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Stability Analysis of a Rigid Body with Attached Geometrically Nonlinear Rod by the Energy–Momentum Method *

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Abstract

This paper applies the energy-momentum method to the problem of nonlinear stability of relative equilibria of a rigid body with attached flexible appendage in a uniformly rotating state. The appendage is modeled as a geometrically exact rod which allows for finite bending, shearing and twist in three dimensions. Application of the energy-momentum method to this example depends crucially on a special choice of variables in terms of which the second variation block diagonalizes into blocks associated with rigid body modes and internal vibration modes respectively. The analysis yields a nonlinear stability result which states that relative equilibria are nonlinearly stable provided that; (i) the angular velocity is bounded above by the square root of the minimum eigenvalue of an associated linear operator and, (ii) the whole assemblage is rotating about the minimum axis of inertia.

§1. Introduction

This paper discusses the application of the energy-momentum method to the case of a rotating rigid body with an attached, flexible appendage. The model for the appendage we have chosen is referred to a a geometrically exact rod model and

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is discussed in detail in SIMO [1985] and SIMO, MARSDEN, & KRISHNAPRASAD [1988]. Because the formulation satisfies exactly all the invariance requirements under superposed rigid body motions, exactly captures without simplification all the dynamic effects, and places no restrictions on the degree of allowable deformations, the rod model is said to be *geometrically exact*. Use of this class of models avoids the potential for unphysical results which may appear in more *ad-hoc* linearized models; see SIMO & VU-QUOC [1988b].

For our stability analysis we use the *energy-momentum method*, introduced in SIMO, POSBERGH & MARSDEN [1989]. This method constitutes a systematic application of the relative equilibrium theorem (see ARNOLD [1978] or ABRAHAM & MARSDEN [1978]), and represents an extension of the *energy-Casimir method* introduced by ARNOLD [1966] and further developed in HOLM, MARSDEN, RATIU & WEINSTEIN [1985]. The energy-Casimir method was applied to rigid bodies with flexible attachments by KRISHNAPRASAD & MARSDEN [1987] using rod models accounting for extension and shear, but precluding bending deformation. The energy-momentum method was applied to this example in POSBERGH & SIMO [1988].

In contrast to the energy-Casimir method, for a Hamiltonian system with symmetry we work directly in the material representation as opposed to the convective (or reduced) representation. Thus, instead of using Casimirs, one employs directly the momentum map as the conserved quantity The success of the method relies crucially on the choice of a particular set of variables, introduced in SIMO, POSBERGH & MARSDEN [1989], which block diagonalizes the second variation $\delta^2 H_\xi$. Conceptually, this choice of variables enforces automatically conservation of the momentum constraint along with gauge symmetries, and separates the overall infinitesimal rigid body modes from the internal vibration modes (including shear and torsion) of the rod. Further geometric aspects underlying this parameterization are examined in the paper of MARSDEN, SIMO, LEWIS & POSBERGH [1989] in this proceedings.

§2. The Energy-Momentum Method

In this section we give a brief outline of the energy-momentum method; for further details see SIMO, POSBERGH & MARSDEN [1989].

§2A General Formulation and Relative Equilibria

We consider a mechanical system with configuration manifold Q and phase space $P = T^*Q$, where T^*Q is the cotangent bundle. The Hamiltonian $H: P \to \mathbf{R}$ corresponds to the total energy of the system. Let $X_H: P \to TP$ denote the Hamiltonian vector field associated with H; *i.e.*,

$$dH(z) \cdot \delta z = \Omega(X_H(z), \delta z), \qquad \text{for all } z \in P \text{ and } \delta z \in T_z P, \tag{2.1}$$

where Ω is the canonical symplectic two-form on P. Hamilton's equations are then formulated abstractly as $\dot{z} = X_H(z)$.

In addition we have a symmetry group G which acts on P by canonical transformations, along with the corresponding Lie algebra $\mathcal{G}$. The action of the group G on Q will be denoted $\Psi: G \times Q \rightarrow Q$. Associated with this action is the corresponding infinitesimal generator

$$\xi_Q(q) := \frac{d}{dt}\bigg|_{t=0} \Psi(\exp(t\xi), q), \qquad q \in Q. \tag{2.2}$$

The action Ψ on Q induces, by cotangent lift, a symplectic action on P.

The momentum map for the action of G on P is denoted by $J: P \longrightarrow \mathcal{G}^*$. We recall that associated with this G-action, for any $\xi \in \mathcal{G}$ one has a Hamiltonian vector field $X_{J(\xi)}: P \rightarrow TP$ with Hamiltonian function $J(\xi): P \rightarrow \mathbf{R}$ defined in terms of the momentum map by the relation

$$J(\xi)(z) = \langle J(z), \xi \rangle \qquad \xi \in \mathcal{G}, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathcal{G}$ and $\mathcal{G}^*$. The function $J(\xi)$ is then given by the standard formula:

$$J(\xi)(\alpha_q) = \langle \alpha_q, \xi_Q(q) \rangle, \qquad \alpha_q \in P, \tag{2.4}$$

which as a special case reproduces the usual linear and angular momentum. We denote by q an element in configuration space Q, and by p an element in $T_q^* Q$, the cotangent space for a particular configuration. Thus $z := (q, p) \in P$.

Following terminology due to Poincaré, a point $z_e \in P$ is a *relative equilibrium* if the trajectory of Hamilton's equations through z_e is given by

$$z(t) = \exp[t\xi] \cdot z_e, \qquad \text{for some} \qquad \xi \in \mathcal{G}, \tag{2.5}$$

a condition which states that the dynamic orbit through z_e equals the group orbit through z_e. A basic result exploited below is that the relative equilibria of a mechanical system with Hamiltonian H and momentum map J for the symplectic action of a Lie group G on the phase space P are the critical points of the *energy-momentum functional* $H_\xi: P \rightarrow \mathbf{R}$ defined as

$$H_\xi := H - \langle J - \mu_e, \xi \rangle \tag{2.6}$$

where $\mu_e = J(z_e)$ is the value of the momentum map at the sought relative equilibrium. In mechanical terms this result, known as the relative equilibrium theorem, provides a variational characterization of the relative equilibrium as the stationary point of the energy (Hamiltonian) subject to the side constraint of constant momentum. Within the context of this constrained optimization problem, formal stability of a relative equilibria is then concluded by examining the definiteness of the second variation $\delta^2 H_\xi$ restricted to the subspace defined by the side constraint $J(z_e) - \mu_e = 0$ modulo neutral directions due to group invariance. This subspace is isomorphic to the quotient space

$$\mathcal{S} \cong \ker[T_{z_e} J(z_e)] \Big/ T_{z_e}(G_{\mu_e} \cdot z_e), \tag{2.7}$$

where $G_{\mu_e} \cdot z_e$ denotes the orbit of the (isotropy) subgroup G_{μ_e} of G that leaves μ_e invariant (under the coadjoint action of G on $\mathcal{G}^*$), and $T_{z_e}(G_{\mu_e} \cdot z_e)$ is the

Box 2.1. The Energy-Momentum Method

1. (*First variation*) Construct $H_\xi = H - [J(\xi) - \langle \mu_e, \xi \rangle]$ and find $z_e \in P$ and $\xi \in \mathcal{G}$ such that

$$dH_\xi(z_e) \cdot \delta z = 0, \qquad \text{and} \quad J(z_e) - \mu_e = 0,$$

for all $\delta z \in T_{z_e} P$ (No restrictions placed on δz at this stage).
2. (*Admissible variations for second variation test*) Choose a linear subspace $\mathcal{S} \subset T_{z_e} P$ such that
   i. $dJ(\xi)(z_e) \cdot \delta z = 0$ for all $\delta z \in \mathcal{S}$.
   ii. $\mathcal{S}$ complements $T_{z_e}(G_{\mu_e} \cdot z_e)$ in [ker $dJ(\xi)(z_e)$]; *i.e.*, every variation $\delta z \in T_{z_e} P$ satisfying i. is uniquely written as

$$\delta z = v + \underbrace{\chi_P(z_e)}_{\textit{tangent to orbit}},$$

for some $v \in \mathcal{S}$ and $\chi \in \mathcal{G}_{\mu_e}$ (so that $\chi_P(z_e) \in T_{z_e}(G_{\mu_e} \cdot z_e)$).
3. Test for definiteness of the second variation $\delta^2 H_\xi$ on $\mathcal{S}$; *i.e.*

$$\delta^2 H_\xi(z_e) \cdot (v, v) > 0,$$

for all $v \in \mathcal{S}$. Definiteness implies *formal stability* of $z_e \in P$.

Box 2.1. Procedure for Stability Analysis

tangent space at z_e to this orbit. The criterion for formal stability then takes the following form

$$\boxed{z_e \in P \;\text{ formally stable } \;\Leftrightarrow\; \delta^2 H(z_e) \cdot (\delta z, \delta z) > 0 \;\text{ for }\; \delta z \in \mathcal{S}.} \tag{2.8}$$

Below, in equation (2.27) we will show how to specifically choose $\mathcal{S}$.

In this paper, we apply this method to the stability analysis of relative equilibria of a uniformly rotating rigid body coupled to a rod. For this problem

$G = SO(3)$ and $\mathcal{G} \in so(3) \cong \mathbf{R}^3$. Moreover, we have

$$\left.\begin{aligned} G_{\mu_e} &= \{\exp[t\hat{\xi}] \in SO(3) \mid t \in \mathbf{R}\}, \\ \mathcal{G}_{\mu_e} &= \{\xi \in \mathbf{R} \ \text{ with } \ \xi \times \mu_e = 0\}. \end{aligned}\right\} \tag{2.9}$$

That is, G_{μ_e} is the group of rotations about ξ, and $\mathcal{G}_{\mu_e}$ is the line along ξ (equivalently, the one dimensional space of infinitesimal rotations about ξ). Note that $\dim[\mathcal{G}_{\mu_e}(z_e)] = 1$.

