# EXPONENTIALLY SMALL ESTIMATES FOR SEPARATRIX SPLITTINGS


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**ABSTRACT** This paper reviews our previous estimates and gives an example exhibiting a new phenomenon. In problems involving asymptotics beyond all orders in a perturbation parameter $\varepsilon$, it is a common *assumption* that the quantity being studied (such as a separatrix splitting distance or angle, a solitary wave mismatch, etc.) can be "estimated" by an expression of the form $a\varepsilon^b e^{-c/\varepsilon}$ as $\varepsilon \to 0$. Here, $a, b$ and $c$ are constants (where $b$ can be negative and $c$ is "sharp", often the distance from the real axis to a pole in the complex plane). The main purpose of our example is to show that this assumption can be wrong. The example, which concerns the splitting of separatrices in a rapidly forced system with a heteroclinic orbit shows that even the estimate from above (using the sharp value of $c$) can be incorrect. We argue that this situation is not isolated or particular, but happens rather generally. We especially note that in situations involving asymptotics beyond all orders, when an estimate of the form $a\varepsilon^b e^{-c/\varepsilon}$ is assumed, it needs to be justified.

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# 1 INTRODUCTION

In HOLMES , MARSDEN, and SCHEURLE [1988] and SCHEURLE [1989], both upper and lower exponentially small estimates for separatrix splitting in rapidly forced planar systems were obtained. Although the results are rather general for analytic reversible planar systems, we recall them for the pendulum:

**UPPER ESTIMATE** *Consider*

$$\ddot{\varphi} + \sin\varphi \;=\; \delta \sin(t/\varepsilon). \tag{1}$$

*For any* $\eta > 0$ *there is a* $\delta_0 > 0$ *and a constant* $C(\eta, \delta_0)$ *such that for all* $\varepsilon$ *and* $\delta$ *satisfying* $0 < \varepsilon \leq 1, 0 < \delta \leq \delta_0$,*we have*

$$|\text{separatrix splitting}| \;\leq\; C(\eta, \delta_0) e^{-(\frac{\pi}{2}-\eta)/\varepsilon} \tag{2}$$

**LOWER ESTIMATE AND SHARP UPPER ESTIMATE** *Consider*

$$\ddot{\varphi} + \sin\varphi \;=\; \varepsilon^p \delta \sin(t/\varepsilon) \tag{3}$$

*and assume that* $p > 8$. *Then there is a* $\delta_0 > 0$ *and constants* $C_1$ *and* $C_2$ *such that for all* $\varepsilon, \delta$ *satisfying* $0 < \varepsilon \leq 1, 0 < \delta \leq \delta_0$, *we have*

$$C_2 \varepsilon^p \delta e^{-\pi/2\varepsilon} \;\leq\; |\text{separatrix splitting}| \;\leq\; C_1 \varepsilon^p \delta e^{-\pi/2\varepsilon} \tag{4}$$

## Remarks

1. Estimates similar to our upper estimates were obtained by NIESHTADT [1984]. The coefficient in the exponent for our estimate, namely $\pi/2$, is the distance from the real axis to the closest pole of the homoclinic orbit in the complex $t$-plane. For NIESHTADT, the exponent is related to the width of a strip in the complex plane on which the angle variable in action-angle coordinates is analytic. The exact relation between these two approaches would be interesting to explore further. NIESHTADT also makes the interesting remark that a variant of KAM theory can be used to bound the whole stochastic layer between exponentially close KAM curves.

2. There are similar upper estimates for *mappings* in FONTICH and SIMO [1990].

3. The upper estimate (1.2) shows that the splitting distance is *beyond all orders in* $\varepsilon$ (without any assumption on $p$, as in the second result). An analyticity argument shows that in (1.1), splitting *does* occur (with at most discrete exceptions) as $\varepsilon \to 0$, but it does not provide an estimate.

