Contemporary Mathematics
Volume 97, 1989

Block Diagonalization and the Energy-Momentum Method

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Dedicated to Roger Brockett on the occasion of his 50th birthday

Abstract

We prove a geometric generalization of a block diagonalization theorem first found by the authors for rotating elastic rods. The result here is given in the general context of simple mechanical systems with a symmetry group acting by isometries on a configuration manifold. The result provides a choice of variables for linearized dynamics at a relative equilibrium which block diagonalizes the second variation of an augmented energy - these variables effectively separate the rotational and internal vibrational modes. The second variation of the effective Hamiltonian is block diagonal, separating the modes completely, while the symplectic form has an off diagonal term which represents the dynamic interaction between these modes. Otherwise, the symplectic form is in a type of normal form. The result sets the stage for the development of useful criteria for bifurcation as well as the stability criteria found here. In addition, the techniques should apply to other systems as well, such as rotating fluid masses.

* Research partially supported by AFOSR/DARPA contract F49620-87-C0118 and MSI at Cornell University.

** Research partially supported by AFOSR contract 2-DJA-544 and 2-DJA-771.

*** Research partially supported by MSI at Cornell University.

AMS Subject Classification 58F, 70H

§1 *Introduction*

In this paper we present a block diagonalization theorem which is designed for the analysis of stability and bifurcation of rotating systems, or more generally, of relative equilibria. The context of the discussion is the energy-momentum method for mechanical systems with symmetry. Simo, Posbergh and Marsden [1989] and Lewis and Simo [1989] discovered crucial special cases of the block diagonalization theorem for uniformly rotating systems, including general nonlinear elasticity and geometrically exact rods. Our purpose is to abstract these examples and prove a general geometric theorem. We expect these general results will be important for rotating gravitational fluid masses as well.

For rotating systems the result says that a splitting of coordinates can be *explicitly* found on a linearized level which represent the rotational and internal vibrational modes. In these coordinates, the second variation of an augmented Hamiltonian is block diagonal. Of course coordinates can always be found in principle to do this, but we are able to do it explicitly enough to give useful stability and, we believe, bifurcation criteria. On the other hand, the symplectic form does *not* block diagonalize, indicating that the rotational and internal modes are in fact dynamically coupled. However, for purposes of the stability calculation, block diagonalization of the augmented energy is what is important. The off diagonal terms in the symplectic form (sometimes called *Coriolis coupling terms*) are, however, sufficiently simple that they should be useful for studying the dynamic interaction of the rotational and internal vibrational modes.

For rotating pseudo-rigid bodies, Lewis and Simo [1989] noticed that the computation of the definiteness of the second variation is considerably simplified by our result - in this case the simplification saves considerable computation time. In their case, the symbolic and numerical manipulation needed would normally require testing a full 14×14 matrix for definiteness; block diagonalization techniques, however, reduces this to testing a 6×6 matrix for nonisotropic bodies or a 3×3 matrix for the isotropic case.

According to Jellinek and Li [1989], "the general problem of separation and characterization of the overall rotation in any (not necessarily rigid or near rigid) N-body system is among the few still unsolved problems of traditional classical mechanics." Jellinek and Li are able to achieve results for the N-body problem by elimination of the coupling from the expression for the energy in an instantaneous fashion. We have been able to achieve a similar result for the general case of rotating structures, be they N-body systems, coupled rigid bodies, or elastic or fluid structures. This flexibility is achieved through a general geometric approach.

Acknowledgements We thank Tony Bloch, P.S. Krishnaprasad, Richard Montgomery, George Patrick, and Tudor Ratiu for helpful comments.

§2 *The Energy-Momentum Method*

We begin our work in the context of standard mechanical systems with symmetry before any reductions have taken place. In other words, we begin with a *symplectic* manifold (P, Ω) rather than a *Poisson* manifold. In fact, shortly we shall specialize to the case of $P = T^*Q$ and a Hamiltonian of the form kinetic plus potential.

Let G be a Lie group acting symplectically on P with an equivariant momentum mapping

$$J : P \to \mathfrak{g}^* \tag{1}$$

(see Abraham and Marsden [1978], Marsden [1981] or Marsden, Weinstein, Ratiu, Schmid, and Spencer [1983] for the standard definitions and results used here).

Let $H : P \to \mathbb{R}$ be a given G-invariant Hamiltonian. A point $z_e \in P$ is called a ***relative equilibrium*** if there is a $\xi \in \mathfrak{g}$, the Lie algebra of G, such that the curve

$$z(t) := \exp(t\xi)\, z_e \tag{2}$$

is the dynamical orbit of the Hamiltonian vector field X_H of H, with initial condition $z(0) = z_e$. Let $\mu_e = J(z_e)$, the momentum at equilibrium. The energy-momentum method relies on the following result.

