t)A(t)^{-1}\underline{x} \\ &= A(t)^{-1}\dot{A}(t)\underline{X} = A(t)^{-1}\underline{V}(\underline{X},t) = A(t)^{-1}\underline{v}(\underline{x},t).\end{aligned} \tag{2.13}$$

Let $\rho_0(\underline{X})$ denote the density of the body in the reference configuration. Then the kinetic energy at time t is, by (2.11), (2.12), (2.13), and the invariance of the Euclidean norm under SO(3),

$$K(t) = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\underline{X})\|\underline{V}(\underline{X},t)\|^2\, d^3\underline{X} \text{ (material)} \tag{2.14}$$

$$= \frac{1}{2}\int_{A(t)\mathcal{B}} \rho_0(A(t)^{-1}\underline{x})\|\underline{v}(\underline{x},t)\|^2\, d^3\underline{x} \quad \text{(spatial)} \tag{2.14}$$

$$= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\underline{X})\|\mathcal{V}(\underline{X},t)\|^2\, d^3\underline{X} \quad \text{(body)} . \tag{2.16}$$

Differentiating $A(t)^TA(t) = \text{Identity}$ and $A(t)\,A(t)^T = \text{Identity}$, it follows that both $A(t)^{-1}\dot{A}(t)$ and $\dot{A}(t)A(t)^{-1}$ are skew-symmetric. Moreover, by (2.12), (2.13), (2.5) and the classical definition of angular velocity, it follows that the vectors $\underline{\omega}_S(t)$ and $\underline{\omega}_B(t)$ in $\mathbb{R}^3$ defined by

$$\hat{\underline{\omega}}_S(t) = \dot{A}(t)A(t)^{-1} \tag{2.17}$$

$$\hat{\underline{\omega}}_B(t) = A(t)^{-1}\dot{A}(t) \tag{2.18}$$

are the spatial and body angular velocities of the top. Note that $\underline{\omega}_S(t) = A(t)\underline{\omega}_B(t)$, or as matrices, $\hat{\underline{\omega}}_S = \mathrm{Ad}_A\hat{\underline{\omega}}_B = A\hat{\underline{\omega}}_BA^{-1}$. Using the Euler angle parametrization (2.2) of SO(3), (2.17), and (2.18), $\underline{\omega}_S$ and $\underline{\omega}_B$ have the following expressions

$$\underline{\omega}_S = \begin{bmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\sin\theta \\ \dot{\theta}\sin\phi - \dot{\psi}\cos\phi\sin\theta \\ \dot{\phi} + \dot{\psi}\cos\theta \end{bmatrix}, \quad \underline{\omega}_B = \begin{bmatrix} \dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta \\ -\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\sin\theta \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix} \tag{2.19}$$

Due to the fact that in (2.14) and (2.16), ρ_0 is independent of time, the kinetic energy can be expressed in a simple manner in the material and reference configurations. We have by (2.16) and (2.5),

$$K(t) = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\underline{X})\|\underline{\omega}_B(t) \times \underline{X}\|^2 \, d^3\underline{X}. \tag{2.20}$$

Using (2.19), the kinetic energy of the body is a function of $(\phi, \psi, \theta, \dot{\phi}, \dot{\psi}, \dot{\theta})$ or of $\underline{\omega}_B$. To give it a more familiar expression, introduce the following inner product on $\mathbb{R}^3$,

$$\langle\!\langle \underline{a}, \underline{b} \rangle\!\rangle : = \int_{\mathcal{B}} \rho_0(\underline{X})(\underline{a} \times \underline{X})\cdot(\underline{b} \times \underline{X}) d^3\underline{X}, \tag{2.21}$$

completely determined by the density $\rho_0(\underline{X})$ of the body. Then (2.20) becomes

$$K(\underline{\omega}_B) = \frac{1}{2} \langle\!\langle \underline{\omega}_B, \underline{\omega}_B \rangle\!\rangle \tag{2.22}$$

Now define the linear isomorphism $I:\mathbb{R}^3 \to \mathbb{R}^3$ by $I\underline{a}\cdot\underline{b} = \langle\!\langle \underline{a}, \underline{b} \rangle\!\rangle$ for all $\underline{a}, \underline{b} \in \mathbb{R}^3$; this is possible and uniquely determines I, since both the dot product and $\langle\!\langle\, , \,\rangle\!\rangle$ are nondegenerate bilinear forms.† It is clear that I is symmetric with respect to the dot product and is positive. To gain a physical interpretation of I we compute its matrix. Let $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$ be an orthonormal basis for material coordinates. Thus,

$$I_{ij} = (I\underset{\sim}{E}_j)\cdot\underset{\sim}{E}_i = \langle\!\langle \underset{\sim}{E}_j, \underset{\sim}{E}_i \rangle\!\rangle = \begin{cases} -\int_{\mathcal{B}} \rho_0(\underline{X}) X^i X^j d^3\underline{X}, & \text{if } i \neq j \\ \int_{\mathcal{B}} \rho_0(\underline{X})(\|\underline{X}\|^2 - (X^i)^2) d^3\underline{X}, & \text{if } i = j \end{cases} \tag{2.23}$$

which are the expressions of the matrix of the <u>inertia tensor</u> from classical mechanics. Thus I represents the inertia tensor. Since it is symmetric, it can be diagonalized; the basis in which it is diagonal is a <u>principal axis body frame</u> and the diagonal elements I_1, I_2, I_3 are the <u>principal moments of inertia</u> of the rigid body. In what follows we work in a principal axis body frame.

To get from (2.22) a function defined on $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ we must take into account that $\mathfrak{so}(3)^*$ and $\mathbb{R}^3$ are identified by the dot product and not by $\langle\!\langle\, , \,\rangle\!\rangle$. Consequently, the linear functional $\langle\!\langle \underline{\omega}_B, \cdot \rangle\!\rangle$ on $\mathfrak{so}(3) \cong \mathbb{R}^3$ is identified with $I\underline{\omega}_B := \underline{m} \in \mathfrak{so}(3)^* \cong \mathbb{R}^3$ since $\underline{m}\cdot\underline{a} = \langle\!\langle \underline{\omega}_B, \underline{a} \rangle\!\rangle$ for all $\underline{a} \in \mathbb{R}^3$. Hence (2.22) becomes, for $I = \mathrm{diag}(I_1, I_2, I_3)$,

$$K(\underline{m}) = \frac{1}{2}\, \underline{m}\cdot I^{-1}\underline{m} = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right) \tag{2.24}$$

†Assuming the rigid body is not concentrated on a line.

which represents the expression of K on $\mathfrak{so}(3)^*$; note that $\underline{m} = I\underline{\omega}_B$ is the angular momentum in the body frame.

By the second formula in (2.19) and the definition of $\underline{m}$ for $I = \mathrm{diag}(I_1, I_2, I_3)$, it follows that

$$\underline{m} = \begin{bmatrix} I_1(\dot{\phi} \sin\theta \sin\psi + \dot{\theta}\cos\psi) \\ I_2(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ I_3(\dot{\phi}\cos\theta + \dot{\psi}) \end{bmatrix} \tag{2.25}$$

This expresses $\underline{m}$ in terms of coordinates on $T(SO(3))$. Since $T(SO(3))$ and $T^*(SO(3))$ are to be identified by the metric defined as the left translate at every point of $\langle\langle\, ,\, \rangle\rangle$, the canonically conjugate variables $(p_\phi, p_\psi, p_\theta)$ to (ϕ, ψ, θ) are given by the Legendre transformation $p_\phi = \partial K/\partial\dot{\phi}$, $p_\psi = \partial K/\partial\dot{\psi}$, $p_\theta = \partial K/\partial\dot{\theta}$ of the kinetic energy on $T(SO(3))$ which is obtained by plugging (2.25) into (2.24). We get the standard formulas

$$\begin{aligned} p_\phi &= I_1(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\sin\theta\sin\psi \\ &\quad + I_2(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\sin\theta\cos\psi + I_3(\dot{\phi}\sin\theta + \dot{\psi})\cos\theta \\ p_\psi &= I_3(\dot{\phi}\cos\theta + \dot{\psi}) \\ p_\theta &= I_1(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\cos\psi - I_2(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\sin\psi \end{aligned} \tag{2.26}$$

whence by (2.25),

$$\underline{m} = \begin{bmatrix} [(p_\phi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi]/\sin\theta \\ [(p_\phi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi]/\sin\theta \\ p_\psi \end{bmatrix} \tag{2.27}$$

and so by (2.24) we get the coordinate expression of the kinetic energy in the material picture to be

$$K(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta) = \frac{1}{2}\left\{\frac{[(p_\phi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi]^2}{I_1\sin^2\theta} + \frac{(p_\phi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi]^2}{I_2\sin^2\theta} + \frac{p_\psi^2}{I_3}\right\}. \tag{2.28}$$

The potential energy V for a heavy top is determined by the height of the center of mass over a horizontal plane in the spatial coordinate system.

Let $\ell\underline{\chi}$ denote the vector determining the center of mass in the reference configuration (i.e. the body frame at $t = 0$), where $\underline{\chi}$ is a unit vector along the straight line segment of length ℓ connecting the fixed point with the center of mass. Thus, if $M = \int_{\mathbb{R}^3} d\mu(x)$ is the total mass of the body, g is the gravitational acceleration, and $\underline{k}$ denotes the unit vector along the spatial Oz axis, the potential energy at time t is

$$V(t) = Mg\underline{k}\cdot A(t)\ell\underline{\chi} = Mg\ell A^{-1}\underline{k}\cdot\underline{\chi} = Mg\ell\underline{\gamma}\cdot\chi = Mg\ell\underline{k}\cdot\underline{\lambda}$$

where $\underline{\gamma} = A^{-1}\underline{k}$ and $\underline{\lambda} = A\underline{\chi}$. Consequently,

$$\begin{aligned} V &= Mg\ell\underline{k}\cdot A\underline{\chi} && \text{(Lagrangian or material)} && (2.29)\\ &= Mg\ell\underline{k}\cdot\lambda && \text{(Eulerian or spatial)} && (2.30)\\ &= Mg\ell\underline{\gamma}\cdot\underline{\chi} && \text{(convective or body)} && (2.31)\end{aligned}$$

Thus, by (2.24) the Hamiltonian has the following expressions

$$H(\underline{m},\underline{\gamma}) = \frac{1}{2}\sum_{j=1}^{3}\frac{m_j^2}{I_j} + Mg\ell\gamma_3 \qquad \text{(body)} \qquad (2.32)$$

$$H = \frac{1}{2}\left\{\frac{[(p_\phi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi]^2}{I_1\sin^2\theta} + \frac{[(p_\phi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi]^2}{I_2\sin^2\theta} + \frac{p_\psi^2}{I_3}\right\} + Mg\ell\cos\theta \quad \text{(material)} \qquad (2.33)$$

The table at the end of this subsection (which appears in Holmes and Marsden [1983]) summarizes and completes the relations between m, $\underline{\gamma}$, ϕ, ψ, θ, $\dot\phi$, $\dot\psi$, $\dot\theta$, p_ϕ, p_ψ, p_θ.

We close with a study of the invariance properties on H on $T^*(SO(3))$. By (2.7) (2.18), (2.24), and (2.29), the Hamiltonian in the material configuration equals

$$H = -\frac{1}{4}\,\mathrm{Tr}(IA^{-1}\dot A A^{-1}\dot A) + Mg\ell\underline{k}\cdot A\underline{\chi}\,. \qquad (2.34)$$

Consequently, if B is a constant matrix in SO(3) and we replace A by BA (left translation), it is easily seen that the kinetic energy does not depend on B, i.e. it is invariant under the maps $A \mapsto BA$. The potential energy however is only invariant if $B\underline{k} = \underline{k}$, i.e. under rotations about the spatial Ox^3 axis. The corresponding conserved quantity is, by Hamilton's equation written in terms of $(\phi,\psi,\theta,\, p_\phi,\, p_\psi,\, p_\theta)$, $p_\phi = \underline{m}\cdot\underline{\gamma}$, since $\dot{p}_\phi = -\partial H/\partial\phi = 0$ by (2.33). There is one more conserved quantity in body coordinates namely $\|\underline{\gamma}\|^2 = 1$. The importance of $\underline{m}\cdot\underline{\gamma}$ and $\|\underline{\gamma}\|^2$ will become apparent in 2.6 and in §4.

Finally, let us note that H depends on the parameter $Mg\underline{\ell k}$. What happens if this parameter is changed will be explained in Sections 4 and 5.

$$m_1 = [(p_\phi - p_\psi \cos\theta)\sin\psi + p_\theta \sin\theta\cos\psi]/\sin\theta = I_1(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)$$

$$m_2 = [(p_\phi - p_\psi \cos\theta)\cos\psi - p_\theta \sin\theta\sin\psi]/\sin\theta = I_2(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)$$

$$m_3 = p_\psi = I_3(\dot\phi\cos\theta + \dot\psi)$$

$$\gamma_1 = \sin\theta\sin\psi$$

$$\gamma_2 = \sin\theta\cos\psi$$

$$\gamma_3 = \cos\theta$$

$$p_\phi = \underline{m}\cdot\underline{\gamma} = I_1(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\sin\theta\sin\psi + I_2(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)\sin\theta\cos\psi + I_3(\dot\phi\sin\theta + \dot\psi)\cos\theta$$

$$p_\psi = m_3 = I_3(\dot\phi\cos\theta + \dot\psi)$$

$$p_\theta = (\gamma_2 m_1 - \gamma_1 m_2)/\sqrt{1-\gamma_3^2} = I_1(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\cos\psi - I_2(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)\sin\psi$$

$$\dot\phi = \frac{1}{I_1}\,\frac{m_1\gamma_1}{1-\gamma_3^2} + \frac{m_2\gamma_2}{1-\gamma_3^2}$$

$$\dot\psi = \frac{m_3}{I_3} - \frac{m_3 m_1\gamma_1}{I_1(1-\gamma_3^2)} - \frac{m_3 m_2\gamma_2}{I_2(1-\gamma_3^2)}$$

$$\dot\theta = \frac{m_1\gamma_2}{I_1\sqrt{1-\gamma_3^2}} - \frac{m_2\gamma_1}{I_2\sqrt{1-\gamma_3^2}}$$

2.5 Equations of Motion. Hamilton's canonical equations

$$\dot\phi = \frac{\partial H}{\partial p_\phi},\qquad \dot\psi = \frac{\partial H}{\partial p_\psi},\qquad \dot\theta = \frac{\partial H}{\partial p_\theta}$$
$$\dot p_\phi = -\frac{\partial H}{\partial\phi},\qquad \dot p_\phi = -\frac{\partial H}{\partial\psi},\qquad \dot p_\theta = \frac{\partial H}{\partial\theta} \tag{2.35}$$

in a chart of $T^*(SO(3))$ with H given by (2.33) become after a lengthy computation in which $\dot\phi,\dot\psi,\dot\theta,\, p_\phi,\, p_\psi,\, p_\theta$ are replaced by $(\underline{m},\underline{\gamma})$, the Euler-Poisson equations

$$\left\{\begin{aligned}\dot{m}_1 &= a_1 m_2 m_3 + Mg\ell(\chi_3\gamma_2 - \chi_2\gamma_3),\ a_1 = \frac{1}{I_3} - \frac{1}{I_2}\\ \dot{m}_2 &= a_2 m_1 m_3 + Mg\ell(\chi_1\gamma_3 - \chi_3\gamma_1),\ a_2 = \frac{1}{I_1} - \frac{1}{I_3}\\ \dot{m}_3 &= a_1 m_1 m_2 + Mg\ell(\chi_2\gamma_1 - \chi_1\gamma_2),\ a_3 = \frac{1}{I_2} - \frac{1}{I_1}\end{aligned}\right. \tag{2.36}$$

$$\left\{\begin{aligned}\dot{\gamma}_1 &= \frac{m_3\gamma_2}{I_3} - \frac{m_2\gamma_3}{I_2}\\ \dot{\gamma}_2 &= \frac{m_1\gamma_3}{I_1} - \frac{m_3\gamma_1}{I_3}\\ \dot{\gamma}_3 &= \frac{m_2\gamma_1}{I_2} - \frac{m_1\gamma_2}{I_1}\ .\end{aligned}\right.$$

Note that these equations are on $\mathfrak{so}(3)^* \times \mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ whereas the canonical equations were on $T^*SO(3))$. This is an instance of a general fact that will be explained in section 4.