4. There are similar estimates for the splitting angle.

5. Based on the example below, we conjecture that for situations of most interest in KAM theory, perhaps even for $\ddot{\varphi} + \sin\varphi = \delta\varepsilon\sin(t/\varepsilon)$, there *is no estimate of the form*

$$C_2 \varepsilon^b \delta e^{-c/\varepsilon} \;\leq\; |\text{splitting distance}| \;\leq\; C_1 \varepsilon^b \delta e^{-c/\varepsilon}$$

with a sharp constant $c/\varepsilon$ (like $\pi/2$ for the pendulum) for any constant $b$ (positive or negative), for all small $\delta$ and $\varepsilon$. In fact, in problems like this, it would seem that *one cannot avoid essential singularities that develop in the estimate*. We will see this explicitly in the example.

6. On the other hand, what we *do* believe occurs in examples like the $p = 1$ pendulum and related maps like the ones in FONTICH and SIMO [1990], is a convergent expression of the form

$$|\text{splitting distance}| = \left(\cdots + a_2\varepsilon^2 + a_1\varepsilon + a_0 + \frac{a_{-1}}{\varepsilon} + \frac{a_{-2}}{\varepsilon^2} + \cdots\right) e^{-c/\varepsilon}$$

for some constants $a_i$. Which power of $\varepsilon$ to put in front of $e^{-c/\varepsilon}$ depends on how small $\varepsilon$ is. Papers giving numerical evidence for a *fixed* power of $\varepsilon$ as $\varepsilon \to 0$ need to confront the example in this paper in which there is *no upper estimate of the form* $a\varepsilon^b e^{-\pi/2\varepsilon}$ as $\varepsilon \to 0$. However, it *suggests* that $a\varepsilon^b e^{-\pi/2\varepsilon}$ may give a good *approximation*, but that $b$ may need to be adjusted as $\varepsilon \to 0$. Despite this extreme delicacy with the sharp exponent $-\pi/2\varepsilon$, the upper estimate, with $\pi/2$ replaced by a slightly smaller exponent, remains valid.

# 2 THE EXAMPLE

We consider the following family of planar systems.

$$\begin{aligned} \dot{x} &= 1 - x^2 \\ \dot{y} &= [2x - (\alpha + 2\beta x)(1 - x^2)]y + \delta\cos(t/\varepsilon) \end{aligned} \tag{5}$$

where $\alpha, \beta, \delta, \varepsilon$ are constants. For $\delta = 0$, this system has the heteroclinic orbit

$$\Gamma : \quad x = \tanh t, \quad y = 0$$

joining $(-1, 0)$ to $(1, 0)$. We are interested in the splitting of this orbit for small $\alpha, \beta, \delta, \varepsilon$ and $\beta, \delta, \varepsilon$ non-zero. Before proceeding with the example, we make a series of remarks.

### Remarks

1. The system (2.1) is chosen so the *variables separate* (the first equation is independent of $y$). This enables one to perform explicit and *exact* calculations, but does not seem to be essential for the phenomenon we want to illustrate.

2. Many systems of interest in the splitting of separatrices are Hamiltonian (see HOLMES , MARSDEN, and SCHEURLE [1988] for instance), so we make some remarks on this structure. First, consider the Hamiltonian

$$H(q, p) = \frac{1}{2m}p^2 + V(q) + pf(q) \tag{6}$$

where $V$ is a potential and $pf(q)$ is a *gyroscopic term*. The variables separate in the limit $m \to \infty$ and Hamilton's equations become

$$\begin{aligned} \dot{q} &= f(q) \\ \dot{p} &= -V'(q) - pf'(q) \end{aligned} \tag{7}$$

For $f(q) = 1-q^2$, (2.3) has a heteroclinic orbit joining $(-1, -\frac{1}{2}V'(-1))$ to $(1, \frac{1}{2}V'(1))$. With the addition of $\delta\cos(t/\varepsilon)$ to the $p$-equation, one can readily compute the splitting and one finds that it is given *exactly* by

$$\text{splitting} \;=\; \int_{-\infty}^{\infty} \frac{\delta e^{it/\varepsilon}}{\cosh^2 t}\,dt \;=\; \frac{\delta\pi}{\varepsilon\sinh(\pi/2\varepsilon)} \tag{8}$$

and so the system (2.3) is too simple to illustrate what we want. That is, one *does* have, for this especially simple situation, a valid prefactor $\varepsilon^b$. Note that if $V = 0$ and $f(q) = 1 - q^2$, then (2.3) reduces to (2.1) with $\alpha = 0, \beta = 0$. However, (2.1) exhibits the behavior we want to illustrate only for $\alpha$ or $\beta$ non-zero. If in (2.2), $m$ is finite, it seems difficult to calculate the splitting directly; however, we suspect the same phenomena can happen as in the nearby system (2.1)—that is, we suspect that there is no prefactor $\varepsilon^b$ one can use (along with the sharp exponential factor $e^{-\pi/2}$) to get an estimate.

