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GAUGED LIE-POISSON STRUCTURES

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ABSTRACT. A global formula for Poisson brackets on reduced cotangent bundles of principal bundles is derived. The result bears on the basic constructions for interacting systems due to Sternberg and Weinstein and on Poisson brackets involving semi-direct products for fluid and plasma systems. The formula involves Lie-Poisson structures, canonical brackets, and curvature terms.

1. INTRODUCTION. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The right (resp. left) reduction of T^*G by G produces the + (resp. -) Lie-Poisson structure on $\mathfrak{g}^*$:

$$\{F,G\}(\mu) = \pm\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \rangle .$$

This construction is now well-known and has been reviewed in the lectures of Weinstein, Ratiu and Morrison in these proceedings. This paper concerns the Poisson structure on the reduction of T^*B, where $\pi:B \to X$ is a principal bundle. The Poisson structure on the (right) reduced space $G\backslash T^*B$ is a mixture of Lie-Poisson and canonical structures and will be computed explicitly.

There are several motivations for considering the constructions presented here. First of all, these reduced spaces occur in the construction of phase spaces for interacting systems: see Sternberg [1977] and Weinstein [1978] for a particle in a Yang-Mills field and Marsden and Weinsten [1982] for the Maxsell-Vlasov equation. The link between the approaches of Sternberg and Weinstein and the physicist's equations (Wong's equations) was given in Montgomery [1983] and provides a basis for this paper.

The second motivation was to better understand the role of Lie-Poisson structures associated with semi-direct products of groups $G \ltimes H$. The way these arise in examples was first systematically explored by Guillemin and Sternberg [1980] and Ratiu [1980,1981,1982]. The symmetry breaking mechanism

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behind their occurrence is now well-understood for examples whose underlying configuration space is a Lie group such as the heavy top, compressible fluids and MHD (see Marsden, Ratiu and Weinstein [1983] and Ratiu's lecture in these proceedings). However, semi-direct products occur in somewhat more mysterious ways as well; for example in the last section of Marsden, Ratiu and Weinstein [1983], it is observed that in momentum representation the brackets for the Maxwell-Vlasov equations and for multifluid plasmas, involve semi-direct products. This paper in fact began on the road to Boulder as an attempt to better this understanding.

The third motivation is to provide a setting for understanding limits of Poisson structures and for averaging. For example, the limit $c \to \infty$ in the Maxwell-Vlasov to Poisson-Vlasov transition can be understood as rescaling the bracket so the motion on the base X freezes (electrodynamics becomes electrostatics) leaving only Lie-Poisson motion in the fiber. The discussions and examples in Weinstein [1983] seem to be consistent with this scheme. Also, if one averages the Hamiltonian H over the fiber by the G action, then the average $\overline{H}$ drops to T^*X by reduction. Hopefully, systems where fast time scales can be smeared out can be understood in this context. As is well-known (see Kummer [1981]), this reduction may involve a modification of the Poisson structure by magnetic (or curvature) terms, a phenomenon we shall see explicitly. In particular, we think one can understand the guiding center equations of Littlejohn [1979] in this way, as well as other situations involving averaging, such as MHD and guiding center plasmas.

In the scheme for interacting systems proposed by Sternberg [1977], Weinstein [1978] and used in Marsden and Weinstein [1982], one starts with a phase space of the form

$$T^*B \times \mathfrak{h}^*$$

where $\mathfrak{h}$ is the Lie algebra of a Lie group H and $\pi : B \to X$ is a principal G-bundle, with G acting by a canonical action on $\mathfrak{h}^*$. In the cases of multifluid plasmas and the Maxwell-Vlasov equations, elements of $\mathfrak{h}^*$ represent matter fields, while T^*B represents the pure fields (Maxwell or Yang-Mills fields). After reduction by G, the coupling manifests itself in the Poisson structure on the reduced space

$$T^*B \times_G \mathfrak{h}^*.$$

An important idea in this paper is to think of $T^*B \times \mathfrak{h}^*$ as $T^*(B \times H)$ reduced by H. As in Guillemin and Sternberg [1980], $G \ltimes H$ acts on $B \times H$ making it a principal $G \ltimes H$ bundle, so