2.1 Relative Equilibrium Theorem *A point z_e is a relative equilibrium iff there is a $\xi \in \mathfrak{g}$ such that z_e is a critical point of $H_\xi : P \to \mathbb{R}$, where*

$$H_\xi(z) = H(z) - \langle J(z) - \mu_e, \xi \rangle . \tag{3}$$

In (3), the Lie algebra element $\xi \in \mathfrak{g}$ may be regarded as a Lagrange multiplier. Since J is conserved by the flow of X_H, the set $J - \mu_e = 0$ is preserved, so one may regard it as a (non-holonomic) constraint set. It also follows that $\xi \in \mathfrak{g}_{\mu_e}$, the isotropy algebra of μ_e (with respect to the coadjoint action). Thus,

$$\delta H_\xi(z_e) = 0 \tag{4}$$

may be regarded as a (constrained) *variational principle for relative equilibria.*

The relative equilibrium theorem is readily verified. Of course it has a long history, going back to Lagrange and Poincaré for rotating systems. Like many basic results, it has been rediscovered in a number of contexts by various authors. Early references in our context are Arnold [1966], Smale [1970] and Marsden and Weinstein [1974]. As we shall state below, the relative equilibrium theorem sometimes specializes to the ***principle of symmetric criticality*** (Palais [1979]).

The energy-momentum test for formal stability of a relative equilibrium z_e proceeds as follows (see Holm *et al.* [1985] for the meaning of formal stability and related references).

Energy-Momentum Method

1. Choose $\xi \in \mathfrak{g}$ such that $\delta H_\xi(z_e) = 0$
2. Choose a linear subspace $\mathcal{S} \subset T_{z_e}P$ such that
   - i $\mathcal{S} \subset \ker TJ(z_e)$ and
   - ii $\mathcal{S}$ complements $T_{z_e}(G_{\mu_e} \cdot z_e)$ in $\ker TJ(z_e)$, where $G_{\mu_e} \subset G$ is the isotropy subgroup of μ_e.
3. Test $\delta^2 H_\xi(z_e)$ for definiteness as a bilinear form on $\mathcal{S}$.

The energy-momentum method "covers" the energy-Casimir method (Holm *et al.* [1985]) in the sense that if the latter applies and gives formal stability, so does the former. One difficulty with the energy-Casimir method is that on the reduced space P/G there may not be enough Casimirs to make the method effective; in particular, it may not be possible to obtain the analogue $\delta(H + C)(z_e) = 0$ of (4). This difficulty is genuine for the case of geometrically exact rods, for instance. See Simo, Posbergh and Marsden [1989] for further details.

The fact that $\delta^2 H_\xi(z_e)$ drops to the reduced space follows from the next lemma.

2.2 Gauge Invariance Lemma

$$\delta^2 H_\xi(z_e)(\eta_P(z_e), \delta z) = 0 \tag{5}$$

for all $\delta z \in \ker TJ(z_e)$ and $\eta \in \mathfrak{g}$, where η_P denotes the infinitesimal generator of the group action on P.

This follows readily from invariance of H and equivariance of J. One can view (5) as a block diagonalization result on the unconstrained tangent space $T_{z_e}P$, but it does not yield block diagonalization *within* the constrained subspace $\mathcal{S}$ in the energy-momentum method. It is the latter that we are concerned with here.

One can identify any choice of $\mathcal{S}$ with the tangent space to the ***reduced space***

$$P_{\mu_e} = J^{-1}(\mu_e)/G_{\mu_e}$$

at $[z_e]$ (assuming, as we shall, that μ_e is a regular and generic value; c.f. Weinstein [1984]), but it is easier to do our analysis directly on $T_{z_e}T^*Q$ rather than on the quotient space. This is the usual situation found in constrained optimization problems although we shall see that keeping in mind the geometry of the quotient space will play a useful role.

§3 *Simple Mechanical Systems*

Let Q be a configuration manifold and $P = T^*Q$ the associated phase space with its canonical symplectic structure. In the finite dimensional case, we denote cotangent coordinates on T^*Q by (q^i, p_i). (When we use coordinates, we assume Q is finite dimensional, although the results are *not* restricted to this case.) Coordinates on the velocity phase space TQ are similarly denoted $(q^i, \dot q^i)$.

Let g denote a Riemannian metric on Q; in coordinates, we write the components of g as g_{ij} as usual, and we write g^{ij} for the inverse tensor. Let $K : TQ \to \mathbf{R}$ denote the corresponding kinetic energy,

$$K(q, \dot q) = \frac{1}{2} g_{ij}(q)\, \dot q^i \dot q^j , \tag{1}$$

and let $V : Q \to \mathbf{R}$ be a given potential.

Assume G acts on Q (by a left action) and hence on T^*Q by the cotangent lift, so the equivariant momentum map is given by

$$\langle J, \xi\rangle(\alpha_q) = \langle \alpha_q, \xi_Q(q)\rangle. \tag{2}$$

In coordinates, we define the ***action coefficients*** $A^i_a(q)$ by writing

$$[\xi_Q(q)]^i = A^i_a(q)\, \xi^a \tag{3}$$

where $a, b, c, \ldots$ denote coordinate indices for the Lie algebra $\mathfrak{g}$. Thus (2) becomes

$$J_a(q, p) = p_i A^i_a(q). \tag{4}$$

We assume that G acts on Q by isometries and that the potential V is G-invariant. For elasticity, for instance, this is the requirement of material frame indifference. Note that (3) of §2 reads

$$H_\xi(q, p) = \frac{1}{2} g^{ij} p_i p_j + V(q) - p_i A^i_a(q)\, \xi^a + \langle \mu_e, \xi\rangle . \tag{5}$$

