A COMPARISON THEOREM FOR HAMILTONIAN VECTOR FIELDS

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ABSTRACT. The question of completeness of Hamiltonian systems is investigated for a class of potentials not necessarily bounded below. The result generalizes previous work of W. Gordon and D. Ebin.

This paper extends the completeness theorem of Ebin [1] to include certain potential functions V not necessarily bounded below. The condition on V is essentially the same as a condition for a corresponding quantum mechanical theorem. See [3] and Remark 3 below. We shall prove a comparison theorem which reduces the general case to the one-dimensional case, so we begin with the latter.

1. **The one-dimensional case.** Let R^+ be the nonnegative reals and $V_0: R^+ \to R$ a nonincreasing C^1 function. Consider the Hamiltonian system with the usual kinetic energy and potential V_0; i.e. if $c(t)$ is a solution curve we have

$$c''(t) = -\frac{dV_0}{dx}(c(t)).$$

By monotonicity of V_0, if $c'(0) \geq 0$ then $c'(t) \geq 0$ for all $t \geq 0$. Thus if $H = [c'(t)/2]^2 + V_0(c(t))$ is the constant total energy,

$$c'(t) = 2(H - V_0(c(t)))^{1/2}.$$

DEFINITION. The potential V_0 is *positively complete* iff

$$\int_{x_1}^x \frac{dx}{(2(H - V_0(x)))^{1/2}} \to \infty \quad \text{as } x \to \infty$$

for all $x_1 \geq 0$ and H such that $V_0(x_1) < H$.

It is easy to see that if this holds for some x_1, H, such that $V_0(x_1) < H$ then it holds for all such x_1, H (use the fact that improper integrals with asymptotic integrands are simultaneously convergent or divergent).

Since the above integral is just the time required for $c(t)$ to move from x_1 to x we see that V_0 is positively complete iff all integral curves $c(t)$ with $c(0) \geqq 0$, $c'(0) \geqq 0$ are defined for all $t \geqq 0$. (The case when $c'(0) = 0$ is easily disposed of.)

Below we will use the notation $\tilde{c}(x_0, H)(t)$ for the integral curve with $\tilde{c}(x_0, H)(0) = x_0$ and energy H.

EXAMPLE. The function $-x^\alpha$ for $\alpha \geqq 0$ is positively complete iff $\alpha \leqq 2$. The same is true for

$$-x[\log(x+1)]^\alpha, \quad -x\log(x+1)[\log(\log(x+1)+1)]^\alpha \quad \text{etc.}$$

2. The general case.

THEOREM. *Let M be a complete Riemannian manifold (possibly infinite-dimensional) and let V be a C^1 function on M. Suppose there is a point $p \in M$ and a positively complete V_0 on R^+ such that for all $m \in M_1$ [with $d(m, p)$ sufficiently large], $V(m) \geqq V_0(d(m, p))$ where d is the Riemannian distance on M. Then the flow on TM of the Hamiltonian vector field with (the usual kinetic energy $K(v) = \langle v, v \rangle / 2$ and) potential V is a complete flow (i.e. integral curves are defined for all $t \in R$).*

EXAMPLES. If $V(m) \geqq -(\text{Constant})d(m, p)^2$ for sufficiently large $d(m, p)$ the conditions hold. This is satisfied if $\|\operatorname{grad} V(m)\| \leqq (\text{Constant})d(m, p)$ (for sufficiently large $d(m, p)$).

PROOF OF THEOREM. Let $c: [0, b) \to TM$ be an integral curve, $0 < b < \infty$. By the arguments in [1], it will suffice to show that the curve $c_0(t)$, the projection of $c(t)$ on M, remains in a bounded set for all $t \in [0, b)$ (a similar argument holds for $t \in (-b, 0]$).

Let $n = c_0(0)$ and H the energy of $c(t)$. Let

$$f_1(t) = d(c_0(t), p) \quad \text{and} \quad f_2(t) = \tilde{c}(d(n, p), H)(t)$$

(notation as in §1). Now

$$f_1(t) \leqq d(p, n) + \int_0^t \|c(s)\| ds$$

$$= d(p, n) + \int_0^t (2[H - V(c_0(s))])^{1/2} ds$$

$$\leqq d(p, n) + \int_0^t (2[H - V_0(f_1(s))])^{1/2} ds.$$

Also we have

$$f_2(t) = d(p, n) + \int_0^t (2[H - V_0(f_2(s))])^{1/2} ds.$$

It follows from these and monotonicity of V_0 that

$$f_1(t) \leqq f_2(t) \leqq \bar{c}(d(n, p), H)(b)$$

for all $t \in [0, b)$. (See $[2$, p. $29]$ for the proof of a similar result.) We conclude that $f_1(t) = d(c_0(t), p)$ remains bounded for $t \in [0, b)$ and so the result follows.

REMARKS. (1) The completeness for $t \geqq 0$ is preserved if a dissipative vector field Y is added to the Hamiltonian vector field (i.e. Y is vertical $[T\pi(Y) = 0]$ and $Y \cdot K \leqq 0$ where K is the kinetic energy). This is easy to see.

(2) This proof also gives an estimate for the growth of $d(c_0(t), p)$ in terms of V_0; for example if $V_0 = -x^2$ then $d(c_0(t), p) - d(n, p)$ grows like e^t.

(3) In $[3]$ (and earlier papers) it is shown that on R^n for potentials V satisfying essentially the condition here (of bounded below by a positively complete function), the corresponding quantum mechanical flows are "complete" i.e. the operator is essentially selfadjoint. In fact the result may be generalized to arbitrary complete Riemannian manifolds. (See P. Chernoff and J. Marsden, *Hamiltonian systems and quantum mechanics* (in preparation).)

(4) See $[4]$ for some related results.

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