2.6 Poisson Bracket in Body Coordinates. For $F, G\colon\ T^*(SO(3)) \to \mathbb{R}$, i.e., F, G are functions of $(\phi,\psi,\theta, p_\phi, p_\psi, p_\theta)$, the canonical Poisson bracket is given by

$$\{F,G\}(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta) = \frac{\partial F}{\partial\phi}\frac{\partial G}{\partial p_\phi} - \frac{\partial F}{\partial p_\phi}\frac{\partial G}{\partial\phi} + \frac{\partial F}{\partial\psi}\frac{\partial G}{\partial p_\psi} - \frac{\partial F}{\partial p_\psi}\frac{\partial G}{\partial\psi} + \frac{\partial F}{\partial\theta}\frac{\partial G}{\partial p_\theta} - \frac{\partial F}{\partial p_\theta}\frac{\partial G}{\partial\theta} \tag{2.37}$$

This bracket becomes after the change of variables

$$(\phi,\psi,\theta, p_\phi, p_\psi, p_\theta) \mapsto (\underline{m},\underline{\gamma}), \tag{2.38}$$

$$\{F,G\}(\underline{m},\underline{\gamma}) = -\underline{m}\cdot(\nabla_{\underline{m}}F \times \nabla_{\underline{m}}G) - \underline{\gamma}\cdot(\nabla_{\underline{m}}F \times \nabla_{\underline{\gamma}}G + \nabla_{\underline{\gamma}}F \times \nabla_{\underline{m}}G), \tag{2.39}$$

where $\nabla_{\underline{m}}$ and $\nabla_{\underline{\gamma}}$ denote the gradients with respect to $\underline{m}$ and $\underline{\gamma}$. Clearly (2.39) defines a bilinear, skew-symmetric operation on functions of $(\underline{m},\underline{\gamma})$. A computation shows that it also satisfies the Jacobi identity, i.e. (2.39) is a Poisson bracket on $\mathbb{R}^3 \times \mathbb{R}^3$. Moreover

$$\dot{F} = \{F,H\}$$

with H given by (2.32) yields, for F equal to m_i, γ_i, $i = 1,2,3$, the equations of motion (2.36). Note that whereas (2.37) is non-degenerate, i.e., $\{F,G\} = 0$ for all G implies F = constant, the bracket in (2.39) is degenerate. It is in fact easy to see that

$$\{F,G\}(\underline{m},\underline{\gamma}) = 0$$

if $G(\underline{m},\underline{\gamma}) = \Phi(\|\underline{\gamma}\|^2)$ or $G(\underline{m},\underline{\gamma}) = \Psi(\underline{m}\cdot\underline{\gamma})$ for arbitrary real valued functions Φ, Ψ of a real variable. Note that unlike the case of the free rigid body where the bracket consists only of the first term of (2.39), an arbitrary function of $\|\underline{m}\|^2$ does not commute with every function of $\underline{m}$ and $\underline{\gamma}$. Recall also that $\underline{m}\cdot\underline{\gamma}$ and $\|\underline{\gamma}\|^2$ are the only conserved quantities for the heavy top, if no other symmetries are present. The geometric reason of (2.39) and the existence of the above two functions will be given in Section 4.

In Section 5 we shall discuss the equations in the spatial picture.

3. IDEAL COMPRESSIBLE ADIABATIC FLUIDS

3.1 Configuration Space. Let Ω be a compact region in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, filled with a moving fluid free of external forces. A configuration of the fluid is chosen and called the reference or Lagrangian configuration; its points, called material or Lagrangian points, are denoted by $\underline{X} = (X^1,X^2,X^3)$; X^i are referred to as material or Lagrangian coordinates. A configuration of the fluid is an orientation preserving diffeomorphism η of Ω with certain smoothness properties.* We shall not be specific here about the correct choices of function spaces and refer the reader to Ebin and Marsden [1970] and Marsden [1976] where this is discussed in great detail for incompressible fluids; obvious changes have to be made for the compressible case. The manifold Ω, thought of as the target space of a configuration η, i.e. a configuration of the fluid at a different time, is called the spatial or Eulerian configuration, whose points, called spatial or Eulerian points, are denoted by lower case letters $\underline{x}$. A motion of the fluid is a time dependent family of diffeomorphisms, written

$$\underline{x} = \eta(\underline{X},t) = \eta_t(\underline{X}),$$

or simply $\underline{x}(\underline{X},t)$.

Given the mass density $\rho_0(\underline{X})$ and entropy $\sigma_0(\underline{X})$ of the fluid in the reference configuration, both functions of $\underline{X}$, denoting by $J_{\eta_t}(\underline{X})$ the

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*In principle, one can develop the theory of fluids as we did for rigid bodies in Section 3.1, considering the fluid and the containing space as two different manifolds. The configuration space is then a space of mappings from the fluid manifold to the container manifold, and it becomes a group only when a reference configuration is chosen. Although this viewpoint is actually necessary in elasticity theory, we have used the more conventional approach here, in which the fluid particles are identified with their positions in space at $t = 0$.

Jacobian determinant $d\underline{x}/d\underline{X}$ of the motion η_t at $\underline{X}$, we shall see in 3.3 that the mass and entropy density satisfy

$$\rho(\underline{x},t)J_{\eta_t}(\underline{X}) = \rho_t(\underline{x})J_{\eta_t}(\underline{X}) = \rho_0(\underline{X}) \quad \text{and} \quad \sigma(\underline{x},t) = \sigma_t(\underline{x}) = \sigma_0(\underline{X}).$$

Consequently, the Eulerian mass density and entropy ρ and σ are completely determined by the motion and ρ_0 and σ_0 respectively. Hence, <u>the configuration space of compressible fluid flow with a given mass and entropy density in the reference configuration is the group of diffeomorphisms $\mathrm{Diff}(\Omega)$ of Ω. Consequently the phase space is the cotangent bundle</u> $T^*(\mathrm{Diff}(\Omega))$.

There are two problems with this approach. First, the configuration space requires a choice of ρ_0 and σ_0. But ρ_0 and σ_0 have to be changed in accordance with the choices of initial conditions. How this is done will be explained abstractly in Section 4. The change of ρ_0 and σ_0 is akin to the change of the parameter $Mg\ell\underline{k}$ in the heavy top problem. We shall think of ρ_0 and σ_0 (exactly as we did of $Mg\ell\underline{k}$ in the previous section) as a parameter. The second problem is much more serious. We think of the fluid as moving nicely in Ω, at any time filling Ω. However, under certain conditions, shocks and cavitation can occur. The present approach cannot deal with such problems and represents a serious limitation.

For a motion $\underline{x} = \eta_t(\underline{X})$ one defines three velocities:

a) the <u>material</u> or <u>Lagrangian velocity</u>

$$\underline{V}(\underline{X},t) = \underline{V}_t(\underline{X}) = \partial\eta(\underline{X},t)\partial t; \tag{3.2}$$

b) the <u>spatial</u> or <u>Eulerian velocity</u>

$$\underline{v}(\underline{x},t) = \underline{v}_t(\underline{x}) = \underline{V}(\underline{X},t), \text{ i.e. } \underline{v}_t\circ\eta_t = \underline{V}_t; \tag{3.3}$$

c) the <u>convective</u> or <u>body velocity</u>

$$\underline{\mathcal{V}}(\underline{X},t) = \underline{\mathcal{V}}_t(\underline{X}) = -\partial\underline{X}(\underline{x},t)/\partial t = -\partial\eta_t^{-1}(\underline{x})/\partial t. \tag{3.4}$$

Taking the derivative of $(\eta_t\circ\eta_t^{-1})(\underline{x}) = \underline{x}$ and denoting by $T_{\underline{X}}\eta_t$ the Jacobian matrix $d\underline{x}/d\underline{X}$ of η_t at $\underline{X}$, we get

$$\underline{\mathcal{V}}_t(\underline{X}) = (T_{\underline{X}}\eta_t)^{-1}\underline{v}_t(\underline{X}) \tag{3.5}$$

i.e.

$$\underline{\mathcal{V}}_t = \eta_t^*\underline{v}_t$$

Note that both $\underline{V}_t$ and $\underline{v}_t$ are tangent to Ω at $\underline{x} = \eta_t(\underline{X})$. This means that $\underline{v}_t$ is a time dependent vector field on Ω. On the other hand, tangency of $\underline{V}_t(\underline{X})$ and $\eta_t(\underline{X})$ says that $\underline{V}_t$ is a <u>vector field over</u> η_t <u>on</u> Ω, i.e. $\underline{V}_t$ is a map from Ω to the tangent bundle $T\Omega$ such that $\underline{V}_t(X)$ is tangent to Ω not at $\underline{X}$, but at $\eta_t(\underline{X})$. Finally, notice that $\underline{\mathcal{V}}_t$ is a tangent vector at $\underline{X}$, i.e. $\underline{\mathcal{V}}_t$ is also a time dependent vector field on Ω.

We summarize the relations between $\underline{V}$, $\underline{v}$, and $\underline{\mathcal{V}}$ in the following commutative diagram, in which vertical arrows mean vector fields.

$$\begin{tikzcd}
T\Omega \arrow[r, "T\eta_t"] & T\Omega \\
\Omega \arrow[u, "\underline{\mathcal{V}}_t"] \arrow[r, "\eta_t"'] \arrow[ur, "\underline{V}_t"'] & \Omega \arrow[u, "\underline{v}_t"']
\end{tikzcd}$$

Figure 2

Let $Z(\underline{X},t)$ be a material quantity, i.e. a given function of $(\underline{X},t)$ and let $z(\underline{x},t) = Z(\underline{X},t)$ be the same quantity expressed in spatial coordinates. Then by the chain rule

$$\frac{\partial Z}{\partial t} = \frac{\partial z}{\partial t} + (\underline{v}\cdot\underline{\nabla})z, \text{ i.e. } \frac{\partial Z}{\partial t} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x^j} v^j . \tag{3.6}$$

In particular, if Z represents different components of a vector $\underline{Z}$, we have

$$\frac{\partial \underline{Z}}{\partial t} = \frac{\partial \underline{z}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{z}, \text{ i.e. } \frac{\partial Z^i}{\partial t} = \frac{\partial z^i}{\partial t} + \frac{\partial z^i}{\partial x^j} v^j . \tag{3.7}$$

The right hand side of (3.6) or of (3.7) is called the <u>material derivative</u> of z or $\underline{z}$ and is usually denoted by $\dot{z} = Dz/Dt$ or $\dot{\underline{z}} = D\underline{z}/Dt$; it represents the time-derivative of z holding the material point $\underline{X}$ fixed. As opposed to that, the usual partial derivative $\partial z/\partial t$ represents the time-derivative of z holding the spatial point $\underline{x}$ fixed. One can develop analogous formulas for the convective velocity. We will return to this point in §5.

We shall determine the phase space $T^*(\mathrm{Diff}(\Omega))$ and elementary Lie group operations on $\mathrm{Diff}(\Omega)$, on its Lie algebra, and its dual.

3.2 <u>The Lie Group Diff(Ω)</u>. There are two ways in which Diff(Ω) can be made into a Lie group. The most obvious one is to consider only C^∞ diffeomorphisms. It turns out that in this way Diff(Ω) becomes a Fréchet manifold, i.e. its model space is a locally convex, Hausdorff, complete vector space. Composition of diffeomorphisms and taking the inverse are smooth operations, so Diff(Ω) becomes a Fréchet Lie group (see e.g. Leslie [1967], and Omori [1975]). The main drawback of this approach is that in Fréchet spaces special hypotheses are needed for inverse function theorems to hold; the same is true of existence and uniqueness theorems for integral curves of differential equations. Use of the Nash-Moser theory is not necessary.

The second approach is to use diffeomorphisms of Sobolev or Hölder class. It turns out that if the Sobolev class $W^{s,p}$ or Holder class $C^{k+\alpha}$ is high enough so that such diffeomorphisms are at least C^1, then they form a C^∞ Banach manifold and one has the usual existence and uniqueness theorems for solutions of differential equations. Unfortunately only right translation is smooth whereas left translation and taking inverses are only continuous. Thus $W^{s,p}$-Diff(Ω) (or $C^{k+\alpha}$-Diff(Ω)) is now a topological group which is a Banach manifold on which right translation is smooth. One may now make Diff(Ω) into a "Lie" group by taking the inverse limit as the differentiability class goes to ∞ (Ebin and Marsden [1970], Omori [1975]).

We next determine the tangent space $T_\eta(\mathrm{Diff}(\Omega))$ of Diff(Ω) at η. Let $t \mapsto \eta_t$ be a smooth curve with $\eta_0 = \eta$. Then $(d\eta_t/dt)|_{t=0}$ is, by definition, a tangent vector at η to Diff(Ω). If $\underline{X} \in \Omega$, then $t \mapsto \eta_t(\underline{X})$ is a smooth curve in Ω through $\eta(\underline{X})$ and thus

$$\left.\frac{d\eta_t(\underline{X})}{dt}\right|_{t=0} \in T_{\eta(\underline{X})}\Omega,$$

where $T_{\eta(\underline{X})}\Omega$ is the tangent space to Ω at $\eta(\underline{X})$. Consequently we have a map $\underline{X} \in \Omega \mapsto (d\eta_t(\underline{X})/dt)|_{t=0} \in T_{\eta(\underline{X})}\Omega$, i.e. $(d\eta_t/dt)|_{t=0}$ is a vector field over η. Thus

$$T_\eta(\mathrm{Diff}(\Omega)) = \{\underline{V}_\eta : \Omega \to T\Omega \,|\, \underline{V}_\eta(\underline{X}) \in T_{\eta(\underline{X})}\Omega\}. \tag{3.8}$$

In coordinates, if $\underline{x} = \eta(\underline{X})$, $\underline{V}_\eta(\underline{X}) = V^i(\underline{X})(\partial/\partial x^i)$.

