Asymptotic Stability for Equilibria of Nonlinear Semiflows with Applications to Rotating Viscoelastic Rods, Part I

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1 Introduction

This paper establishes abstract results, which extend those of Potier–Ferry and Sobolevskii, on global existence and stability of solutions to quasilinear equations near an equilibrium point whose spectrum lies in the strict left half plane. The result may be regarded as a version of the linearization

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principle for quasilinear systems in a context where the main difficulty is to show that near the equilibrium shocks are suppressed by small damping. In the second part to this work, applications are be made to the dynamics of rods undergoing uniform rotation and satisfying the formal stability criteria based on the energy-momentum method of Simo, Posbergh, and Marsden.

The stability of relative equilibria of dissipationless geometrically exact rods moving in space was analyzed by Simo, Posbergh, and Marsden [1990]. Applying the energy-momentum method, they obtained sufficient conditions for the formal stability of these relative equilibria. For these partial differential equations the theory only gives conditional stability since basic existence and uniqueness questions remain a difficulty due to the quasilinear nature of the equations and the associated problem of shock formation.

In this paper we prove that in the presence of dissipation (viscoelastic dissipation, for instance), formal stability also ensures the global existence of smooth solutions and nonlinear asymptotic dynamical stability for relative equilibria of geometrically exact rods (shells, etc.) moving in space. Since the system is free to rotate, the stability results are modulo appropriate rotations.

Early work in this direction was done by Browne [1978], who considered the problem of existence, uniqueness and stability for the quasilinear partial differential equations governing the motion of nonlinearly viscoelastic one-dimensional bodies.

Results Obtained. This study will consist of two parts. In the first part, we shall look at the fixed points of semiflows in a Banach space. We will prove an abstract version of the linearization principle type which states that

> *if some modest continuity conditions are satisfied and if the linearized systems have eigenvalues all with negative real parts, then these fixed points are locally asymptotically stable in their neighborhoods, and we have global existence for solutions in these neighborhoods.*

Our result generalizes the linearization principle of Potier–Ferry [1981] and is more convenient for the kind of applications we intend, which adopt the geometrical formulation developed in Simo, Marsden, and Krishnaprasad [1988].

The above result will be applied to the fixed points of evolution equations in a Banach space. Sobolevskii [1966] established some basic results about the existence and continuity of solutions to Cauchy problems for equations of parabolic type in a Banach space. We will make use of these results to find conditions on the evolution equation that guarantee the asymptotic stability of fixed points and global existence of solutions in the neighborhood of fixed points.

In Part II, we shall analyze some relative equilibria of viscoelastic rods moving in space, using the two director Cosserat rod model. This model satisfies the invariance requirements under superposed rigid body motions and imposes no restrictions on the degree of allowable deformations. By a ***relative equilibrium*** we mean a dynamical solution $z(t)$ which is also a group orbit: $z(t) = \exp(t\xi) \cdot z_e$ for some Lie algebra element ξ. In our situation relative equilibria are uniformly rotating solutions. Stability itself is, as we have already stated, taken relative to group orbits, and in our case taken modulo rotations about the axis of rotation of the equilibrium solution.

Part II will prove that

> *the equations of motion for geometrically exact rods with dissipation and linearized at a relative equilibrium generate an exponentially decaying holomorphic semigroup.*

We do this by modifying the techniques of Potier–Ferry [1982], which in turn are essentially based on Sobolevskii's theory of equations of parabolic type in a Banach space and are used to prove the stability of static equilibria of elastic bodies moving freely in space.

Finally, we write the equations of motion for hyperelastic geometrically exact rods moving with a viscoelastic dynamical response in the abstract form

$$\frac{du}{dt} = G(u).$$

These equations have the form of Hamiltonian equations with dissipation and the potential energy used is the augmented stored energy potential. Applying our abstract result on the fixed points of semiflows in a Banach space to this evolution system, we prove that

> *the relative equilibria of hyperelastic rods in the presence of viscoelastic dissipation are asymptotically stable if they are formally stable, and that the solutions to the equations of motion in the neighborhood of a relative equilibrium exist and are smooth for all time and decay exponentially to the relative equilibrium.*

We believe the approach in this work also applies to the case of thermoelasticity as well as elastic shells and three-dimensional elastic bodies.

2 Stability of Fixed Points of Semiflows

In this section we consider the stability of the equilibria of semiflows (and flows) in a Banach space.

2.1 Notation

Let $\mathbf{E}$ be a Banach space and $\mathbf{U}$ an open subset of $\mathbf{E}$. Let $\mathbf{V}$ be a neighborhood of $\mathbf{U} \times \{0\}$ in $\mathbf{U} \times \mathbb{R}$ (or $\mathbf{U} \times \mathbb{R}_+$) such that for each $x \in \mathbf{V}$, we have

1. $(\{x\} \times \mathbb{R}) \cap \mathbf{V} = \{x\} \times (a, b)$ for some open interval (a, b) containing 0.

2. In the case of $\mathbf{U} \times \mathbb{R}_+$, $(\{x\} \times \mathbb{R}) \cap \mathbf{V} = \{x\} \times [0, a)$ for some $a > 0$.

We will write $F_t \equiv F(\cdot, t)$ for any map $F : \mathbf{V} \to \mathbf{E}$. We call a map $F : \mathbf{V} \to \mathbf{E}$ or F_t a ***flow*** or ***semiflow*** if F_t satisfies

1. $F_0 = \mathrm{Id}$ (the identity map), and

2. $F_t \circ F_s = F_{t+s}$ whenever F_t, F_s and F_{t+s} are all defined.

A point $u_0 \in \mathbf{U}$ is said to be a ***fixed point*** of the flow or semiflow F_t if $F_t(u_0) = u_0$, for all t (for which F_t is defined). The time for which $F_t(u)$ exists will be called the ***lifetime*** of u.

We shall denote the space derivative by D_u, or by D.

2.2 Boundedness and Joint Continuity of Space Derivatives

Let F_t be a semiflow on a Banach space E. Assume that

A–I u_0 is a fixed point of the semiflow;

A–II there exist $T_0 > 0$ and a neighborhood U_0 of u_0 such that each $u \in \mathbf{U}$ has a positive lifetime $T_u \geq T_0$;

A–III $F_t(u)$ is continuous in t for $t > 0$ and fixed u over $U_0 \times [0, T_0]$;

A–IV $D_u F_t(u)$ is norm-continuous in u for fixed $t \in (0, T_0]$;

A–V $D_u F_t(u)$ is strongly continuous in t for fixed $u \in U_0$.

The following lemma is a modification of Lemma 8A.4, p.260 of Marsden and McCracken [1976], which in turn is based on Chernoff and Marsden [1972].

Lemma 2.1. *Let $u_n \to u_0$ in $\mathbf{E}$ and $\delta > 0$. There exists a dense subset G of $[\delta, T]$ such that if $t_m \to t_0 \in G$, then*

$$(a) \qquad \lim_{m,n\to\infty} \|DF_{t_m}(u_n) - DF_{t_m}(u_0)\| = 0;$$

$$(b) \qquad \lim_{m,n\to\infty} DF_{t_m}(u_n)x = DF_{t_0}(u_0)x,$$

for fixed $x \in E$.

Proof. For $\varepsilon > 0$, set

$$G_{n,\varepsilon} = \{t \in [\delta, T] \mid \|DF_t(u_l) - DF_t(u_l)\| \le \varepsilon \text{ for all } l \ge n\}.$$

The set $G_{n,\varepsilon}$ is closed because $DF_t(u)$ is strongly continuous in t. Assume $\tilde{t}_n \in G_{n,\varepsilon}$ and $\tilde{t}_n \to \tilde{t}$. Let x be an arbitrary unit vector and $l \ge n$ and x. It is obvious that

$$\|DF_{\tilde{t}}(u_l)x - DF_{\tilde{t}}(u_0)x\| = \lim_{n\to\infty} \|DF_{\tilde{t}_n}(u_l)x - DF_{\tilde{t}_n}(u_0)x\| \le \varepsilon.$$

Hence, $\|DF_{\tilde{t}}(u_l) - DF_{\tilde{t}}(u_0)\| \le \varepsilon$ since x is arbitrary. Thus, $\tilde{t} \in G_{n,\varepsilon}$.