$$T^*B \times_G \mathfrak{h}^* = G \ltimes H\backslash T^*(B \times H)$$

which reduces the study of $T^*B \times_G \mathfrak{h}^*$ to the case $G\backslash T^*B$. (Warning. As explained in Ratiu's lecture in these proceedings, the right reduction of $T^*(B \times H)$ to $T^*B \times \mathfrak{h}^*$ by H is not simply by projection if G acts on $\mathfrak{h}^*$ on the left, but it is if G acts on $\mathfrak{h}^*$ on the right).

In this paper we shall describe the reduced brackets on $G\backslash T^*B$ in both the Weinstein and Sternberg representations. (See Marsden [1981] for a synopsis of the two viewpoints.) On the Weinstein side we deal directly with $G\backslash T^*B$ where G acts by the cotangent lift. On the Sternberg side one selects a connection A to split T^*B into horizontal and vertical covectors before reduction. The main new results of this paper are formulas for the Poisson bracket on the Sternberg side (see §4).

In a more comprehensive paper in preparation we shall

a. Give an intrinsic proof of the global formula in §4;

b. show how the semi-direct bracket formulas in §5 apply to fluids and plasmas, and

c. obtain a formula for the brackets for free boundary problems and for Yang-Mills fluids and plasmas in reduced variables (the analogs of E and B).

In future publications, we hope to apply the ideas herein to study limits of Poisson structures and averaging, continuing the program begun by Weinstein [1983].

2. BRACKETS IN THE WEINSTEIN REPRESENTATION. Let $\pi: B \to X$ be a principal (right) G bundle. We are interested in the bracket structure on the reduced space

$$W = G\backslash T^*B$$

in a local trivialization. This is essential for understanding the Sternberg side. To begin then, assume $B = X \times G$, so $T^*B = T^*X \times T^*G$ and we can identify

$$\begin{aligned} G\backslash T^*B &= T^*X \times G\backslash T^*G \\ &= T^*X \times \mathfrak{g}^*_+. \end{aligned}$$

The second equality occurs because T^*G right trivialized is canonically isomorphic to $\mathfrak{g}^*$ with its + Lie-Poisson structure. Thus, in this choice of trivialization, the Poisson structure on the Weinstein side is canonical on T^*X and Lie-Poisson on $\mathfrak{g}^*_+$:

$$\{F,G\}(x,p,\mu) = \frac{\delta F}{\delta x}\cdot\frac{\delta G}{\delta p} - \frac{\delta G}{\delta x}\cdot\frac{\delta F}{\delta p} + \langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta F}{\delta \mu}\right] \rangle$$

The first two terms denote, of course, the canonical bracket on T^*X. For computational purposes later we will need to make this a bit more precise. Assume in the trivialization that X is also a coordinate neighborhood, so without loss of generality, X is Banach space. Then $T^*X = X \times X^*$, with $(x,p) \in X \times X^*$. So $\frac{\delta F}{\delta x}$ means the first partial derivative $d_1F(x,p,\mu)$: $X \to \mathbb{R}$, an element of X^*. Likewise $\frac{\delta F}{\delta p} \in X^{**}$ and we assume it lies in X, just as one assumes $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$.

3. BRACKETS IN THE STERNBERG REPRESENTATION, LOCAL VERSION. Our version of the Sternberg space is

$$S = \tilde{B} \times_G \mathfrak{g}^*$$

where $\tilde{B}$ is the pullback bundle of B to T^*X:

$$\begin{tikzcd} \tilde{B} \arrow[r, "\tilde{\tau}"] \arrow[d, "\tilde{\pi}"'] & B \arrow[d, "\pi"] \\ T^*X \arrow[r, "\tau"] & X \end{tikzcd}$$