To define the constraint subspace $\mathcal{S} \subset T_{z_e}P$, we first will need to enforce the condition i in Box 2.1, *i.e.*, $TJ(z_e) \cdot \delta z = 0$, for any $(z_e; \delta z) \in \ker[T_{z_e}J(z_e)]$. This condition places *three* restrictions on the variations in $T_{z_e}P$. The *additional constraint* ii in Box 2.1 that variations in $\ker[T_{z_e}J(z_e)]$ be taken modulo the *one dimensional* subspace $\mathcal{G}_{\mu_e}(z_e)$ introduces another restriction and leads to the dimension count

$$\operatorname{codim}[\mathcal{S}] = 4. \tag{2.10}$$

To perform the second variation test in Box 2.1 we introduce a decomposition of the of the constraint subspace $\mathcal{S}$ of the form

$$\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT} \tag{2.11}$$

which results in a block diagonal structure of the second variation of the energy-momentum functional H_ξ restricted to $\mathcal{S}$,

$$\delta^2 H_\xi \Big|_{\mathcal{S}\times\mathcal{S}} = \begin{bmatrix} \begin{bmatrix} 2\times 2 \text{ rigid} \\ \text{body block} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \text{Internal vibration} \\ \text{block} \end{bmatrix} \end{bmatrix}. \tag{2.12}$$

We outline below the basic steps involved in the construction of this decomposition following the construction given in SIMO, POSBERGH & MARSDEN [1989]. We direct the reader to this later reference for further details. Abstract and geometric aspects underlying this construction are examined in MARSDEN, SIMO, LEWIS & POSBERGH [1989].

§2B The Block Diagonalization Theorem and the Second Variation Test

Assume a Hamiltonian function H of the form $H = V + K$, where $V: Q \to \mathbf{R}$ is the potential energy, and $K: P \to \mathbf{R}$ is the kinetic energy of the system. We further assume that K defines on Q an inner product denoted by

$$\langle\cdot,\cdot\rangle_g : TQ \times TQ \to \mathbf{R}. \tag{2.13}$$

For instance, for finite dimensional Hamiltonian systems, we have

$$K = \tfrac{1}{2}\langle p^\sharp, p^\sharp\rangle_g = \tfrac{1}{2} p_i g^{ij}(q) p_j, \tag{2.14}$$

where $g(q) = g_{ij}\, dq^i \otimes dq^j$ is a given Riemannian metric on Q.

Step 1. *Reformulation of the energy-momentum functional.* Define a modified potential $V_\xi : Q \to \mathbf{R}$ by the expression

$$\left.\begin{aligned} V_\xi(q) &:= V(q) + L_\xi(q), \\ L_\xi(q) &:= -\tfrac{1}{2}\langle \xi_Q(q), \xi_Q(q)\rangle_g . \end{aligned}\right\} \tag{2.15}$$

It can be shown that the critical points of V_ξ are precisely the relative equilibrium configurations $q_e \in Q$ (see MARSDEN, SIMO, LEWIS & POSBERGH [1989]). Accordingly, if $\delta V_\xi / \delta q$ denotes the functional derivative of V_ξ defined in the standard fashion as

$$dV_\xi(q) \cdot \delta q = \langle \delta q, \frac{\delta V_\xi}{\delta q} \rangle, \qquad \text{for all } q \in Q, \tag{2.16}$$

we have the critical point condition

$$\left. \frac{\delta V_\xi}{\delta q} \right|_{(q_e)} = 0. \tag{2.17}$$

For stationary rotations about $\xi \in \mathcal{G}$, the term L_ξ gives the potential energy associated with the centrifugal force.

Next, define a potential function $K_\xi : P \to \mathbf{R}$ by the expression

$$K_\xi(z) := \tfrac{1}{2} \| p - \mathsf{FL}(\xi_Q(q)) \|^2_{g^{-1}}, \qquad z = (q, p) \in P, \tag{2.18}$$

where $\mathsf{FL}: TQ \to P$ is the Legendre transformation and $\| \cdot \|_{g^{-1}}$ is the norm induced by (2.14) on T_q^*Q. It is evident that K_ξ also has critical points at the relative equilibria $z_e \in P$. Furthermore, we have

$$\boxed{H_\xi = V_\xi + K_\xi + \langle \mu_e, \xi \rangle.} \tag{2.19}$$

Finally, observe that the second variations of V_ξ and K_ξ make *intrinsic sense* at a relative equilibrium $z_e \in P$.

Step 2. *The tangent space of admissible variations for V_ξ.* Recall that G acts on Q by *isometries*, and that V is *left invariant* under the full group G at *any* configuration $q \in Q$ (for $G = SO(3)$ this is the condition of frame indifference). The term L_ξ, on the other hand, is left invariant under the full G *only at a relative equilibrium* $q_e \in Q$ (*i.e.*, the Lie derivative of V_ξ in the direction of any η_Q, evaluated at q_e vanishes). Thus, in general, L_ξ and consequently V_ξ, is invariant only under the action of the isotropy subgroup $G_{\mu_e} \subset G$. Let $\mathcal{G}_{\mu_e} \subset \mathcal{G}$ be the corresponding Lie subalgebra; *i.e.*,

$$\mathcal{G}_{\mu_e} := \{ \zeta \in \mathcal{G} \mid ad^*_{\mu_e}(\zeta) = 0 \}. \tag{2.20}$$

The space of admissible variations $\mathcal{V} \subset T_qQ$ for V_ξ at q, is then the tangent space to the orbit space Q/G_{μ_e} which can be realized as

$$\boxed{\mathcal{V} := T_q Q \Big/ T_q(G_{\mu_e} \cdot z_e) \cong \{ \delta q \in T_q Q \mid \langle \delta q, \zeta_Q(q) \rangle_g = 0, \ \zeta \in \mathcal{G}_{\mu_e} \}.} \tag{2.21}$$

Step 3. *Split of $\mathcal{V}$: Block-diagonalization of V_ξ.* Construct a decomposition of $\mathcal{V}$,

$$\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT}, \tag{2.22}$$

into infinitesimal rigid body variations, and 'deformation' variations as follows. Let $\mathcal{G}_{\mu_e}^{\perp} \subset \mathcal{G}$ be the g-dependent orthogonal complement of $\mathcal{G}_{\mu_e}$ in the kinetic energy inner product; *i.e.*,

$$\mathcal{G}_{\mu_e}^{\perp} := \{ \eta \in \mathcal{G} \mid \langle \eta_Q(q), \zeta_Q(q) \rangle_g = 0, \ \zeta \in \mathcal{G}_{\mu_e} \}, \tag{2.23}$$

so that $\mathcal{G} = \mathcal{G}_{\mu_e} \oplus \mathcal{G}_{\mu_e}^{\perp}$. Recall that a superposed rigid body variation is of the form $\eta_Q(q) \in T_q Q$, with $\eta \in \mathcal{G}$. Thus, in view of (2.21) and (2.22) we set

$$\boxed{\mathcal{V}_{RIG} := \{ \eta_Q(q_e) \in T_{q_e} Q \mid \eta \in \mathcal{G}_{\mu_e}^{\perp} \} \subset \mathcal{V}.} \tag{2.24}$$

Note that the requirement $\eta \in \mathcal{G}_{\mu_e}^{\perp}$ furnishes the condition which ensures that indeed $\mathcal{V}_{RIG} \subset \mathcal{V}$.

To construct $\mathcal{V}_{INT}$, recall that V is G-left invariant whereas V_ξ is only G_{μ_e}-left invariant. However, V_ξ *is* infinitesimally $\mathcal{G}$ invariant, but the *body force* $\delta V_\xi / \delta q$ need not be. The quantity capturing the lack of invariance of $\delta V_\xi / \delta q$ under G/G_{μ_e} is

$$\begin{aligned} \mathcal{L}_{\eta_Q(q_e)} \frac{\delta V_\xi}{\delta q}(q_e) &= \mathcal{L}_{\eta_Q(q_e)} \frac{\delta L_\xi}{\delta q}(q_e) \\ &:= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Psi_{\exp[-\epsilon\eta]} \left(\frac{\delta L_\xi}{\delta q} \left(\Psi_{\exp[\epsilon\eta]}(q_e) \right) \right) \qquad \text{for all } \eta \in \mathcal{G}_{\mu_e}^{\perp}, \end{aligned} \tag{2.25}$$

where $\mathcal{L}_a b$ denotes the Lie derivative of b in the direction a. We define $\mathcal{V}_{INT} \subset \mathcal{V}$ by the condition

$$\boxed{\mathcal{V}_{INT} := \{ \delta q \in \mathcal{V} \mid \langle \delta q, \mathcal{L}_{\eta_Q(q_e)} \frac{\delta L_\xi}{\delta q}(q_e) \rangle_g = 0, \ \text{for } \eta \in \mathcal{G}_{\mu_e}^{\perp} \}.} \tag{2.26}$$

Note that the number of constraints in (2.26) equals $\dim[\mathcal{G}_{\mu_e}^{\perp}] = \dim[\mathcal{V}_{RIG}]$. Furthermore, by construction $\mathcal{V}_{INT} \cap \mathcal{V}_{RIG} = \{0\}$ so that (2.22) holds.