In particular, if e denotes the identity map of Ω, $T_e(\mathrm{Diff}(\Omega)) = \mathfrak{X}(\Omega)$, the Lie algebra of vector fields on Ω. It turns out that the Lie algebra bracket of $\mathfrak{X}(\Omega)$ is minus the usual Lie bracket of vector fields, i.e. $[U,V]^i = V^j(\partial U^i/\partial X^j) - U^j(\partial V^i/\partial X^j)$. Thus <u>the Lie algebra</u> of Diff(Ω) <u>may be identified with</u> $\mathfrak{X}(\Omega)$, with the negative of the usual Lie algebra structure.

To determine the dual of $\mathfrak{X}(\Omega)$ and the cotangent bundle of Diff(Ω), we take a geometric point of view. Instead of considering the functional analytic dual of all linear continuous functionals on $\mathfrak{X}(\Omega)$, we will be content to find another vector space $\mathfrak{X}(\Omega)^*$ and a weakly non-degenerate pairing

$$\langle\ \ \rangle : \mathfrak{X}(\Omega)^* \times \mathfrak{X}(\Omega) \to \mathbb{R};$$

this means that $\langle\ ,\ \rangle$ is a bilinear mapping such that if $\langle \underline{M}, \underline{V} \rangle = 0$ for all $\underline{V} \in \mathfrak{X}(\Omega)$, then $\underline{M} = 0$. Clearly $\mathfrak{X}(\Omega)^*$ is a subspace of the functional analytic dual. With this definition, it is easy to see that $\mathfrak{X}(\Omega)^*$ consists of all one-form densities on Ω, i.e.

$$\mathfrak{X}(\Omega)^* = \Lambda^1(\Omega) \otimes |\Lambda^3(\Omega)|. \tag{3.9}$$

The notation in (3.9) is the standard one: $\Lambda^i(\Omega)$ denotes the set of all exterior i-forms on Ω and $|\Lambda^3(\Omega)|$ denotes the densities on Ω. Thus a one-form density is of the form $\alpha d^3\underline{X}$ with α a one-form on Ω, so locally it is $(\alpha_i(\underline{X})\, dX^i)\, d^3\underline{X}$. The pairing $\langle\ ,\ \rangle$ between $\mathfrak{X}(\Omega)^*$ and $\mathfrak{X}(\Omega)$ is $\langle \alpha d^3\underline{X}, \underline{V} \rangle = \int_\Omega \alpha(\underline{V})(\underline{X})d^3\underline{X}$ or in local coordinates, $\int_\Omega \alpha_i(\underline{X})V^i(\underline{X})d^3\underline{X}$.

Finally, in view of (3.9), $T^*(\mathrm{Diff}(\Omega))$ consists of all one-form densities over η, i.e.,

$$T^*_\eta(\mathrm{Diff}(\Omega)) = \{\underline{\alpha}_\eta : \Omega \to T^*\Omega \otimes |\Lambda^3(\Omega)|\,|\,\underline{\alpha}_\eta(\underline{X}) \in T^*_{\eta(\underline{X})}\Omega \otimes |\Lambda^3_{\underline{X}}(\Omega)|\}. \tag{3.10}$$

This means that $\underline{\alpha}_\eta = \underline{\xi}_\eta d^3\underline{X}$, where $\underline{\xi}_\eta$ is a one-form over η on Ω, i.e. $\underline{\xi}_\eta(\underline{X}) \in T^*_{\eta(\underline{X})}\Omega$. Locally, $\underline{\alpha}_\eta = (\xi_i(\underline{X})dx^i)d^3\underline{X}$, where $(x^i) = \underline{x} = \eta(\underline{X})$ and $\xi_\eta(\underline{X}) = \xi_i(\underline{X})dx^i$. We shall denote the action of one-forms $\underline{\xi}$ over η on vector fields $\underline{V}_\eta$ over η by $\underline{\xi}_\eta(\underline{V}_\eta)$; the result is a function of $\underline{X}$ which locally equals $\xi_i V^i$. The pairing $\langle\ ,\ \rangle$ between $T^*_\eta(\mathrm{Diff}(\Omega))$ and $T_\eta(\mathrm{Diff}(\Omega))$ is given by $\langle \underline{\alpha}_\eta, \underline{V}_\eta \rangle = \int_\Omega \underline{\xi}_\eta(\underline{V}_\eta)(\underline{X})d^3\underline{X}$ if $\underline{\alpha}_\eta = \underline{\xi}_\eta d^3\underline{X}$; locally this has the expression $\int_\Omega \xi_i(\underline{X})\underline{V}^i(\underline{X})d^3\underline{X}$.

Left and right translations are defined by

$$L_\eta : \mathrm{Diff}(\Omega) \to \mathrm{Diff}(\Omega),\ L_\eta(\phi) = \eta\circ\phi$$
$$R_\eta : \mathrm{Diff}(\Omega) \to \mathrm{Diff}(\Omega),\ R_\eta(\phi) = \phi\circ\eta$$

for $\eta,\phi \in \mathrm{Diff}(\Omega)$. Both are diffeomorphisms of the Lie group $\mathrm{Diff}(\Omega)$. It is easy to see that their derivatives have the following expressions:

$$T_\phi L_\eta : T_\phi(\mathrm{Diff}(\Omega)) \to T_{\eta\circ\phi}(\mathrm{Diff}(\Omega));\quad T_\phi L_\eta(\underline{V}_\phi) = T\eta\circ\underline{V}_\phi \tag{3.11}$$

and

$$T_\phi R_\eta : T_\phi(\mathrm{Diff}(\Omega)) \to T_{\phi\circ\eta}(\mathrm{Diff}(\Omega));\quad T_\phi R_\eta(\underline{V}_\phi) = \underline{V}_\phi\circ\eta \tag{3.12}$$

for $\underline{V}_\phi \in T_\phi(\mathrm{Diff}(\Omega))$. The physical interpretation of these formulas is the following. Think of ϕ as a relabelling or rearrangement of the particles in Ω and of η as a motion. Then (3.11) says that the material derivative of the motion η followed by the relabelling ϕ equals $T\eta\circ\underline{V}_\phi$. In local coordinates, if $\phi(\underline{X}) = \underline{Y}$ and $\eta(\underline{Y}) = \underline{y}$, then $\underline{V}_\phi(\underline{X}) = V^i(\underline{X})(\partial/\partial Y^i)$ and

$$(T\eta\circ\underline{V}_\phi)^i(\underline{X}) = \frac{\partial x^i}{\partial y^j}(\underline{Y})V^j(\underline{X})\frac{\partial}{\partial y^i}. \tag{3.13}$$

On the other hand, (3.12) says that the material derivative of the relabelling ϕ followed by the motion η equals $\underline{V}_\phi\circ\eta$. In local coordinates, if $\eta(\underline{X}) = \underline{x}$, $\phi(\underline{x}) = \underline{y}$, then $\underline{V}_\phi(\underline{X}) = V^i(\underline{X})(\partial/\partial y^i)$ and

$$(\underline{V}_\phi\circ\eta)^i(\underline{X}) = (V^i\circ\eta)(\underline{X})(\partial/\partial y^i). \tag{3.14}$$

Simply put, left translation by η transforms $V_\phi(\underline{X})$, a vector at $\phi(\underline{X})$ to a vector at $\eta(\phi(\underline{X}))$ whereas right translation merely changes the argument from $\underline{X}$ to $\eta(\underline{X})$.

By (3.12), the derivative of right translation is again right translation, so R_η is C^∞. However, if η and ϕ are diffeomorphisms of a given finite Sobolev class, $T\eta$ looses one derivative. (This is basically the reason why left translation is only continuous in $W^{s,p}$-$\mathrm{Diff}(\Omega)$.) In C^∞-$\mathrm{Diff}(\Omega)$ however (with differentiability suitably interpreted), left translation is C^∞.

As an application, note that by (3.3) and (3.5), the material velocity $\underline{V}_t$ is the right translate of the spatial velocity $\underline{v}_t$ and the left translate of the convective velocity $\underline{\mathcal{V}}_t$.

If $\underline{V} \in \mathfrak{X}(\Omega)$, a diffeomorphism $\eta \in \mathrm{Diff}(\Omega)$ acts on $\underline{V}$ by the adjoint action, the analogue of conjugation for matrices. The definition combined with (3.11) and (3.12) gives

$$\mathrm{Ad}_\eta\underline{V} := T_e(L_\eta\circ R_{\eta^{-1}})\underline{V} = T_{\eta^{-1}}L_\eta(T_eR_{\eta^{-1}}(\underline{V}))$$

$$= T\eta\circ V\circ\eta^{-1} = \eta_*\underline{V},$$

i.e. the adjoint action of η on $\underline{V}$ is the push-forward of vector fields:

$$\mathrm{Ad}_\eta\underline{V} = \eta_*\underline{V}\ . \tag{3.15}$$

For example, by (3.3) and (3.5), $\underline{v}_t = \mathrm{Ad}_{\eta_t}\underline{\mathcal{V}}_t$, which is similar to the formula which relates $\hat{\underline{\omega}}_B$ to $\hat{\underline{\omega}}_S$ in the previous section. Finally, we compute the coadjoint $\mathrm{Ad}^*_{\eta^{-1}}\alpha$ action of η on $\underline{\alpha} \in \mathfrak{X}(\Omega)^*$. By the change of variables formula we have

$$\langle Ad^*_{\eta^{-1}}\underline{\alpha},\underline{V}\rangle := \langle\underline{\alpha},Ad_{\eta^{-1}}\underline{V}\rangle = \int_\Omega \underline{\alpha}\cdot\eta^*\underline{V} = \int_\Omega \eta_*\underline{\alpha}\cdot\underline{V};$$

here $\underline{\alpha}\cdot\underline{V}$ in the integrand signifies the pairing between one-form densities and vector fields so that $\underline{\alpha}\cdot\underline{V}$ is a density on Ω. Thus

$$Ad^*_{\eta^{-1}}\underline{\alpha} = \eta_*\underline{\alpha}; \tag{3.16}$$

$\eta_*\underline{\alpha}$ is the push-forward of the one-form density $\underline{\alpha}$; the push-forward operates separately on the one-form and the density.

3.3 Equations of Motion. We review the derivation of the equations of motion in Eulerian coordinates from four principles: conservation of mass, entropy, and momentum. Conservation of energy will follow by imposing the adiabatic equation of state.

a) The principle of conservation of mass stipulates that mass can be neither created or destroyed, i.e.

$$\int_{\eta_t(W)} \rho_t(\underline{x})d^3\underline{x} = \int_W \rho_0(\underline{X})d^3\underline{X}$$

for all compact W with non-empty interior having smooth boundary. Changing variables, this becomes

$$\eta_t^*(\rho_t(\underline{x})d^3\underline{x}) = \rho_0(\underline{X})d^3\underline{X} \quad\text{or}\quad (\eta_t^*\rho_t)J(\eta_t) = \rho_0, \tag{3.17}$$

where $J(\eta_t) = |d\underline{x}/d\underline{X}|$ is the Jacobian determinant of η_t and η_t^* is pull-back of forms or functions as the case may be. Using the relation between Lie derivatives and flows, (3.17) is equivalent to the continuity equation

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho_t\underline{v}) = 0. \tag{3.18}$$

b) By the principle of conservation of entropy, the heat content of the fluid cannot be altered, i.e.

$$\int_{\eta_t(W)} \sigma_t(\underline{x})\rho_t(\underline{x})d^3\underline{x} = \int_W \sigma_0(\underline{X})\ \rho_0(\underline{X})d^3\underline{X}$$

for all compact W with non-empty interior having smooth boundary. By a change of variables this becomes

$$\eta_t^*(\sigma_t(\underline{X})\rho_t(\underline{x})d^3\underline{x}) = \sigma_0(\underline{X})\rho_0(\underline{X})d^3X$$

and by (3.17) we get

$$\eta_t^*(\sigma_t(\underline{x})) = \sigma_0(\underline{X}), \quad \text{or} \quad \frac{\partial\sigma}{\partial t} + \underline{v}\cdot\underline{\nabla}\sigma_t = 0; \tag{3.19}$$

the second relation follows by taking the time derivative of the first. The last relation says that no heat is exchanged across flow-lines.

c) Balance of momentum is described by Newton's second law: the rate of change of momentum of a portion of the fluid equals the total force applied to it. Since we assume that no external forces are present, the only forces acting on the fluid are forces of stress. The assumption of an ideal fluid means that the force of stress per unit area exerted across a surface element at $\underline{x}$, with outward unit normal $\underline{n}$ at time t, is $-p(\underline{x},t)\underline{n}$ for some function $p(\underline{x},t)$ called the pressure. With this hypothesis, the balance of momentum becomes Euler's equations of motion:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{1}{\rho}\underline{\nabla}\cdot p \tag{3.2}$$

with the boundary condition $\underline{v}$ parallel to $\partial\Omega$ (no friction exists between fluid and boundary) and the initial condition $\underline{v}(\underline{x},0) = \underline{v}_0(\underline{x})$ on Ω.

The proof of conservation of energy is standard. The kinetic energy of the fluid is $\frac{1}{2}\int_\Omega \rho(\underline{x})\|\underline{v}(\underline{x})\|^2\, d^3\underline{x}$. The assumption of an adiabatic fluid means that the internal energy of the fluid is $\int_\Omega \rho(\underline{x})w(\rho(\underline{x}), \sigma(\underline{x}))d^3\underline{x}$ with the equation of state $p(\underline{x}) = \rho(\underline{x})^2(\partial w/\partial\rho)(\underline{x})$ satisfying $\partial w/\partial\rho > 0$. In the next computation the following two vector identities will be needed:

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = \underline{\nabla}\left(\frac{\|\underline{v}\|^2}{2}\right) + \underline{\omega}\times\underline{v}, \text{ where } \underline{\omega} = \text{curl } \underline{v}, \text{ the vorticity}$$

$$\underline{\nabla}(w + \rho\partial w/\partial\sigma) = \underline{\nabla}p/\rho + (\partial w/\partial\sigma)\underline{\nabla}\sigma.$$

We have by (3.18), (3.19), and (3.20)

$$\frac{\partial}{\partial t}\left(\frac{\|\underline{v}\|^2}{2} + \rho w(\rho,\sigma)\right) = -\text{div}(\rho\underline{v})\left[\frac{1}{2}\|\underline{v}\|^2 + w(\rho,\sigma) + \rho\frac{\partial w}{\partial\rho}\right] - \rho\underline{v}\cdot[(v\cdot\nabla)\underline{v} + \frac{1}{\rho}\underline{\nabla}p + \frac{\partial w}{\partial\sigma}\underline{\nabla}\sigma]$$

$$= -\text{div}\left(\rho\underline{v}\left[\frac{1}{2}\|\underline{v}\|^2 + w + \rho\frac{\partial w}{\partial\rho}\right]\right)$$

Consequently, the total energy

$$H(\underline{v},\rho,\sigma) = \frac{1}{2}\int_\Omega \rho(\underline{x})\|\underline{v}\|^2 d^3\underline{x} + \int_\Omega \rho(\underline{x})w(\rho(\underline{x}), \sigma(\underline{x}))\, d^3\underline{x}, \tag{3.21}$$

which represents the Hamiltonian of the system, is conserved.