Define the ***moment of inertia tensor*** $\mathbf{I}$ for the system ***locked at*** $q \in Q$ by

$$I_{ab}(q) = g_{ij}(q)\, A^i_a(q)\, A^j_b(q) \tag{6}$$

(alternatively, in terms of the q-dependent inner product $\langle \xi, \eta\rangle := \langle \xi_Q(q), \eta_Q(q)\rangle$ on $\mathfrak{g}$ we have $\langle \xi, \eta\rangle = I_{ab}(q)\xi^a\eta^b$), define the ***augmented potential*** V_ξ by

$$V_\xi(q) = V(q) - \frac{1}{2} I_{ab}(q)\, \xi^a \xi^b. \tag{7}$$

We note that the function V_ξ is not the same as the ***amended*** potential in the sense of Smale [1970]. This is the function $V_\mu(q) = V(q) + \frac{1}{2} I^{ab}(q)\, \mu_a \mu_b$ which also plays an important role in this story, but will be the subject of other investigations.

One can readily verify the following (see Abraham and Marsden [1978] and Palais [1979]) by writing out the conditions $\delta H_\xi = 0$ in **2.1**. A more elegant argument is, however, given below.

3.1 Principle of Symmetric Criticality *A point* $z_e = (q_e , p_e) = (q^i, p_i)$ *is a relative equilibrium if and only if there is a* $\xi \in \mathfrak{g}_{\mu_e}$ *such that*

i $\quad p_i = g_{ij} A^j_a \xi^a$ (*i.e.*, p_e *is the Legendre transform of* $\xi_Q(q_e)$) (8a)

and

ii $\quad q_e$ *is a critical point of* V_ξ. (8b)

This is useful for carrying out the computations that follow. We also observe that V_ξ is G_ξ-invariant, and so induces a function on Q/G_ξ. Define the one-form A^ξ on Q by $A^\xi = A^\xi_i dq^i$, where

$$A^\xi_i(q) = g_{ij}(q)\, A^j_a(q)\, \xi^a \tag{9}$$

or abstractly, $A^\xi(q) = [\xi_Q(q)]^\flat$, where $^\flat$ denotes the index lowering operation with respect to the metric g_{ij}. In other words, $A^\xi(q)$ is the Legendre transform of $\xi_Q(q)$. We remark that A may be viewed as a G-connection for the bundle $Q \to Q/G$ and that this connection plays an important role in Berry's phase; cf. Marsden, Montgomery and Ratiu [1989]. Now notice that at equilibrium, (8a) says

$$p_e = A^\xi(q_e) . \tag{10}$$

Also note that

$$H_\xi(q,p) = K_\xi(q,p) + V_\xi(q) + \langle \mu_e, \xi \rangle\,, \tag{11}$$

where $K_\xi(q,p) = \frac{1}{2}\| p - A^\xi(q) \|^2$, and V_ξ is given by (7). By (10), K_ξ has a critical point at z_e. *Thus, (8b) is a consequence of the relative equilibrium theorem and (11).*

In the energy-momentum method we shall use a special choice of $\mathcal{S}$, namely

$$\mathcal{S} = \left\{ v_{z_e} \in T_{z_e}T^*Q \mid T\pi_Q \cdot v_{z_e} \text{ is g-orthogonal to } T(G_{\mu_e} \cdot q_e) \text{ and } v_{z_e} \in \ker[TJ(z_e)] \right\}, \tag{12a}$$

where $\pi_Q : T^*Q \to Q$ is the canonical projection. Letting coordinates on $T(T^*Q)$ be denoted

$$(q^i, p_i, \delta q^i, \delta p_i),$$

(12) reads, with the help of (8a),

$$\mathcal{S} = \Big\{ (q^i, p_i, \delta q^i, \delta p_i) \mid \; g_{ij}(\delta q)^i A^j_a \chi^a = 0 \text{ for all } \chi \in \mathfrak{g}_{\mu_e} \text{ and } (\delta p)_i A^i_a + g_{ij} A^j_b \xi^b \frac{\partial A^i_a}{\partial q^k} (\delta q)^k = 0 \Big\} \tag{12b}$$