Step 4. *Split of $\mathcal{S}$: Block-diagonalization of H_ξ.* Conditions i and ii in Box 2.1 lead to the following concrete realization of $\mathcal{S}$ as a (constrained) subspace of $T_{z_e}P$:

$$\mathcal{S} := \{\delta z = (\delta q, \delta p) \in T_{z_e}P \mid T_{z_e}J(z_e)\cdot\delta z = 0, \text{ and } \delta q \in \mathcal{V},\ \zeta \in \mathcal{G}_{\mu_e}\}. \quad (2.27)$$

The split (2.22) then induces a split $\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$ via the Legendre transformation as follows. $\mathcal{S}_{RIG}$ will now be identified with the tangent space at z_e of superposed G/G_{μ_e}-motions on motions starting at z_e. (For $G = SO(3)$, these are superposed infinitesimal rigid body motions modulo motions about μ_e).

Let $t \mapsto z(t) = (q(t), p(t)) \in P$ be a motion starting at $z(t)|_{t=0} = z_e$. Consider a superposed G/G_{μ_e}-motion which, by definition, is given by

$$\left.\begin{aligned} q^+(t) &= \Psi_{g(t)}(q(t)), \\ p^+(t) &= \mathbb{F}L\Big(\frac{d}{dt}\Psi_{g(t)}(q(t))\Big), \end{aligned}\right\} \quad (2.28)$$

where $t \mapsto g(t)$ is a motion in G/G_{μ_e}. Here Ψ is the action induced by $g(t)$ on Q. The tangent space at $z_e \in P$ to all superposed G/G_{μ_e}-motions is obtained by linearizing (2.28) at $t = 0$ as follows. Consider the one parameter family of G/G_{μ_e}-motions given, for $t \mapsto \eta(t) \in \mathcal{G}_{\mu_e}^{\perp}$, by

$$\epsilon \mapsto g(\epsilon, t) := \exp[\epsilon\eta(t)] \in G\big/G_{\mu_e}. \quad (2.29)$$

Then define $\Delta z = (\Delta q, \Delta p) \in T_{z_e}P$ by the expressions

$$\Delta q := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Psi_{\exp[\epsilon\eta(t)]}(q(t))\bigg|_{t=0} = \eta_Q(q_e), \quad (2.30)$$

so that $\Delta q \in \mathcal{V}_{RIG}$, and

$$\Delta p := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathbb{F}L\left(\frac{d}{dt}\Psi_{\exp[\epsilon\eta(t)]}(q(t))\right). \quad (2.31)$$

It can be shown that $\Delta z = (\Delta q, \Delta p)$ given by formulae (2.30) and (2.31) actually lies in $T_{z_e}P/T_{z_e}(G_{\mu_e}\cdot z_e)$. Since $\mathcal{V}_{RIG} \subset \mathcal{V}$ it follows from (2.27) that the restriction to $\ker[T_{z_e}J(z_e)]$ completes the construction of $\mathcal{S}_{RIG}$; *i.e.*,

$$\boxed{\mathcal{S}_{RIG} := \{\Delta z = (\Delta q, \Delta p) \mid T_{z_e}J(z_e)\cdot\Delta z = 0\}.} \quad (2.32)$$

One can show that $\mathcal{S}_{RIG}$ defined by (2.32) is parametrized solely in terms of elements $\eta \in \mathcal{G}_{\mu_e}^{\perp}$; hence,

$$\dim[\mathcal{S}_{RIG}] = \dim[\mathcal{V}_{RIG}] = \dim[\mathcal{G}_{\mu_e}^{\perp}]. \quad (2.33)$$

Finally, we define $\mathcal{S}_{INT}$ by setting

$$\boxed{\mathcal{S}_{INT} := \{\delta z = (\delta q, \delta p) \in \mathcal{S} \mid \delta q \in \mathcal{V}_{INT}\}.} \quad (2.34)$$

One can easily show that $\mathcal{S}_{RIG} \cap \mathcal{S}_{RIG} = \{0\}$ so that one indeed has

$$\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}. \tag{2.35}$$

With this construction in hand we have the following basic result

The block-diagonalization theorem. *Let $z_e = (q_e, p_e) \in P$ be a relative equilibrium. Further, let $\mathcal{V}_{RIG}, \mathcal{V}_{INT} \subset \mathcal{V}$ and $\mathcal{S}_{RIG}, \mathcal{S}_{INT} \subset \mathcal{S}$ be constructed as above. Then*

$$\text{i.} \qquad \delta^2 V_\xi(q_e) \cdot (\eta_Q(q_e), \delta q) = 0, \tag{2.36a}$$

$$\text{ii.} \qquad \delta^2 K_\xi(z_e) \cdot (\Delta z, \delta z) = 0, \tag{2.36b}$$

for all $\eta_Q(q_e) \in \mathcal{V}_{RIG}$, $\delta q \in \mathcal{V}_{INT}$ and $\Delta z \in \mathcal{S}_{RIG}$, $\delta z \in \mathcal{S}_{INT}$. Consequently, in view of (2.19) we also have

$$\text{iii.} \qquad \delta^2 H_\xi(z_e) \cdot (\Delta z, \delta z) = 0. \tag{2.36c}$$

■

Remarks

1. The condition $\delta^2 H_\xi(z_e)|_{\mathcal{S}_{RIG} \times \mathcal{S}_{RIG}} > 0$ leads to stability requirements that generalize the classical stability conditions for a rigid body in stationary rotation.

2. From expression (2.18) and the fact that at a relative equilibrium one has

$$p_e = \mathbb{F}L(\xi_Q(q_e)), \tag{2.37}$$

it is easily concluded that

$$\delta^2 K_\xi(z_e)\Big|_{\mathcal{S}_{INT} \times \mathcal{S}_{INT}} > 0. \tag{2.38}$$

Consequently, one has the estimate

$$\delta^2 H_\xi(z_e)\Big|_{\mathcal{S}_{INT} \times \mathcal{S}_{INT}} > \delta^2 V_\xi(q_e)\Big|_{\mathcal{V}_{INT} \times \mathcal{V}_{INT}}. \tag{2.39}$$

Thus, positive definiteness of $\delta^2 V_\xi(q_e)|_{\mathcal{V}_{INT} \times \mathcal{V}_{INT}}$ ensures positive definiteness of $\delta^2 H_\xi(z_e)|_{\mathcal{S}_{INT} \times \mathcal{S}_{INT}}$.

3. The proof of the block-diagonalization result (2.36a) follows from the identity

$$\left\langle \delta q, \nabla_{\eta_Q(q_e)} \frac{\delta V_\xi}{\delta q}(q_e) \right\rangle_g = d^2 V_\xi(q_e) \cdot (\eta_Q(q_e), \delta q), \tag{2.40}$$

the defining condition in (2.26), and relation (2.25). ■

With the aid of the block diagonalization theorem the second variation test for formal stability in Box 2.1 takes a remakably simple form: Formal stability of

$z_e = (q_e, p_e)$ is implied by the conditions

$$\boxed{\begin{array}{ll} \text{i.} & \delta^2 H_\xi(z_e)\Big|_{S_{RIG}\times S_{RIG}} > 0, \\ \text{ii.} & \delta^2 V_\xi(q_e)\Big|_{V_{INT}\times V_{INT}} > 0. \end{array}} \tag{2.41}$$

Condition ii requires that the lowest eigenvalue of $\delta^2 V_\xi(z_e)$ restricted to V_{INT} be positive. It can be shown (see SIMO, POSBERGH & MARSDEN [1989]) that this condition is in turn implied by the requirement that $|\xi|$ be less that the lowest natural frequency of the system at the relative equilibrium configuration $q_e \in Q$.

§3. The Rigid Body

In this section we outline the notation and mechanical setup for the rigid body. For a more complete discussion we refer to ABRAHAM & MARSDEN [1978] or ARNOLD [1978]. Most of the concepts and notation will be used again for the geometrically exact rod in the next section. For an application of the energy momentum method to the rigid body alone see SIMO, POSBERGH & MARSDEN [1989].

§3A. *Notation for the Rotation Group*

The rotation group $SO(3)$ consists of all orthogonal linear transformations of Euclidean three space to itself which have determinant one. Its Lie algebra, denoted $so(3)$ consists of all 3×3 skew matrices, which we identify with $\mathbf{R}^3$ by the isomorphism $\hat{}: \mathbf{R}^3 \to so(3)$ defined by

$$\Omega \mapsto \hat{\Omega} = \begin{bmatrix} 0 & -\Omega^3 & \Omega^2 \\ \Omega^3 & 0 & -\Omega^1 \\ -\Omega^2 & -\Omega^1 & 0 \end{bmatrix}, \tag{3.1}$$

where $\Omega = (\Omega^1, \Omega^2, \Omega^3)$. One checks that for any vector r, $\hat{\Omega} r = \Omega \times r$ and, $\hat{\Omega}\hat{\Theta} - \hat{\Theta}\hat{\Omega} = (\Omega \times r)\hat{}$. These give the usual identification of the Lie algebra $so(3)$ with $\mathbf{R}^3$ and the Lie algebra bracket with the cross product of vectors.

Given $\Lambda \in SO(3)$, let $\hat{v}_\Lambda$ denote an element of the tangent space to $SO(3)$ at Λ. Since $SO(3)$ is a submanifold of $GL(3)$, the general linear group, we can identify $\hat{v}_\Lambda$ with a 3×3 matrix, which we denote with the same letter. Linearizing the defining (submersive) condition $\Lambda\Lambda^T = 1$, gives

$$\Lambda \hat{v}_\Lambda^T + \hat{v}_\Lambda \Lambda^T = 0, \tag{3.2}$$

which defines $T_\Lambda SO(3)$. We can identify $T_\Lambda SO(3)$ with $so(3)$ by the following isomorphism: Given $\hat{\theta} \in so(3)$ and $\Lambda \in SO(3)$ we define $(\Lambda, \hat{\theta}) \mapsto \hat{\theta}_\Lambda \in T_\Lambda SO(3)$ through right translations by setting

$$\hat{\theta}_\Lambda := T_e R_\Lambda \cdot \hat{\theta} \cong (\Lambda, \hat{\theta}\Lambda). \tag{3.2b}$$

Thus $\hat{\theta}_\Lambda$ is the right invariant extension of $\hat{\theta}$.