The bundle $\tilde{B}$ has a concrete realization as the subbundle of T^*B which annihilates vertical vectors in TB. Here, τ is the cotangent projection, and $\tilde{\tau}$ is the restriction of the cotangent projection $T^*B \to B$. The map $\tilde{\pi}$ is defined by

$$\tilde{\pi}(\alpha_b)\cdot v_x = \alpha_b \cdot V_b$$

where

$$\alpha_b \in \tilde{B}_b \subseteq T_b^*B, \ \pi(b) = x \in X,$$

and

$$V_b \in T_bB \text{ is any vector with } T\pi_b \cdot V_b = v_x \in T_xX .$$

Note that $\tilde{B}$ is a principal bundle over T^*X where the G action is the restriction to $\tilde{B}$ of the (lifted) G action on T^*B and G acts on $\mathfrak{g}^*$ by the coadjoint action. S is then a vector bundle over T^*X. It is an

associated bundle, also known as the coadjoint bundle to $\tilde{B}$. It consists of G orbits in $\tilde{B} \times \mathfrak{g}^*$ where the action is

$$(\alpha_b, \mu)\cdot g = (TR^*_{g^{-1}}\alpha_b,\ Ad_g{}^*\mu).$$

To define a Poisson structure on S we need a connection A on B. Such a connection can be viewed as an equivariant splitting of TB into horizontal and vertical vectors, or dually, as an equivariant splitting of T^*B:

$$\tilde{B} \times \mathfrak{g}^* \xrightarrow{\ \sim\ } T^*B$$

$$(\alpha_b, \mu) \longmapsto \alpha_b + A_b^*\mu$$

where $A_b : T_bB \to \mathfrak{g}$ is the connection one-form. We use this isomorphism to pull back the canonical symplectic structure on T^*B in order to get an A-dependent symplectic structure on $\tilde{B} \times \mathfrak{g}^*$. If we now mod out by G, we get a Poisson isomorphisim $S \xrightarrow{\sim} W$.

As before, we are interested in the Poisson brackets in a local trivialization. So we will assume $B = X \times G$ with X a Banach space. Then

$$\tilde{B} = X \times X^* \times G \hookrightarrow T^*X \times T^*G = T^*B$$

where G is embedded as the zero section in T^*G. And

$$S = X \times X^* \times G \backslash (G \times \mathfrak{g}^*) \supset X \times X^* \times \mathfrak{g}^*.$$

In the previous section we showed that the same trivialization induces an identification of W with $X \times X^* \times \mathfrak{g}^*$ also. It was shown in Montgomery [1983] that the isomorphism $S \xrightarrow{\sim} W$ is then given by

$$(x, p, \mu) \longmapsto (x, p + \underline{A}(x)^*\mu)$$

where $\underline{A}$ is the $\mathfrak{g}$-valued one-form on X induced by the trivialization. Since this is a Poisson isomorphism we can now calculate the

<u>Local formula for the Sternberg bracket:</u>

$$\{F, G\}(x, p, \mu) = \frac{\delta F}{\delta x}\cdot\frac{\delta G}{\delta p} - \frac{\delta G}{\delta x}\cdot\frac{\delta F}{\delta p} +$$

$$\langle \mu, -\left[\underline{A}(x)\cdot\frac{\delta F}{\delta p}, \frac{\delta G}{\delta \mu}\right] + \left[\underline{A}(x)\cdot\frac{\delta G}{\delta p}, \frac{\delta F}{\delta \mu}\right]\rangle + \langle \mu, \underline{\Omega}(x)\left(\frac{\delta F}{\delta p}, \frac{\delta G}{\delta p}\right)\rangle$$

$$+ \langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \rangle$$

Here $\underline{\Omega}$ is the local expression for Ω, the curvature of A.

Proof. Let

$$\overline{F}(x,p,\nu) = F(x,p - \underline{A}(x)^*\nu,\nu)$$

denote the pushforward of the function F on S to $\overline{F}$ on W. Then

$$\{F,G\}_S(x,p,\nu) = \{\overline{F},\overline{G}\}_W(x,p + \underline{A}(x)^*\nu,\nu) = \frac{\delta \overline{F}}{\delta x}\cdot\frac{\delta \overline{G}}{\delta p} - \frac{\delta \overline{G}}{\delta x}\cdot\frac{\delta \overline{F}}{\delta p}$$

$$+ \langle \mu, \left[\frac{\delta \overline{F}}{\delta \mu}, \frac{\delta \overline{G}}{\delta \mu}\right] \rangle$$