The physical problem to be solved now consists of the continuity equation (3.18), entropy convection (3.19), and Euler's equations (3.20) with $p = \rho^2 \partial w/\partial\rho$, where the <u>internal energy density</u> $w(\rho,\sigma)$ is a known function; the boundary condition says that $\underline{v}$ is tangent to $\partial\Omega$ and the initial condition is $\underline{v}(x,0) = \underline{v}_0(\underline{x})$, $\underline{v}_0$ a given vector field on Ω.

Recall that $\partial p/\partial\rho$ is the square of the sound speed, so that $\partial p/\partial\rho > 0$ represents a very reasonable physical condition. We also mention that $\partial p/\partial\rho > 0$ is exactly the condition needed to prove local existence and uniqueness of solutions.

3.4 <u>Hamiltonian in Lagrangian Coordinates</u>. The equations of motion just described are not on $T^*(\mathrm{Diff}(\Omega))$ which is the phase space of the problem. To describe the dynamics in $T^*(\mathrm{Diff}(\Omega))$ using Hamilton's equations, the Hamiltonian (3.21) must be expressed on $T^*(\mathrm{Diff}(\Omega))$, i.e. in material coordinates.

We start with the potential energy. Perform the change of variables $\underline{x} = \eta_t(\underline{X})$ in the potential energy and use (3.17) to get

$$\int_\Omega \rho(\underline{x}) w(\rho(\underline{x}),\sigma(\underline{x}))d^3\underline{x} = \int_\Omega \rho_0(\underline{X}) w(\rho_0(\underline{X}) J_{\eta_t}^{-1}(\underline{X}),\ \sigma_0(\underline{X}))d^3\underline{X}. \tag{3.22}$$

The right hand side is a function of η_t and hence defined on $\mathrm{Diff}(\Omega)$ so that by lifting we get a function on $T^*(\mathrm{Diff}(\Omega))$.

To express the kinetic energy as a function on the cotangent bundle, we need first its expression in terms of the material velocity. This is accomplished by performing the same change of variables $\underline{x} = \eta_t(\underline{X})$. We have by (3.3) and (3.17)

$$\frac{1}{2}\int_\Omega \rho(\underline{x})\|\underline{v}_t(\underline{x})\|^2 d^3\underline{x} = \frac{1}{2}\int_\Omega \rho_0(\underline{X})\|\underline{V}_t(\underline{X})\|^2 d^3\underline{X} \tag{3.23}$$

But $\underline{V}_t \in T_{\eta_t}(\mathrm{Diff}(\Omega))$ so that (3.23) represents the expression of the kinetic energy on the tangent bundle. Note that the mapping

$$\langle\!\langle \underline{V}_\eta, \underline{W}_\eta \rangle\!\rangle \int_\Omega \rho_0(\underline{X})\underline{V}_\eta(\underline{X})\cdot\underline{W}_\eta(\underline{X})d^3\underline{X} \tag{3.24}$$

for $\underline{V}_\eta, \underline{W}_\eta \in T_\eta(\mathrm{Diff}(\Omega))$ and the dot in the integrand signifying the metric on Ω (in our case the usual dot product), defines a weak Riemannian metric on $\mathrm{Diff}(\Omega)$ and (3.23) is its kinetic energy.

In finite dimensions, a metric on a manifold induces a bundle metric on the cotangent bundle as we have seen in Section 2. In infinite dimensions, as in the present case, this bundle metric does not exist in general and in examples it must be constructed explicitly. Let $\underline{\alpha}_\eta, \underline{\beta}_\eta \in T^*_\eta(\mathrm{Diff}(\Omega))$, i.e. $\underline{\alpha}_\eta = \underline{\xi}_\eta d^3\underline{X}$, $\beta_\eta = \zeta_\eta d^3\underline{X}$ with $\underline{\xi}_\eta$ and $\underline{\zeta}_\eta$ one-forms over η. Consequently, $\underline{\alpha}_\eta/(\rho_0\, d^3\underline{X}) = \underline{\xi}_\eta/\rho_0$ $\beta_\eta/(\rho_0\, d^3X) = \underline{\zeta}_\eta/\rho_0$ are one-forms over η, so evaluated at $\underline{X}$ they are elements of $T^*_{\eta(X)}\Omega$. But Ω is a finite dimensional Riemannian manifold (with the Euclidean metric in our case) so to every one-form at $\eta(\underline{X})$ there exists a unique vector at $\eta(\underline{X})$ associated by the metric. Explicitly, if $\underline{u}_x \in T_x\Omega$, the one-form $\underline{u}_x^\flat \in T^*_x\Omega$ is defined by $\underline{u}_x^\flat(\underline{w}_x) = \underline{u}_x \cdot \underline{w}_x$ for all $\underline{w}_x \in T_x\Omega$. In this way, the <u>index lowering action</u> $\flat : T\Omega \to T^*\Omega$ is a bundle isomorphism. The inverse of $\flat$ is denoted by $\# : T^*\Omega \to T\Omega$ and is called the <u>index raising action</u>. In coordinates, if $g = (g_{ij})$ is the metric and (g^{ij}) is the inverse matrix of (g_{ij}), we have for $\underline{u} = u^i(\partial/\partial x^i)$, $\underline{\alpha} = \alpha_i dx^i$,

$$\underline{u}^\flat = g_{ij}u^j dx^i, \quad \underline{\alpha}^\# = g^{ij}\alpha_j(\partial/\partial x^i).$$

Now define the bundle metric on $T^*(\mathrm{Diff}(\Omega))$ by

$$(\underline{\alpha}_\eta, \underline{\beta}_\eta) = \int_\Omega \rho_0(\underline{X})\underline{V}_\eta(\underline{X})\cdot\underline{W}_\eta(\underline{X})d^3\underline{X} \tag{3.25}$$

for $\underline{V}_\eta = (\underline{\alpha}_\eta/\rho_0 d^3\underline{X})^\#$, $\underline{W}_\eta = (\underline{\beta}_\eta/\rho_0 d^3X)^\# \in T_\eta(\mathrm{Diff}(\Omega))$. Denote by $\|\cdot\|$ the bundle norm defined by the metric (3.25) and let

$$\underline{M}_\eta = \rho_0\underline{V}_\eta^\flat d^3\underline{X} \in T^*_\eta(\mathrm{Diff}(\Omega)) \tag{3.26}$$

be the <u>material momentum density</u> of the fluid. With this notation, (3.23) becomes $\|\underline{M}_\eta\|^2/2$ and so by (3.22) the expression of the Hamiltonian on $T^*(\mathrm{Diff}(\Omega))$ becomes

$$H(\underline{M}_\eta) = \frac{1}{2}\|\underline{M}_\eta\|^2 + \int_\Omega \rho_0(\underline{X})w(\rho_0(\underline{X})J_\eta^{-1}(\underline{X}), \sigma_0(\underline{X}))d^3\underline{X}\,. \tag{3.27}$$

We want to investigate the symmetry properties of H. We shall prove that H is right invariant under the subgroup

$$\mathrm{Diff}(\Omega)_{\rho_0,\sigma_0} = \{\phi \in \mathrm{Diff}(\Omega) \mid (\rho_0\circ\phi)J_\phi = \rho_0, \sigma_0\circ\phi = \sigma_0\}. \tag{3.28}$$

For the potential energy this is easily seen, for if one replaces η by $\eta\circ\phi$ with ϕ as in (3.28), both arguments of w do not change. To right translate $\underline{M}_\eta$ by ϕ means to compute the dual map of (3.12). Let ψ be an arbitrary diffeomorphism and $\underline{M}_\eta = \underline{\xi}_\eta d^3\underline{X}$. By a change of variables, we have for any ψ

$$\langle T^*_{\eta\circ\psi^{-1}}R_\psi(\underline{M}_\eta), \underline{V}_{\eta\circ\psi^{-1}}\rangle := \langle \underline{M}_\eta, T_{\eta\circ\psi^{-1}}R_\psi(\underline{V}_{\eta\circ\psi^{-1}})\rangle = \langle \underline{M}_\eta, \underline{V}_{\eta\circ\psi^{-1}}\circ\psi\rangle$$

$$= \int_\Omega \underline{\xi}_\eta(\underline{X})\left(\underline{V}_{\eta\circ\psi^{-1}}(\psi(\underline{X}))\right)\underline{X}\; d^3\underline{X}$$

$$= \int_\Omega (\underline{\xi}_\eta\circ\psi^{-1})(\underline{Y})(\underline{V}_{\eta\circ\psi^{-1}}(\underline{Y}))J_{\psi^{-1}}(\underline{Y})d^3\underline{Y} = \langle J_{\psi^{-1}}(\underline{\xi}_\eta\circ\psi^{-1})d^3\underline{X}, \underline{V}_{\eta\circ\psi^{-1}}\rangle .$$

Consequently, if

$$\underline{M}_\eta = \underline{\xi}_\eta d^3\underline{X}, \quad \text{then} \quad T^*_{\eta\circ\psi^{-1}}R_\psi(\underline{M}_\eta) = J_{\psi^{-1}}(\underline{\xi}_\eta\circ\psi^{-1})d^3\underline{X} . \tag{3.29}$$

Thus, if $\underline{M}_\eta = \rho_0\underline{V}_\eta^\flat d^3\underline{X}$ and ϕ satisfies (3.28), then $T^*_{\eta\circ\phi^{-1}}R_\phi(\underline{M}_\eta) = \rho_0(\underline{V}_\eta\circ\phi^{-1})^\flat d^3\underline{X}$, so that by (3.25), a change of variables, and (3.29) we have

$$\|T^*_{\eta\circ\phi^{-1}}R_\phi(M_\eta)\|^2 = \int_\Omega \rho_0(\underline{X})\|\underline{V}_\eta(\phi^{-1}(\underline{X})\|^2\; d^3\underline{X}$$

$$= \int_\Omega \rho_0(\phi(\underline{Y})\|\underline{V}_\eta(\underline{Y})\|^2 J_\phi(\underline{Y})d^3\underline{Y} = \int_\Omega \rho_0(\underline{Y})\|\underline{V}_\eta(\underline{Y})\|^2\; d^3\underline{Y}$$

$$= \|\underline{M}_\eta\|^2,$$

i.e. the kinetic energy is invariant by right translations with diffeomorphisms of the form (3.28).

3.5 Poisson Bracket in Eulerian (Spatial) Coordinates. The dynamics of the Hamiltonian (3.27) on $T^*(\mathrm{Diff}(\Omega))$ is equivalent to

$$\dot{F} = \{F,H\} \tag{3.29}$$

for F an arbitrary function on $T^*(\mathrm{Diff}(\Omega))$ and $\{\ ,\ \}$ the canonical Poisson bracket of the cotangent bundle. If $\underline{E}$ is a function space on Ω modelling the manifold $\mathrm{Diff}(\Omega)$, then $\underline{E}\times\underline{E}^*$ models $T^*(\mathrm{Diff}(\Omega))$; the dual $\underline{E}^*$ has to be taken in the same geometric manner as we discussed in 3.3. If $\underline{n}\in E$, $\underline{\nu}\in\underline{E}^*$, the Poisson bracket (3.29) is given by

$$\{F,G\}(\underline{n},\underline{\nu}) = \int_\Omega\left(\frac{\delta F}{\delta\underline{n}}\frac{\delta G}{\delta\underline{\nu}} - \frac{\delta F}{\delta\underline{\nu}}\frac{\delta G}{\delta\underline{n}}\right)d^3\underline{X} \tag{3.30}$$

where the functional derivatives $\delta F/\delta\underline{n}\in\underline{E}^*$ and $\delta F/\delta\underline{\nu}\in\underline{E}$ are defined by

$$D_{\underline{n}}F(\underline{n},\underline{\nu})\cdot\underline{n}' = \int_\Omega \frac{\delta F}{\delta \underline{n}}(\underline{n}')\, d^3\underline{X}, \quad \text{for any } \underline{n}' \in \underline{E}$$

$$D_{\underline{\nu}}F(\underline{n},\underline{\nu})\cdot\underline{\nu}' = \int_\Omega \underline{\nu}'\left(\frac{\delta F}{\delta \underline{\nu}}\right) d^3\underline{X}, \quad \text{for any } \underline{\nu}' \in \underline{E}^*$$

where $D_n F$, $D_\nu F$ denote the Fréchet derivatives of F with respect to $\underline{n}$ and $\underline{\nu}$. By (3.26), (3.3), (3.17), and (3.29), the expression of $\underline{M}_\eta$ in Eulerian coordinates $\underline{x} = \eta(\underline{X})$ is

$$\underline{M}_\eta(\underline{X}) = \rho_0(X)\underline{V}_\eta(\underline{X})^\flat d^3\underline{X} = \rho_0(\underline{X})\underline{v}(\underline{x})^\flat J_{\eta^{-1}}(\underline{x})d^3\underline{X}$$
$$= \rho(\underline{x})v(\underline{x})^\flat d^3\underline{x} = T^*_\eta R_{\eta^{-1}}(\underline{M})(\underline{X}),$$

where the quantity

$$\underline{M}(\underline{x}) = \rho(\underline{x})v(\underline{x})^\flat d^3\underline{x} = T^*_e R_\eta(M_\eta)(\underline{X}) \tag{3.31}$$

is called the Eulerian momentum density of the fluid. Consider the map

$$\underline{M}_\eta(\underline{X}) \mapsto (\underline{M}(\underline{x}), \rho(\underline{x})d^3\underline{x}, \sigma(\underline{x})) \tag{3.32}$$

from Lagrangian to Eulerian coordinates, where $\underline{M}$ is given by (3.31) and

$$\rho = J_{\eta^{-1}}(\rho_0\circ\eta^{-1}); \quad \sigma = \sigma_0\circ\eta^{-1} . \tag{3.33}$$

Note that (3.33) is simply a rewriting of (3.17). Then a computation shows that the bracket (3.30) via the change of variables (3.33) becomes

$$\{F,G\}(\vec{M},\rho) = \int_\Omega \vec{M}\cdot\left[\left(\frac{\delta G}{\delta \vec{M}}\cdot\underline{\nabla}\right)\frac{\delta F}{\delta \vec{M}} - \left(\frac{\delta F}{\delta \vec{M}}\cdot\underline{\nabla}\right)\frac{\delta G}{\delta \vec{M}}\right] d^3\underline{x}$$
$$+ \int_\Omega \rho\left[\frac{\delta G}{\delta \vec{M}}\cdot\left(\underline{\nabla}\frac{\delta F}{\delta \rho}\right) - \frac{\delta F}{\delta \vec{M}}\cdot\left(\underline{\nabla}\frac{\delta G}{\delta \rho}\right)\right] d^3\underline{x}$$
$$+ \int_\Omega \sigma\left[\frac{\delta G}{\delta \vec{M}}\cdot\left(\underline{\nabla}\frac{\delta F}{\delta \sigma}\right) - \frac{\delta F}{\delta \vec{M}}\cdot\left(\underline{\nabla}\frac{\delta G}{\delta \sigma}\right)\right] d^3\underline{x} \tag{3.34}$$

where $\vec{M}(\underline{x}) = \rho(\underline{x})\,\underline{v}(\underline{x})$ is identified with $\underline{M}$ in (3.31). The computation that transforms the bracket (3.30) via (3.31), (3.32), (3.33) to (3.34) is tedious; see Kaufmann's lecture in these proceedings for a different example where such a computation is carried out. An even longer computation shows that (3.34) which is bilinear and skew-symmetric, also satisfies the Jacobi identity. In §4 we shall give an abstract theorem which includes these results and allows one to efficiently bypass such computations yet obtain the correct answers.