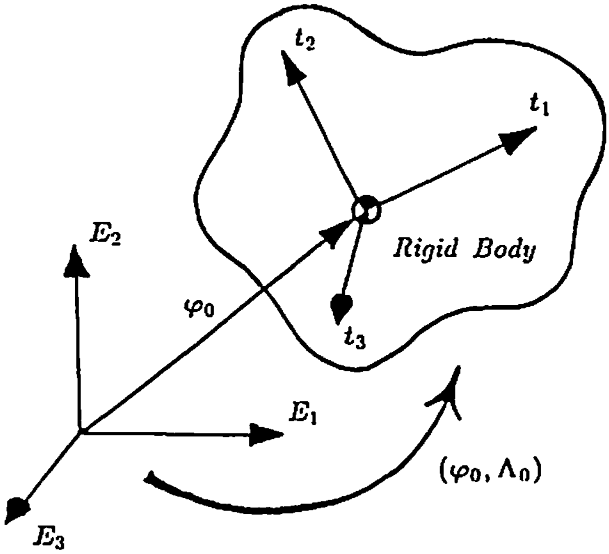

Figure 3.1. Rigid Body

Often, the base point is omitted and with an abuse in notation we write $\hat{\theta}\Lambda$ for $\hat{\theta}_\Lambda$.

The dual of the Lie algebra $so(3)$ is identified with $\mathbb{R}^3$ via the standard dot product:

$$\pi \cdot \theta = \tfrac{1}{2}\mathrm{tr}[\hat{\pi}^T \hat{\theta}]. \tag{3.3}$$

This extends to the *left-invariant* pairing on $T_\Lambda SO(3)$ given by

$$\langle \hat{\pi}_\Lambda, \hat{\theta}_\Lambda \rangle = \tfrac{1}{2}\mathrm{tr}[\hat{\pi}_\Lambda^T \hat{\theta}_\Lambda] = \tfrac{1}{2}\mathrm{tr}[\hat{\pi}^T \hat{\theta}] = \pi \cdot \theta. \tag{3.4}$$

We shall, thereby, write elements of $so(3)^*$ as $\hat{\pi}$ with $\pi \in \mathbb{R}^3$ and elements of $T^*_\Lambda SO(3)$ in spatial representation as

$$\hat{\pi}_\Lambda = (\Lambda, \hat{\pi}\Lambda). \tag{3.5}$$

§3B. *Rigid Body Dynamics*

In general, any configuration of the rigid body is described by a position, $\varphi_0 \in \mathbb{R}$, and an orientation, $\Lambda_0 \in SO(3)$ in ambient space (see Figure 3.1). Thus, we let

$$C_0 := \left\{ \Phi_0 := (\varphi_0, \Lambda_0) \in \mathbb{R}^3 \times SO(3) \right\} \tag{3.6}$$

be the configuration manifold for the rigid body. Associated with this configuration manifold we have the collection of tangent spaces $T_{\Phi_0}C_0$ for $\Phi_0 \in C_0$ defined as

$$T_{\Phi_0}C_0 := \{\mathrm{v}_{\Phi_0} := (v_{\varphi_0}, \hat{\omega}_{\Lambda_0}) \in \mathbf{R}^3 \times T_{\Lambda_0}SO(3)\}. \tag{3.7}$$

The *phase space* for the rigid body is the cotangent bundle $T^*C_0 = \{\cup_{\Phi_0 \in C_0} T^*_{\Phi_0}C_0\}$ where

$$T^*_{\Phi_0}C_0 := \{\mathrm{p}_{\Phi_0} := (p_{\varphi_0}, \hat{\pi}_{\Lambda_0}) \in \mathbf{R}^3 \times T^*_{\Lambda_0}SO(3)\}. \tag{3.8}$$

The tangent space $T_{\Phi_0}C_0$ and the cotangent space $T^*_{\Phi_0}C_0$ are in duality through the pairing given by

$$\langle \mathrm{v}_{\Phi_0}, \mathrm{p}_{\Phi_0} \rangle = p_{\varphi_0} \cdot v_{\varphi_0} + \tfrac{1}{2}\mathrm{tr}(\hat{\pi}^T_{\Lambda_0} \hat{w}_{\Lambda_0}). \tag{3.9}$$

The Hamiltonian is the mapping $H: P \to \mathbf{R}$ corresponding to the kinetic energy of a free rigid body. Thus

$$H = \tfrac{1}{2}\pi_0 \cdot \mathbf{I}_B{}^{-1}\pi_0 + \tfrac{1}{2}M_B^{-1}\|p_0\|^2. \tag{3.10}$$

where π_0 is the spatial angular momentum vector, p_0 is the spatial linear momentum vector, M_B is the mass of the rigid body and $\mathbf{I}_B := \Lambda_0 \mathbf{J}_B \Lambda_0^T$ is the *time dependent* inertia tensor (in spatial coordinates) and $\mathbf{J}_B$ is the *constant inertia dyadic* given by

$$\mathbf{J}_B = \int_B \rho_{ref}(X)[\|X\|^2 1 - X \otimes X]\, d^3X. \tag{3.11}$$

Here, $B \subset \mathbf{R}^3$ is the reference configuration of the rigid body and $\rho_{ref} : B \to \mathbf{R}$ the reference density.

§4. Geometrically Exact Rod Models

In this section we outline the geometrically exact rod model used in this paper. For a more complete discussion see SIMO [1985]. For application of the energy-momentum method to rods see SIMO, POSBERGH & MARSDEN [1989].

§4A. Kinematic Idealization. Canonical Phase Space

We assume that the placement in $\mathbf{R}^3$ at time t of a rod-like body is defined as the set

$$\mathcal{S}_t := \{x \in \mathbf{R}^3 \mid x = \varphi(S,t) + \sum_{\alpha=1}^{2} \xi^\alpha t_\alpha(S,t);\ \text{where}\ 0 \le S \le L \ \text{and}\ (\xi^1, \xi^2) \in A\}, \tag{4.1}$$

where $A \subset \mathbb{R}^2$ is a given compact set. The map $\varphi_t: [0, L] \to \mathbb{R}^3$ given by $\varphi_t(S) = \varphi(S, t)$ defines the position at time $t \in \mathbb{R}_+$ of a curve referred to as the *line of centroids*. The vector fields $\boldsymbol{t}_\alpha(S, t)$, $\alpha = 1, 2$, are the director fields which in the special (restricted) theory of Cosserat rods, are subject to the constraints

$$\|\boldsymbol{t}_\alpha(S, t)\| = 1, \qquad \alpha = 1, 2, \qquad \text{and} \qquad \boldsymbol{t}_1(S, t) \cdot \boldsymbol{t}_2(S, t) = 0. \tag{4.2}$$

In addition, the admissible motions are required to satisfy the condition

$$\boldsymbol{t}_3(S, t) \cdot \frac{\partial}{\partial S}\varphi(S, t) > 0, \qquad \text{where} \qquad \boldsymbol{t}_3(S, t) := \boldsymbol{t}_1(S, t) \times \boldsymbol{t}_2(S, t) \neq 0. \tag{4.3}$$

We refer to $\mathcal{S}_t(S) := \{x \in \mathbb{R}^3 | [x - \varphi(S, t)] \cdot \boldsymbol{t}_3(S, t) = 0\}$ as the *placement of a cross-section of the rod* at time t. Condition (4.3) limits the amount of *shearing* experienced by each cross-section.

Consequently, at each time $t \in \mathbb{R}_+$ and $S \in [0, L]$, we have an *orthogonal frame* $\{\boldsymbol{t}_I(S, t)\}_{I=1,2,3}$ referred to as a *director frame* in the sequel. Given any *fixed (inertial)* frame $\{\boldsymbol{E}_I(S, t)\}_{I=1,2,3}$ for example, the standard basis in $\mathbb{R}^3$, there is a *unique orthogonal* transformation $\Lambda_t: [0, L] \to SO(3)$, defined for each time t such that

$$\boldsymbol{t}_I(S, t) = \Lambda(S, t)\boldsymbol{E}_I(S, t), \qquad (I = 1, 2, 3),\ S \in [0, L], \tag{4.4}$$

where $\Lambda(S, t) = \Lambda_t(S)$ for $t \in \mathbb{R}_+$.

Thus, the rod is described by a curve with values in the special orthogonal group which at each point orient the director frame. Abstractly, this latter view leads to the configuration manifold

$$\boxed{C := \{\Phi = (\varphi, \Lambda) : [0, L] \to \mathbb{R}^3 \times SO(3)\},} \tag{4.5}$$

suitably restricted by prescribed boundary conditions to be specified below. A motion is a curve of configurations $t \mapsto \Phi_t = (\varphi_t, \Lambda_t) \in C$. Associated with any such motion, there is a sequence of placements $\mathcal{S}_t \subset \mathbb{R}^3$ of a physical body defined by (4.1).

Associated with the configuration manifold one has the collection of tangent spaces $T_\Phi C$ for $\Phi \in C$, defined as

$$T_\Phi C := \{\mathbf{v}_\Phi = (v_\varphi, \hat{\omega}_\Lambda): [0, L] \to \mathbb{R}^3 \times T_\Lambda SO(3)\}, \tag{4.6}$$

where, in accordance with the earlier notation, the tangent field $S \in [0, L] \to \hat{\omega}_\Lambda(S) \in T_{\Lambda(S)}SO(3)$ admits the following *right* realization:
Given $\hat{\omega}: [0, L] \to so(3)$, set

$$\hat{\omega}_\Lambda(S) := (\Lambda(S), \hat{\omega}(S)\Lambda(S)), \qquad \text{for} \quad S \in [0, L]. \tag{4.7}$$

We refer to the right realization as *spatial* representation. Following continuum mechanics conventions we have used lowercase letters for the representation.