Also, we have

$$\bigcup_{n=1}^{\infty} G_{n,\varepsilon} = [\delta, T],$$

since $DF_t(u)$ is norm-continuous in u for fixed t. It now follows from the Baire Category Theorem that some of the $G_{n,\varepsilon}$'s have nonempty interiors. Thus,

$$G_\varepsilon = \bigcup_{n=1}^{\infty} \mathrm{Int}(G_{n,\varepsilon})$$

is nonempty. We claim that G_ε *is dense in* $[\delta, T]$.

Otherwise, there would be at least one closed interval $[a, b] \subset [\delta, T]$ with the property that $[a, b] \cap G_\varepsilon = \varnothing$. Applying the same argument to $[a, b]$, one gets a nonempty open subset $G_\varepsilon^{[a,b]}$ of $[a, b]$ contained in G_ε, which is a contradiction.

Next, we set

$$G = \bigcap_{k=1}^{\infty} G_{1/k},$$

where $G_{1/k}$ is constructed like G_ε. Since it is a countable intersection of open dense subsets of $[\delta, T]$, G is itself dense in $[\delta, T]$. Pick any $t_0 \in G$. Since each $G_{1/k}$ is open, there is a neighborhood U_k of t_0 contained in $G_{n_k,k}$ for some n_k. For $n \ge n_k$ and large m such that $t_m \in U_k$,

$$\|DF_{t_m}(u_n) - DF_{t_m}(u_0)\| \le \frac{1}{k}.$$

Therefore, (a) is true. As for (b), for any fixed $x \in \mathbf{E}$, we can find $M > 0$ such that

$$\|DF_{t_m}(u_0)x - DF_{t_0}(u_0)x\| \leq 1/k$$

and $t_m \in U_k$ for $m \geq M$. Hence,

$$\begin{aligned}
&\|DF_{t_m}(u_n)x - DF_{t_0}(u_0)x\| \\
&\qquad \leq \|DF_{t_m}(u_n) - DF_{t_m}(u_0)\| \cdot \|x\| + \|DF_{t_m}(u_0) - DF_{t_0}(u_0)\| \cdot \|x\| \\
&\qquad \leq \frac{1}{k}(\|x\| + 1),
\end{aligned}$$

for all $n \geq n_k$ and $m \geq M$. ■

Another basic property we will need is:

A–VI $DF_t(u_0)$ is norm-continuous in t for $t \in (0, T_0]$, i.e.,

$$\lim_{t \to t_0} \|DF_t(u_0) - DF_{t_0}(u_0)\| = 0$$

for any $t_0 \in (0, T_0]$.

Proposition 2.2. *If the semiflow also satisfies* A–VI*, and if $u_n \to u_0$ and $t_n \to t_0 > 0$, then the limit*

$$Tx = \lim_{n \to \infty} DF_{t_n}(u_n)x$$

defines a bounded operator T on E and

$$\lim_{n \to \infty} \|T - DF_{t_n}(u_n)\| = 0.$$

Proof. The assertion follows if we can show that $DF_{t_n}(u_n)$ is a Cauchy sequence (by the Banach-Steinhaus Theorem, see e.g., Theorem I.1.8 p.55 of Dunford and Schwarz [1953]).

Let G be constructed as in Lemma 2.1. Pick $\tilde{t} \in G$ such that $0 < \tilde{t} < t_0$ and let $\tau_n := t_n - t_0 + \tilde{t}$. We write

$$\varphi_t(u) \equiv DF_t(u), u^t \equiv F_t(u).$$

By (a) of Lemma 2.1,

$$\lim_{m,n \to \infty} \|\varphi_{\tau_m}(u_n) - \varphi_{\tau_m}(u_0)\| = 0. \tag{2.2.1}$$

Now

$$\begin{aligned}
&\|\varphi_{t_m}(u_m) - \varphi_{t_n}(u_n)\| \\
&\leq \|\varphi_{t_m}(u_m) - \varphi_{t_m}(u_0)\| + \|\varphi_{t_m}(u_0) - \varphi_{t_n}(u_0)\| + \|\varphi_{t_n}(u_0) - \varphi_{t_n}(u_n)\|
\end{aligned}$$

and

$$\|\varphi_{t_m}(u_m) - \varphi_{t_m}(u_0)\| \le \|\varphi_{t_0-\tilde{t}}(u_m^{\tau_m}) \circ (\varphi_{\tau_m}(u_m) - \varphi_{\tau_m}(u_0))\| + \|[\varphi_{t_0-\tilde{t}}(u_m^{\tau_m}) - \varphi_{t_0-\tilde{t}}(u_0)] \circ \varphi_{\tau_m}(u_0)\|.$$

Assumptions A–III and A–IV on $F_t(u)$ ensure that $F_t(u)$ is separately continuous in $u \in U_0$ and $t > 0$, hence also jointly continuous in u and t for $(u,t) \in U \times (0, T_0]$. (See Marsden and McCracken [1976], Theorem 8A.3, p.260.) We thus have

$$u_m^{\tau_m} \to u_0$$

and hence

$$\|\varphi_{t_0-\tilde{t}}(u_m^{\tau_m}) - \varphi_{t_0-\tilde{t}}(u_0)\| \to 0.$$

Also $\|\varphi_{t_m}(u_0)\| \to \|\varphi_{t_0}(u_0)\|$ by A–VI. Therefore, noting (2.2.1), we can find $N_1 > 0$ such that

$$\|\varphi_{t_m}(u_m) - \varphi_{t_m}(u_0)\| < \varepsilon/3,$$

for all $m > N_1$, where $\varepsilon > 0$ is given.

Similarly, we find $N_2 > N_1$ such that

$$\|\varphi_{t_n}(u_0) - \varphi_{t_n}(u_n)\| < \varepsilon/3,$$

for all $n > N_2$.

Finally, by A–VI one finds $N > N_2$ such that if $m, n > N$,

$$\|\varphi_{t_m}(u_0) - \varphi_{t_n}(u_0)\| < \varepsilon/3,$$

for all $m, n > N$. It follows from the above inequalities that for $m, n > N$,

$$\|\varphi_{t_m}(u_m) - \varphi_{t_n}(u_n)\| < \varepsilon,$$

and hence, $\varphi_{t_n}(u_n)$ is a Cauchy sequence. ■

The next basic property we need is,

A–VII Given any $x \in E$, there exists $M_x > 0$, $\varepsilon > 0$, and a neighborhood U_x of u_0 such that

$$\|DF_t(u)x - DF_0(u)x\| \equiv \|DF_t(u)x - x\| \le M_x,$$

for all $0 \le t < \varepsilon$ and $u \in U_x$.

Proposition 2.3. *Assume in addition that $F_t(\cdot)$ satisfies* A–VII. *Then, there exist $\delta > 0$, $M > 0$, and a neighborhood $\tilde{U}$ of u_0 such that*

$$\|DF_t(u)\| \le M,$$

for all $u \in \tilde{U}$ and $t \in [0, \delta]$.

Proof. This is a consequence of A–VII and the Uniform Boundedness Principle. For the purpose of contradiction, suppose that $\|F_t(u)\|$ is unbounded over any $U \times [0, \delta]$, where U is a neighborhood of u_0. Thus, for any $n \in \mathbb{N}$, there exist $u_k^{(n)}, k = 1, 2, \ldots$ and $t_{k,m}^{(n)}, m = 1, 2, \ldots$ satisfying

$$\lim_{m\to\infty} t_{k,m}^{(n)} = 0, \quad r_k^{(n)} \equiv \|u_k^{(n)} - u_0\| \searrow 0, \quad \text{and} \quad \|DF_{t_{k,m}^{(n)}}(u_k^{(n)})\| \geq n.$$

Taking subsequences, one gets sequences u_n and t_n that satisfy

$$r_n \equiv \|u_n - u_0\| \searrow 0, \qquad t_n \searrow 0,$$

and

$$\|DF_{t_n}(u_n)\| \geq n.$$

It is obvious from A–VII that $\|DF_{t_n}(u_n)x - x\| \to 0$ as $n \to \infty$. Hence, $\{DF_{t_n}(u_n)x\}$ is bounded for any given $x \in E$, and by the Uniform Boundedness Principle, there is some $M > 0$ such that, for all n,

$$\|DF_{t_n}(u_n)\| < M,$$

contradicting what we deduced from our supposition. ■

2.3 Exponential Decay of the Spatial Derivative

Proposition 2.4. *Let $F_t(\cdot)$ be a semiflow satisfying* A–I *through* A–VI. *If*

$$\|DF_t(u_0)\| \leq \exp(-\sigma t)$$

for $t > 0$, and for some $\sigma > 0$, then for any given $\delta \in (0, T_0]$ and $0 < \sigma' < \sigma$ one can find a neighborhood U of u_0, where

$$\|DF_t(u)\| \leq \exp(-\sigma' t),$$

for all $u \in U_0$ and $t \in (\delta, T_0]$.