The equations of motion (3.18), (3.19), (3.20) in terms of $\vec{M}$, ρ and σ can be obtained from (3.34) and the dynamics $\dot{F} = \{F,H\}$ by taking for F the functions $\int_\Omega \rho(\underline{x})\, d^3\underline{x}$, $\int_\Omega \rho(\underline{x})\, \sigma(\underline{x})\, d^3\underline{x}$, $\int_\Omega M_i(\underline{x})\, d^3\underline{x}$, $i = 1,2,3$.

Also, note that the map (3.32) is defined on $T^*(\mathrm{Diff}(\Omega))$ with values in $\mathfrak{X}^*(\Omega) \times \mathcal{F}^*(\Omega) \times \mathcal{F}(\Omega)$, where $\mathfrak{X}^*(\Omega)$ denotes the one-form densities on Ω in accordance with (3.9), $\mathcal{F}(\Omega)$ denotes the space of smooth functions on Ω, and $\mathcal{F}^*(\Omega)$ denotes the geometric dual of $\mathcal{F}(\Omega)$, the densities on Ω. The pairing between $\mathcal{F}(\Omega)$ and $\mathcal{F}^*(\Omega)$ is integration of the product.

If $\underline{\omega} = \mathrm{curl}\ \underline{v}$ denotes the Eulerian vorticity, the Eulerian potential vorticity is defined by

$$\Omega: = \underline{\omega}\cdot\underline{\nabla}\sigma/\rho. \tag{3.35}$$

From the equations of motion, it is easy to see that σ and Ω are conserved. In fact, a computation shows that any functional on $\mathcal{F}^*(\Omega) \times \mathcal{F}^*(\Omega) \times \mathcal{F}(\Omega)$ commutes (using the bracket (3.34)) with

$$F_\Phi(\sigma,\Omega) = \int_\Omega \rho(\underline{x})\Phi(\sigma(\underline{x}),\Omega(\underline{x}))d^3\underline{x}, \tag{3.36}$$

where Φ is an arbitrary real valued smooth function of two real variables. Consequently, (3.34) is a degenerate bracket, unlike the canonical bracket (3.30). The significance of the functionals F_Φ will be explained in the next section.

4. MECHANICAL SYSTEM ON DUALS OF SEMIDIRECT PRODUCT LIE ALGEBRAS

4.1 Poisson Manifolds and Momentum Maps. Throughout this section we employ the following standard notations and conventions. For a smooth manifold P, $\mathcal{F}(P)$ and $\mathfrak{X}(P)$ denote the ring of functions and the Lie algebra of vector fields on P respectively. The Lie algebra bracket of $\mathfrak{X}(P)$ is minus the usual Lie bracket for vector fields i.e. minus the bracket given by

$$[\underline{X},\underline{Y}]^i = X^j(\partial Y^i/\partial x^j) - Y^j(\partial X^i/\partial x^j). \tag{4.1}$$

A Lie group G is a smooth manifold which is a group in which multiplication and taking inverses are smooth maps. The tangent space T_eG at the identity $e \in G$ has a bracket operation obtained in the following way. For $\xi,\eta \in T_eG$, one defines vector fields $\underline{X}_\xi(g) = T_eL_g(\xi)$, $\underline{X}_\eta(g) = T_eL_g(\eta)$, where $L_g: G \to G$, $L_g(h) = gh$ is left translation and $T_eL_g: T_eG \to T_gG$ is the derivative of L_g, a linear map from T_eG to the tangent space T_gG to G at g. Then $[\xi,\eta] = [\underline{X}_\xi,\underline{X}_\eta](e)$. With this bracket, T_eG becomes a Lie algebra, called the left Lie algebra of G, or simply Lie algebra of G, and is denoted by $\mathfrak{g}$. Of course the same construction can be performed with right translations $R_g(h) = hg$, and again a Lie algebra structure on T_eG would result. The latter structure is anti-isomorphic to $\mathfrak{g}$, i.e. its bracket has the opposite sign of that of $\mathfrak{g}$.

$\mathfrak{X}(P)$ is the right Lie algebra of Diff(P); (see Ebin and Marsden [1970], or Abraham and Marsden [1978], ex. 4.1.G, page 274.) Since Lie algebras are usually thought of as left Lie algebras of Lie groups, we must introduce certain minus signs in the definitions that follow, in order to obtain the standard formulas in the literature; in particular, the left Lie algebra bracket of $\mathfrak{X}(P)$ is minus the usual Lie bracket of vector fields.

Let P be a smooth manifold. A Poisson bracket on P is a multiplication $\{\ ,\ \}$ on $\mathcal{F}(P)$ making $(\mathcal{F}(P), \{\ ,\ \})$ into a Lie algebra and a map $f \mapsto X_f \in \mathfrak{X}(P)$ such that $X_f(g) = \{g,f\}$ that is a Lie algebra homomorphism of $\mathcal{F}(P)$ into $\mathfrak{X}(P)$, i.e. $X_{\{f,g\}} = -[X_f,X_g]$. A manifold P endowed with a Poisson bracket is called a Poisson manifold. A map $\alpha:(P_1,\{\ ,\ \}_1) \to (P_2,\{\ ,\ \}_2)$ between Poisson manifolds is called canonical, if

$$\alpha^*\{f,g\}_2 = \{\alpha^*f,\alpha^*g\}_1 \tag{4.2}$$

for any $f,g \in \mathcal{F}(P_2)$, where the upper star denotes the pull-back operation. Functions $C \in \mathcal{F}(P)$ such that $\{C,f\} = 0$ for all $f \in \mathcal{F}(P)$ are called Casimir functions. Note that a canonical map $\alpha:P_1 \to P_2$ takes trajectories of $h \in \mathcal{F}(P_1)$ into trajectories of $\alpha^*h \in \mathcal{F}(P_2)$.

A Lie group action on a manifold P is a group homomorphism $\Phi:G \to \mathrm{Diff}(P)$, where Diff(P) denotes the group of diffeomorphisms of P, such that the map $(g,p) \mapsto \Phi_g(p)$ is smooth. If P is a Poisson manifold, Φ is called canonical if all the diffeomorphisms Φ_g, $g \in G$, are canonical maps of P. A Lie algebra action (infinitesimal generator) on a manifold P is a Lie algebra anti-homomorphism $\phi:\mathfrak{g} \to \mathfrak{X}(P)$ such that the map $(\xi,p) \mapsto \phi(\xi)(p)$ is smooth. If $\mathfrak{g}$ happens to be the (left) Lie algebra of a Lie group G acting on P, then $\phi = \Phi'$, where the upper prime denotes the Lie algebra homomorphism induced by Φ i.e. $\Phi' = T_e\Phi$. If P is a Poisson manifold, the Lie algebra action ϕ is said to be canonical if for any $\xi \in \mathfrak{g}$ and f_1, $f_2 \in \mathcal{F}(P)$,

$$\phi(\xi)\{f_1,f_2\} = \{\phi(\xi)f_1,f_2\} + \{f_1,\phi(\xi)f_2\}. \tag{4.3}$$

If the Lie group G with Lie algebra $\mathfrak{g}$ acts canonically on the Poisson manifold P, a momentum mapping $J:P \to \mathfrak{g}^*$ is a map satisfying

$$\phi(\xi) = X_{\hat{J}(\xi)} \tag{4.4}$$

for all $\xi \in \mathfrak{g}$, where $\hat{J}(\xi) \in \mathcal{F}(P)$ is defined by $\hat{J}(\xi)(p) = \langle J(p), \xi \rangle$, where $\langle \ , \ \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. J is said to be equivariant, if

$$J \circ \Phi_g = \mathrm{Ad}^*_{g^{-1}} \circ J \tag{4.5}$$

for all $g \in G$; here $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ denotes the adjoint action of G on $\mathfrak{g}$ and $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual map. If we deal with a canonical Lie algebra action ϕ of $\mathfrak{g}$ on P, the definition of the momentum mapping is unchanged, but equivariance is replaced by

$$T_p J(\phi(\xi)(p)) = -(\mathrm{ad}\ \xi)^*(J(p)) \tag{4.6}$$

for all $\xi \in \mathfrak{g}$, $p \in P$; here $T_p J : T_p P \to \mathfrak{g}^*$ denotes the tangent map (differential) of J at $p \in P$. Lie group (algebra) actions on a Poisson manifold admitting equivariant momentm maps are called Hamiltonian actions.

In duals of the Lie algebras, a Casimir function is characterized by the property of being invariant under the coadjoint action. This means that $C \in \mathcal{F}(\mathfrak{g}^*)$ is a Casimir function if and only if

$$C \circ \mathrm{Ad}^*_{g^{-1}} = C, \quad \text{or} \quad C(\mathrm{Ad}^*_{g^{-1}}\mu) = C(\mu)$$

for all $g \in G$ and $\mu \in \mathfrak{g}^*$.

We now give examples of the concepts above. Any symplectic manifold, in particular any cotangent bundle, is a Poisson manifold, the Poisson bracket being defined by the symplectic 2-form. The Casimir functions are constants. As we shall see in the next subsection, duals of Lie algebras are Poisson manifolds. If $G = SO(3)$ and $P = \mathbb{R}^3$, an example of an action of G on P is $\Phi_A(\underline{X}) = A\underline{X}$, where $A \in SO(3)$ and $\underline{X} \in \mathbb{R}^3$. If $G = \mathrm{Diff}(\Omega)$ and $P = \mathfrak{X}(\Omega)$, $\mathcal{F}(\Omega)$ or $\mathcal{F}^*(\Omega)$ (the densities on Ω), an action of P on G is given by push-forward. The adjoint action of $SO(3)$ on $\mathfrak{so}(3)$ is conjugation, and the adjoint action of $\mathrm{Diff}(\Omega)$ an $\mathfrak{X}(\Omega)$ is push-forward.

The momentum maps used in this section are all defined by actions which are cotangent lifts. This means that G acts on the manifold Q and one considers the induced action on T^*Q. Thus, if $\Phi : G \to \mathrm{Diff}(Q)$ is a (left or right) action, then

$$(g, \alpha_q) \mapsto T^*_{\Phi_g(q)} \Phi_{g^{-1}}(\alpha_q) \tag{4.7}$$

is also a (left or right) action; here $g \in G$, $\alpha_q \in T^*_q Q$, where $T^*_q Q$ denotes the cotangent space at q to Q and $T^*_{\Phi_g(q)} \Phi_{g^{-1}}$ is the dual of the tangent map (derivative) of $\Phi_{g^{-1}}$. In particular, if $Q = G$, then the left and right translations L_g and R_g can be lifted to left and right actions also denoted by L_g and R_g, of T^*G by (4.7), namely

$$L:G \times T^*G \to T^*G,\ L_g(\alpha_h) := T^*_{gh}L_{g^{-1}}(\alpha_h) \tag{4.8}$$

$$R:T^*G \times G \to T^*G,\ R_g(\alpha_h) := T^*_{hg}R_{g^{-1}}(\alpha_h). \tag{4.9}$$

Noether's theorem gives a formula for the momentum map of the actions (4.7). If ξ_Q denotes the infinitesimal generator of the action of G on Q, $\xi \in \mathfrak{g}$, then the equivariant momentum map

$$J:T^*Q \to \mathfrak{g}^*$$

is
$$J(\alpha_q)(\xi) = \langle \alpha_q, \xi_Q(q) \rangle \tag{4.10}$$

for $\alpha_q \in T^*_qQ$, $\xi \in \mathfrak{g}$, and $\langle\ ,\ \rangle$ the pairing between T^*Q and TQ. In particular, the two commuting actions (4.8) and (4.9) have the equivariant momentum maps

$$J_L:T^*G \to \mathfrak{g}^*,\quad J_L(\alpha_g) = (T_eR_g)^*(\alpha_g) \quad \text{for } L \tag{4.11}$$

$$J_R:T^*G \to \mathfrak{g}^*,\quad J_R(\alpha_g) = (T_eL_g)^*(\alpha_g) \quad \text{for } R. \tag{4.12}$$

4.2 Duals of Lie Algebras. The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a Poisson manifold with respect to the $\pm$ Lie-Poisson bracket given by

$$\{f,g\}_\pm(\mu) = \pm\left\langle \mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right] \right\rangle , \tag{4.13}$$

for $\mu \in \mathfrak{g}^*$ and f,g functions on $\mathfrak{g}^*$; here $\langle\ ,\ \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The "functional derivative" $\delta f/\delta\mu \in \mathfrak{g}$ is the derivative $Df(\mu)$ regarded as an element of $\mathfrak{g}$ rather than $\mathfrak{g}^{**}$, i.e.

$$Df(\mu)\cdot\nu = \left\langle \nu, \frac{\delta f}{\delta\mu} \right\rangle \tag{4.14}$$

for $\mu,\nu \in \mathfrak{g}^*$. (For infinite dimensional $\mathfrak{g}$, the pairing is with respect to a weakly non-degenerate form and the existence of $\delta f/\delta\mu$ is a bona fide hypothesis on f.) The space $\mathfrak{g}^*$ endowed with the $\pm$ Lie-Poisson bracket is denoted by $\mathfrak{g}^*_\pm$. The Hamiltonian vector field defined by the function h on $\mathfrak{g}^*$ is given by

$$X_h(\mu) = \mp \operatorname{ad}\left(\frac{\delta h}{\delta \mu}\right)^*(\mu), \tag{4.15}$$

where $\operatorname{ad}(\xi)\cdot\eta = [\xi,\eta]$ is the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ and $(\operatorname{ad}(\xi))^*:\mathfrak{g}^* \to \mathfrak{g}^*$ its dual map.