3. The equations (2.1) are Hamiltonian when $\delta = 0$, with Hamiltonian function

$$H \;=\; e^{\alpha x+\beta x^2}[(1-x^2)y]$$

and symplectic form

$$\Omega \;=\; e^{\alpha x+\beta x^2}dx\wedge dy$$

as is readily checked. For $\delta \neq 0$ the equations are still Hamiltonian in the non-autonomous sense.

4. A formal calculation of the Melnikov function for (2.1) gives the function

$$M_\varepsilon(t_0) = \delta\int_{-\infty}^{\infty}(1-x^2)\cos((t+t_0)/\varepsilon)dt = \frac{\delta\pi\cos(t_0/\varepsilon)}{\varepsilon\sinh(\pi/2\varepsilon)}, \tag{9}$$

whose magnitude coincides with the exact result (2.4). This occurs because the evolution equation along the heteroclinic orbit for $\alpha = \beta = 0$ is linear, so that the iteration procedure of Holmes, Marsden, and Scheurle [1988] terminates after the first term. Note, however, that the formal result (2.5) is obtained for *all* $\alpha$ and $\beta$. As we shall see, the $\varepsilon$ dependence of this "leading" term is in general incorrect for $\beta = \varepsilon^p$.

5. Essentially the same equation (2.1) with $\alpha = \beta = 0$ with forcing, but not rapid forcing, was presented by Bountis, Papageorgiou, and Bier [1987] as an example which is integrable in the sense that it is separable, so can be explicitly integrated, but which nonetheless exhibits separatrix splitting.

6. A key point that leads to the invalidity of the assumption of a prefactor of the form $\varepsilon^b$ for (2.1) is the *essential singularity* in the resulting formula for the splitting distance in (3.7) below. It should be noted that we did not put in this essential singularity by hand–it arises naturally even though there are no obvious essential singularities in the given problem. From the proofs of the splitting estimates, one sees that one should expect this most of the time, even in simple problems.

## 3 THE SPLITTING DISTANCE FORMULA

An interesting feature of the equations (2.1) is that there is a relatively explicit formula for the exact splitting distance. From $\dot{x} = 1 - x^2$, we find that the first component of the stable and unstable manifolds near $\Gamma$ are given by $x = \tanh t$. Substituting in the second equation, we get

$$\begin{aligned} \dot{y} &= [2x - (\alpha + 2\beta x)(1 - x^2)]y + \delta \cos(t/\varepsilon) \\ &= [2 \tanh t - (\alpha + 2\beta \tanh t)(\mathrm{sech}^2 t))]y + \delta \cos(t/\varepsilon) \end{aligned} \tag{10}$$

Let $u = y/\cosh^2 t$ so that

$$\dot{u} = \frac{\dot{y}}{\cosh^2 t} - \frac{2y \sinh t}{\cosh^3 t} = \frac{\dot{y} - 2y \tanh t}{\cosh^2 t}$$

i.e.,

$$\dot{u} = -\frac{(\alpha + 2\beta \tanh t)}{\cosh^2 t} u + \frac{\delta \cos(t/\varepsilon)}{\cosh^2 t} \tag{11}$$

Since an indefinite integral of $\alpha \mathrm{sech}^2 t + 2\beta \mathrm{sech}^2 t \tanh t$ is $\alpha \tanh t - \beta \mathrm{sech}^2 t$, we get

$$u(t) = e^{-\alpha \tanh t + \beta \mathrm{sech}^2 t} \left\{ e^{-\beta} u(0) + \int_0^t e^{\alpha \tanh s - \beta \mathrm{sech}^2 s} \frac{\delta \cos(s/\varepsilon)}{\cosh^2 s} ds \right\}$$

i.e.,

$$y(t) = \cosh^2 t \, e^{-\alpha \tanh t + \beta \mathrm{sech}^2 t} \left\{ e^{-\beta} y(0) + \int_0^t e^{\alpha \tanh s - \beta \mathrm{sech}^2 s} \frac{\delta \cos(s/\varepsilon)}{\cosh^2 s} ds \right\} \tag{12}$$