Proof. First we observe the following three points:

(i) Since $\exp(-\sigma' t) > \exp(-\sigma t)$ for $t > 0$ and $DF_t(u)$ is norm-continuous in u when t is fixed, there exists $r_t \in (0, \infty)$ or $r_t = \infty$ such that $\|u - u_0\| < r_t$ implies $\|DF_t(u)\| < \exp(-\sigma' t)$ and for finite r_t, one can find at least one u_t satisfying

$$\|u_t - u_0\| = r_t, \quad \text{and} \quad \|DF_t(u_t)\| \geq \exp(-\sigma' t).$$

(ii) The existence of U in this proposition is equivalent to

$$\tilde{r} = \inf_{t\in[\delta,T_0]} \{r_t\} > 0 \tag{2.2.2}$$

(iii) Suppose $\tilde{r} = 0$. Then, one would be able to find a sequence $t_n \in [\delta, T_0]$ with corresponding

$$r_n \equiv r_{t_n} \searrow 0,$$

and $\{u_n\} \subset U_0$ satisfying

$$\|u_n - u_0\| = r_n, \qquad \|DF_{t_n}(u_n)\| \geq \exp(-\sigma' t_n).$$

Here, without loss of generality, by passing to a subsequence if necessary, we can assume $t_n \to t_0 \in [\delta, T_0]$.

We will prove that $\tilde{r} = 0$ leads to a contradiction. Let t_n, u_n, r_n be as in (iii). By Proposition 2.2,

$$T = \lim_{n\to\infty} DF_{t_n}(u_n) \in \mathcal{B}(E)$$

(the space of bounded operators on E) and

$$\|T\| > \exp(-\sigma t_0),$$

since $\|DF_{t_n}(u_n)\| \geq \exp(-\sigma' t_n)$, $t_n \to t_0 > 0$, and $\exp(-\sigma t_0) > \exp(-\sigma' t_0)$. Hence, for large n one can find some $\epsilon_0 > 0$ such that $\|F_{t_n}(u_n)\| \geq \exp(-\sigma t_0) + \varepsilon_0$. On the other hand, starting with $[\delta', T]$, $0 < \delta' < \delta$, we obtain a dense subset G of $[\delta', T]$ as in Lemma 2.1. Pick $\tilde{t} \in G$ such that $\delta' < \tilde{t} < t_0$ and set

$$\tau_n \equiv t_n - t_0 + \tilde{t}.$$

Then, $\tau_n \in [\delta', T_0]$ for large n and $\tau_n \to \tilde{t} \in G$.

The assumptions A–III and A–IV on $F_t(u)$ guarantee the joint continuity of $F_t(u)$ at $(u,t) \in U_0 \times (0, T_0]$. Hence

$$\lim_{n\to\infty} F_{\tau_n}(u_n) = F_{\tilde{t}}(u_0) = u_0$$

Therefore,

$$DF_{t_0-\tilde{t}}(F_{\tau_n}(u_n)) \to DF_{t_0-\tilde{t}}(u_0)$$

in norm as $n \to \infty$. Now pick any x in E. Then,

$$DF_{\tau_n}(u_n)x \to DF_{\tilde{t}}(u_0)x$$

in norm by virtue of part (b) of Lemma 2.1. It follows that

$$\begin{aligned}\lim_{n\to\infty} \|DF_{t_n}(u_n)x\| &= \lim_{n\to\infty} \|DF_{t_0-\tilde{t}}(F_{\tau_n}(u_0)) \circ DF_{\tau_n}(u_n) \cdot x\| \\ &= \|DF_{t_0-\tilde{t}}(u_0) \circ DF_{\tilde{t}}(u_0) \cdot x\| \\ &= \|DF_{t_0}(u_0) \cdot x\| \leq \|DF_{t_0}(u_0)\| \cdot \|x\| \\ &\leq \exp(-\sigma t_0)\|x\|.\end{aligned}$$

Thus, since x is arbitrary,

$$\|T\| = \| \lim_{n\to\infty} DF_{t_n}(u_n)\| \le \exp(-\sigma t_0),$$

a contradiction. ■

For the next proposition, we need the following lemma on the upper-continuity of the spectrum of a bounded operator on a Banach space. (see Theorem 3.1 and Remark 3.3 of p.208 of Kato [1977]).

Lemma 2.5. *The spectrum $\sigma(T)$ is an upper semicontinuous function of $T \in \mathcal{B}(E)$, that is, for any $T \in \mathcal{B}(E)$ and $\varepsilon > 0$, there exists a $\delta > o$ such that*

$$\operatorname{dist}(\sigma(S), \sigma(T)) \equiv \sup_{\lambda\in\sigma(S)} (\lambda, \sigma(T)) < \varepsilon$$

if $\|S - T\| < \delta$.

Proposition 2.6. *Let F_t be a semiflow on E satisfying A–I through A–VI. Assume also that the spectrum of $DF_t(u_0)$ lies inside and at a positive distance away from the unit circle for any $t \in (0, T_0]$. Then, given any $0 < \delta < T_0$, there is an equivalent norm $|\cdot|$ on E and $\sigma > 0$ such that*

$$|DF_t(u_0)| < \exp(-\sigma t),$$

for all $t \in [\delta, T_0]$.

Proof. Since F_t is a semiflow and u_0 is a fixed point, denoting $DF_t(u_0)$ by φ_t, we get from the Chain Rule

$$\varphi_{t+s}(x) = \varphi_t \circ \varphi_s(x),$$
$$\varphi_0 = \mathrm{Id},$$

for all $x \in E$ and t, s such that φ is defined. Also, by assumption

$$\text{s-}\lim_{t\searrow 0} \varphi_t = \varphi_0 = \mathrm{Id}\,.$$

Let $t_1 + t_2 = t'_1 + t'_2, t_i, t'_i \in [0, T_0]$, i = 1, 2. It is easy to verify that

$$\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t'_1} \circ \varphi_{t'_2}.$$

Moreover, both $\varphi_{t_1} \circ \cdots \circ \varphi_{t_n}$ and $\varphi_{t'_1} \circ \cdots \circ \varphi_{t'_n}$ are well-defined if

$$\sum_{i=1}^n t_i = \sum_{i=1}^n t'_i \qquad \text{for } t_i, t'_i \in [0, T_0],$$

where $i = 1, 2, \cdots, n$ and

$$\varphi_{t_1} \circ \cdots \circ \varphi_{t_n} = \varphi_{t'_1} \circ \cdots \circ \varphi_{t'_n}.$$

Therefore, we can extend φ_t to $[0, \infty)$ by defining

$$\varphi_t = \varphi_{t_1} \circ \cdots \circ \varphi_{t_n},$$

where $t_i \in [0, T_0]$ and $t_1 + t_2 + \cdots + t_n = t$. Thus, φ_t is a C_0-semigroup of linear operators on E.