The *canonical phase space* for our geometrically exact rod model is the *cotangent bundle* $T^*C := \{\bigcup_{\Phi \in C} T^*_\Phi C\}$, where

$$T^*_\Phi C := \{\mathbf{p}_\Phi = (p_\varphi, \hat{\pi}_\Lambda) \colon [0, L] \to \mathbf{R}^{3*} \times T^*_\Lambda SO(3)\}. \tag{4.8}$$

As above, one also has the *right* representation

$$\hat{\pi}_\Lambda(S) := (\Lambda(S), \hat{\pi}(S)\Lambda(S)), \tag{4.9}$$

for all $S \in [0, L]$, $\hat{\pi} \colon [0, L] \to so(3)^*$. Finally, we recall (see SIMO, MARSDEN & KRISHNAPRASAD [1988]) that for *pure displacement boundary conditions* $T_\Phi C$ and $T^*_\Phi C$ are in *duality* through the standard L_2 pairing

$$\langle \mathbf{v}_\Phi, \mathbf{p}_\Phi \rangle := \int_0^L [p_\varphi(S) \cdot v_\varphi(S) \langle \pi_\Lambda(S), \theta_\Lambda(S) \rangle] \, dS. \tag{4.10}$$

where $\langle \pi_\Lambda(S), \theta_\Lambda(S) \rangle$ is defined analogously to the rigid body case.

More generally, definition (4.10) needs to be modified by appending additional boundary terms to accommodate (natural) stress free boundary conditions.

As a function on the phase space $P = T^*C$, the kinetic energy $K \colon P \to \mathbf{R}$ takes the form

$$K = \int_0^L [\rho_A^{-1} p \cdot p + \pi \cdot \mathbf{I}^{-1} \pi] \, dS \tag{4.11}$$

$$= \int_0^L [\rho_A^{-1} p \cdot p + \pi \cdot \Lambda \mathbf{J}^{-1} \Lambda \pi] \, dS, \tag{4.12}$$

where p and π are the linear and angular velocity respectively, ρ_A is the mass per unit length of the rod, and

$$\mathbf{I} = \Lambda \mathbf{J} \Lambda^T, \qquad \mathbf{J} = \int_A \xi^\alpha \xi^\beta \rho_{ref} [\delta_{\alpha\beta} \mathbf{1}_3 - \mathbf{E}_\alpha \otimes \mathbf{E}_\beta] \, dA, \tag{4.13}$$

where ρ_{ref} is the density in the reference configuration. Here ξ^α are the integration variables, such that $\xi^3 \in [0, L]$, and $(\xi^1, \xi^2) \in \Omega$.

The potential energy V as a function on $P = T^*C$ is expressed in terms of a stored energy function $\psi \colon \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ as

$$V = \int_0^L \psi(\Lambda^T(\varphi' - t_3), \Lambda^T \omega) \, dS, \qquad \hat{\omega} := \Lambda' \Lambda^T, \tag{4.14}$$

where $' = \frac{\partial}{\partial S}$. The spatial strains are $\gamma = \varphi' - t_3$ and ω. When the rod is deformed the potential energy gives rise to internal stress resultants defined as

$$n = \frac{\partial \psi}{\partial \gamma}; \qquad m = \frac{\partial \psi}{\partial \omega}. \tag{4.15}$$

Consequently, the Hamiltonian $H \colon T^*C \to \mathbf{R}$ is given as $H = K + V$.

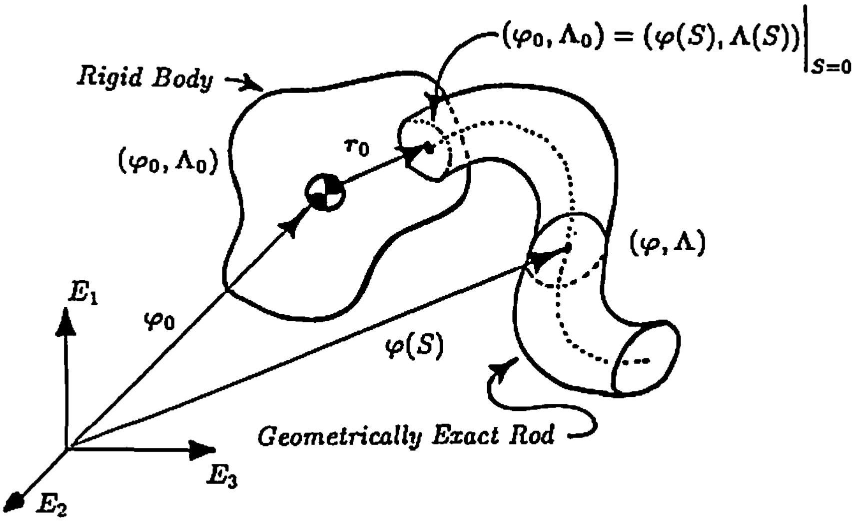

Figure 5.1. Rigid Body With Attached Rod

§5. Coupled Rigid Body – Geometrically Exact Rod

In this section we consider the stability analysis of the relative equilibria for a coupled rigid body and flexible appendage modeled as a fully nonlinear geometrically exact rod. It appears that this development represents one of the first examples of a rigorous nonlinear stability analysis of a realistic, fully nonlinear coupled structural system. The basic configuration is illustrated in figure 5.1. We assume the base of the rod is fixed at a point located a distance r_0 from the center of mass of the rigid body. We model this configuration by imposing suitable boundary conditions on the rigid body and geometrically exact rod.

§5A. Boundary Conditions

For the rigid body with the attached, flexible rod we require that

$$\varphi_0(t) + r_0(t) = \varphi(S,t)\Big|_{S=0} ; \qquad \Lambda_0(t) = \Lambda(S,t)\Big|_{S=0} \tag{5.1}$$

For a rigid body clamped to the base of the rod we also require that the linear and angular velocities match; *i.e.*,

$$\dot{\varphi} + w_0 \times r_0 = \dot{\varphi}\Big|_{S=0} ; \qquad \hat{w}_0 = \hat{w}\Big|_{S=0} \tag{5.2}$$

These boundary conditions impose further conditions on the admissible variations:

Lemma 5.1. *For the rigid body coupled to a geometrically exact rod, let* $(\delta\varphi_0, \delta\theta_0) \in T_{\Phi_0}P_0$ *and* $(\delta\varphi, \delta\theta) \in T_{\Phi}P$. *Then, the variations of the configuration satisfy the constraints*

$$\delta\varphi_0(t) + \delta\theta_0(t) \times r_0(t) = \delta\varphi(S,t)\Big|_{S=0}; \qquad \delta\theta_0(t) = \delta\theta(S,t)\Big|_{S=0}, \tag{5.3}$$

and the variations in angular momentum satisfy the constraint

$$\mathrm{I}_B^{-1}(\delta\pi_0 + \pi_0 \times \delta\theta_0) = \mathrm{I}^{-1}(\delta\pi + \pi \times \delta\theta)\Big|_{S=0}. \tag{5.4}$$

Proof: Conditions (5.3) follow by direct computation while conditions (5.4) follow by direct computation and application of the Legendre transformation (5.2). ■

We make the assumption that the tip of the rod is stress free. However, at the base of the rod

$$n(S,t)\Big|_{S=0} = n_0; \qquad m(S,t)\Big|_{S=0} = m_0, \tag{5.5}$$

corresponding to the force and moment balance at the boundary.

§5B. *Tangent and Cotangent Spaces: Rigid Body and Rod*

We now outline the appropriate configuration space and the associated tangent bundle and phase space for the problem of a rigid body fixed to one end of a geometrically exact rod and free to move in space.

We set $\Phi_0 = (\varphi_0, \Lambda_0)$, $\Phi = (\varphi, \Lambda)$ and define the configuration manifold of the rigid body with attached rod as

$$\boxed{Q = \left\{q := (\Phi_0, \Phi) \in \mathcal{C}_0 \times \mathcal{C} \;\middle|\; \beta(\Phi_0) = \Phi(S)\Big|_{S=0}\right\}.} \tag{5.6}$$

where $\beta(\Phi_0) = (\varphi_0 + r_0, \Lambda_0)$. Furthermore, we let $\dot{q} := (\dot{\Phi}_0, \dot{\Phi})$ where $\dot{\Phi}_0 = (\dot{\varphi}_0, \hat{w}_0\Lambda_0)$, and $\dot{\Phi} = (\dot{\varphi}, \hat{w}\Lambda)$.

In view of lemma 5.1 , the tangent bundle associated with this configuration space is given by

$$TQ = \left\{(q;\dot{q}) \mid q \in Q, \dot{\beta}(\Phi_0) = \dot{\Phi}\Big|_{S=0}\right\} \tag{5.7}$$

where $\dot{\beta}(\Phi_0) = (\dot{\varphi}_0 + w_0 \times r_0, \hat{w}_0\Lambda_0)$.

Application of the Legendre transformation to (5.7) with $p = (\mathrm{P}_0, \mathrm{P})$ then yields the phase space

$$\boxed{P := T^*Q = \left\{(q;p) \mid q \in Q,\, p \in \mathsf{FL}TQ\right\}.} \tag{5.8}$$

where $\mathsf{FL}: TQ \to T^*Q$ denotes the Legendre transformation. Elements of the phase space will be denoted as $z := (q,p) \in P$.

§5C. Hamiltonian and Momentum Maps

As a function on P, the kinetic energy in material representation $K: P \to \mathbf{R}$ takes the form

$$K = \tfrac{1}{2} M_B^{-1} p_0 \cdot p_0 + \tfrac{1}{2} \mathbf{I}_B^{-1} \pi_0 \cdot \pi_0 + \int_0^L [\rho_A^{-1} p \cdot p + \pi \cdot \mathbf{I}^{-1} \pi] \, dS. \qquad (5.9)$$

Here K is the sum of the kinetic energy of the rigid body, and that of the geometrically exact rod.