§4 *Rigid Variations*

One version of the cotangent bundle reduction theorem (see Abraham and Marsden [1978] and Kummer [1981], Montgomery [1986] and references therein) states that the reduced space $(T^*Q)_{\mu_e}$ is a symplectic bundle over $T^*(Q/G)$ with fiber the coadjoint orbit through μ_e. Thus there is an isomorphism

$$T_{[z_e]}(T^*Q)_{\mu_e} \cong \mathfrak{g}/\mathfrak{g}_{\mu_e} \times T_{[z_e]}(T^*(Q/G)) \cong \mathfrak{g}/\mathfrak{g}_{\mu_e} \times (\mathcal{V}_{\rm INT} \times \mathcal{V}^*_{\rm INT})$$

where $\mathcal{V}_{\rm INT}$ is a model space for Q/G. For $G = SO(3)$, $\mathcal{V}_{\rm INT}$ *models the configuration space for the internal modes*, while $\mathfrak{g}/\mathfrak{g}_e \cong T_{\mu_e}\mathcal{O}_{\mu_e}$ *models the phase space for rigid modes*. Our goal is to realize this decomposition explicitly, in such a way that $\delta^2 H_\xi(z_e)$ block diagonalizes. The bundle $(T^*Q)_\mu \to T^*(Q/G)$ with fiber $\mathcal{O}_\mu$ also has a natural connection (Montgomery [1986]). Unfortunately, our decomposition is not simply the horizontal-vertical split for this connection. We shall need a construction which is somewhat more sophisticated, but is similar in spirit.

We will define two subspaces $\mathcal{S}_{\rm RIG}$ and $\mathcal{S}_{\rm INT}$ of $\mathcal{S}$ and further subspaces $\mathcal{W}_{\rm INT}$ (isomorphic with $\mathcal{V}_{\rm INT}$) and $\mathcal{W}^*_{\rm INT}$ (isomorphic with $\mathcal{V}^*_{\rm INT}$) of $\mathcal{S}_{\rm INT}$ such that

$$\mathcal{S} = \mathcal{S}_{\rm RIG} \oplus \mathcal{S}_{\rm INT} = \mathcal{S}_{\rm RIG} \oplus (\mathcal{W}_{\rm INT} \oplus \mathcal{W}^*_{\rm INT}), \tag{1}$$

relative to which $\delta^2 H_\xi(z_e)$ will be block diagonal. As above, the first component $\mathcal{S}_{\rm RIG} \cong \mathfrak{g}/\mathfrak{g}_{\mu_e}$ of $\mathcal{S}$ is isomorphic to the tangent space to the coadjoint orbit through μ_e. As we shall see, this component also carries the coadjoint orbit symplectic structure.

The first component $\mathcal{S}_{\rm RIG}$ will be defined in terms of rigid variations. This will be done by going back to Q temporarily, defining rigid variations there, and then using the Legendre transformation to transfer the information over to the cotangent bundle. To carry this out, let

$$\mathfrak{g}_Q = \{ \eta_Q(q) \in TQ \mid \eta \in \mathfrak{g} \text{ and } q \in Q \} \tag{2}$$

and let $T\mathfrak{g}_Q \subset T(TQ)$ be its tangent bundle.

4.1 Definition *Let* $V_{\rm RIG} = s(T\mathfrak{g}_Q)$ *where* $s : T^2Q \to T^2Q$ *is the canonical involution. Alternatively,* $V_{\rm RIG}$ *consists of double tangents of curves denoted by* $\Delta\dot q$ *(identified with velocity variations of superposed rigid body motions in the case of* $SO(3)$).

$$\Delta\dot q = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{d}{dt}\Big|_{t=0} \exp(\varepsilon\eta(t))\, q(t),$$

where $\eta(t)$ *is a curve in* $\mathfrak{g}$ *with* $\eta(0) = \eta$ *and* $q(t)$ *is a curve in* Q. (The canonical involution in effect swaps the order of differentiation.)

In coordinates, if we write elements of $V_{\rm RIG}$ as

$$(q^i, \dot q^i, \Delta q^i, \Delta\dot q^i), \tag{3a}$$

then we find that

$$\Delta q^i = A^i_a \eta^a \quad\text{and}\quad \Delta\dot q^i = \frac{\partial A^i_a}{\partial q^k} \dot q^k \eta^a + A^i_a \zeta^a. \tag{3b}$$

An intrinsic way of writing the split (3b) and hence the definition of $V_{\rm RIG}$ is the following:

$$V_{\rm RIG} = T_{v_e}(G \cdot v_e) \oplus \mathrm{vert}_{v_e}(\mathfrak{g}_Q) \tag{3c}$$

where $v_e = (q_e, \dot{q}_e)$ is the relative equilibrium in TQ. Here, $FL(v_e) = z_e$, where $FL : TQ \to T^*Q$ is the Legendre transform given by

$$p_i = g_{ij}\,\dot{q}^j \tag{4}$$

and $\mathrm{vert}_{v_e}(\mathfrak{g}_Q)$ denotes the vertical lift of the bundle $\mathfrak{g}_Q$ at the point v_e.

Next, let $TFL : T(TQ) \to T(T^*Q)$ be the tangent map to the Legendre transformation (4), and set

$$\mathcal{S}_{RIG} = TFL \cdot V_{RIG} \cap \mathcal{S}, \tag{5}$$

where $\mathcal{S}$ is defined by (12) of §3. If we let $\mathfrak{g}_{\mu_e}^{\perp}$ denote the (q-dependent) orthogonal complement of $\mathfrak{g}_{\mu_e}$ in the metric I_{ab}, then one finds that $\mathcal{S}_{RIG}$ is parametrized by elements $\eta \in \mathfrak{g}_{\mu_e}^{\perp}$ as follows: we write elements of $\mathcal{S}_{RIG}$ as