The unstable manifold of the periodic point near $(-1, 0)$ starting at $(0, y(0))$ at $t = 0$ is characterized by choosing $y(0) = y_u(0)$ so that $y(t)$ is bounded as $t \to -\infty$. Then (3.3) gives

$$y_u(0) = \delta e^{\beta} \int_{-\infty}^0 e^{\alpha \tanh s - \beta \mathrm{sech}^2 s} \frac{\cos(s/\varepsilon)}{\cosh^2 s} ds \tag{13}$$

Subtracting an analogous formula for $y_s(0)$ gives

$$y_u(0) - y_s(0) = \delta e^{\beta} \int_{-\infty}^{\infty} e^{\alpha \tanh s - \beta \mathrm{sech}^2 s} \frac{\cos(s/\varepsilon)}{\cosh^2 s} ds \tag{14}$$

The formula with a starting time $t_0$ and position $(0, y(t_0))$ similarly gives

$$y_u(t_0) - y_s(t_0) = \delta e^{\beta} \int_{-\infty}^{\infty} e^{\alpha \tanh s - \beta \mathrm{sech}^2 s} \frac{\cos(s + t_0)/\varepsilon}{\cosh^2 s} ds \tag{15}$$

Thus, the *splitting distance* is the absolute value

$$d = \left| \delta e^{\beta} \int_{-\infty}^{\infty} e^{\alpha \tanh s - \beta \mathrm{sech}^2 s} \frac{e^{is/\varepsilon}}{\cosh^2 s} ds \right| \tag{16}$$

The integrand has an essential singularity at $s = i\pi/2$. If we shift the contour to $s = i\pi + w$, we get

$$\delta e^{\beta} \left| \int_{i\pi-\infty}^{i\pi+\infty} e^{\alpha \tanh w - \beta \mathrm{sech}^2 w} \frac{e^{iw/\varepsilon}}{\cosh^2 w} dw \right| e^{-\pi/\varepsilon}$$

Thus,

$$d = \left| \frac{\delta e^{\beta}}{1 - e^{-\pi/\varepsilon}} \oint_{C'} e^{\alpha \tanh w - \beta \mathrm{sech}^2 w} \frac{e^{iw/\varepsilon}}{\cosh^2 w} dw \right|$$

where the integration contour $C'$ encloses the point $i\pi/2$. Now make the change of variables $z = w - i\pi/2$ and use $\cosh(z + i\pi/2) = i \sinh z$ to get

$$d = \left| \frac{\delta e^{\beta}}{1 - e^{-\pi/\varepsilon}} \oint e^{\alpha \coth z + \beta / \sinh^2 z} \frac{e^{iz/\varepsilon}}{\sinh^2 z} dz \right| e^{-\pi/2\varepsilon} \tag{17}$$

where the integration contour encloses the origin (and none of the other singularities at $(2n + 1)\pi i/2$). Formulas (3.7) and (3.8) are the splitting distance formulas we shall work with.

The main problem we now pose is this: If $\beta = -\varepsilon$ and $\alpha = 0$, can one estimate (3.8) above by an expression of the form $a\varepsilon^b e^{-\pi/2\varepsilon}$ for constants $a$ and $b$ (not depending on $\varepsilon$) as $\varepsilon \to 0$ ? As we shall see in the next section, the answer is NO.

## Remarks

1. The main difficulty with (3.8) is the presence of the essential singularity of the function

$$g(z) = e^{\alpha \coth z + \beta / \sinh^2 z} \frac{e^{iz/\varepsilon}}{\sinh^2 z} \tag{18}$$

at $z = 0$. One can ask if essential singularities occur typically or are a peculiarity of this example. For the pendulum example, one sees from SCHEURLE [1989] that there is an uncontrollable accumulation of poles as the iteration procedure is carried out. This accumulation is the counterpart of the essential singularity of $g$, so one can expect a similar consequence. A similar phenomenon seems to occur in the bifurcation example in HOLMES, MARSDEN and SCHEURLE [1988].