Let $\delta \in (0, T_0)$ be given. Choose $\delta_0 > 0$ such that $\delta_0 < \delta$ and $m\delta_0 = T_0$ for some positive integer m. If $t' \in [\delta_0, T_0]$, then

$$r(\varphi_{t'}) \leq \exp(-\varepsilon_t)$$

for some positive ε_t by our assumption on the spectrum of $DF_t(u_0)$. Pick $0 < \sigma'_{t'} < \sigma_{t'}$, where $\sigma_{t'} \cdot t' = \varepsilon_t$. In view of hypothesis A–VI and Lemma 2.5, there exists a $\delta_{t'}$ such that

$$r(\varphi_t) \leq \exp(-\delta_{t'} \cdot t),$$

for all $t \in (t' - \delta_{t'}, t' + \delta_{t'})$. Hence, we have a cover of $[\delta_0, T_0]$ of the form

$$\{(t - \delta_t, t + \delta_t) \mid t \in [\delta_0, T_0]\}$$

with the δ_t's chosen in a manner similar to the above $\delta_{t'}$. Furthermore, $[\delta_0, T_0]$ being compact, a finite subcover exists, say,

$$(t_1 - \delta_{t_1}, t_1 + \delta_{t_1}), \cdots, (t_n - \delta_{t_n}, t_n + \delta_{t_n})$$

with corresponding $\sigma'_1, \cdots, \sigma'_n > 0$. Setting $\sigma = \min\{\sigma'_1, \cdots, \sigma'_n\}$, we now have

$$r(\varphi_t) \leq \exp(-\sigma t),$$

for all $t \in [\delta_0, T_0]$.

If, in choosing $\delta_0 = T_0/m$, we always pick an even $m > 2$, any $t \in [\delta_0, \infty)$ can be written as $t = 2n_0\delta_0 + t'$ with $t' \in [\delta_0, T_0]$, and $n_0 \in \mathbb{N}$, the natural numbers. It follows that

$$\begin{aligned} r(\varphi_t) &= \lim_{n\to\infty} \|\varphi^n_{2n_0\delta_0+t'}\|^{1/n} \\ &\leq \lim_{n\to\infty} \|\varphi^n_{2\delta_0}\|^{1/n} \cdots \|\varphi^n_{2\delta_0}\|^{1/n} \cdot \|\varphi^n_{t'}\|^{1/n} \\ &\leq e^{-\sigma t}. \end{aligned}$$

Thus, $r(\varphi_t) \leq e^{-\sigma t}$ holds for $t \in [\delta_0, \infty)$, and

$$\|\varphi^n_t\|/e^{-n\sigma t},$$

is uniformly bounded from above for all $t \geq \delta$ and for all $n \in \mathbb{N}$,. This allows us to define a new norm on E as follows (cf. Abraham, Marsden, and Ratiu [1988], Lemma 4.3.8, p.301):

$$|x| = \sup_{n\geq 0, t\geq\delta} \left(\frac{\|\varphi^n_t(x)\|}{e^{-n\sigma t}} \right)$$

for $x \in E$. Clearly this defines a norm, and the two norms $\|\cdot\|$ and $|\cdot|$ are equivalent because

$$\|x\| \leq |x| \leq \left(\sup_{n\geq 0, t\geq \delta} (\|\varphi_t^n\|/e^{-n\sigma t} \right) \cdot \|x\|$$

for any $x \in E$. When estimating $|\varphi_{t_0}(x)|$ we need to consider two cases. If the supremum is assumed at $n = 0$,

$$\sup_{n\geq 0, t\geq \delta} (\, \|\varphi_t^n(\varphi_{t_0}(x)\,)\,\|/e^{-n\sigma t}) = \|\varphi_{t_0}(x)\|$$

we get

$$\begin{aligned} |\varphi_{t_0}(x)| &= \|\varphi_{t_0}(x)\| \\ &\leq e^{-\sigma t_0} \left(\sup_{n\geq 0, t\geq \delta} (\|\varphi_t^n(x)\|/e^{-n\sigma t} \right) \\ &= e^{-\sigma t_0}|x| \end{aligned}$$

Otherwise, we have

$$\begin{aligned} |\varphi_{t_0}(x)| &= \sup_{n\geq 0, t\geq \delta} \left(\|\varphi_t^n(\varphi_{t_0}(x))\|/e^{-n\sigma t} \right) \\ &\leq e^{-\sigma t_0} \left(\sup_{n\geq 0, t'\geq \delta} (\|\varphi_{t'}^n(x)\|/e^{-n\sigma t'} \right) \\ &= e^{-\sigma t_0}|x| \end{aligned}$$

Thus, under this new norm

$$|\varphi_t| < e^{-\sigma t},$$

for all $x \in [\delta, \infty)$. ∎

2.4 The Main Theorem

There are two versions of the main result. We begin with the following preparatory case.

Proposition 2.7. *Assume that the semiflow $F_t(\cdot)$ satisfies* A–I *through* A–VII, *and $\|DF_t(u_0)\| \leq \exp(-\sigma t)$, for all $t \in \mathbb{R}_+$ and some $\sigma > 0$. We have*

(a) *Global existence of integral curves in a neighborhood of u_0; that is, there exists a neighborhood U of u_0 such that every $u \in U$ has infinite positive lifetime.*

(b) *Asymptotic stability at u_0:*

$$\lim_{t\to\infty} \|F_t(u) - u_0\| = 0,$$

for all $u \in U$.

Proof. By Proposition 2.3, there are $\delta > 0$, $M > 0$ and a neighborhood U_1 of u_0 such that

$$\|DF_t(u)\| \leq M,$$

for all $u \in U_1$ and $t \in [0, \delta]$. Fix $\delta < T_0/2$ and U_1. We find a neighborhood U_2 of u_0 as in Proposition 2.4 such that

$$\|DF_t(u)\| \leq \exp(-\delta' t),$$

for all $t \in [\delta, T_0]$, and $u \in U_2$ for some δ'. Both U_1 and U_2 are chosen to be subsets of some U_0, where A–II is satisfied. Now let $U \subset U_1$ be a neighborhood of u_0 such that

$$M(u - u_0) \in U_2,$$

for all $u \in U_0$. Taking note of the estimate

$$\begin{aligned}
\|F_t(u) - u_0\| &= \|F_t(u) - F_t(u_0)\| \\
&= \left\| \int_0^1 DF_t(su + (1-s)u_0) \cdot (u - u_0)\, ds \right\| \\
&\leq \begin{cases} M\|u - u_0\| & \text{for } 0 \leq \delta, \\ \exp(-\sigma' t)\|u - u_0\| & \text{for } t \in [\delta, T_0], \end{cases}
\end{aligned}$$

we know $F_t(u) \in U_1 \cap U_2$, hence can be extended in time by at least T_0.

For $t > 0$, write

$$t = n(t)T_0 + t',$$

where $T_0 > t' \geq 0$, and $n(t) \in \mathbf{N}$. By induction, one gets the estimate

$$\begin{aligned}
\|F_t(u) - u_0\| &= \|F_{n(t)T_0} \circ F_t(u) - F_{n(t)T_0} \circ F_{t'}(u_0)\| \\
&= \left\| \int_0^1 DF_{(n(t)-1)T_0}(\cdots) \circ DF_{T_0}(F_{t'}(u) - u_0)\, ds \right\| \\
&\leq M \exp(-n(t)\sigma' T_0) \cdot \|u - u_0\|.
\end{aligned}$$

Without loss of generality $M \geq 1$ is assumed here. Thus, the semiflow can be extended infinitely in time for every u in U and

$$\lim_{t\to\infty} \|F_t(u) - u_0\| = 0,$$

since $n(t) \to \infty$ as $t \to \infty$. ∎

Combining Propositions 2.2 through 2.7, we obtain the following theorem on the asymptotic stability of the fixed points of flows (semiflows) in a Banach space and the global existence of (semi-)flows in their neighborhood.

Theorem 2.8. *Let F_t be a semiflow (flow) in a Banach space E. Assume that F_t satisfies the hypotheses* A–I *through* A–VII. *Assume also that the spectrum $\sigma(DF_t(u_0))$ lies uniformly inside the unit circle for $t \in (0, T_0]$. Then, there exists a neighborhood U of u_0 such that*

I *Global existence: each $u \in U$ has infinite lifetime,*

II *Asymptotic stability at u_0:*

$$\lim_{t \to \infty} \|F_t(u) - u_0\| = 0,$$

for all $u \in U$.