$$(q^i, p_j, \Delta q^i, \Delta p_j), \tag{6a}$$

where

$$\Delta q^i = A_a^i\,\eta^a, \tag{6b}$$

and

$$\Delta p_i = -\frac{\partial A_a^k}{\partial q^i}\eta^a\,p_k + g_{ij}\,A_a^j\,\zeta^a, \tag{6c}$$

where $\eta \in \mathfrak{g}_{\mu_e}^{\perp}$ and $\zeta \in \mathfrak{g}$; the condition that (6a) belongs to $\ker(T_{z_e}J)$ is equivalent to the relation

$$\zeta^a = I^{ab}(\mathrm{ad}_\eta^*\,\mu_e)_b \tag{7}$$

i.e., $\zeta^b = \mathrm{ad}_\eta^*\mu_e$, so ζ is determined by η. One checks that $\zeta \in \mathfrak{g}_{\mu_e}^{\perp}$ as well.

§5 *The Internal Vibration Space*

Now we define a complement to $\mathcal{S}_{RIG}$ in $\mathcal{S}$. We will do this by a constructive procedure that can be effectively carried out in examples. To define this complement, to be denoted $\mathcal{S}_{INT}$, we first describe $\mathcal{V}_{INT}$.

Recall that the amended potential V_ξ is given by

$$V_\xi = V + L_\xi, \tag{1a}$$

where

$$L_\xi(q) = -\tfrac{1}{2}\langle \xi_Q(q), \xi_Q(q)\rangle. \tag{1b}$$

For mechanical systems undergoing stationary rotations about ξ, i.e., $G = SO(3)$ and G_{μ_e} = rotations about the axis μ_e, which is parallel to ξ, we note that L_ξ gives the potential of the centrifugal force. Now define $\mathcal{V}_{INT}$ as the subspace on which V_ξ or, equivalently, L_ξ ***looks objective*** in the sense of nonlinear elasticity (cf. Marsden and Hughes [1983]). More precisely:

5.1 Definition

$$\mathcal{V}_{INT} = \Big\{\delta q \in T_{q_e}Q \mid \big\langle \delta q, (\pounds_{\eta_Q} dL_\xi)(q)\big\rangle = 0 \quad \text{for all} \quad \eta \in \mathfrak{g}_{\mu_e}^{\perp} \quad \text{and} \quad \langle \delta q, \chi_Q(q_e)\rangle = 0 \quad \text{for all} \quad \chi \in \mathfrak{g}_{\mu_e}\Big\}, \tag{2}$$

where the first pairing is the natural pairing between vectors and one forms while the second is the metric inner product, and $\pounds$ *denotes the Lie derivative.*

Since V_ξ has a critical point at q_e (by the principal of symmetric criticality) and V is G-invariant, we find that

$$\big\langle \delta q, (\pounds_{\eta_Q} dL_\xi)(q)\big\rangle = \delta^2 V_\xi(q_e)(\delta q, \eta_Q(q)) \tag{3}$$

and so we see that the geometric condition (2) is exactly what is needed to block diagonalize $\delta^2 V_\xi(q_e)$ within $\mathcal{S}$. In coordinates, the first condition on δq^i defining $\mathcal{V}_{INT}$ is the geometric condition

$$\delta q^i\,\eta^a\,\xi^b\,\xi^c\,\frac{\partial}{\partial q^i}\Big[A_a^k\,\frac{\partial}{\partial q^k}(A_b^\ell\,A_c^m\,g_{\ell m})\Big] = 0; \tag{2'}$$

the second condition is just the defining condition on $\mathcal{S}$. Notice that the space $\mathcal{V}_{INT}$ is a model space for the quotient Q/G. Now we are ready to define $\mathcal{S}_{INT}$.

5.2 Definition $\mathcal{S}_{INT} = \{\delta z \in T_{z_e}T^*Q \mid \delta q \in \mathcal{V}_{INT}$ *and* $\delta z \in \ker[TJ(z_e)]\} \subset \mathcal{S}$. (4)

Assuming that the quadratic form in η and $\overline{\eta}$ given by

$$A^i_d \eta^d \overline{\eta}^a \xi^b \xi^c \frac{\partial}{\partial q^i}\left[A^k_a \frac{\partial}{\partial q^k}(A^\ell_b A^m_c g_{\ell m})\right] \tag{2''}$$

is nondegenerate, we get the following result.

5.3 Proposition $\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$.

In fact, the condition of nondegeneracy of the form (2″) implies that $\mathcal{S}_{RIG} \cap \mathcal{S}_{INT} = \{0\}$, the spanning follows from the dimension count $\dim \mathcal{S}_{RIG} = \dim(\mathfrak{g}/\mathfrak{g}_\mu)$ and the fact that $\mathcal{S}_{INT}$ is determined by $\dim(\mathfrak{g}/\mathfrak{g}_\mu)$ equations.

As we shall see, the condition of nondegeneracy of (2″) is the same as the condition that the second variation of the Hamiltonian H_ξ restricted to $\mathcal{S}_{RIG}$ is nondegenerate. In fact, for stability, we want to assume that this is positive definite.