The potential energy as a function on $P = T^*Q$ is the same as for the rod and is expressed in terms of a stored energy function $\psi: \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ as

$$V = \int_0^L \psi(\Lambda^T(\varphi' - t_3), \Lambda^T \omega) \, dS, \qquad \hat{\omega} := \Lambda' \Lambda^T. \qquad (5.10)$$

Note that the stored energy function depends on the configuration of the rod alone. More generally it may also depend on the configuration of the rigid body. This would be the case, for example, were the rigid body attached to the rod by an elastic hinge. Again, the Hamiltonian is given as $H = K + V$.

Next we turn our attention to the invariance properties of H under group actions. We first compute the *momentum maps* corresponding to the group of rotations and translations.

We have the following group invariance properties.

i. *Left $SO(3)$-Invariance.* We note that the reduced expression for the stored energy function ψ appearing in (5.10) is *constructed precisely* to satisfy this invariance property. Thus $G = SO(3)$, Q as defined in (5.6), and $\mathcal{G} = so(3)$, so that

$$\Psi_1(A, (\varphi_0, \Lambda_0, \varphi, \Lambda)) := (A\varphi_0, A\Lambda_0, A\varphi, A\Lambda), \qquad \text{for all } A \in SO(3), \quad (5.11)$$

is the action induced by $SO(3)$ on Q. Given any $\hat{\xi} \in so(3)$, the infinitesimal generator is

$$\begin{aligned} \xi_Q((\varphi_0, \Lambda_0, \varphi, \Lambda)) &= \frac{d}{dt}\bigg|_{t=0} (\exp(t\hat{\xi})\varphi_0, \exp(t\hat{\xi})\Lambda_0, \exp(t\hat{\xi})\varphi, \exp(t\hat{\xi})\Lambda) \\ &= (\hat{\xi}\varphi_0, \hat{\xi}\Lambda_0, \hat{\xi}\varphi, \hat{\xi}\Lambda). \end{aligned} \qquad (5.12)$$

From the relation $\langle J_1(\alpha_q), \xi \rangle = J_1(\xi)(\alpha_q)$ there follows

$$\boxed{J_1(p_{\varphi_0}, \pi_{\Lambda_0}, p_\varphi, \pi_\Lambda) := \pi_0 + \varphi_0 \times p_0 + \int_0^L (\pi + \varphi \times p) \, dS.} \qquad (5.13)$$

ii. $\mathbb{R}^3$-*translational invariance.* The reduced expressions for the stored energy function ψ in (4.14) is also *invariant* under $\mathbb{R}^3$-translations. Consequently, $G = \mathbb{R}^3$, $\mathcal{G} \cong \mathbb{R}^3$, and Q is as defined in (5.6) so that

$$\Psi_2(\mathbf{c}, (\varphi_0, \Lambda_0, \varphi, \Lambda)) := (\varphi_0 + \mathbf{c}, \Lambda_0, \varphi + \mathbf{c}, \Lambda), \qquad \text{for all } \mathbf{c} \in \mathbb{R}^3, \tag{5.14}$$

is the action induced by $\mathbb{R}^3$ on Q. Given $\xi \in \mathbb{R}^3$, the infinitesimal generator is

$$\xi_Q(\varphi_0, \Lambda_0, \varphi, \Lambda) = \left.\frac{d}{dt}\right|_{t=0} (\varphi_0 + t\xi, \Lambda_0, \varphi + t\xi, \Lambda) = (\xi, 0, \xi, 0). \tag{5.15}$$

Again, we use $\langle J_2(\alpha_q), \xi \rangle = J_2(\xi)(\alpha_q)$ the corresponding momentum map $J_2 : P \to \mathbb{R}^3$ as

$$\boxed{J_2(p_{\varphi_0}, \pi_{\Lambda_0}, p_\varphi, \pi_\Lambda) := p_0 + \int_0^L p\, dS.} \tag{5.16}$$

The first step of the energy-momentum method then requires the construction of the energy-momentum functional which in the present context takes the form

$$\begin{aligned} H_{\xi,u} &= K + V - \langle J_1 - \mu_e, \xi \rangle - \langle J_2 - v_e, u \rangle \\ &= V_{\xi,u} + K_{\xi,u} + \langle \mu_e, \xi \rangle + \langle v_e, u \rangle, \end{aligned} \tag{5.17a}$$

where

$$\begin{aligned} V_{\xi,u} = V - \tfrac{1}{2}\Big\{ & M_B \|\xi \times \varphi_0 + u\|^2 + \mathbf{I}_B \xi \cdot \xi \\ & + \int_0^L \rho_A \|\xi \times \varphi + u\|^2\, dS + \int_0^L \mathbf{I}\xi \cdot \xi\, dS \Big\}, \end{aligned} \tag{5.17b}$$

and

$$\begin{aligned} K_{\xi,u} = \tfrac{1}{2}\Big\{ & \|M_B^{-\frac{1}{2}} p_0 - M_B^{\frac{1}{2}}(\xi \times \varphi_0 + u)\|^2 + \|\mathbf{I}_B^{-\frac{1}{2}} \pi_0 - \mathbf{I}_B^{\frac{1}{2}} \xi\|^2 \\ & + \int_0^L \|\rho_A^{-\frac{1}{2}} p - \rho_A^{\frac{1}{2}}(\xi \times \varphi + u)\|^2\, dS + \int_0^L \|\mathbf{I}^{-\frac{1}{2}} \pi - \mathbf{I}^{\frac{1}{2}} \xi\|^2\, dS \Big\}. \end{aligned} \tag{5.17c}$$

§5D. Energy-momentum functional: First Variation

As pointed out in §2, the computation of the relative equilibria exploits the crucial fact that relative equilibria are critical points of the energy-momentum functional. We have the following result

Proposition 5.2. *(First variation of $H_{\xi,u}$). The Euler-Lagrange equations associated with the critical points of $H_{\xi,u}$ are*

$$\left.\begin{array}{ll}
\text{Rigid Body:} & \text{Flexible Appendage:} \\
M_B^{-1} p_{0,e} = \xi \times \varphi_{0,e} + u, & \rho_A^{-1} p_e = \xi \times \varphi_e + u, \\
\mathbf{I}_{B,e}^{-1} \pi_{0,e} = \xi, & \mathbf{I}_e^{-1} \pi_e = \xi, \\
\xi \times p_{0,e} = n_0, & \xi \times p_e = n_e', \\
\mathbf{I}_{B,e}^{-1} \pi_{0,e} \times \pi_{0,e} = m_{0,e} + \varphi_{0,e} \times n_{0,e}. & \mathbf{I}_e^{-1} \pi_e \times \pi_e = m_e' + \varphi_e' \times n_e
\end{array}\right\} \tag{5.18}$$

where n_0, and m_0 are as defined in (5.5), *and n, and m are as defined in* (4.15).

Proof: Follows by a direct computation of

$$\delta H_{\xi,u} = \delta V_{\xi,u} + \delta K_{\xi,u}. \tag{5.19a}$$

From (5.17b) we find

$$\begin{aligned}
\delta V_{\xi,u} = \delta V &- M_B(\xi \times \varphi_0 + u) \cdot \xi \times \delta\varphi_0 - \mathbf{I}_B \xi \times \xi \cdot \delta\theta_0 \\
&- \int_0^L \rho_A(\xi \times \varphi + u) \cdot \xi \times \delta\varphi \, dS - \int_0^L \mathbf{I}\xi \times \xi \cdot \delta\theta \, dS;
\end{aligned} \tag{5.19b}$$

whereas from (5.17c) we obtain

$$\begin{aligned}
\delta K_{\xi,u} = {} & M_B^{-1}(p_0 - M_B(\xi \times \varphi_0 + u)) \cdot (\delta p_0 - M_B(\xi \times \delta\varphi_0)) \\
& + (\mathbf{I}_B^{-1}\pi_0 \times \pi_0 + \mathbf{I}_B \xi \times \xi) \cdot \delta\theta_0 + (\mathbf{I}_B^{-1}\pi_0 - \xi) \cdot \delta\pi_0 \\
& + \int_0^L \rho_A^{-1}(p - \rho_A(\xi \times \varphi + u)) \cdot (\delta p - \rho_A(\xi \times \delta\varphi)) \, dS \\
& + \int_0^L (\mathbf{I}^{-1}\pi \times \pi + \mathbf{I}\xi \times \xi) \cdot \delta\theta + (\mathbf{I}^{-1}\pi - \xi) \cdot \delta\pi \, dS.
\end{aligned} \tag{5.19c}$$

Furthermore, for δV after integration by parts and using (5.3) we obtain the result that

$$\begin{aligned}
\delta V = {} & -\delta\theta_0 \cdot (m_0 + r_0 \times n_0) - \delta\varphi_0 \cdot n_0 \\
& + \int_0^L \left[-(m' + \varphi' \times n) \cdot \delta\theta - n' \cdot \delta\varphi \right] dS.
\end{aligned} \tag{5.20}$$

By standard arguments in the calculus of variations we conclude that the stationarity of the first variation requires conditions (5.18) to hold. ■

We note that the relative equilibrium conditions (5.18) are precisely those one would expect to obtain by a 'bare hands' computation using the definition of a relative equilibrium and the the laws of motion. These results are of course in complete agreement with those obtained by the relative equilibrium theorem.

The following theorem describes several properties of a relative equilibrium of the rod.