3 The Evolution Equation

3.1 Introduction

Now we investigate the semiflows generated by evolution equations in a Banach space. We shall apply the results we obtained in Section 2 to flows of evolution equations. More specifically, we shall find conditions on the evolution

$$\frac{du}{dt} = G(u), \tag{3.3.1}$$

which, when satisfied, will guarantee that the equilibrium u_0 is asymptotically stable. In (3.3.1), G is a map from $\mathcal{Y}$ to $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}$ are Banach spaces and $\mathcal{Y}$ is continuously and densely included in $\mathcal{X}$. Consistent with our applications, we shall assume that G has the form

$$G(u) = A(u)u + g(u),$$

where $A(u)$ is a closed linear operator, and $g(u)$ a C^1 nonlinear mapping. Taylor-expanding $g(u)$ at u_0 and combining $D_uG(u_0)$ with $T(u)$, we can assume that $G(u)$ is in the form

$$G(u) = A(u)u + g(u),$$

where $A : \mathcal{Y} \to \mathcal{X}$ is a closed linear operator and g is a nonlinear map from $\mathcal{Y}$ to $\mathcal{X}$ having the property $\|g(u)\|_{\mathcal{X}} = o(\|u - u_0\|_{\mathcal{X}})$, when we consider the equation in a neighborhood of u_0.

3.2 Notation and Terminology

Let us recall some definitions and notation to be used. A ***continuous local semiflow*** on a Banach space $\mathcal{Y}$ is a continuous map $F : \mathcal{D} \subset \mathcal{Y} \times R^+ \to \mathcal{Y}$, where $\mathcal{D}$ is an open subset, satisfying

- $\mathcal{Y} \times \{0\} \subset \mathcal{D}$,
- $F(x, 0) = x$,
- if $F(x,t) \in \mathcal{D}$ and $(F(x,t), s) \in \mathcal{D}$, then $F(x, t+s) \in \mathcal{D}$ and
$$F(x, t+s) = F(F(x,t), s).$$

We say G ***generates the semiflow*** $F(x,t)$ if $F(x,t)$ is t-differentiable for $t \geq 0$ and $x \in \mathcal{Y}$, and

$$\frac{d}{dt} F(x,t) = G(F(x,t)).$$

When G depends explicitly on time, we replace local semiflow $F(x,t)$ by an ***evolution operator*** $F_{t,s} : \mathcal{Y} \to \mathcal{Y}$, satisfying

- $F_{t,t} = \mathrm{Id}$;
- $F_{t,s} \circ F_{s,r} = F_{t,r}$, when $0 \leq r \leq s \leq t \leq T$, for some T;
- $\dfrac{d}{dt} F_{t,s}(x) = G(F_{t,s}(x), t)$.

Let $\{U(t) \mid t \geq 0\}$ be a (C^0) semigroup on a Banach space $\mathcal{X}$, A its ***infinitesimal generator*** defined by

$$Ax = \lim_{t \searrow 0} \frac{U(t)x - x}{t},$$

on the domain $\mathcal{D}(A)$, that is, the set of those $x \in \mathcal{X}$ for which the above limit exists. We say $A \in \mathcal{G}(\mathcal{X}, M, \beta)$ if $\|U(t)\| \leq M e^{-t\beta}$. We will also use the following notation,

$$\Sigma(\omega, \beta) = \{\lambda \in \mathcal{C} \mid |\operatorname{Arg}\lambda| \leq \pi/2 + \omega \quad \text{or} \quad \operatorname{Re}\lambda \geq -\beta\}.$$

Note that the following two conditions are equivalent;

- the spectrum of a linear operator lies uniformly to the left of of the imaginary axis.
- there are positive ω and β such that $\Sigma(\omega, \beta)$ is contained in the resolvent set of the operator.

3.3 Sobolevskii's Results on Parabolic Equations in Banach Spaces

We will make use of the following results Sobolevskii [1966] obtained for equations of parabolic type in a Banach space.

Theorem 3.1. *Let the operator $A(t)$, $t \in [0,T]$, act in E and have an everywhere dense domain of definition D not depending on t. For any $t, r, s \in [0,T]$ suppose*

$$\|[A(t) - A(\tau)]\, A^{-1}(s)\| \leq C|t-\tau|^{\epsilon}$$

for some $\epsilon \in (0,1]$.

For any λ with $\operatorname{Re}\lambda \geq 0$, assume the operator $A(t) + \lambda I$ has a bounded inverse and

$$\|[A(t) + \lambda I]^{-1}\| \leq C[|\lambda| + 1]^{-1}.$$

Then, there exists an evolution operator $U(t,\tau)$ which is defined and strongly continuous, for all t and τ such that $0 \leq \tau \leq t \leq T$. Also, $U(t,\tau)$ is uniformly differentiable in t for $t > \tau$, and

$$\frac{\partial U(t,\tau)}{\partial t} + A(t)\, U(t,\tau) = 0.$$

For $v_0 \in E$,

$$v(t) = U(t,0)\, v_0$$

defines a unique solution to the Cauchy problem

$$\frac{dv}{dt} + A(t)\, v = 0, \qquad (0 < t < T)$$
$$v(0) = v_0,$$

which is continuous for all $t \in [0,T]$ and continuously differentiable for $t > 0$. If $v_0 \in D$, then $v(t)$ is continuously differentiable for $t = 0$.

Theorem 3.2. *Assume $f(t)$ satisfies the Hölder condition*

$$\|f(t) - f(s)\| \leq C|t-s|^{\delta}$$

for some $\delta \in (0,1]$. Then, the variation of constants formula

$$v(t) = U(t,0)\, v_0 + \int_0^t U(t,s)\, f(s)\, ds$$

gives a unique solution to the nonhomogeneous equation

$$\frac{dv}{dt} + A(t)\, v = f(t),$$

which is continuous for all $t \in [0, T]$ *and continuously differentiable for* $t > 0$*. If* $v_0 \in D$*, then* $v(t)$ *is continuously differentiable for* $t = 0$*.*

If $f(t)$ *is an operator function, then the formula defines a uniformly continuously differentiable solution.*

Finally we shall need:

Theorem 3.3. *Let* $A_0 = A(0, v_0)$ *be a linear operator whose domain of definition* D *is dense in* E*. Let the operator* A_0^{-1} *be completely continuous in* E *and* $A_0 + \lambda I$ *have a bounded inverse satisfying*

$$\|[A_0 + \lambda I]^{-1}\| \le C[\,|\lambda| + 1]^{-1},$$

for any λ *with* $\operatorname{Re}\lambda \ge 0$*. For some* $\alpha \in [0, 1)$ *and for any* $v \in E, \|v\| \le R$*, assume the operator* $A(t, A_0^{-\alpha} v)$ *is defined in* D *and satisfy*

$$\|[A(t, A_0^{-\alpha} v) - A(\tau, A_0^{-\alpha} w)]\, A_0^{-1}\| \le C(R)[|t - \tau|^{\epsilon} + \|v - w\|^{\rho}]$$

with $\epsilon, \rho \in (0, 1]$*, for any* $0 \le t, \tau \le T$*,* $\|v\| \le R$*,* $\|w\| \le R$*. In this region, suppose*

$$\|[f(t, A_0^{-\alpha} v) - f(\tau, A_0^{-\alpha} w)]\, A_0^{-1}\| \le C(R)[|t - \tau|^{\epsilon} + \|v - w\|^{\rho}].$$

Lastly, for some $\beta > \alpha$*, let* $v_0 \in D(A_0^{\beta})$*, and let* $\|A_0^{\alpha} v_0\| < R$*. Then, there exists at least one solution of the Cauchy problem*

$$\frac{dv}{dt} + A(t, v)v = f(t, v), \tag{3.3.2}$$

$$v(0) = v_0. \tag{3.3.3}$$

which is defined on a segment $[0, t_0)$*, is continuous for* $t \in [0, t_0)$*, and continuously differentiable for* $t > 0$*.*

If $\rho = 1$*, the solution is unique, and we can omit the assumption on the complete continuity of* A_0^{-1}*. The solution can be obtained by the method of successive approximation in this case.*