From the definition of $\overline{F}$ we read off

$$d\overline{F}(x,p,\mu)\cdot(\dot{x},\dot{p},\dot{\mu}) = d_1F\cdot\dot{x} - d_2F\circ d_x(\underline{A}^*\mu)\cdot\dot{x} + d_2F\cdot\dot{p} - d_2F\circ\underline{A}(x)^*\cdot\dot{\mu} + d_3F\cdot\dot{\mu}$$

$$= \left[\frac{\delta F}{\delta x} - d_2F\circ d_x(\underline{A}^*\mu)\right]\cdot\dot{x} + \frac{\delta F}{\delta p}\cdot\dot{p} + \langle \frac{\delta F}{\delta \mu} - \underline{A}(x)\cdot\frac{\delta F}{\delta p}, \dot{\mu} \rangle$$

$$= \frac{\delta \overline{F}}{\delta x}\cdot\dot{x} + \frac{\delta \overline{F}}{\delta p}\cdot\dot{p} + \langle \frac{\delta \overline{F}}{\delta \mu}, \dot{\mu} \rangle$$

Plugging these results into the previous equations we get

$$\{F,G\}_S = \left[\frac{\delta F}{\delta x} - d_2F\circ d_x(\underline{A}^*\mu)\right]\cdot\frac{\delta G}{\delta p} - \left[\frac{\delta G}{\delta x} - d_2G\circ d_x(\underline{A}^*\mu)\right]\cdot\frac{\delta F}{\delta p}$$

$$+ \langle \mu, \left[\frac{\delta F}{\delta \mu} - A(x)\cdot\frac{\delta F}{\delta p}, \frac{\delta G}{\delta \mu} - \underline{A}(x)\cdot\frac{\delta G}{\delta p}\right] \rangle$$

$$= \frac{\delta F}{\delta x}\cdot\frac{\delta G}{\delta p} - \frac{\delta G}{\delta x}\cdot\frac{\delta F}{\delta p}$$

$$+ \langle \mu, -\left[\underline{A}(x)\cdot\frac{\delta F}{\delta p}, \frac{\delta G}{\delta \mu}\right] + \left[\underline{A}(x)\cdot\frac{\delta G}{\delta p}, \frac{\delta F}{\delta \mu}\right] \rangle + \langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \rangle$$

$$+ d_2G\circ d_x(\underline{A}^*\mu)\cdot\frac{\delta F}{\delta p} - d_2F\circ d_2(\underline{A}^*\mu)\cdot\frac{\delta G}{\delta p}$$

$$+ \langle \mu, \left[\underline{A}(x)\cdot\frac{\delta F}{\delta p}, \underline{A}(x)\frac{\delta G}{\delta p}\right] \rangle$$

Comparing this with the alleged local formula, we see that it suffices to prove that the last two terms equal the curvature term in the local formula. Since $\underline{\Omega} = d\,\underline{A} + [\underline{A},\underline{A}]$ we need only show:

$$\langle \mu,\ d\underline{A}(x)\left(\frac{\delta F}{\delta p},\ \frac{\delta G}{\delta p}\right)\rangle =$$

$$d_2G\circ d_x(\underline{A}^*\mu)\cdot\frac{\delta F}{\delta p} - d_2F\circ d_x(\underline{A}^*\mu)\ \frac{\delta G}{\delta p}\ .$$

The left hand side is

$$\langle \mu,\ \frac{\delta F}{\delta p}\left[\underline{A}\cdot\frac{\delta G}{\delta p}\right] - \frac{\delta G}{\delta p}\left[\underline{A}\cdot\frac{\delta F}{\delta p}\right] - \underline{A}\cdot\left[\frac{\delta F}{\delta p},\ \frac{\delta G}{\delta p}\ \right]\rangle$$