An important property of equivariant momentum maps is that they are canonical. More precisely, if $J:P \to \mathfrak{g}^*$ is an equivariant momentum map of a left Lie group or algebra action then $J:P \to \mathfrak{g}^*_+$ is canonical, i.e.

$$\{J^*f, J^*g\} = J^*\{f,g\}_+ \tag{4.16}$$

for all $f,g \in F(\mathfrak{g}^*)$ and $\{\ ,\}$ the Poisson bracket on P. An equivalent formulation is

$$J([\xi,\eta]) = \{\hat{J}(\xi),\hat{J}(\eta)\} \tag{4.17}$$

for all $\xi,\eta \in \mathfrak{g}$. If left actions are replaced by right actions, all the signs in (4.16) and (4.17) have to be changed; $J:P \to \mathfrak{g}^*_-$ is canonical. For example

$$J_L : T^*G \to \mathfrak{g}^*_+$$

and

$$J_R : T^*G \to \mathfrak{g}^*_-$$

given by (4.11) and (4.12) are canonical maps. By (4.11) and (4.12) J_L is right invariant and J_R is left invariant. Another important example is provided by the following.

Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $\alpha:\mathfrak{g} \to \mathfrak{h}$ a linear map. The dual map $\alpha^*:\mathfrak{h}^* \to \mathfrak{g}^*$ is canonical if and only if α is a Lie algebra homomorphism.

For a study of the local structure of Poisson manifolds the reader is referred to Weinstein [1984] and his lecture in this volume.

4.3 Semidirect Products. Let V be a topological vector space and assume that Φ is a left Lie group action on V such that each Φ_g is linear, i.e. $\Phi:G \to \operatorname{Aut}(V)$ is a group homomorphims, where $\operatorname{Aut}(V)$ is the Lie group of all linear continuous isomorphism of V. Such an action is called a (left) representation of G on V. The Lie algebra of $\operatorname{Aut}(V)$ is the space $\operatorname{End}(V)$ of all linear continuous maps of V into itself, with bracket the commutator of linear maps $[A,B] = AB - BA$, $A,B \in \operatorname{End}(V)$. The group representation Φ induces a Lie algebra representation $\Phi':\mathfrak{g} \to \operatorname{End}(V)$, so Φ' is a Lie algebra homomorphism.

Given G, V, and Φ, we define the semidirect product S as the Lie group with underlying manifold $G \times V$ and multiplication

$$(g_1,u_1)(g_2,u_2) = (g_1g_2,\ u_1 + \Phi(g_1)(u_2)) \tag{4.18}$$

where $g_1, g_2 \in G$, $u_1, u_2 \in V$. S is usually denoted by $G \ltimes V$, the action of G on V being known. Let $\mathfrak{s} = \mathfrak{g} \ltimes V$ be the Lie algebra of S; its bracket equals

$$[(\xi_1,v_1),(\xi_2,v_2)] = ([\xi_1,\xi_2],\ \Phi'(\xi_1)v_2 - \Phi'(\xi_2)v_1) \tag{4.19}$$

for $\xi_1, \xi_2 \in \mathfrak{g}$, $v_1, v_2 \in V$. The adjoint and coadjoint actions of S on $\mathfrak{s}$ and $\mathfrak{s}^*$ are given by

$$\mathrm{Ad}_{(g,u)}(\xi,v) = (\mathrm{Ad}_g\xi, \Phi(g)v - \Phi'(\mathrm{Ad}_g\xi)u) \tag{4.20}$$

and

$$[\mathrm{Ad}_{(g,u)^{-1}}]^*(\nu,a) \equiv \mathrm{Ad}^*_{(g,u)^{-1}}(\nu,a) = (\mathrm{Ad}^*_{g^{-1}}\nu + (\Phi_u')^*(\Phi_*(g))a,\ \Phi_*(g)a) \tag{4.21}$$

where $g \in G$, $u,v \in V$, $\nu \in \mathfrak{g}^*$, and $a \in V^*$; $\Phi_u' : \mathfrak{g} \to V$ is given by $\Phi_u'(\xi) = \Phi'(\xi)u$ and $(g,u)^{-1} = (g^{-1}, -\Phi(g^{-1})u)$. Recall that a <u>Casimir function</u> is characterized by being invariant under the coadjoint action. Formula (4.21) will be used in examples to determine whether a given function is a Casimuir.

The $\pm$ Lie-Poisson bracket of $F,H:\mathfrak{s}^* \to \mathbb{R}$ is, by (4.13) and (4.19), equal to

$$\{F,H\}_\pm(\mu,a) = \pm\left\langle\mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta H}{\delta\mu}\right]\right\rangle \pm \left\langle a, \Phi'\left(\frac{\delta F}{\delta\mu}\right)\cdot\frac{\delta H}{\delta a}\right\rangle \mp \left\langle a, \Phi'\left(\frac{\delta H}{\delta\mu}\right)\cdot\frac{\delta F}{\delta a}\right\rangle \tag{4.22}$$

where $\mu \in \mathfrak{g}^*$, $a \in V^*$, and as in (4.15), $\frac{\delta F}{\delta\mu} \in \mathfrak{g}$ and $\frac{\delta F}{\delta a} \in V$. From formula (4.15), we compute the Hamiltonian vector field of $H:\mathfrak{s}^* \to \mathbb{R}$ to be

$$X_H(\mu,a) = \mp\left(\mathrm{ad}\left(\frac{\delta H}{\delta\mu}\right)^*\mu - {\Phi'_{\frac{\delta H}{\delta a}}}^*a,\ \Phi'\left(\frac{\delta H}{\delta\mu}\right)^*a\right) \tag{4.23}$$

where $\Phi'_{\frac{\delta H}{\delta a}} : \mathfrak{g} \to V$ is given by $\Phi'_{\frac{\delta H}{\delta a}}(\xi) = \Phi'(\xi)\cdot\frac{\delta H}{\delta a}$, and ${\Phi'_{\frac{\delta H}{\delta a}}}^*$ is its adjoint.

The left and right translations on S, for $(\alpha_h,v,a) \in T^*_{(h,v)}(G \times V) = T^*_hG \times V \times V^*$, are

$$L((g,u),(\alpha_h,v,a)) = \left((T_{gh}L_{g^{-1}})^*\alpha_h, u + \Phi(g)v, \Phi(g^{-1})^*a\right), \tag{4.24}$$

$$R((g,u),(\alpha_h,v,a)) = \left((T_{hg}R_{g^{-1}})^*\alpha_h - df^a_{\Phi(g^{-1})u}(hg), v + \Phi(h)u, a\right) \tag{4.25}$$

where $f_u^a(g)$ is the "matrix element" $\langle a,\Phi(g)u\rangle$ and $df_u^a(g)$ its differential. The corresponding momentum mappings are by (4.11) and (4.12)

$$J_L:T^*S \to \mathfrak{s}_+^*,\ J_L(\alpha_g,v,a) = (T_{(e,0)}R_{(g,v)})^*(\alpha_g,v,a) = ((T_eR_g)^*\alpha_g + (\Phi_v')^*a,a) \tag{4.26}$$

and

$$J_R:T^*S \to \mathfrak{s}_-^*,\quad J_R(\alpha_g,v,a) = (T_{(e,0)}L_{(g,v)})^*(\alpha_g,v,a) = ((T_eL_g)^*\alpha_g,\ \Phi(g)^*a). \tag{4.28}$$

Recall that J_R is left invariant and J_L is right invariant; both are canonical.

4.4 The Theorems. In many physical examples a Hamiltonian system on T^*G is given whose Hamiltonian function H_a depends smoothly on a parameter $a \in V^*$. In addition, H_a is left invariant under the stabilizer $G_a = \{g \in G|\Phi(g^{-1})^*a = a\}$ whose Lie algebra is $\mathfrak{g}_a = \{\xi \in \mathfrak{g}|\Phi'(\xi)^*a = 0\}$. We can think of this Hamiltonian also as a function $H:T^*G \times V^* \to \mathbb{R}$, $H_a(\alpha_g) = H(\alpha_g,a)$, where $T^*G \times V^*$ has the direct sum Poisson structure: the bracket of two functions on $T^*G \times V^*$ is their bracket on T^*G. (If V were a Lie algebra, a case not discussed in this lecture, V^* would be endowed with its own Lie-Poisson structure.) We wish to study the motion determined by H on a "flat" space whithout losing any information about the original motion on $T^*G \times V^*$. The key to this approach is the momentum maps (4.26) and (4.27).

We start with the left action of the semi-direct product $S = G \ltimes V$ on T^*S. The momentum map J_R is invariant under the left action and the subgroup $V \subset S$ acting on the left on T^*S has a momentum map given by the second component of J_L (see (4.26)), i.e. $(\alpha_g,u,a) \mapsto a$. This is a canonical map if V^* is thought of as having the + Lie-Poisson structure, which is trivial since V is an abelian Lie algebra. Moreover, the canonical projection $T^*S \to T^*G$ is clearly canonical, so that the map

$$P_L:T^*S \to T^*G \times V^*,\quad P_L(\alpha_g,u,a) = (\alpha_g,a) \tag{4.28}$$

is canonical. Now it is easily seen that J_R factors through P_L:

$$\tilde{J}_R:T^*G \times V^* \to \mathfrak{s}_-^*,\quad \tilde{J}_R(\alpha_g,a) = (T_e^*L_g(\alpha_g),\Phi(g)^*a) \tag{4.29}$$

i.e. the following diagram commutes

$$\begin{tikzcd} & T^*S \arrow[dl, "P_L"'] \arrow[dr, "J_R"] & \\ T^*G \times V^* \arrow[rr, "\tilde{J}_R"'] & & \mathfrak{s}^*_- \end{tikzcd}$$

Consequently, $\tilde{J}_R$ is a canonical map.

A similar situation occurs when one considers the right action of S on T^*S. The momentum map J_L is right invariant, and the subgroup $V \subset S$ acts on the right in a Hamiltonian manner on T^*S with momentum map given by the second component of (4.27), i.e. $(\alpha_g, u, a) \mapsto \Phi(g)^*a$. This map is therefore canonical. Moreover, the map

$$(\alpha_g, u, a) \mapsto \alpha_g + df^a_{\Phi(g^{-1})(u)}(g) = \alpha_g + T^*_g R_{g^{-1}}(\Phi'_u)^* a$$

being a projection followed by a translation with an exact differential on the fibers, is a canonical map $T^*S \to T^*G$. Consequently

$$P_R : T^*S \to T^*G \times V^*$$
$$P_R(\alpha_g, u, a) = (\alpha_g + T^*_g R_{g^{-1}}(\Phi'_u)^* a, \Phi(g)^*a) \tag{4.30}$$

is canonical. It is easily shown that J_L factors through P_R:

$$\tilde{J}_L : T^*G \times V \to \mathfrak{s}^*_+, \quad \tilde{J}_L(\alpha_g, a) = (T^*_e R_g(\alpha_g), \Phi(g^{-1})^*a) \tag{4.31}$$

i.e. the following diagram commutes

$$\begin{tikzcd} & T^*S \arrow[dl, "P_R"'] \arrow[dr, "J_L"] & \\ T^*G \times V^* \arrow[rr, "\tilde{J}_L"'] & & \mathfrak{s}^*_+ \end{tikzcd}$$

Consequently, $\tilde{J}_L$ is the reduction of J_L by V, so is a canonical map.

A few comments are in order regarding the difference between right and left in the previous construction. The space $T^*G \times V^*$ is diffeomorphic to the orbit space of T^*S by the left or right V-action. The explicit diffeomorphisms are $[\alpha_g, u, a] \mapsto (\alpha_g, a)$ for the left V-action and $[\alpha_g, u, a] \mapsto \left(\alpha_g + df^a_{\Phi(g^{-1})}(g), \Phi(g)^*a\right)$ for the right V-action, where $[\alpha_g, u, a]$ denotes the left or right V-orbit through (α_g, u, a). Via these diffeomorphisms the canonical projections become P_L and P_R respectively. (As we remarked in the introduction, the asymmetry between left and right is because we have chosen a left action of G on V.). We refer the reader to Marsden, Ratiu, Weinstein [1983] for an analysis of the symplectic leaves of $T^*G \times V^*$. We summarize the results in the following.

Theorem 1. *The maps*

$$\tilde{J}_L, \tilde{J}_R : T^*G \times V^* \to s^*_\pm\ ,\ \tilde{J}_L(\alpha_g, a) = (T_e^*R_g(\alpha_g),\ \Phi(g^{-1})^*a)$$

$$\tilde{J}_R(\alpha_g, a) = (T_e^*L_g(\alpha_g),\ \Phi(g)^*a)$$

are canonical; in fact, these maps are reductions of the momentum maps by the action of V *and are themselves momentum maps for the action (left or right) of* $G \ltimes V$ *on the Poisson manifold* $T^*G \times V^*$.

See Holm, Kupershmidt, and Levermore [1983] for a direct verification of the canonical nature of $\tilde{J}_L$ in some examples.

After this kinematic theorem we turn our attention to dynamics. Let $H : T^*G \times V^* \to \mathbb{R}$ be a Hamiltonian and assume that the function $H_a : T^*G \to \mathbb{R}$, $H_a(\alpha_g) = H(\alpha_g, a)$, $a \in V^*$, is invariant under the lift to T^*G of the left action of the stabilizer G_a on G. Then it is easily seen that H induces a Hamiltonian function $H_L : s^*_- \to \mathbb{R}$ defined by $H_L \circ \tilde{J}_R = H$, i.e. $H_L(T_e^*L_g(\alpha_g),\ \Phi(g)^*a) = H(\alpha_g, a)$. For right invariant Hamiltonians interchange "left" by "right", and "-" by "+". However, since the maps $\tilde{J}_R$ and $\tilde{J}_L$ are different, we have $H_R \circ \tilde{J}_L = H$, i.e. $H_R(T_e^*R_g(\alpha_g),\ \Phi(g^{-1})^*a) = H(\alpha_g, a)$.

It is of interest to investigate the evolution of $a \in V^*$ in s^*_-; we work now with a left action. Let $c_a(t) \in T^*G$ denote an integral curve of H_a and let $g_a(t)$ be its projection on G. Then $t \mapsto (c_a(t), a)$ is an integral curve of H on $T^*G \times V^*$ so that the curve $t \mapsto \tilde{J}_R(c_a(t), a)$ is an integral curve of H_L on s^*_-. Thus $t \mapsto \Phi(g_a(t))^*a$ is the evolution of the initial condition a in s^*_-. For right actions, if $c_a(t)$ and $g_a(t)$ are as above, the curve $t \mapsto \tilde{J}_L(c_a(t),\ \Phi(g_a(t)^{-1})^*a)$ is an integral curve of H on $T^*G \times V^*$ so that $t \mapsto \tilde{J}_L(c_a(t),\ \Phi(g_a(t)^{-1})^*a)$ is an integral curve of H_R on s^*_+. Hence $t \mapsto \Phi(g_a(t)^{-1})^*a$ is again the evolution of the variable a in s^*_+. The difference between the integral curves of H for left and right actions is due to the different formulas for P_L and P_R. We have proved the following.