3.4 Continuity of Solutions and Evolution Systems

Proposition 3.4. *Let* $F_t(u)$ *be the solution of* (3.3.2) *corresponding to the initial condition*

$$F_0(u) = u \in D.$$

Then, under the conditions of Theorem 3.2 and

B–I $(A'(u)\cdot v)\,x$ *is Lipschitz continuous in* u*, i.e.,*

$$\|(A'(u_1)-A'(u_2))\cdot v)x\| \le C\|u_1-u_2\|\,\|v\|\,\|x\|_D,$$

and the Lipschitz estimate

B–II

$$\|f(u_1)-f(u_2)\| \le C\|u_1-u_2\|,$$

the solution $F_t(u)$ *is continuous in* u*, uniformly in* $t\in[0,t_1]$ *for some* $t_1<t_0$*.*

Proof. Let $u(t)=F_t(u_0)$, $\tilde u(t)=F_t(\tilde u_0)$, $\Delta(t)=u(t)-\tilde u(t)$. Then

$$\begin{aligned}
&\|(A(u(t))-A(\tilde u(t)))\cdot x\| \\
&\quad = \left\| \int_0^1 [A'(\lambda u(t)+(1-\lambda)\tilde u(t))\cdot(u(t)-\tilde u(t))\cdot x]\;d\lambda \right\| \\
&\quad \le \|A'(\lambda u(t)+(1-\lambda)\tilde u(t))\|\,\|u(t)-\tilde u(t)\|\,\|x\|_D.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d\,\Delta(t)}{dt} &= -A(u(t))\cdot(u(t)-\tilde u(t)) + \\
&\qquad (A(u(t))-A(\tilde u(t)))\cdot\tilde u(t) + (f(u(t)-f(\tilde u(t))).
\end{aligned}\tag{3.3.4}$$

It follows that

$$\begin{aligned}
\frac{d}{dt}\|\Delta(t)\| &\le \|A(u(t))\|\,\|\Delta(t)\| + \\
&\qquad \|A'(\lambda u(t)+(1-\lambda)\tilde u(t))\|\,\|\Delta(t)\|\,\|\tilde u(t)\|_D + C(\|\Delta(t)\|) \\
&\le \tilde C\|\Delta(t)\|
\end{aligned}\tag{3.3.5}$$

where $\tilde C$ is a positive constant dependent on $u_0,\tilde u_0$, and t_0. To get this estimate, we have used the fact that $u(t)$ and $\tilde u(t)$ are continuous in t, and $A'(u)$ is continuous in u. Moreover, the constant in the estimate can be chosen to depend only on t_0 and $\tilde u_0$. Since $[0,t_1]$, $[0,1]$, and $\{(t,\lambda)\mid 0\le t\le t_1, 0\le\lambda\le 1\}$ are compact, $\lambda u(t)+(1-\lambda)\tilde u(t)$, $\tilde u(t)$, and $u(t)$ are continuous, there exists a connected finite subcover of

$$\{\lambda u(t)+(1-\lambda)\tilde u(t)\mid\; 0\le t\le t_1, 0\le\lambda\le 1\}$$

consisting of open balls contained in D. Also recall that the solution to (3.3.2) and (3.3.3) is the fixed point of the map $u(t)\mapsto v(t)$ given by

$$v(t)=U(t,0)u_0+\int_0^t U(t,s)f(u(s))\,ds.$$

Hence,

$$\begin{aligned}\|u_{n+1}(t)-u_0\| &= \left\|(U(t,0)-U(0,0))u_0+\int_0^t U(t,s)f(u_n(s))\,ds\right\| \\ &\le (1+e^{-\beta t})\|u_0\| \\ &\quad + \left(\max_{0\le t\le t_0}\|u(t)-u_0\|+\|f(u_0)\|\right)\cdot\frac{1}{\beta}\left(1-e^{-\beta t}\right)\end{aligned}$$

and

$$\max_{0\le t\le t_0}(\|u_{n+1}(t)-u_0\|)\le B(u_0,t_0)+\max_{0\le t\le t_0}(\|u_n(t)-u_0\|)\cdot\left[\frac{1}{\beta}(1-e^{-\beta t})\right].$$

where $B(u_0,t_0)$ is continuous in u_0 and t_0. It follows that

$$\max_{0\le t\le t_0}(\|u_n(t)-u_0\|)\le e^{Bt_0}+\|u_0\|\cdot\left[\frac{1}{\beta}(1-e^{-\beta t})\right]^n.$$

and the solution $u(t)$ satisfies

$$\max_{0\le t\le t_0}(\|u(t)-u_0\|)\le C(u_0,t_0).$$

Thus, if we start with v_0 close to u_0, for the constant in (3.3.5) we can find a constant $\tilde{C}$ dependent on $\tilde{u}_0$ and t_0 only, such that

$$\frac{d}{dt}\|\Delta(t)\|\le\tilde{C}\|\Delta(t)\|.$$

Therefore, by Gronwall's inequality,

$$\max_{0<t<t_1}\|u(t)-\tilde{u}(t)\|\to 0$$

when $\|u_0-\tilde{u}_0\|\to 0$, as claimed in the proposition. ■

The above proof gives the following estimate which will be used later.

Corollary 3.5. *Under the same conditions as in Proposition 2.4,*

$$\|\Delta(t)\|\le C(\tilde{u}_0,t_0)\,\|\Delta(0)\|\,,$$

for all $t\in[0,t_1]$.

Proposition 3.6. *Let* $F_t(u)$ *be the solution to* (3.3.2) *satisfying the initial condition* $F_0(u)=u$. *Let* $u(t)=F_t(u_0)$, $v(t)=F_t(v_0)$, $A_1(t)=A(u(t))$, *and* $A_2(t)=A(v(t))$. *Assume* $\|u_0-v_0\|=\delta$, *then under the same conditions as in Proposition 2.2 there exist* t_1, *a constant* C *and some* $\theta\in(0,1)$ *such that*

$$\|(A_1(t)-A_2(t))x\|\le C\delta\|x\|_D, \tag{3.3.6}$$

$$\|(A_1(t)-A_2(t)+A_1(s)-A_2(s))x\|\le C\delta|t-s|\|x\|_D, \tag{3.3.7}$$

for all $0\le s\le t\le t_1$.

Proof. Let $T(\lambda,t) = A(\lambda u(t) + (1-\lambda)v(t))$, $\Delta(t) = u(t) - v(t)$. The first inequality is obvious from the proof of Proposition 3.6. As for (3.3.7), rewrite

$$\begin{aligned}
&(A_1(t) - A_2(t) - A_1(s) + A_2(s))x \\
&\qquad = \int_0^1 \frac{\partial}{\partial\lambda}[T(\lambda,t) - T(\lambda,s)]\,x\,d\lambda \\
&\qquad = \int_0^1 \{[T_\lambda(\lambda,t) - T_\lambda(\lambda,s)]\cdot\Delta(t) + T_\lambda(\lambda,s)\cdot[\Delta(t) - \Delta(s)]\}x\,d\lambda\,.
\end{aligned}$$

and we have the estimate

$$\begin{aligned}
&\|[T_\lambda(\lambda,t) - T_\lambda(\lambda,s)]\cdot\Delta(t)\,x\| \\
&\qquad = \|[A'(\lambda u(t) + (1-\lambda)v(t)) - A'(\lambda u(s) + (1-\lambda)v(s))]\cdot\Delta(t)\,x\| \\
&\qquad \le C_1\|\lambda(u(t) - u(s)) + (1-\lambda)(v(t) - v(s))\|\|\Delta(t)\|\,\|x\|_D \\
&\qquad \le C \max_{0\le t\le t_1}\left(\left\|\frac{du(t)}{dt}\right\|,\left\|\frac{dv(t)}{dt}\right\|\right)|t-s|\|\Delta(t)\|\,\|x\|_D \\
&\qquad \le C\delta|t-s|\|x\|_D
\end{aligned}$$

by the Corollary, where C is some generic constant. Similarly we get the estimates

$$\|\Delta(t) - \Delta(s)\| \le \max_{0\le t\le t_1}\left(\left\|\frac{d\,\Delta(t)}{dt}\right\|\right)|t-s| \le C_1\delta|t-s|$$

and

$$\|T_\lambda(\lambda,s)\cdot(\Delta(t) - \Delta(s))x\| \le C_2C_3|t-s|\|x\|_D,$$

where the bound C_2 on $\|T_\lambda(\lambda,s)\|$ results from $(A'(u)\cdot v)x$ being continuous in u. Combining these estimates, we get (3.3.7). ■

With the above inequalities established, we can now estimate how close $U^x(t,s)$ and $U^y(t,s)$, the evolution operators corresponding to the solutions $F_t(x)$ and $F_t(y)$, are to each other. For this end we need the following lemma. The proof given below follows Sobolevskii [1966] and Potier–Ferry [1982].