Now

$$\langle \mu,\ \frac{\delta F}{\delta p}\left[A\cdot\frac{\delta G}{\delta p}\right]\rangle = \frac{\delta F}{\delta p}\left[\langle \mu,\ \underline{A}\cdot\frac{\delta G}{\delta p}\rangle\right]$$

$$= \frac{\delta F}{\delta p}\left[\langle \underline{A}^*\mu,\ \frac{\delta G}{\delta p}\rangle\right]$$

$$= \langle d_x(\underline{A}^*\mu)\cdot\frac{\delta F}{\delta p},\ \frac{\delta G}{\delta p}\rangle + \langle \underline{A}^*\mu,\ d_x\ \frac{\delta G}{\delta p}\cdot\frac{\delta F}{\delta p}\rangle$$

$$= d_2G\circ d_x(\underline{A}^*\mu)\cdot\frac{\delta F}{\delta p} + \langle \mu,\ \underline{A}\left(d_x\left(\frac{\delta G}{\delta p}\right)\cdot\frac{\delta F}{\delta p}\right)\rangle$$

Subtracting the similar expression with F and G switched and recalling the local expression for the Lie bracket of the vector fields $\left[\frac{\delta F}{\delta p},\ \frac{\delta G}{\delta p}\right]$ yields the result. ■

4. BRACKETS IN THE STERNBERG REPRESENTATION, GLOBAL VERSION. The global formula for these brackets requires some more terminology. In this section

$$\pi : S \to T^*X$$

denotes the vector bundle projection. Using the trivialization of the previous section, π is given by

$$(x,p,\mu) \mapsto (x,p)$$

(See Montgomery [1983] for this calculation). Let

$$\tilde{A} = \tilde{\tau}^* A$$

denote the pullback connection on $\tilde{B}$. (It is trivial in the momentum directions of T^*X.) Since S is an associated vector bundle to $\tilde{B}$, we have a horizontal lift $h = h_{\tilde{A}}$ of vectors on T^*X to vectors on S. Using this, we define the covariant differential of a function $F:S \to \mathbb{R}$ at $s \in S$ to be that covector $d_{\tilde{A}}F(s)$ at $p = \pi(s) \in T^*X$ given by

$$d_{\tilde{A}}F(s)\cdot v_p = dF(s)\cdot(h(s)\cdot v_p)$$

This may be thought of as the horizontal part of $dF(s)$. The vertical part may be thought of as an element in the dual bundle to S which is the adjoint bundle

$$\frac{\delta F}{\delta v}(s) \in (\tilde{B} \times_B \mathfrak{g})_p = S_p^* .$$

It is given by

$$\frac{\delta F}{\delta v}(s)\cdot s' = \frac{d}{dt}\bigg|_{t=0} F(s + ts') .$$

The curvature of $\tilde{A}$ is

$$\tilde{\Omega} = \tilde{\tau}^*\Omega.$$

where Ω is the curvature of A. We may consider $\tilde{\Omega}$ to be a two-form on T^*X with values in S^* by the mapping

$$(v_p,w_p) \mapsto [(b, \tilde{\Omega}(b)(hv_p, hw_p)]_G .$$

Here $b \in \tilde{B}$, $\tilde{\pi}(b) = p$, h denotes horizontal lift to $\tilde{B}$, and the brackets denote an equivalence class in $\tilde{B} \times \mathfrak{g}$ under the G-action. From the transformation law for curvatures, this equivalence class is independent of which $b \in \tilde{B}_p$ is picked. [In the case where B is the frame bundle this formulation of the curvature is the usual Riemann tensor.]

Finally, if β is a covector at p on T^*X, then $\beta^{\#}$ denotes the symplectically dual vector at p given by

$$\beta = \omega(p)(\cdot,\beta^{\#}) ,$$

where ω is the canonical two-form on T^*X.

Global bracket formula on S

$$\{F,G\}(s) = \omega(p)(d_{\tilde{A}}F(s)^{\#}, d_{\tilde{A}}G(s)^{\#}) + \langle s, \tilde{\Omega}(p)(d_{\tilde{A}}F(s)^{\#}, d_{\tilde{A}}G(s)^{\#}) \rangle$$

$$+ \langle s, \left[\frac{\delta F}{\delta v}(s), \frac{\delta G}{\delta v}(s)\right] \rangle$$

where $\pi(s) = p$.