Theorem 2. *Let* $H : T^*G \times V^* \to \mathbb{R}$ *be left invariant under the action on* T^*G *of the stabilizer* G_a *for every* $a \in V$. *Then* H *induces a Hamiltonian* $H_L \in F(s^*_-)$ *defined by* $H(T_e^*L_g(\alpha_g),\ \Phi(g)^*a) = H(\alpha_g, a)$, *thus yielding Lie-*

Poisson equations on s_-^*. *The curve* $c_a(t) \in T^*G$ *is a solution of Hamilton's equations defined by* H_a *on* T^*G *if and only if* $\tilde{J}_R(c_a(t),a)$ *is a solution of the Hamiltonian system defined by* H_L *on* s_-^*. *In particular, the evolution of* $a \in V^*$ *is given by* $\Phi(g_a(t))^*a$ *where* $g_a(t)$ *is the projection of* $c_a(t)$ *on* G. *For right invariant systems, interchange everywhere "left" by "right," "-" by "+", set* $H_R \in F(s_+^*)$, $H_R(T_e^*R_g(\alpha_g)$, $\Phi(g^{-1})^*a) = H(\alpha_g,a)$, *and the evolution of* a *is given by* $\Phi(g_a(t)^{-1})^*a$.

We conclude this section with some general remarks. In many examples one is given the phase space T^*G, but it is not obvious a priori what V and Φ should be. The phase space T^*G is often interpreted as 'material' or 'Lagrangian' coordinates, while the equations of motion may be partially or wholly derived in 'spatial' ('Eulerian') or 'convective' ('body') coordinates. This means that the Hamiltonian might be given directly on a space of the form $\mathfrak{g}^* \times V^*$, where the evolution of the V^* variable is by 'dragging along' or 'Lie transport' i.e. it is of the form $t \mapsto \Phi(g(t))^*a$ for left invariant systems (or $t \mapsto \Phi(g(t)^{-1})^*a$ for right invariant ones), where $a \in V^*$ and $g(t)$ is the solution curve in the configuration space G. This then determines the representation Φ and shows whether one should work with left or right actions. The relation between H_L (or H_R) and H_a in Theorem 2 uniquely determines H_a, which is automatically G_a-invariant, and (4.20), (4.21) give the corresponding Lie-Poisson bracket and equations of motion. The parameter $a \in V^*$ often appears in the form of an initial condition on some physical variable of the given problem.

5. APPLICATIONS.

In this section we shall consider the heavy top and the adiabatic fluid equations in both space and body coordinates. The convective picture for the heavy top and the Eulerian picture for fluids are classical. The other two pictures are less common but are also interesting; see Guillemin and Sternberg [1980] for some indications in this direction.

5.1. Heavy Top in Body Coordinates. We shall apply the theorems of the previous section first directly and then backwards.

The direct approach starts with the Lagrangian picture. The Hamiltonian (2.34) (or (2.33) in terms of Euler angles) is invariant under rotations about the spatial Oz-axis. This means that we deal with the standard left representation of SO(3) on $\mathbb{R}^3$,

$$\Phi(A)\underline{x} = A\underline{x}, \tag{5.1}$$

$A \in SO(3)$, $\underline{x} \in \mathbb{R}^3$. By Theorem 2, H defines a Hamiltonian H_L on $e(3)^*$, where $e(3) = so(3) \ltimes \mathbb{R}^3$ is the Euclidean Lie algebra. The Lagrangian to body (convective) map $\tilde{J}_R : T^*(SO(3)) \times \mathbb{R}^3 \to e(3)^*$ is given in this case by

$$\tilde{J}_R(\underline{\alpha}_A, \underline{X}) = (A\underline{\alpha}_A, A^{-1}\underline{X}). \tag{5.2}$$

To gain a physical interpretation of this map, we must determine $\underline{\alpha}_A$, if $t \mapsto A(t)$ is a solution of the problem. If $\mathring{A}(t)$ is the tangent vector to $SO(3)$ at $A(t)$, then $\hat{\underline{\omega}}_B(t) = A(t)^{-1}\mathring{A}(t) \in so(3)$ and $\underline{m} = I\underline{\omega}_B \in so(3)^*$. Now recall that by definition when working in body coordinates, $\underline{\alpha}_A = T^*_A L_{A^{-1}}(\underline{m})$, i.e. $\underline{\alpha}_A$ is the momentum in the material picture. Thus, if $\underline{\alpha}_A$ is as above,

$$\tilde{J}_R(\underline{\alpha}_A, Mg\ell\underline{k}) = (\underline{m}, Mg\ell\underline{\gamma}).$$

In coordinates this is the map (2.38). Thus, the Hamiltonian H_L has the familiar expression (2.32). By Theorem 2, the evolution of $\underline{k}$ is given by $A(t)^{-1}\underline{k}$, where $A(t)$ is the solution of the problem in the configuration space $SO(3)$. Note that $\underline{\gamma} = A(t)^{-1}\underline{k}$ is the dynamic variable in $e(3)^*$ in accordance to the general theory. It is clear that $Mg\ell\underline{k}$ is a parameter in the problem. It represents the direction of gravity and the momentum of the body around the fixed point.

By the general theory, H_L given by (2.32) defines Lie-Poisson equations on $e(3)^*$. The bracket is given by (4.20) and the equations of motion by (4.21). To write the bracket and the Lie-Poisson equations explicitly, we note first that $\Phi' : so(3) \to \mathrm{End}(\mathbb{R}^3)$ is given by $\Phi'(\xi)\underline{x} = \xi\underline{x}$, $\xi \in so(3)$, $\underline{x} \in \mathbb{R}^3$. For $F, G : e(3)^* \to \mathbb{R}$, $\frac{\delta F}{\delta \underline{m}} = (\nabla_{\underline{m}}F)^\wedge$, $\frac{\delta F}{\delta \underline{\gamma}} = \nabla_{\underline{\gamma}}F$ ($\nabla_{\underline{m}}$, $\nabla_{\underline{\gamma}}$ denote the usual gradients with respect to $\underline{m}, \underline{\gamma} \in \mathbb{R}^3$) and hence

$$\Phi'\left(\frac{\delta F}{\delta \underline{m}}\right) \cdot \frac{\delta G}{\delta \underline{\gamma}} = (\nabla_{\underline{m}}F)^\wedge \nabla_{\underline{\gamma}} G = \nabla_{\underline{m}} F \times \nabla_{\underline{\gamma}} G,$$

$$\Phi'\left(\frac{\delta F}{\delta \underline{m}}\right)^* \cdot \gamma = -\nabla_{\underline{m}} F \times \underline{\gamma}, \quad \Phi'\left(\frac{\delta F}{\delta \underline{\gamma}}\right)^* \underline{\gamma} = \nabla_{\underline{\gamma}} F \times \underline{\gamma}, \text{ and } \quad \mathrm{ad}\left(\frac{\delta F}{\delta \underline{m}}\right)^* \cdot \underline{m} = -\nabla_{\underline{m}} F \times \underline{m}.$$

With these formulas, the bracket (4.20) becomes (2.30). The Lie-Poisson equations (4.23) become for this case

$$\begin{cases} \dot{\underline{m}} = -\nabla_{\underline{m}} H \times \underline{m} - \nabla_{\underline{\gamma}} H \times \underline{\gamma} \\ \dot{\underline{\gamma}} = -\nabla_{\underline{m}} H \times \underline{\gamma} \end{cases}$$

or, explicitly taking into account that $\nabla_{\underline{m}} H = \frac{m_1}{I_1}, \frac{m_2}{I_2}, \frac{m_3}{I_3}$ and $\nabla_{\underline{\gamma}} H = Mg\ell\underline{\chi}$, this system is (2.36).

Theorem 2 of the previous section can also be applied backwards. Then one starts with the Hamiltonian (2.32) on $\mathbb{R}^3 \times \mathbb{R}^3$ and the equations of motion (2.36). The last three equations say that $\underline{\gamma}$ is dragged along by the group action. So $\underline{\gamma} = A(t)^{-1}\underline{k}$ where $A(t)$ is the solution of the problem in the configuration space $SO(3)$. This implies by Theorem 2 that we are dealing with a left invariant system and the standard representation of $SO(3)$ on $\mathbb{R}^3$. Consequently, one easily computes J_R and H which, of course, turns out to be (2.34). Thus, once again, (2.36) are Hamiltonian with respect to the bracket (2.39).

To determine the Casimir functions, note that $\|\underline{\gamma}\|^2$ and $\underline{m}\cdot\underline{\gamma}$ are conserved by (2.36). By (4.21) applied to $e(3)^*$, the coadjoint action is given by

$$\mathrm{Ad}^*_{(A,\underline{u})^{-1}}(\underline{m},\underline{\gamma}) = (A\underline{m} + \underline{u} \times A\underline{\gamma},\ A\underline{\gamma}).$$

With this formula, it is easy to see that

$$C_1(\underline{m},\underline{\gamma}) = \Phi(\|\underline{\gamma}\|^2), \quad C_2(\underline{m},\underline{\gamma}) = \psi(\underline{m}\cdot\gamma)$$

are invariant under the coadjoint action, for arbitrary real-valued functions of a real variable Φ and Ψ. In other words, C_1 and C_2 are Casimir functions.

5.2 Heavy Top in Space Coordinates. To study the motion of the heavy top in space coordinates, we again apply the theorems of the previous section. As remarked at the end of §4, we first have to investigate the invariance properties of H under right translations $A \mapsto AB$, $B \in SO(3)$, a constant matrix. By (2.34),

$$H(AB) = -\frac{1}{4}\,\mathrm{Tr}(BIB^{-1}A^{-1}\dot{A}A^{-1}\dot{A}) + Mg\ell\underline{k}\cdot AB\underline{\chi}\ ,$$

so that $H(AB) = H(A)$ if and only if

$$BIB^{-1} = I, \quad B\underline{\chi} = \underline{\chi}.$$

Thus, the parameter a in the general theory is, in this case, the pair $(I,\underline{\chi})$. So far, we have thought of I as being a diagonal matrix, which was consistent with the body coordinate approach: an observer sitting on the body perceives I as constant, so he can choose once and for all a body coordinate system in which I is diagonal. However, an observer who is spatially fixed sees I moving. Thus, even if I is initially diagonal, it will not stay so; i.e. I must be a general symmetric covariant (indices down) two tensor, the vector space of all them being denoted by $S_2(\mathbb{R}^3)$. We see that $(I,\underline{\chi})$ belongs to $S_2(\mathbb{R}^3) \times \mathbb{R}^3 = V^*$. The dual of $S_2(\mathbb{R}^3)$ is $S^2(\mathbb{R}^3)$, the space of contravariant symmetric two-tensors on $\mathbb{R}^3$, with the pairing given by contraction on both indices, i.e. the trace of the product. Consequently $V = S^2(\mathbb{R}^3) \times \mathbb{R}^3$, and the action of $SO(3)$ on V is conjugation in the first factor and the standard action on the second. The Lagrangian to Eulerian map $\tilde{J}_L : T^*(SO(3)) \times S_2(\mathbb{R}^3) \times \mathbb{R}^3 \to (\mathfrak{so}(3) \ltimes (S^2(\mathbb{R}^3) \times \mathbb{R}^3))^*_+$ is given by

$$\tilde{J}_L(\underline{\alpha}_A, L, \underline{X}) = (T^* R_A(\underline{\alpha}_A), ALA^{-1}, A\underline{X}) .$$

To gain a physical interpretation of this map, recall that I was computed in a body frame. The spatial frame is obtained by a rotation by $A(t)^{-1}$, where $t \mapsto A(t)$ is the trajectory of the motion. Consequently, the moment of inertia I_s in space coordinates is $I_s = AIA^{-1}$. Thus, the map $\tilde{J}_L$ becomes

$$\tilde{J}_L(\underline{\alpha}_A, I, Mg\ell\underline{\chi}) = (\underline{m}_s, I_s, Mg\ell\underline{\lambda})$$

where $\underline{\lambda} = A\underline{\chi}$ and $\underline{m}_s = I_s\underline{\omega}_s = A\underline{m}$. The Hamiltonian becomes

$$H_s(\underline{m}_s, I_s, \underline{\lambda}) = \underline{m}_s \cdot I_s^{-1}\underline{m}_s + Mg\ell\underline{k}\cdot\underline{\lambda}$$

and thus

$$\frac{\delta H_s}{\delta \underline{m}_s} = I_s^{-1}\underline{m}_s \quad , \quad \frac{\delta H_s}{\delta \underline{\lambda}} = Mg\ell k \quad , \quad \frac{\delta H_s}{\delta I_s} = \underline{\omega}_s \otimes \underline{\omega}_s$$

where $\underline{a} \otimes \underline{b}$ represents the symmetric matrix whose entries are $a_i b_j$. The + Lie-Poisson bracket on $(\mathfrak{so}(3) \times (S^2(\mathbb{R}^3) \times \mathbb{R}^3))^*$ is by (4.21) equal to

$$\{F,G\}(\underline{m}_s, I_s, \underline{\lambda}) = \underline{m}_s \cdot (\nabla_{\underline{m}_s} F \times \nabla_{\underline{m}_s} G) + \mathrm{Tr}\left(I_s\left(\left[\left(\frac{\delta F}{\delta \underline{m}_s}\right)^{\wedge}, \frac{\delta G}{\delta I_s}\right] - \left[\left(\frac{\delta G}{\delta \underline{m}_s}\right)^{\wedge}, \frac{\delta F}{\delta I_s}\right]\right)\right)$$
$$+ \underline{\lambda}\cdot(\nabla_{\underline{m}_s} F \times \nabla_{\underline{\lambda}} G + \nabla_{\underline{m}_s} G \times \nabla_{\underline{\lambda}} F).$$

The equations of motion are by (4.22)

$$\begin{cases} \dot{\underline{m}}_s = \nabla_{\underline{m}_s} H_s \times \underline{m}_s - \left[\dfrac{\delta H_s}{\delta I_s}, I_s\right]^{\vee} + \nabla_{\underline{\lambda}} H_s \times \underline{\lambda} \\ \dot{I}_s = \left[I_s, \left(\dfrac{\delta H}{\delta \underline{m}_s}\right)^{\wedge}\right] \\ \dot{\underline{\lambda}} = \nabla_{\underline{m}_s} H_s \times \underline{\lambda} \end{cases}$$

where ${}^{\vee}: so(3) \to \mathbb{R}^3$ is the inverse of $\hat{}$. Using $\hat{\underline{u}} \cdot \underline{v} = \underline{u} \times \underline{v}$ and the "back-cab" identity $\underline{A} \times (\underline{B} \times \underline{C}) = \underline{B}(\underline{A} \cdot \underline{C}) - \underline{C}(\underline{A} \cdot \underline{B})$, one sees that for our Hamiltonian the first two terms of the $\dot{\underline{m}}_s$ equation cancel. Thus, we have

$$\begin{cases} \dot{\underline{m}}_s = Mg\ell \underline{k} \times \underline{\lambda} \\ \dot{I}_s = [I_s, \hat{\underline{\omega}}_s] \\ \dot{\underline{\lambda}} = \underline{\omega}_s \times \underline{\lambda} \end{cases}$$

where $\underline{m}_s = I_s \underline{\omega}_s$. This is the spatial form of the heavy top equations and they are thus in Euler-Poisson form; i.e. Lie-Poisson for a semi-direct product.