Now we want to insert the space $\mathcal{V}_{INT}$ into the space $\mathcal{S}_{INT}$. To do so, we shall use the condition $TJ(z_e) \cdot \delta z = 0$. This condition gives us a way of determining δp in terms of δq in such a way that the corresponding δz lies in the space $\ker TJ(z_e)$. This condition in coordinates is as follows:

$$\delta p_i A^i_a(q) + p_i \frac{\partial A^i_a}{\partial q^k} \delta q^k = 0. \tag{5a}$$

Now, let $\mathcal{W}^*_{INT}$ be defined as the set of vertical variations $(\delta q, \delta p) = (0, \delta p)$ which annihilate the infinitesimal action; i.e., $\delta p_i A^i_a(q) = 0$. Clearly, $\mathcal{W}^*_{INT} \subset \mathcal{S}_{INT}$. Next, let

$$\mathcal{T} = \{\delta p_i \in T^*_{q_e}Q \mid g^{ij} dp_i \pi_j = 0 \text{ for all } \pi_i \text{ such that } \pi_i A^i_a(q) = 0\}$$

$$= \{\delta p_i \in T^*_{q_e}Q \mid \delta p_i = g_{ij} A^j_b \chi^b \text{ for some } \chi \in \mathfrak{g}\}.$$

Now given $\delta q^i \in \mathcal{V}_{INT}$, we can *uniquely solve* (5a) for $\delta p_i \in \mathcal{T}$. In fact, we find that

$$\chi^b = -\mathbb{I}^{ba} p_i \frac{\partial A^i_a}{\partial q^k} \delta q^k. \tag{5b}$$

Using (5b), construct the corresponding pair $(\delta q^i, \delta p_i)$ (using the metric and vertical lift makes this intrinsic). This defines the space $\mathcal{W}_{INT}$. By construction, $\mathcal{W}_{INT} \subset \mathcal{S}_{INT}$ and $\mathcal{W}_{INT}$ is isomorphic to $\mathcal{V}_{INT}$. Thus, we have acheived the split

$$\mathcal{S}_{INT} = \mathcal{W}_{INT} \oplus \mathcal{W}^*_{INT}. \tag{6}$$

Remark This way of injecting $\mathcal{V}_{INT}$ into $\ker TJ(z_e)$ is closely related to the map

$$\alpha_{\mu_e} : Q \to J^{-1}(\mu_e);\; q \mapsto (\mathbb{I}(q)^{-1}\mu_e)_Q(q)^\flat$$

which occurs in the cotangent bundle reduction theorem (Smale [1970], Abraham and Marsden [1978], Kummer [1981]).

We remark here that even if G is abelian (for instance, $G = S^1$ in the case of planar coupled rigid bodies), the decompositions are not trivial; while $\mathcal{S}_{RIG} = \{0\}$ in this case, $\mathcal{S}_{INT} = \mathcal{W}_{INT} \oplus \mathcal{W}^*_{INT}$ is still not a trivial decomposition.

Next, we give a characterization of $\mathcal{V}_{INT}$ in terms of superposed motions.

5.4 Proposition *Let* $q_\varepsilon \in Q$ *be a curve tangent to* δq *at* q_e, *let* $\eta \in \mathfrak{g}^\perp_{\mu_e}$ *and let* $\eta_\varepsilon = \mathrm{Ad}_{\exp(\varepsilon\xi)}\eta$. *Then* $\mathcal{V}_{INT}$ *is characterized by those* δq *orthogonal to* $T_{q_e}(G_{\mu_e} \cdot q_e)$ *and satisfying*

$$\frac{d}{d\varepsilon}\left\langle \xi_Q(q_\varepsilon), (\eta_\varepsilon)_Q(q_\varepsilon)\right\rangle\Big|_{\varepsilon=0} = 0 \tag{7}$$

or, equivalently,

$$\frac{d}{d\varepsilon}\left\langle \xi_Q(\exp(\varepsilon\xi)q_\varepsilon),\ \eta_Q(\exp(\varepsilon\xi)q_\varepsilon)\right\rangle\Big|_{\varepsilon=0} = 0. \tag{8}$$

This is verified by a direct coordinate calculation. We can lift this expression to get an alternative characterization of $\mathcal{S}_{INT}$. We consider the momentum map J restricted to $\mathfrak{g}^\perp_{\mu_e}$ and regarded as a function on TQ. In other words, for $\zeta \in \mathfrak{g}^\perp_{\mu_e}$, set

$$J(\zeta)(\delta q) = \langle \zeta_Q(q), \delta q\rangle = g_{ij} A^i_a \zeta^a (\delta q)^j. \tag{9}$$

In what follows, we shall assume the following condition is satisfied, which is automatically true for $SO(3)$: $\zeta \in \mathfrak{g}^\perp_{\mu_e} \Rightarrow [\xi, \zeta] \in \mathfrak{g}^\perp_{\mu_e}$. (This property is only needed for a few alternative formulas and will be investigated for general groups in a future publication.) Now consider the condition

$$\frac{d}{dt} J(\zeta)(\delta q)\Big|_{t=0} = 0, \tag{10}$$

where ζ is to evolve as $\dot{\zeta} = [\xi, \zeta]$ which is consistent with (8a) and $\zeta \in \mathfrak{g}^\perp_{\mu_e}$; here ξ is the Lie algebra element giving the relative equilibrium. Equation (10) defines a condition on $T(TQ)$. We shall regard it as a condition on $T_{z_e}(T^*Q)$ via the Legendre transform. For simplicity we still write the resulting condition as $\dot{J} = 0$.