In a future paper a global proof will be presented. Here we will prove the formula by showing that it agrees locally with the formula given above:

In the local trivialization of the previous section,

$$d_{\tilde{A}}F(x,p,\nu) = \left(\frac{\delta F}{\delta x} - \langle \nu, \left[\frac{\delta F}{\delta \nu}, \underline{A}(x)\cdot\right]\rangle, \frac{\delta F}{\delta p}\right) .$$

This is seen by considering F as a G-invariant function on $\tilde{B} \times \mathfrak{g}^*$, which we will denote $\hat{F}$. Then

$$d_{\tilde{A}}F(\dot{x},\dot{p},\nu)\cdot(x,p) = d\hat{F}(x,p,e,\nu)\cdot(h(\dot{x},\dot{p}),0)$$

where $h(\dot{x},\dot{p}) = (\dot{x},\dot{p}, -\underline{A}(\dot{x})\cdot\dot{x})$ is the horizontal lift of $(\dot{x},\dot{p})$ to $(x,p,e) \in \tilde{B}$. By the G invariance of $\hat{F}$,

$$d\hat{F}(\dot{x},\dot{p}, -A(x)\cdot\dot{x},0) = d\hat{F}(\dot{x},\dot{p},0, \mathrm{ad}_{A(x)\cdot\dot{x}}{}^*\nu)$$

$$= \frac{\delta F}{\delta x}\dot{x} + \langle \nu, \left[\underline{A}(x)\cdot\dot{x}, \frac{\delta F}{\delta \nu}\right]\rangle + \frac{\delta F}{\delta p}\cdot\dot{p} .$$

The covector bracket on T^*X is

$$\omega(\alpha^{\#},\beta^{\#}) = \alpha_1\cdot\beta_2 - \beta_1\cdot\alpha_2$$

where $\alpha = (\alpha_1,\alpha_2) \in T_p^*(T^*X) = X^* \times X$, and likewise with β. If we set $\alpha = d_{\tilde{A}}F$, $\beta = d_{\tilde{A}}G$ and use the formula for $d_{\tilde{A}}$ we find the first term of the global bracket formula equals the first two terms of the local bracket.

To check that the curvature terms of the two formulas match, note that

$$\tilde{\Omega}(p)(d_{\tilde{A}}F(s)^{\#},\ d_{\tilde{A}}G(s)^{\#}) = \Omega(x)(T\tau\cdot d_{\tilde{A}}F(s)^{\#},\ T\tau\cdot d_{\tilde{A}}G(s)^{\#})$$

and that

$$T\tau\cdot d_{\tilde{A}}F(s)^{\#} = \frac{\delta F}{\delta p}\ .$$

Finally, it is clear that the last, pure Lie-Poisson terms of the two formulas are equal.

5. COUPLING AND SEMI-DIRECT PRODUCTS. Suppose $\tilde{\rho}^*$ is a canonical <u>right</u> action of G on the "matter-fields" $\mathfrak{h}_+^*$. Then we can reduce the total phase space $T^*B \times \mathfrak{h}_+^*$ by the canonical G action:

$$(\alpha_b,\mu)\cdot g = (T_bR_{g^{-1}}{}^*\alpha_b,\ \tilde{\rho}^*(g)\mu).$$

We want an expression for the brackets on this reduced phase space and also one for the Sternberg side, namely on $\tilde{B} \times_G (\mathfrak{g}^* \times \mathfrak{h}^*)_+$.

Assume $\tilde{\rho}^*$ is induced by an action

$$\tilde{\rho}:G \to \text{Lie algebra automorphisms of } \mathfrak{h}$$

which in turn is induced by a right action

$$\rho:G \to \text{Aut } H$$

of G on H by automorphisms. That is

$$\tilde{\rho}(g) = T_e(\rho(g)):\ \mathfrak{h} = T_eH \to \mathfrak{h},$$

and

$$\tilde{\rho}^*(g) = \tilde{\rho}(g)^*:\mathfrak{h}^* \to \mathfrak{h}^*$$

is its dual. If one thinks of $\mathfrak{h}_+^*$ as a reduced Poisson space; i.e.

$$\mathfrak{h}_+^* = H\backslash T^*H,$$

then $\tilde{\rho}^*$ is the action induced on $\mathfrak{h}^*$ by the lift of ρ acting on T^*H.