The coadjoint action is given by

$$\mathrm{Ad}^*_{(A,J,\underline{u})}(\underline{m}_s, I_s, \underline{\lambda}) = (A\underline{m}_s + \underline{u} \times A\underline{\lambda} + [J, AI_sA^{-1}], AI_sA^{-1}, A\underline{\lambda}).$$

Let π_1, π_2, π_3 be the three invariants of the matrix I_s. Since they are invariant under conjugation, they are invariant under the above coadjoint action. Consequently, these give Casimirs. There are in fact six in all:

$$\begin{aligned} C_1(\underline{m}_s, I_s, \underline{\lambda}) &= \Phi_1(\pi_1) \\ C_2(\underline{m}_s, I_s, \underline{\lambda}) &= \Phi_2(\pi_2) \\ C_3(\underline{m}_s, I_s, \underline{\lambda}) &= \Phi_3(\pi_3) \\ C_4(\underline{m}_s, I_s, \underline{\lambda}) &= \Phi_4(\|\underline{\lambda}\|^2) \\ C_5(\underline{m}_s, I_s, \underline{\lambda}) &= \Phi_5((I_s\underline{\lambda}) \cdot \underline{\lambda}) \\ C_6(\underline{m}_s, I_s, \underline{\lambda}) &= \Phi_6(\|I_s \lambda\|^2) \end{aligned}$$

The generic orbit in our twelve dimensional Lie-Poisson space $(so(3) \ltimes (S^2(\mathbb{R}^3) \ltimes \mathbb{R}^3))^*$ is six dimensional (the coadjoint action has, at each point, a six dimensional isotropy subgroup with A = Identity and $\underline{u} = I_s\underline{\lambda}$, $J = \lambda \otimes \lambda$, (one dimension); $\underline{u} = \underline{0}$, $[J, I_3] = 0$ (three dimensions) and $\underline{u} \perp \underline{\lambda}$, $J = 0$ (two dimensions)) which is consistent with the existence of six Casimirs. There is, in addition, the constant of motion $\underline{m}_s \cdot \underline{k}$ for our special Hamiltonian corresponding to invariance under rotations about the z-axis. Thus, we can reduce again getting back to the four dimensional reduced phase space (T^*S^2) of the heavy top. (For the Lagrange top there is, of course, an additional conserved quantity).

The above shows concretely the duality between the spatial and body descriptions of a heavy top. In fact they form a dual pair (Weinstein [1983]). We shall see a similar situation for fluids in Sections 5.3 and 5.4 following.

5.3 Ideal Compressible Adiabatic Fluids in Eulerian Coordinates. The Hamiltonian (3.27) was shown in §3 to be right invariant under the subgroup of $\mathrm{Diff}(\Omega)$ given by (3.28). This means that we deal with the representation of $\mathrm{Diff}(\Omega)$ on $\mathcal{F}(\Omega) \times \mathcal{F}^*(\Omega)$ by push-forward, i.e.

$$\Phi(\eta)(f,\mu) = (\eta_* f, \eta_* \mu)$$

for $f \in \mathcal{F}(\Omega)$, $\mu \in \mathcal{F}^*(\Omega)$. The induced Lie algebra representation is by minus the Lie derivative. The Lagrangian to Eulerian map

$$\tilde{J}_L : T^*(\mathrm{Diff}(\Omega)) \times \mathcal{F}^*(\Omega) \times \mathcal{F}(\Omega) \to \left(\mathfrak{X}(\Omega) \ltimes (\mathcal{F}(\Omega) \times \mathcal{F}^*(\Omega))\right)^*_+$$

is given in this case by

$$\tilde{J}_L(\underline{\alpha}_\eta, \mu, f) = ((\underline{\xi}_\eta \circ \eta^{-1}) J_{\eta^{-1}} d^3\underline{x},\ \eta^*\mu,\ \eta^* f),$$

where $\underline{\alpha}_\eta(\underline{X}) = \xi_\eta(\underline{X}) d^3\underline{X}$, $\underline{x} = \eta(\underline{X})$. Thus, if $\underline{V}_\eta$ is the material velocity, (3.26) gives the material momentum density, $\underline{M}_\eta = \rho_0 \underline{V}^b d^3\underline{X}$ and the above formula becomes

$$\tilde{J}_L(\underline{M}_\eta,\ \rho_0 d^3\underline{X}, \sigma_0) = (\underline{M},\ \rho d^3\underline{x}, \sigma)$$

where $\underline{M}(\underline{x}) = \rho(\underline{x}) v^b(\underline{x}) d^3\underline{x}$. This is exactly the map (3.32). By Theorem 2 the evolution of $\rho_0 d^3\underline{X}$ and σ_0 is given by $t \mapsto \eta_t^*(\rho_0 d^3\underline{X})$, $t \mapsto \eta_t^*\sigma_0$, where η_t is the solution curve in $\mathrm{Diff}(\Omega)$. The Lie-Poisson bracket given by (4.21) is easily seen to equal (3.34) and the equations of motion (4.22) are (3.18), (3.19) and (3.20). The change of the parameters ρ_0 and σ_0 corresponds to choosing different initial conditions.

The coadjoint action of $\mathrm{Diff}(\Omega) \propto (\mathcal{F}(\Omega) \times \mathcal{F}^*(\Omega))$ on $(\mathfrak{X}(\Omega) \ltimes (\mathcal{F}^*(\Omega) \times \mathcal{F}(\Omega))^*$ is given by (4.20), (3.16)

$$\mathrm{Ad}^*_{(\eta, f, \mu)^{-1}}(\underline{M},\ \rho d^3\underline{x}, \sigma) = (\eta_* \underline{M} + f\eta_*(\rho d^3\underline{x}) + \mu\eta_*\sigma,\ \eta_*(\rho d^3\underline{x}),\ \eta_*\sigma).$$

Let now $\underline{\omega} = \mathrm{curl}\ \underline{v}$ be the vorticity and denote by $\Omega = \underline{\omega}\cdot\underline{\nabla}\sigma/\rho = \mathrm{div}(\sigma\underline{\omega})/\rho$. It is then easy to see that the functional

$$C(\underline{M}, \rho, \sigma) = \iiint_\Omega \rho(\underline{x}) \Phi(\sigma(\underline{x}), \Omega(\underline{x})) d^3\underline{x},$$

for an arbitrary real-valued function Φ of two real variables, is invariant under the Ad^*-action. Consequently, the functional C is a Casimir function.

The theorems of §4 can also be applied backwards in order to interpret (3.18), (3.19), (3.20) as Lie-Poisson equations. Start with the configuration space $\mathrm{Diff}(\Omega)$, the physical energy function $H(\underline{M},\rho,\sigma)$ given by (3.21), $\underline{M}(\underline{x}) = \rho(\underline{x})v^b(\underline{x})d^3\underline{x}$, conservation of mass and entropy (3.18), (3.19) and balance of momentum (3.20) with equation of state $p = \rho^2\partial w(\rho,\sigma)/\partial\rho$. Then remark that (3.18), (3.19) are equivalent to $L_{\underline{v}}(\rho(\underline{x})d^3\underline{x}) = 0$, $L_{\underline{v}}(\sigma(\underline{x})) = 0$, i.e. $\eta_t^*(\sigma(\underline{x})d^3\underline{x}) = \sigma_0(\underline{X})d^3\underline{X}$, $\eta_t^*(\sigma(\underline{x})) = \sigma_0(\underline{X})$, for ρ_0, σ_0 the initial mass and entropy density. Hence the dual of the representation space is $\mathcal{F}^*(\Omega) \times \mathcal{F}(\Omega)$ so that $V = \mathcal{F}(\Omega) \times \mathcal{F}^*(\Omega)$. Moreover, the prior push-forward formulas show that the left representation of $\mathrm{Diff}(\Omega)$ on V^* is push-forward so that by Theorem 2, the representation of $\mathrm{Diff}(\Omega)$ on V is also push-forward. Then, again by Theorem 2, since H_{ρ_0,σ_0} is invariant under $\mathrm{Diff}(\Omega)_{\rho_0,\sigma_0}$, equations (3.18), (3.19), (3.20) are + Lie-Poisson equations on $(\mathfrak{X}(\Omega) \ltimes (\mathcal{F}(\Omega) \times \mathcal{F}^*(\Omega))^*_+$.

5.4 Ideal Compressible Adiabatic Fluids in Convective Coordinates. To study the motion in convective coordinates we have to investigate the invariance properties of H under left translations $\eta \mapsto \psi\circ\eta$, ψ a time independent (orientation preserving) diffeomorphism. Since

$$T^*_{\eta\circ\psi^{-1}}L_\psi(\underline{M}_\eta) = T^*\psi(\underline{\xi}_\eta)d^3\underline{X}$$

for $\underline{M}_\eta = \underline{\xi}_\eta d^3\underline{X}$, formula (3.27) yields

$$H(T^*_{\eta\circ\psi^{-1}}L_\psi(\underline{M}_\eta)) = \frac{1}{2}\,\|T^*_{\eta\circ\psi^{-1}}L_\psi(\underline{M}_\psi)\| + \int_\Omega \rho_0(\underline{X})w(\rho_0(\underline{X})J^{-1}_{\psi\circ\eta}(\underline{X}),\ \sigma_0(\underline{X}))d^3\underline{X}.$$

If g is the metric on Ω and $\dagger$ denotes adjoints with respect to g, then if $\underline{M}_\eta = \rho_0\underline{V}^b_\eta d^3\underline{X}$, using the definitions in §3.4 we see that

$$(T^*\psi(\underline{\xi}_\eta))^\# = (T\psi)^\dagger\,\underline{V}_\eta,$$

so that

$$(T^*_{\eta\circ\psi^{-1}}L_\psi(\underline{M}_\eta)/\rho_0 d^3X)^\# = (T\psi)^\dagger\underline{V}_\eta.$$

Thus the Hamiltonian becomes

$$H(T^*_{\eta\circ\psi^{-1}}L_\psi(\underline{M}_\eta)) = \frac{1}{2}\int \rho_0(\underline{X})\|(T\psi)^\dagger V_\eta(\underline{X})\|^2 d^3\underline{X} + \int \rho_0(\underline{X})w\Big(\rho_0(\underline{X})J^{-1}_\eta(\underline{X})J^{-1}_\psi(\eta(\underline{X})),\ \sigma_0(\underline{X})\Big)d^3\underline{X}\ .$$

This expression coincides with the one for $H(\underline{M}_\eta)$ if and only if $(T\psi)^\dagger = T\psi$ and hence $T\psi$ is an isometry, and $\rho(\underline{x})\, J_\psi^{-1}(\underline{x}) = \rho(\underline{x})$, where $\rho(\underline{x}) = \rho_0(\underline{X}) J^{-1}(\underline{X})$. If ψ is an isometry, then $J_\psi = 1$; consequently, H is left invariant under the group

$$\mathrm{Diff}(\Omega)_g = \{\psi \in \mathrm{Diff}(\Omega) \,|\, \psi \text{ is an isometry for } g\}.$$

Thus, the parameter a in the general theory is in this case g. The space of all the g's does not form a vector space, but it is an open cone in the vector space of all symmetric covariant 2-tensors $S_2(\Omega)$. The dual of $S_2(\Omega)$ is $S^2(\Omega) \otimes |\Lambda^3(\Omega)|$ the vector space of contravariant symmetric two-tensor densities. The convected ρ and σ have trivial equations of motion corresponding to the dependence of a only on g. However, we can include them for comparison with the spatial case (this is analogous to adding the trivial equations $\dot{I} = 0$ to the heavy top equations in body coordinates.) Thus, we take $V^* = S_2(\Omega) \times \mathcal{F}^*(\Omega) \times \mathcal{F}(\Omega)$ and hence $V = S^2(\Omega) \otimes |\Lambda^3(\Omega)| \times \mathcal{F}(\Omega) \times \mathcal{F}(\Omega)^*$. The representation of $\mathrm{Diff}(\Omega)$ on V is by push-forward in every factor.

The Lagrangian to convective map $\tilde{J}_R : T^*(\mathrm{Diff}(\Omega)) \times S_2(\Omega) \times \mathcal{F}^*(\Omega) \times \mathcal{F}(\Omega) \to (\mathrm{Diff}(\Omega) \ltimes (S^2(\Omega) \otimes |\Lambda^3(\Omega)| \times \mathcal{F}(\Omega) \times \mathcal{F}(\Omega)^*))^*_-$ is given by

$$J_R(\underline{M}_\eta, g, \rho, \sigma) = (T_e L_\eta(\underline{M}_\eta),\ \eta^* g,\ \eta^*\rho\ \eta^*\sigma) := (\underline{\mathcal{M}}, \mathcal{G}, \mathcal{R}, \mathcal{S})$$

where $\underline{\mathcal{M}}$ is the convected momentum density, related to the spatial momentum density $\underline{M}$ by

$$\underline{\mathcal{M}} = \eta^* \underline{M}$$

Using the identity $L_{\underline{v}} \underline{v}^\flat = (\nabla_{\underline{v}} \underline{v})^\flat + \frac{1}{2} d\|\underline{v}\|^2$, we find that the equations of motion for $(\underline{\mathcal{M}}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ are

$$\frac{\partial \underline{\mathcal{M}}}{\partial t} = \frac{1}{2} \mathcal{R} d(\|\underline{\mathcal{M}}\|^2_{\mathcal{G}}) - dP$$

$$\frac{\partial \mathcal{G}}{\partial t} = L_{\underline{\mathcal{V}}} \mathcal{G}$$

$$\frac{\partial \mathcal{R}}{\partial t} = 0, \quad \frac{\partial \mathcal{S}}{\partial t} = 0$$

where $\|\underline{\mathcal{M}}\|^2_{\mathcal{G}}$ is the length of $\underline{\mathcal{M}}$ in the metric $\mathcal{G}$, $P = \eta^* p = \mathcal{R}^2 \partial W / \partial \mathcal{R}$ is the convected pressure and $\underline{\mathcal{V}}$ is the convected velocity: $\underline{\mathcal{M}} = \mathcal{R} \underline{\mathcal{V}}^\flat$. By the general theory, these equations are Lie-Poisson on the space of tuples $(\underline{\mathcal{M}}, \mathcal{G}, \mathcal{R}, \mathcal{S})$; of course $\mathcal{R}$ and $\mathcal{S}$ are 'cyclic' variables. As in the heavy top, what were Casimirs in the spatial picture now become special constants of the motion, and new Casimirs appear (integrals of functions of $\mathcal{G}$, $\mathcal{R}$ and

S). Note finally that G, analogous to $\underline{\gamma}$ for the heavy top, is advected in the convective (body) picture, but is static in the spatial (Eulerian) picture. Likewise R and S, analogous to I are static in the convective picture and dynamic in the spatial picture.

As mentioned in the introduction, the duality between the spatial and convective pictures and its relationship to the stress formulas of Doyle-Ericksen-Simo-Marsden (see Simo and Marsden [1983]) as well as to covariance of energy balance under body (right) and spatial (left) diffeomorphisms will be the subject of a future publication.

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