2. If $\beta = C\varepsilon^p$ where $p \geq 3$, then the essential singularity can be controlled, as in the pendulum case, and in this case the upper and lower estimates are valid.

3. Equations (2.1) can be modified to

$$\begin{aligned} \dot{x} &= 1 - x^2 \\ \dot{y} &= [2x - (\alpha + 2\beta x)(1 - x^2)]y - \mu[2x - (\alpha + 2\beta x)(1 - x^2) \sin(t/\varepsilon) \\ &\quad + \left(\delta + \frac{\mu}{\varepsilon}\right) \cos(t/\varepsilon), \end{aligned} \tag{19}$$

which may be regarded as a perturbation of (2.1) with the new parameter $\mu$. However, for $\delta = 0$ and all $\alpha, \beta$ and $\mu$, this system is integrable, since it is transformed back to the autonomous Hamiltonian system (2.1) with $\delta = 0$ under the change of variables

$$x = \tilde{x}, \qquad y = \tilde{y} + \mu \sin(t/\varepsilon),$$

and so the splitting is zero. In this case, there is still an essential singularity in the splitting distance formula, but there is in fact no splitting. Examples like this show how delicate the splitting condition can be!

# 4 EXPONENTIALLY SMALL ESTIMATES

We now turn to estimates on $d$ given by (3.7) and (3.8). We shall focus on the region in parameter space where $\alpha = 0, \beta = -\varepsilon$ and $\varepsilon > 0$. However, there are many possible ways of expanding the integral, so we leave $\beta$ independent of $\varepsilon$ for the moment.

Let

$$D = \int_{-\infty}^{\infty} e^{-\beta/\cosh^2 s} \frac{e^{is\varepsilon}}{\cosh^2 s} ds \tag{20}$$

As in (3.8), we get

$$D = \frac{-1}{2\sinh(\pi/2\varepsilon)} \oint e^{\beta/\sinh^2 z} \frac{e^{iz/\varepsilon}}{\sinh^2 z} dz \tag{21}$$

Thus, with $\alpha = 0$, we get $d = \delta e^{\beta} D$. Now we expand the first exponential in (4.1), giving

$$D = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{(-\beta)^n}{n!} \frac{e^{is/\varepsilon}}{\cosh^{2n+2} s} ds \tag{22}$$

Observe that the (inverse) Fourier transform of $1/\cosh^2 s$ gives

$$\int_{-\infty}^{\infty} \frac{e^{is\eta}}{\cosh^2 s} ds = \frac{\eta}{\sinh(\pi\eta/2)} \geq 0 \tag{23}$$

With $\eta = 1/\varepsilon$ in (4.4), we see that each term in the expansion (4.3) is a multiple convolution of the *non-negative* function $\eta/\sinh(\pi\eta/2)$ and therefore it is clear that these terms are postive. Therefore we get an inequality if we discard the tail of the series. Thus, we have proved the following:

**Lemma 1** *If $\beta < 0$ and $N$ is a positive integer, then*

$$D \geq \sum_{n=0}^{N} \frac{(-\beta)^n}{n!} \int_{-\infty}^{\infty} \frac{e^{is/\varepsilon}}{\cosh^{2n+2} s} ds \tag{24}$$

Next, rewrite (4.5), as we did in (4.2), to give

$$D \geq \frac{1}{2\sinh(\pi/2\varepsilon)} \sum_{n=0}^{N} \frac{(-\beta)^n}{n! i^{2n+2}} \oint \frac{e^{iz/\varepsilon}}{\sinh^{2n+2} z} dz. \tag{25}$$