Lemma 3.7. *Under the same conditions as in Proposition 3.4,*

$$\|[A(F_t(x))U^x(t,s)A^{-1}(F_s(x)) - A(F_t(y))U^y(t,s)A^{-1}(F_s(y))\| \le C\delta\,.$$

Proof. Let

$$\begin{aligned}
Q(t,s) &= A(F_t(x))U^x(t,s)A^{-1}(F_s(x)) =: A(t)U(t,s)A^{-1}(s), \\
\delta Q(t,s) &= A(F_t(x))U^x(t,s)A^{-1}(F_s(x)) - A(F_t(y))U^y(t,s)A^{-1}(F_s(y)),
\end{aligned}$$

where we have abbreviated the notation for convenience, and adopted the convention that δ in front of a quantity denotes the variation of that quantity brought about when x is perturbed to y. Then, the operator

$$\phi(r) = e^{-(t-r)A(t)}U(r,s)A^{-1}(s)$$

is strongly differentiable, and integration of $\phi'(r)$ from s to t shows that $W(t,s)$ is the solution to the Volterra integral equation

$$\begin{aligned} Q(t,s) = {} & A(t)e^{-(t-s)A(t)}A^{-1}(s) \\ & + \int_s^t A(t)e^{-(t-r)A(t)}[A(t)-A(r)]A^{-1}(r)Q(r,s)\,dr\,. \end{aligned} \tag{3.3.8}$$

It follows that

$$\begin{aligned} \delta Q(t,s) = {} & \delta A(t)e^{-(t-s)A(t)}A^{-1}(s) \\ & + \int_s^t \delta\{A(t)e^{-(t-r)A(t)}\}[A(t)-A(r)]A^{-1}(r)Q(r,s)\,dr \\ & + \int_s^t A(t)e^{-(t-r)A(t)}\delta\{[A(t)-A(r)]\}A^{-1}(r)Q(r,s)\,dr \\ & + \int_s^t A(t)e^{-(t-r)A(t)}[A(t)-A(r)]\delta\{A^{-1}(r)\}Q(r,s)\,dr \\ & + \int_s^t A(t)e^{-(t-r)A(t)}[A(t)-A(r)]A^{-1}(r)\delta Q(r,s)\,dr\,. \end{aligned} \tag{3.3.9}$$

Since the semigroup generated by $A(t)$ is holomorphic, we have the estimates

$$\|\delta A^{-1}(s)x\| \le C_1\delta\|x\|_D, \tag{3.3.10}$$

$$\|\delta\{A(t)e^{-(t-s)A(t)}\}\| \le C_2\delta/(t-s), \tag{3.3.11}$$

$$\|\delta\{A(t)e^{-(t-s)A(t)}A^{-1}(s)\}\| \le C_3\delta. \tag{3.3.12}$$

and the resulting inequality

$$\begin{aligned} \|\delta Q(t,s)\| \le {} & C_3\delta + C_2\delta\int_s^t \|[A(t)-A(r)]A^{-1}(r)Q(r,s)\|\,dr \\ & + C\delta\int_s^t \|A(t)e^{-(t-r)A(t)}A^{-1}(r)q(r,s)\|\,dr \\ & + C_1\delta\int_s^t \|A(t)e^{-(t-r)A(t)}[A(t)-A(r)]Q(r,s)\|\,dr \\ & + \int_s^t \|A(t)e^{-(t-r)A(t)}[A(t)-A(r)]A^{-1}(r)\|\|\delta Q(r,s)\|\,dr. \end{aligned}$$

Since the relevant functions inside the integrals are continuous, we obtain

$$\|\delta Q(t,s)\| \le C\delta + B\int_s^t \sup_{s\le r\le t} \|\delta Q(t,r)\|.$$

Hence,

$$\|\delta Q(t,s)\| \le \tilde{C}\delta. \qquad \blacksquare$$

Proposition 3.8. *Assume* B–I *and* B–II. *Then,* $U^u(t,s)$*, the evolution systems for the Cauchy problem* (3.3.2) *and* (3.3.3)*, is norm-continuous in* u*, where* u *is the initial condition in* (3.3.3)*.*

Proof. From Theorem 3.3 it follows that

$$\frac{\partial}{\partial r}(U^x(r,s) - U^y(r,s)) = -[A(F_r(x))U^x(r,s) + A(F_r(y))U^y(r,s)].$$

Integration from s to t yields

$$\begin{aligned} U^x(t,s) - U^y(t,s) = -\int_s^t &[A(F_r(x))U^x(r,s)A^{-1}(F_s(x))A(F_s(x)) \\ &- A(F_r(y))U^y(r,s)A^{-1}(F_s(y))A(F_s(y))]dr \end{aligned}$$

Hence,

$$\begin{aligned} \|[U^x(t,s) - U^y(t,s)]v\| \le \int_s^t &\|\delta Q(r,s)A(F_s(x))\,v\| \\ &+ \|Q(r,s)[A(F_s(x)) - A(F_s(y))]\,v\|\,dr \end{aligned}$$

The first term has the estimate,

$$\|\delta Q(r,s)A(F_s(x))\,v\| \le C\delta\|A(F_s(x))\,v\|,$$

for some constant C by Lemma 2.1, and from Proposition 3.6 (inequality (3.3.6)) we get

$$\|A(F_r(x))U^x(r,s)A^{-1}(F_s(y))[A(F_s(x)) - A(F_s(y))]\,v\| \le C_1\delta\|v\|_D\,,$$

where $Q(r,s)$ and $\delta Q(r,s)$ are as in Lemma 3.7, and $\delta = \|x-y\|$. Therefore,

$$\lim_{\|x-y\|\to 0} \|U^x(t,s) - U^y(t,s)\| = 0\,. \qquad \blacksquare$$

3.5 Spatial Differentiability of Solutions

Returning to the evolution equation (3.3.1),

$$\frac{du}{dt} = G(u,t) = A(u,t)u + g(u,t),$$

we examine the spatial derivatives of its solutions. The following results will be used in the proof:

Bounded Perturbation Theorem. *If $A \in \mathcal{G}(\mathcal{X}, M, \beta)$ (the space of generators on $\mathcal{X}$ with bounds M and β, as defined in Section 3.2) and $B \in \mathcal{B}(\mathcal{X})$, then $A + B \in \mathcal{G}(\mathcal{X}, M, \beta + \|B\|M)$.* (See Kato [1977], p.495.)

Trotter–Kato Theorem. *If $A_n \in \mathcal{G}(\mathcal{X}, M, \beta)$ $(n = 1, 2, \cdots)$, $A \in \mathcal{G}(\mathcal{X}, M, \beta)$ and for λ sufficiently large, $(\lambda - A_n)^{-1} \to (\lambda - A)^{-1}$ strongly, then $e^{tA_n} \to e^{tA}$ strongly, uniform on bounded t-intervals* (See Kato [1977], p.502.)