5.5 Proposition

$$\mathcal{S}_{INT} = \{\dot{J}(z_e) = 0\} \cap \mathcal{S}. \tag{11}$$

Remark The split (6) appears to be not the same as, but related to the complement to the vertical space relative to a natural connection on the coadjoint orbit bundle $(T^*Q)_\mu \to T^*(Q/G)$. In this regard we note that the metric naturally induced on Q/G is Wilson's G-matrix (see Wilson, Decius and Cross [1955]). Our decomposition appears to be finer than the one proposed by Guichardet [1984] and discussed by Iwai [1988]. Notice that we have connections on all levels of this *tower of bundles*

$$T^*Q \supset J^{-1}(\mu) \to (T^*Q)_\mu \to T^*(Q/G)$$

where $J^{-1}(\mu) \to (T^*Q)_\mu$ is regarded as a G_μ bundle and $(T^*Q)_\mu \to T^*(Q/G)$ is regarded as an $\mathcal{O}_\mu$ bundle, where $\mathcal{O}_\mu$ is the coadjoint orbit through μ.

The Guichardet-Iwai results appear to be largely concerned with the bundle $J^{-1}(\mu) \to (T^*Q)_\mu$; the fact that the reduced space $(T^*Q)_\mu$ still has the factor $\mathcal{O}_\mu$ seems to be the reason the connection on the G_μ bundle $J^{-1}(\mu) \to (T^*Q)_\mu$ is not sufficient to completely isolate the vibrational modes from the rotational ones. We believe that the $\mathcal{O}_\mu$ bundle fills this gap.

§ 6 *Block Diagonalization*

Now $H_\xi = K_\xi + V_\xi + \langle \mu_e, \xi \rangle$ and we have arranged for V_ξ to be block diagonal. As far as K_ξ is concerned, we compute in coordinates that

$$K_\xi = \frac{1}{2} g^{ij}(p_i - g_{ik}A^k_a\xi^a)(p_j - g_{jm}A^m_b\xi^b). \tag{1a}$$

Thus, since $p_i = g_{ik} A^k_a \xi^a$ at equilibrium, we get

$$\delta^2 K_\xi(z_e) \cdot (\delta z, \delta\overline{z}) = g^{ij}\, \delta p_i\, \delta\overline{p}_j\,. \tag{1b}$$

Regarding the block diagonalization of $\delta^2 K_\xi$ on $\mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$, we shall use some further interesting identities.

The block diagonalization results for $\delta^2 H_\xi$ follow from two basic formulas:

6.1 Proposition *Let* $\Delta z \in \mathcal{S}_{RIG}$ *and* $\delta z \in T_{z_e}P$. *Then*

$$\delta^2 H_\xi(z_e)(\Delta z, \delta z) = \frac{d}{dt}\langle \zeta_Q(q), \delta q\rangle - \langle [\xi, \eta], \delta J(z_e) \cdot \delta z\rangle\,, \tag{2a}$$

where Δz *has associated* η *and* ζ *as in* (3b) *and* (7) *of* §4.

6.2 Proposition *Let* δz_1 *and* $\delta z_2 \in \mathcal{S}_{INT}$; *then*

$$\delta^2 H_\xi(z_e)(\delta z_1, \delta z_2) = \delta^2 K_\xi(z_e) \cdot (\delta z_1, \delta z_2) + \delta^2 V_\xi(q_e)(\delta q_1, \delta q_2) \tag{2b}$$

Proposition **6.1**, which is proved by direct calculation, shows that $\delta^2 H_\xi(z_e)$ *block diagonalizes* on $\mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$, i.e., if $\Delta z \in \mathcal{S}_{RIG}$ and $\delta z \in \mathcal{S}_{INT}$, then

$$\delta^2 H_\xi(z_e)(\Delta z, \delta z) = 0\,. \tag{3}$$

Proposition **6.2** then follows from our earlier calculations. It also follows that if $\Delta z \in \mathcal{S}_{RIG}$ and $\Delta\overline{z} \in \mathcal{S}_{RIG}$, then

$$\delta^2 H_\xi(z_e)(\Delta z, \Delta\overline{z}) = \frac{d}{dt}\langle \zeta_Q(q), \overline{\zeta}_Q(q)\rangle \tag{4}$$

which is a generalization of the rigid body second variation formula for motion on the coadjoint orbit $\mathcal{O}_{\mu_e}$ with the metric I_{ab}. (Recall that ζ are determined by η and μ_e by equation (7) of §4.) We summarize:

6.3 Theorem *The relative equilibrium* z_e *is formally stable (with* $\delta^2 H_\xi(z_e)$ *on* $\mathcal{S}$ *positive definite) iff*

I $\frac{d}{dt}\langle \zeta_Q(q), \overline{\zeta}_Q(q)\rangle$ *is positive definite on* $\mathcal{S}_{RIG}$

and II $\delta^2 H_\xi(q_e)$ *is positive definite on* $\mathcal{W}_{INT}$.