Making the change of variables $\zeta = z/\varepsilon$, we get

$$\begin{aligned} D &\geq \frac{-\varepsilon}{2\sinh(\pi/2\varepsilon)} \sum_{n=0}^{N} \frac{\beta^n}{n!} \oint \frac{e^{i\zeta}}{\sinh^{2n+2}(\varepsilon\zeta)} d\zeta \\ &= \frac{-\varepsilon}{2\sinh(\pi/2\varepsilon)} \sum_{n=0}^{N} \frac{\beta^n}{n!} \oint \frac{1}{(\varepsilon\zeta)^{2n+2}}\, h_n(\varepsilon\zeta) e^{i\zeta} d\zeta \end{aligned} \tag{26}$$

where $h_n(w) = w^{2n+2}/\sinh^{2n+2} w$, which is analytic near zero. Therefore, by Cauchy's theorem for the derivatives, we get

$$D \geq \frac{-1}{2\sinh(\pi/2\varepsilon)} \sum_{n=0}^{N} \frac{\beta^n}{n!} \frac{2\pi i}{(2n+1)!} \frac{1}{\varepsilon^{2n+1}} \frac{d^{2n+1}}{d\zeta^{2n+1}} [h_n(\varepsilon\zeta) e^{i\zeta}]_{|\zeta=0}. \tag{27}$$

The power series expansion of $h_n(w)$ has the form

$$h_n(w) = 1 + a_{n,2} w^2 + a_{n,4} w^4 + \cdots$$

for constants $a_{n,m}$, and so

$$h_n(\varepsilon\zeta) e^{i\zeta} = \left(1 + a_{n,2}\varepsilon^2\zeta^2 + a_{n,4}\varepsilon^4\zeta^4 + \cdots\right)\left(1 + i\zeta + \frac{(i\zeta)^2}{2!} + \cdots\right)$$

The coefficient of $\zeta^{2n+1}$ has the form

$$\frac{i}{(2n+1)!}\left[(-1)^n + \varepsilon^2 b_{n,2} + \varepsilon^4 b_{n,4} + \cdots + \varepsilon^{2n} b_{n,2n}\right]$$

for real constants $b_{n,2}, \ldots, b_{n,2n}$. Thus, (4.8) gives

$$D \geq \frac{-\pi}{\sinh(\pi/2\varepsilon)} \sum_{n=0}^{N} \frac{\beta^n}{n!} \frac{-1}{(2n+1)!\varepsilon^{2n+1}} \left[(-1)^n + \varepsilon^2 b_{n,2} + \cdots + \varepsilon^{2n} b_{n,2n}\right] \tag{28}$$

For $\varepsilon$ small, and since this is a *finite* sum, the expression (4.9) is dominated by the term from $n = N$; i.e., we have proved that

**Lemma 2** *For $\varepsilon$ sufficiently small and $\beta < 0$, we have*

$$D \geq \frac{1}{2} \frac{\pi}{\sinh(\pi/2\varepsilon)} \frac{|\beta|^N}{N!(2N+1)!} \frac{1}{\varepsilon^{2N+1}} \tag{29}$$

If $|\beta| = \varepsilon^p$, then as in our main lower estimate, for $p \geq 3$, this lower estimate is consistent with a splitting of the form $a\varepsilon^b e^{-\pi/2\varepsilon}$. However, with $\beta = -\varepsilon$, we get

$$D \geq \frac{\pi}{2\sinh(\pi/2\varepsilon)} \frac{1}{N!(2N+1)!} \frac{1}{\varepsilon^{N+1}} \geq \frac{C_N}{\varepsilon^{N+1}} e^{-\pi/2\varepsilon} \tag{30}$$

Putting this all together, we have established our main result:

**Theorem 1** *For any integer $N$, there are constants $c_N > 0$ and $\varepsilon_N > 0$ such that for $\alpha = 0, \beta = -\varepsilon$ and $\delta = \varepsilon$ , we have*

$$d \geq \frac{c_N}{\varepsilon^N} e^{-\pi/2\varepsilon} \tag{31}$$

*for all* $0 < \varepsilon < \varepsilon_N$

## Remarks

1. This result shows, that *no sharp upper estimate of the form* $a\varepsilon^b e^{-\pi/2\varepsilon}$ *can exist*. We emphasize that our "rough" upper estimate still applies, giving $d \leq C_\eta e^{-(\frac{\pi}{2}-\eta)/\varepsilon}$ uniformly as $\varepsilon \to 0$ for any $\eta > 0$, which is possible, despite (4.12).