Theorem 5.3. *At a relative equilibrium configuration of the rigid body with attached, geometrically exact rod, the following conditions hold.*

i. *The spatial angular velocity field is constant for all $S \in [0, L]$; i.e.,*

$$\hat{w}_e := [\dot{\Lambda}_e \Lambda_e^T] \equiv \textit{constant}, \tag{5.21}$$

so that

$$\Lambda_e(S,t) = \exp[t\hat{w}_e]\Lambda_0(S). \tag{5.22}$$

ii. *The center of mass of the rod, defined by $t \mapsto r_e^0 \in \mathbb{R}^3$ where*

$$\left.\begin{aligned} r_e^0 &:= \frac{1}{M}(M_B \varphi_{0,e} + \int_0^L \rho_A(S)\varphi_e(S,t)\,dS), \\ M &:= M_B + \int_0^L \rho_A(S)\,dS, \end{aligned}\right\} \tag{5.23}$$

undergoes uniform motion with constant velocity

$$v_e^0 := \frac{1}{M}(p_0 + \int_0^L p(S)\,dS) = u + \xi \times r_e^0. \tag{5.24}$$

iii. *The fixed spatial rotation axis $\xi \in \mathbb{R}^3$ is an eigenvector of the extended inertia dyadic relative to the center of mass; i.e.,*

$$\mathbb{I}_\infty^0 \xi = \lambda \xi, \tag{5.25}$$

where we define

$$\left.\begin{aligned} \mathbb{I}_\infty^0 &:= \mathbb{I}_\infty + M[\|r_e^0\|^2 1_3 - r_e^0 \otimes r_e^0] \\ \mathbb{I}_\infty &:= \mathbf{I}_B + M_B[\|\varphi_0\|^2 1_3 - \varphi_0 \otimes \varphi_0]) \\ &\quad + \int_0^L (\mathbf{I} + \rho_A[\|\varphi_0\|^2 1_3 - \varphi_0 \otimes \varphi_0])\,dS. \end{aligned}\right\} \tag{5.26}$$

iv. *The total linear and angular momentum at a relative equilibrium satisfy the condition*

$$\xi \times \mu_e + u \times \ell_e = 0. \tag{5.27}$$

Proof: see SIMO, POSBERGH & MARSDEN [1989]. ∎

Recall the first variation (5.20). If we set $u = 0$ (corresponding to a reduction to center of mass) then the computation of the second variation is straight forward. Thus, for $\delta^2 H_\xi(z_e)$ on TP at equilibrium we have

$$\begin{aligned}
\delta^2 H_\xi(z_e)\cdot(\delta z,\Delta z) &= \delta^2 V_\xi(q_e)\cdot(\delta q,\Delta q) + \delta^2 K_\xi(z_e)\cdot(\delta z,\Delta z)\\
&= (\delta\pi_0 + \pi_{0,e}\times\delta\theta_0)\cdot \mathbf{I}_{B,e}^{-1}(\Delta\pi_0 + \pi_{0,e}\times\Delta\theta_0)\\
&\quad + \delta\theta_0\cdot[\xi\times\Delta\pi_0 + \pi_{0,e}\times(\xi\times\Delta\theta_0)] - \delta\pi_0\cdot(\xi\times\Delta\theta_0)\\
&\quad - \delta\varphi_0\cdot\Delta p_0\times\xi + \delta p\cdot[M_B^{-1}\Delta p_0 - \xi\times\Delta\varphi_0]\\
&\quad + \int_0^L (\delta\pi + \pi_e\times\delta\theta)\cdot\mathbf{I}_e^{-1}(\Delta\pi + \pi_e\times\Delta\theta)\\
&\quad + \delta\theta\cdot[\xi\times\Delta\pi + \pi\times(\xi\times\Delta\theta)] - \delta\pi\cdot(\xi\times\Delta\theta)\\
&\quad - \delta\varphi\cdot\Delta p\times\xi + \delta p\cdot[\rho_A^{-1}\Delta p - \xi\times\Delta\varphi]\,dS + \delta^2 V(q_e)
\end{aligned} \tag{5.28}$$

Subsequently, we will set $\Delta z \in \mathcal{S}_{RIG}$. Elements in $\mathcal{S}_{RIG}$ are of the form $\Delta z = (\Delta q, \widetilde{\Delta p})$, where $\Delta q = (\Delta\varphi_0, \Delta\theta_0, \Delta\varphi, \Delta\theta)$ is given by

$$\left.\begin{aligned}
\Delta\varphi_0 &= \eta\times\varphi_0; &\quad \Delta\varphi &= \eta\times\varphi;\\
\Delta\theta_0 &= \eta, & \Delta\theta &= \eta,
\end{aligned}\right\} \tag{5.29a}$$

and $\widetilde{\Delta p} = (\Delta p_0, \Delta\pi_0, \Delta p, \Delta\pi)$ is given by

$$\left.\begin{aligned}
\Delta p_0 &= M_B\zeta\times\varphi + \eta\times p_0; &\quad \Delta p &= \rho_A\zeta\times\varphi + \eta\times p;\\
\Delta\pi_0 &= \mathbf{I}_B\zeta + \eta\times\pi_0, & \Delta\pi &= \mathbf{I}\zeta + \eta\times\pi,
\end{aligned}\right\} \tag{5.29b}$$

for $\eta, \zeta \in \mathbb{R}^3$

To perform the second variation test we will need the following result.

Lemma 5.4. *Let $\Delta q \in \mathcal{V}_{RIG}$, and let $q_e \in Q$ be a configuration in relative equilibrium. Then, in general $\delta^2 V(q_e) \neq 0$; in fact for $\delta^2 V(q_e) : \mathcal{V}_{RIG}\times TQ \to \mathbb{R}$ we find*

$$\begin{aligned}
\delta^2 V(q_e)(\Delta q,\delta q) &= (\xi\times p_{0,e})\times\delta\varphi_0\cdot\eta + (\xi\times\pi_{0,e})\times\delta\theta_0\cdot\eta\\
&\quad - \int_0^L \left[(\xi\times p_e)\times\delta\varphi\cdot\eta + (\xi\times\pi_e)\times\delta\theta\cdot\eta\right] dS.
\end{aligned} \tag{5.30}$$

Proof: Follows by a direct calculation involving integration by parts. For details see SIMO, POSBERGH & MARSDEN [1989]. ∎

§5E. The second variation of the energy-momentum functional

For $\delta^2 H_\xi(z_e) : TP \times \mathcal{S}_{RIG} \to \mathbb{R}$ from (5.28), (5.29) and (5.30) we have

$$
\begin{aligned}
\delta^2 H_\xi(z_e) = \zeta \cdot \Big\{ & \delta\pi_0 + \pi_{0,e} \times \delta\theta_0 + \varphi_{0,e} \times \delta p_0 + p_{0,e} \times \delta\varphi_0 \\
& + \mathrm{I}_B(\delta\theta_0 \times \xi) + \xi \times \mathrm{I}_B \delta\theta_0 \\
& + \int_0^L [\delta\pi + \pi_e \times \delta\theta + \varphi_e \times \delta p + p_e \times \delta\varphi \\
& + \mathrm{I}_e(\delta\theta \times \xi) + \xi \times \mathrm{I}_e \delta\theta] \, dS \Big\} \\
& - (\zeta \times \xi) \cdot \Big\{ \mathrm{I}_B \delta\theta_0 + M_B \varphi_{0,e} \times \delta\varphi_0 + \int_0^L [\mathrm{I}_e \delta\theta + \rho_A \varphi_e \times \delta\varphi] \, dS \Big\} \\
& + (\eta \times \xi) \cdot \Big\{ \delta\pi_0 + \delta\varphi_{0,e} \times p_{0,e} + \varphi_{0,e} \times \delta p_{0,e} \\
& + \int_0^L [\delta\pi + \delta\varphi \times p_e + \varphi_e \times \delta p] \, dS \Big\}
\end{aligned}
\tag{5.31}
$$

We can now state the following Corollary in the case of a geometrically exact rod attached to a rigid body.

Corollary 5.5. *At a relative equilibrium $z_e \in P$, for any $\Delta z \in \mathcal{S}_{RIG}$ and $\delta z \in \mathcal{S}_{INT}$ we have*

$$
\boxed{\delta^2 H_\xi(z_e)(\Delta z, \delta z) = 0.}
\tag{5.32}
$$

so that, restricted to $\mathcal{S} \subset T_{z_e} P$ the second variation of the energy-momentum functional can be written in the uncoupled form

$$
\boxed{\delta^2 H_\xi(z_e)\Big|_{\mathcal{S}} = \delta^2 H_\xi(z_e)\Big|_{\mathcal{S}_{RIG}} + \delta^2 H_\xi(z_e)\Big|_{\mathcal{S}_{INT}}.}
\tag{5.33}
$$

Proof: Equation (5.32) follows from expression (5.31) and the definition of $\mathcal{V}_{INT}$. On the other hand, (5.33) follows immediately from the bilinearity of $\delta^2 H_\xi : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$, and the equilibrium conditions. ■

Proposition 5.6. *Let $\Delta z \in \mathcal{S}_{RIG}$, and let $z_e \in P$ be a (relative) equilibrium configuration. Let the second variation be regarded as a function $\delta^2 H_\xi : \mathcal{S}_{RIG} \times \mathcal{S}_{RIG} \to \mathbb{R}$. Then, at an equilibrium, $\delta^2 H_\xi(q_e)$ on $\mathcal{S}_{RIG} \times \mathcal{S}_{RIG}$ can be written as the following quadratic form*

$$
\boxed{\delta^2 H_\xi(z_e) = (\mu_e \times \eta) \cdot [\mathbb{I}_\infty^{-1} - \lambda \mathbf{1}_3](\mu_e \times \eta)}
\tag{5.34}
$$

Proof: See SIMO, POSBERGH & MARSDEN [1989]. ■

We note that on S_{INT} the situation is the same as for the rod alone. This is because *all the flexibility* is assigned to the rod (*i.e.*, V is a depends only on Φ_e, the configuration of the rod). This would not be the case were the rod attached to the rigid body by an elastic hinge.