Using ρ, one forms the semi-direct product group $G \ltimes H$ with multiplication

$$(g_1,h_1)(g_2,h_2) = (g_1 g_2,\ h_1 \rho(g_1)\cdot h_2)$$

and the semi-direct principal bundle $B \ltimes H$ whose underlying manifold is $B \times H$; it has the right $G \ltimes H$ action given by

$$(b,k)\cdot(g,h) = (bg,\ \rho(g^{-1})\ (kh)).$$

The connection A on B induces the semidirect product connection $\hat{A}$ on $B \ltimes H$. $\hat{A}$ is uniquely determined by the fact that the embedding $B \cong B \times \{e\} \subset B \times H$ maps horizontal subspaces of A onto horizontal subspaces of $\hat{A}$ (see Kobayashi-Nomizu [1963], p. 79). Somewhat lengthy calculations prove the formula

$$\tilde{A}(b,h)\cdot(u_b,v_h) = A_b\cdot u_b + T_h L_{h^{-1}}\cdot v_h + T_h L_{h^{-1}}\cdot \rho'(A_b\cdot u_b)$$

where

$$\rho' : \mathfrak{g} \to \text{Lie algebra of Aut } H \subseteq \mathfrak{X}(H)$$

is given by

$$\rho'(\xi)(h) = \frac{d}{dt}\Big|_{t=0} \rho(\exp t\ \xi)\cdot h \in T_h H.$$

We now apply the results of the previous section with $B \ltimes H$ in place of B. Note that the pullback bundle is

$$\widetilde{B \times H} = \tilde{B} \ltimes H \subseteq T^*(B \ltimes H)$$

where as a manifold $\tilde{B} \ltimes H = \tilde{B} \times H \subseteq T^*B \times TH$ with $H \subseteq T^*H$ as the zero section. Now we reduce by $G \ltimes H$ in two steps, first by the normal subgroup $H = \{e\} \times H$, then by G. Calculations show that this results in the commutativity of:

$$\begin{tikzcd}
\widetilde{B \times H} \times (\mathfrak{g} \times \mathfrak{h})^*_+ = \tilde{B} \times H \times (\mathfrak{g} \times \mathfrak{h})^*_+ \arrow[r, "\text{given by } \hat{A}"] \arrow[d, "\text{reduce by } H"] \arrow[dd, bend right=60, "\text{reduce by } G \ltimes H"'] & T^*B \times T^*H = T^*(B \ltimes H) \arrow[d, "\text{reduce by } H"] \arrow[dd, bend left=60, "\text{reduce by } G \ltimes H"] \\
\tilde{B} \times \mathfrak{g}^*_+ \times \mathfrak{h}^*_+ \arrow[r] \arrow[d, "\text{reduce by } G"] & T^*B \times \mathfrak{h}^*_+ \arrow[d, "\text{reduce by } G"] \\
\widetilde{B \times H} \times_{G \ltimes H} (\mathfrak{g} \times \mathfrak{h})^*_+ = \tilde{B} \times_G (\mathfrak{g}^* \times \mathfrak{h}^*)_+ \arrow[r] & T^*B \times_G \mathfrak{h}^*_+ = G \ltimes H \backslash T^*(B \ltimes H)
\end{tikzcd}$$

where the central horizontal map is the isomorphism $\tilde{B} \times \mathfrak{g}^* \to T^*B$ given by A on these factors, and the identity on the $\mathfrak{h}^*$ factor.