2. The above calculations show that with $\beta = -\varepsilon, \delta = \varepsilon, \alpha = 0$ and $\varepsilon > 0$, the splitting distance has the form

$$d = \left( \cdots + a_2\varepsilon^2 + a_1\varepsilon + a_0 + \frac{a_{-1}}{\varepsilon} + \frac{a_{-2}}{\varepsilon^2} + \cdots \right) e^{-\pi/2\varepsilon} \tag{32}$$

where the Laurent series is *convergent* and infinitely many of $a_{-1}, a_{-2} \ldots$ are non-zero. In other words,

$$d = \varphi(\varepsilon) e^{-\pi/2\varepsilon} \tag{33}$$

where $\varphi$ has an *essential singularity* in $\varepsilon$ at $\varepsilon = 0$. Of course in particular examples, like this one, one can compute, in principle, the coefficients $a_k$. We suspect the asymptotic form (4.13), (4.14) is valid rather generally and that $\varphi$ will often have an essential singularity. Moreover, from (4.11) it is reasonable to suspect that the coefficients $a_k$ get small quickly for large $|k|$, so for $\varepsilon$ not *too* small, a truncation of (4.13) may yield a useful numerical approximation.

## CONCLUSIONS

In this paper we have given an explicit example of a system in the plane of the form $\dot{u} = f_0(u) + \varepsilon f(u, t/\varepsilon)$, where $\dot{u} = f_0(u)$ has a heteroclinic connection, with pole at $i\pi/2$ in the complex $t$-plane, and there is *no* upper estimate of the form $a\varepsilon^b e^{-\pi/2\varepsilon}$ for *any* constants $a, b$ uniformly as $\varepsilon \to 0$ for the separatrix splitting.

This example illustrates phenomena that we believe are generic, and not isolated. In particular, it shows that the assumption that the splitting is obtained by (*i.e.,* estimated above and below by) $a\varepsilon^b e^{-c/\varepsilon}$, where $a, b$ and $c$ are constants and the latter is sharp, is not, in general, correct; in fact, not even the upper estimate is correct. (Note that for reliable numerical work, one would ideally like error bounds corresponding to an estimate both above and below). Rather, it seems to us that the correct splitting is given by a prefactor that is an infinite expansion in powers of $\varepsilon$, both positive and negative. Certain ranges of powers may be useful for numerically estimating the splitting over corresponding ranges of $\varepsilon$, and this would be of interest to explore further.

The source of the basic difficulty that the example illustrates is the essential singularity, which builds up in the iteration process used to give the exact splitting distance; this essential singularity will typically be present in examples, even though the original problem may have only poles in the complex $t$ plane and there is no obvious essential singularity in the given data. In this example, one is able to see the essential singularity explicitly in the exact formula (3.7). The essential singularity corresponds to one that the proof suggests will build up through a successive iteration process in most examples. There may be particular mechanisms that can be used to control the essential singularity, and this is the purpose of the powers $\varepsilon^p$ in equation (1.3). This type of phenomenon also holds in our main example (2.1), with $\beta = C\varepsilon^p$, where the power of $p$ needed is given by inspection in this case and in general by a proof analysis of what is required to control the growth of the pole order in the iteration proccess. Without such special assumptions, one should expect the type of behaviour in our example.

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## REFERENCES

T. Bountis, V. Papageorgiou, and M. Bier [1987] On the singularity analysis of intersecting separatrices in near-integrable dynamical systems *Physica D* **24**, 292–304

P. Holmes, J. Marsden and J. Scheurle [1988] Exponentially small splittings of separatrices with applications to KAM theory and degenerate bifurcations, *Cont. Math., AMS* **81**, 213–244

E. Fontich and C. Simo [1990] The splitting of separatrices for analytic diffeomorphisms, *Ergodic Th. and Dyn. Sys.* **10**, 295–318

A. I. Nieshtadt [1984] The separation of motions in systems with rapidly rotating phase, *PMM USSR* **48**, 133–139

J. Scheurle [1989] Chaos in a rapidly forced pendulum equation, *Cont. Math. AMS* **97**, 411–419