A sufficient condition for II *is*

II* $\delta^2 V_\xi(q_e)$ *is positive definite on* $\mathcal{V}_{INT}$.

We note that condition I is sufficient for the nondegeneracy condition (2″) of §5.

We note that $\delta^2 V_\xi(q_e)$ separates (in coordinates on $\mathcal{V}_{INT}$) into $\delta^2 V(q_e)$ plus a term which is quadratic in ξ. Thus, II is equivalent to a condition of the form $\|\xi\| \le \sqrt{\lambda_{min}}$, where $\|\ \|$ is a suitable norm and λ_{min} is the minimum (non-zero) eigenvalue of $\delta^2 V(q_e)$; one has to take care here because V itself need not have a critical point at q_e, so $\delta^2V(q_e)$ does *not* make intrinsic sense. To see how this works in examples, see Simo, Posbergh and Marsden [1989] and Lewis and Simo [1989].

Interestingly, the effective potential V_μ which was defined earlier, is such that *the conditions in the preceeding theorem become sharp*; that is, they are not only sufficient for stability, they are necessary as well. This point is important for bifurcation analysis, and will be the subject of future investigations.

As far as the symplectic form Ω is concerned, we have

6.4 Theorem *Let* $\Delta z \in \mathcal{S}_{RIG}$ *and* $\delta z \in T_{z_e}P$. *Then*

$$\Omega(z_e)(\Delta z, \delta z) = -\langle \eta, \delta J(z_e)\cdot\delta z\rangle + \langle \zeta_Q(q_e), \delta q\rangle. \tag{5}$$

Notice that in an appropriate sense, $\delta^2H_\xi(z_e)$ on $\mathcal{S}_{RIG}\times T_{z_e}P$ is the time derivative of the symplectic form Ω!

From (5) and (7) of §4 one finds that on $\mathcal{S}_{RIG}\times\mathcal{S}_{RIG}$, Ω gives the coadjoint orbit symplectic form

$$\Omega(z_e)(\Delta z, \Delta\bar{z}) = -\langle \mu_e, [\eta, \bar{\eta}]\rangle, \tag{6}$$

while on $\mathcal{S}_{RIG}\times\mathcal{S}_{INT}$ we have the cross terms

$$\Omega(z_e)(\Delta z, \delta z) = \langle \zeta_Q(q_e), \delta q\rangle, \tag{7}$$

which depend on the δq components alone. In summary, we have

$$\begin{array}{cccc} & \mathfrak{g}/\mathfrak{g}_{\mu_e} & \mathcal{W}_{INT} & \mathcal{W}^*_{INT} \\ \delta^2H_\xi(z_e) = & \left[\begin{array}{c} \left[\begin{array}{c}\text{Generalized}\\ \text{Rigid Body}\\ \text{Second Variation}\end{array}\right] \\ \begin{array}{c}0\\0\end{array}\end{array}\right. & \begin{array}{c} 0 \\ \left[\begin{array}{cc}\delta^2H_\xi(z_e) & 0\\ 0 & \delta^2K_\xi(z_e)\end{array}\right.\end{array} & \left.\begin{array}{c} 0 \\ \right]\end{array}\right] \end{array}$$

for the second variation of the augmented energy. In this matrix, note that $\delta^2H_\xi(z_e)$ is given by the expression (2b) and so positive definiteness of $\delta^2V_\xi(q_e)$ implies postive definiteness of $\delta^2H_\xi(z_e)$ and hence, assuming the rigid body second variation is positive (the rotation is about the shortest axis), positive definiteness of the entire matrix. As far as the symplectic form is concerned, we have the form

$$\begin{array}{cccc} & \mathfrak{g}/\mathfrak{g}_{\mu_e} & \mathcal{W}_{INT} & \mathcal{W}^*_{INT} \\ \Omega = & \left[\begin{array}{c} \left[\begin{array}{c}\text{Coadjoint Orbit}\\ \text{Symplectic Form}\end{array}\right] \\ -\left[\begin{array}{c}\text{Internal-Rigid}\\ \text{Coupling (7)}\end{array}\right] \\ 0 \end{array}\right. & \begin{array}{c} \left[\begin{array}{c}\text{Internal-Rigid}\\ \text{Coupling (7)}\end{array}\right] \\ \left[\begin{array}{c}\text{Canonical symplectic form}\\ \text{plus a "magnetic" term}\end{array}\right] \end{array} & \left.\begin{array}{c} 0 \\ \\ \end{array}\right] \end{array}$$

For information on the "magnetic" term, and its interpretation as a curvature, we refer the reader to Kummer [1981]. Also, we presume that the coupling terms can be interpreted in terms of the curvature of the connection on the coadjoint bundle $T^*Q \to T^*(Q/G)$ that we used to define the splitting of $\mathcal{S}_{INT}$; see Montgomery [1986] and Lewis, Marsden, Montgomery and Ratiu [1986].

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