§5F. *Stability Conditions associated with $\mathcal{V}_{INT}$.*

We complete our stability analysis by deriving conditions which guarantee the definiteness of the component on $\mathcal{V}_{INT}$. According to our outline in §2B, the space $\mathcal{V} \subset T_{q_e}Q$ is given by

$$\mathcal{V} := \{\delta q \in T_{q_e}Q \mid \xi \cdot \int_0^L [\mathrm{I}_e \delta\theta + \rho_A \varphi_e \times \delta\varphi]\, dS = 0\,\}. \tag{5.35}$$

One can show that the evaluation of the Lie derivative condition in (2.26) yields the following explicit expression

$$\begin{aligned}\langle \delta q, \mathcal{L}_{\eta_Q(q_e)} \frac{\delta L_\xi}{\delta q}(q_e)\rangle_g &= \zeta \cdot \int_0^L \left[\pi_e \times \delta\theta + 2p_e \times \delta\varphi + \mathrm{I}_e(\delta\theta \times \xi) + \xi \times \mathrm{I}_e \delta\theta\right] dS \\ &\qquad - (\zeta \times \xi) \cdot \int_0^L [\mathrm{I}_e \delta\theta + \rho_A \varphi_e \times \delta\varphi]\, dS \\ &= 0 \end{aligned} \tag{5.36}$$

for all $\delta\Phi \in \mathcal{V}_{INT}$, and all $\zeta \in \mathbf{R}^3$ such that $\zeta \cdot \xi = 0$. This condition defines $\mathcal{V}_{INT} \subset \mathcal{V}$.

As a quadratic form, the second variation of the effective potential evaluated at an equilibrium configuration can now be written as

$$\delta^2 V_\xi \Big|_e = \delta^2 V(\Phi_e) - \int_0^L \left\{\rho_A \|\xi \times \delta\varphi\|^2 + \delta\theta \cdot \hat{\xi}(\hat{\pi}_e - \mathrm{I}_e \hat{\xi}_e)\delta\theta\right\} dS \tag{5.37}$$

where

$$\delta^2 V(\Phi_e) = \int_0^L \mathsf{A}(\Phi_e)(\delta\Phi, \delta\Phi)\, dS \tag{5.38}$$

and $\mathsf{A}(\Phi_e)(\delta\Phi, \delta\Phi)$ is referred to as the *total elasticity tensor.*

To bound (5.37) from below we introduce the following auxiliary eigenvalue problem.

Eigenvalue Problem. Let $\delta^2 V(\Phi_e)$ be as above. Then, from (5.38) we have

$$\boxed{\delta^2 V(\Phi_e)(\delta\Phi, \delta\Phi) = \langle \delta\Phi, \mathsf{A}(\Phi_e)\delta\Phi\rangle} \tag{5.39}$$

Let $M := \begin{bmatrix} \rho_A 1_3 & 0 \\ 0 & I \end{bmatrix}$ be the matrix form of the kinetic energy Riemannian metric. Consider the symmetric variational eigenvalue problem:

Find $(\lambda, \eta) \in \mathbf{R} \times \mathcal{V}_{INT}$ such that, for all $\delta\Phi \in \mathcal{V}_{INT}$;

$$\boxed{\langle \delta\Phi, \mathsf{A}(\Phi_e)\eta \rangle = \lambda \langle \delta\Phi, M\eta \rangle} \tag{5.40}$$

We note the following:

1. Suppose that Φ_e is the *reference configuration,* so that the geometric term $G(\Phi_e) \equiv 0$. Then, since $\ker[\Xi(\Phi_e)] = \mathcal{V}_{RIG}$ so that $\eta \notin \mathcal{V}_{RIG}$ by construction, it follows that (5.40) has *positive eigenvalues* since M is always positive definite and $C(\Phi)|_{\Phi=\text{identity}}$ is positive definite. The solutions of (5.40) are, in fact, the *natural frequencies* of the *system* at configuration Φ_e.

2. In general, of course, $\mathsf{A}(\Phi_e)$ *cannot* be *positive definite* for *all* Φ_e (restricted to $\mathcal{V}_{INT}$) since this would imply *convexity* of the stored energy function and hence uniqueness; clearly, an unacceptable restriction. See MARSDEN & HUGHES [1983, Chapter 6], OGDEN [1984] and CIARLET [1988] for a detailed discussion of some appropriate restrictions on the stored energy functions.

3. If *$A(\Phi_e)$ is indeed positive definite at $\Phi_e \in C$*, then (5.40) has *positive* eigenvalues. The *lowest* one is given by the Rayleigh quotient

$$\boxed{\lambda_0(\Phi_e) = \inf_{\delta\Phi \in \mathcal{V}_{INT}} \frac{\langle \delta\Phi, \mathsf{A}(\Phi_e)\delta\Phi \rangle}{\langle \delta\Phi, M\delta\Phi \rangle},} \tag{5.41}$$

so that, trivially, one has the inequality

$$\langle \delta\Phi, \mathsf{A}(\Phi_e)\delta\Phi \rangle \quad \geq \quad \lambda_0(\Phi_e)\langle \delta\Phi, M\delta\Phi \rangle > 0, \tag{5.42}$$

for all $\delta\Phi \in \mathcal{V}_{INT}$.

With these observations in mind we have

Theorem 5.7. *Let the relative equilibrium $\Phi_e \in C$ be such that $\mathsf{A}(\Phi_e)$, as defined by (5.38) is positive definite, and let $\lambda_0(\Phi_e)$ be the lowest eigenvalue of problem (5.40) as given by (5.41), so that (5.42) holds. Then, stability on $\mathcal{V}_{INT}$ requires that*

$$\boxed{\delta^2 H_\xi \Big|_e \geq \lambda_0(\Phi_e)\langle \delta\Phi, M\delta\Phi \rangle - \int_0^L \left\{ \rho_A \|\xi \times \delta\varphi\|^2 + \delta\theta \cdot \hat{\xi}[\hat{\pi}_e - \mathrm{I}_e \hat{\xi}_e]\delta\theta \right\} dS.} \tag{5.43}$$

Remark. A sharp estimate of the constant $\lambda_0(\Phi_e) > 0$ in (5.42) can be obtained by solving explicitly the local form of the eigenvalue problem (5.40). Integration by parts of (5.40) and use of standard results in the calculus of variations yields the second order linear eigenvalue problem

$$\Xi^*(\Phi_e)[C(\Phi_e)\Xi(\Phi_e)\eta] = \lambda M\eta \quad \text{in} \quad [0, L], \tag{5.44}$$

along with the (linearized) boundary conditions

$$\Xi(\Phi_e)\eta\Big|_{S=0,L} = 0, \tag{5.45}$$

and the constraint conditions

$$\int_O^L M\delta\Phi\, dS = 0. \tag{5.46}$$

Here, $\Xi^*(\Phi_e)$ is given by the expression

$$(\delta n, \delta m) \mapsto \Xi^*(\Phi_e)(\delta n, \delta m) = \begin{Bmatrix} -\delta n' \\ -\delta m' - \varphi_e' \times \delta n \end{Bmatrix}. \tag{5.47}$$

At a straight reference configuration $\Phi_e = (\varphi_0, 1)$ where $\varphi_0 = Se_3$, for $S \in [0, L]$, and $e_3 = (0, 0, 1)$, equation (5.42) reduces to the classical problem for a Timoshenko beam model if $C(\Phi_e)$ is assumed to be constant and diagonal form of the

$$C(\Phi_e) = \text{diag}[GA_1, GA_2, EA, EI_1, EI_2, GI]. \tag{5.48}$$

Explicitly, setting $\eta = (\eta_1, \eta_2, \eta_3, \theta_1, \theta_2, \theta_3)$, we have

$$-\begin{Bmatrix} GA_1(\eta_1'' - \theta_2') \\ GA_2(\eta_2'' + \theta_1') \\ EA\theta_3'' \\ EI_1\theta_1'' - GA_2(\eta_2' + \theta_1) \\ EI_2\theta_2'' + GA_1(\eta_1' - \theta_2) \\ GJ\theta_3'' \end{Bmatrix} = \lambda_0 \begin{Bmatrix} \rho_A\eta_1 \\ \rho_A\eta_2 \\ \rho_A\eta_3 \\ I_1\theta_1 \\ I_2\theta_2 \\ J\theta_3 \end{Bmatrix}, \tag{5.49}$$

along with the stress free boundary conditions which now take the form

$$\left.\begin{aligned} [\eta_1' - \theta_2]\Big|_{S=0,L} &= [\eta_2' + \theta_1]\Big|_{S=0,L} = \eta_3'\Big|_{S=0,L} = 0 \\ \theta_1'\Big|_{S=0,L} &= \theta_2'\Big|_{S=0,L} = \theta_3'\Big|_{S=0,L} = 0 \end{aligned}\right\} \tag{5.50}$$

and the constraint condition (5.28) which *eliminates* rigid body modes. An explicit solution to this problem can be found in elementary text books, *i.e.* GRAFF [1975].

§6. Conclusions

In this paper we used the energy-momentum method to investigate the stability of the relative equilibria of a uniformly rotating rigid body with an attached flexible appendage. A fundamental decomposition of the space of admissible variations was employed, which decoupled the problem into a 'rigid body' stability problem and an 'internal vibration' problem. The stability conditions on the first of two subspaces, denoted by $\mathcal{S}_{RIG}$, corresponded to that of a classical rigid body. On the complement to this space, denoted by $\mathcal{S}_{INT}$, the stability conditions corresponded to requiring the rate of rotation to be below that of the lowest frequency of excitation of an associated eigenvalue.

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