The Lie bracket on $\mathfrak{g} \times \mathfrak{h}$ is

$$[(\xi_1,\gamma_1),(\xi_2,\gamma_2)] = ([\xi_1,\xi_2],\ [\gamma_1,\gamma_2] + \tilde{\rho}'(\xi_1)\cdot\gamma_2 - \rho'(\xi_2)\cdot\gamma_1$$

where $\rho' : \mathfrak{g} \to \operatorname{der} \mathfrak{h}$ is the derivative of $\tilde{\rho} : G \to \operatorname{Aut} \mathfrak{h}$. In the local Weinstein formula, we replace μ by (μ,ν) and replacement of the bracket there by this bracket leads to the following

<u>Local form of the bracket on</u> $T^*B \times_G \mathfrak{h}^* \simeq X \times X^* \times \mathfrak{g}^* \times \mathfrak{h}^*$.

$$\{F,G\}(x,p,\mu,\nu) = \frac{\delta F}{\delta x}\cdot\frac{\delta G}{\delta p} - \frac{\delta G}{\delta x}\cdot\frac{\delta F}{\delta p} + \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle$$

$$+ \left\langle \nu, \left[\frac{\delta F}{\delta \nu}, \frac{\delta G}{\delta \nu}\right] + \rho'\left(\frac{\delta F}{\delta \mu}\right)\cdot\frac{\delta G}{\delta \nu} - \rho'\left(\frac{\delta G}{\delta \mu}\right)\cdot\frac{\delta G}{\delta \nu}\right\rangle$$

To calculate the brackets on the Sternberg side note that

$$\hat{\underline{A}} = (\underline{A},0), \text{ a 1-form on } X \text{ with values in } \mathfrak{g} \times \mathfrak{h}$$

where $\underline{A}$ is as before and $\hat{\underline{A}}$ is the pullback of $\hat{A}$ on $B \ltimes H = X \times G \times H$ by the identity section $x \mapsto (x,e,e)$. And that

$$\hat{\underline{\Omega}} = (\underline{\Omega},0).$$

Plugging these results into the local formula with hats $(\hat{\ })$ on $\underline{A}$ and $\underline{\Omega}$, μ replaced by (μ,ν), and $\mathfrak{g}$ brackets replaced by $\mathfrak{g} \ltimes \mathfrak{h}$ brackets, we obtain the following

<u>Local form of the coupled brackets on the Sternberg side</u> $\tilde{B} \times_G (\mathfrak{g}^* \times \mathfrak{h}^*) \simeq X \times X^* \times \mathfrak{g}^* \times \mathfrak{h}^*$:

$$\begin{aligned}\{F,G\}(x,p,\mu,\nu) = {} & \frac{\delta F}{\delta x}\cdot\frac{\delta G}{\delta p} - \frac{\delta G}{\delta x}\cdot\frac{\delta F}{\delta p} + \left\langle \mu, -\left[\underline{A}(x)\cdot\frac{\delta F}{\delta p}, \frac{\delta G}{\delta \mu}\right] + \left[\underline{A}(x)\cdot\frac{\delta G}{\delta p}, \frac{\delta F}{\delta \mu}\right] \right\rangle \\ & + \left\langle \nu, \rho'\left(\underline{A}(x)\cdot\frac{\delta G}{\delta p}\right)\cdot\frac{\delta F}{\delta \nu} - \tilde{\rho}'\left(\underline{A}(x)\cdot\frac{\delta F}{\delta p}\right)\cdot\frac{\delta G}{\delta \nu} \right\rangle \\ & + \left\langle \mu, \underline{\Omega}(x)\left(\frac{\delta F}{\delta p}, \frac{\delta G}{\delta p}\right) \right\rangle \\ & + \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle \\ & + \left\langle \nu, \left[\frac{\delta F}{\delta \nu}, \frac{\delta G}{\delta \nu}\right] + \rho'\left(\frac{\delta F}{\delta \mu}\right)\cdot\frac{\delta G}{\delta \nu} - \rho'\left(\frac{\delta G}{\delta \mu}\right)\cdot\frac{\delta F}{\delta \nu} \right\rangle\end{aligned}$$

As we have mentioned in the introduction, these formulas give, in particular, the semidirect product formulas appearing in the last section of Marsden, Ratiu and Weinstein [1983]. Details concerning this and other applications to Yang-Mills fluids and plasmas and to free boundary problems will be the subject of another publication.

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