Theorem 3.9. *Assume conditions of Theorem 3.2 are satisfied. Assume also that* B–I *and* B–II *are satisfied. If $Dg(u_0) = 0$ and Dg is continuous at u_0, then in some neighborhood U of u_0 the solution to* (3.3.1)*, $F_{t,s}(x)$, is differentiable with respect to x, with $DF_{t,s}(x)$ being the solution to*

$$\frac{\partial w}{\partial t} = \left[A(F_t(x)) + Dg(F_t(x))\right] w + A'(F_t(x))\, w\, (F_t(x) + w)\,. \tag{3.3.13}$$

Proof. Let $\Delta(t,s) = F_{t,s}(x) - F_{t,s}(y)$. We have, by construction,

$$\begin{aligned}
\Delta(t,s) &= U^x x - U^y y + \int_s^t \left[U^x g(F(x)) - U^y g(F(y))\right]\, ds \\
\frac{\partial \Delta(t,s)}{\partial t} &= -[A(t,F(x))F(x) - A(t,F(y))F(y)] + g(F(x)) - g(F(y)), \\
\frac{\partial U^x(t,s)}{\partial t} &= -A(t, F_{t,s}(x))U^x(t,s)\,, \\
\frac{\partial U^x(t,r)}{\partial r} &= U^x(t,r)A(r, F_{r,s}(x))\,,
\end{aligned}$$

where we have dropped subscripts or arguments in $F_{t,s}(x)$, $F_{t,s}(y)$, $U^x(t,s)$ and $U^y(t,s)$ respectively in the first two equations.

Let $\Delta_t(h) := F_t(x+h) - F_t(x)$. From Proposition 3.6, and Proposition 3.8, it follows that

$$\begin{aligned}
&\frac{\partial}{\partial t}(F_t(x+h) - F_t(x)) \\
&\quad = A(F_t(x+h))F_t(x+h) + g(F_t(x+h)) - A(F_t(x))F_t(x) - g(F_t(x)) \\
&\quad = A(F_t(x))(\Delta_t(h)) + Dg(F_t(x))(\Delta_t(h)) + o(\Delta_t(h)) + [A'(F_t(x))\Delta_t(h) \\
&\qquad\qquad + o(\Delta_t(h))]F_t(x) + [A'(F_t(x))\Delta_t(h) + +o(\Delta_t(h))]\Delta_t(h) \\
&\quad = [A(F_t(x)) + Dg(F_t(x)) + \rho(x,h)]\Delta_t(h) + [A'(F_t(x))\Delta_t(h) \\
&\qquad\qquad + \tilde{\rho}(x,h)]F_t(x) + [A'(F_t(x))\Delta_t(h) + \tilde{\rho}(x,h)]\Delta_t(h)\ ,
\end{aligned}$$

where $\rho(x,h)$ and $\tilde{\rho}(x,h)$ are operators continuous in x and h, whose norms satisfy

$$\lim_{\|h\|\to 0} \frac{\|\rho(x,h)\|}{\|h\|} = 0\,, \qquad (3.3.14)$$

$$\lim_{\|h\|\to 0} \frac{\|\tilde{\rho}(x,h)\|}{\|h\|} = 0\,,$$

since $\|F_t(x+h) - F_t(x)\| \to 0$ as $\|h\| \to 0$. Fix x and h. Let $\zeta_t(h)$ be the solution to

$$\frac{\partial \zeta_t(h)}{\partial t} = [A(F_t(x)) + Dg(F_t(x)) + \rho(x,h)]\zeta_t(h) + \tilde{g}(\zeta_t(h))\;, \qquad (3.3.15)$$

$$\zeta_0(h) = I, \qquad (3.3.16)$$

where

$$\tilde{g}(w) = [A'(F_t(x))w + \tilde{\rho}(x,h)]F_t(x) + [A'(F_t(x))w + \tilde{\rho}(x,h)]w.$$

Since Dg is continuous at u_0 with $Dg(u_0) = 0$, $F_t(x)$ is strongly continuous in t and $\|e^{A(F_t(x))}\| \leq e^{-t\delta}$ for some $\delta > 0$, there exists $[0,t_0]$ such that for $t \in [0,t_0]$

$$\|e^{A(F_t(x))+Dg(F_t(x))}\| \leq e^{-t\tilde{\delta}} \qquad (3.3.17)$$

for some $\tilde{\delta} > 0$, by the Bounded Perturbations Theorem. By the same argument, we know that there exist $[0,t_0]$ and $\epsilon > 0$, such that for $t \in [0,t_0]$ and $h \leq \epsilon$,

$$\|e^{[A(F_t(x))+Dg(F_t(x))+\rho(x,h)]}\| \leq e^{-t\tilde{\delta}}. \qquad (3.3.18)$$

Therefore, $\zeta_t(h)$ exists over $[0,t_0]$, for all $h \leq \epsilon$, by Theorem 3.2.

Thus, letting $\theta_t(h) := \Delta(x,h,t) - \zeta_t(h)\cdot h$, we have

$$\frac{\partial \theta_t(h)}{\partial t} = [A(F_t(x)) + Dg(F_t(x)) + \rho(x,h)]\theta_t(h) + \tilde{g}(\theta_t(h))\;, \qquad (3.3.19)$$

$$\theta_0(h) = 0\,. \qquad (3.3.20)$$

We now show that $\|\theta_t(h)\|/\|h\| \to 0$ as $\|h\| \to 0$. From the preceding two equalities, we get

$$\frac{\partial \|\theta_t(h)\|}{\partial t} \leq \|\rho(x,h)\|\,\|\theta_t(h)\| + M\|\theta(h)\|$$

where $M > 0$ is a constant independent of h. Hence, by (3.3.14)

$$\frac{\partial}{\partial t}\left(\frac{\|\theta(h)\|}{\|h\|}\right) \leq \epsilon(\|h\|) + M\left(\frac{\|\theta(h)\|}{\|h\|}\right) \qquad (3.3.21)$$

where $\epsilon(\|h\|) \to 0$ as $\|h\| \to 0$. Thus,

$$\lim_{\|h\|\to 0} \frac{\|F_t(x+h) - F_t(x) - \zeta_t(h)\cdot h\|}{\|h\|} = 0\,. \tag{3.3.22}$$

by Gronwall's inequality. Therefore, $D_x F_t(x) = \lim_{h\to 0} \zeta_t(h)$, if the limit exists.

Next, we show that $\lim_{h\to 0} \zeta_t(h)$ exists as a result of the Trotter-Kato Theorem. First, we need to prove that $\zeta_t(h_n)$ is a Cauchy sequence for $h_n \to 0$, which is equivalent to showing that, for any two $h_1,\ h_2 < \epsilon$, $\|\zeta_t(h_1) - \zeta_t(h_2)\|$ can be made arbitrarily small if $h_1,\ h_2$ are small enough.

Since

$$\frac{\partial \zeta_t(h_1)}{\partial t} = [A(F_t(x)) + Dg(F_t(x)) + \rho(x,h_1)]\zeta_t(h_1) + \tilde{g}(\zeta_t(h_1))$$

and

$$\frac{\partial \zeta_t(h_2)}{\partial t} = [A(F_t(x)) + Dg(F_t(x)) + \rho(x,h_2)]\zeta_t(h_2) + \tilde{g}(\zeta_t(h_1))\ ,$$

we have

$$\frac{\partial \|\zeta_t(h_1) - \zeta_t(h_2)\|}{\partial t} \leq M\|\zeta_t(h_1) - \zeta_t(h_2)\| + \|\rho(x,h_1)\zeta_t(h_1) - \rho(x,h_2)]\zeta_t(h_2)\|\ .$$

By (3.3.14), for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$\|\rho(x,h_1)\zeta_t(h_1) - \rho(x,h_2)]\zeta_t(h_2)\| < \epsilon$$

if $h_1,\ h_2 < \delta$. Thus,

$$\|\zeta_t(h_1) - \zeta_t(h_2)\| \leq \epsilon\, t_0\, e^{Mt_0} \to 0 \qquad \text{for any } t \in [0,t_0], \tag{3.3.23}$$

as $h_1,\ h_2 \to 0$.

Before we can apply Trotter-Kato Theorem, we have to verify that for λ sufficiently large, $(\lambda - A_n)^{-1} \to (\lambda - A)^{-1}$ strongly, where A_n stands for

$$A(F_t(x)) + Dg(F_t(x)) + \rho(x,h_n),$$

and A stands for $A(F_t(x)) + Dg(F_t(x))$. Since $A_n \to A$ strongly by (3.3.14), this is a consequence of the *resolvent identity*:

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu\,,$$

where $R_\lambda = (\lambda - A)^{-1},\ R_\mu = (\mu - A)^{-1}$ are the resolvents of $A \in \mathcal{G}(\mathcal{X}, M, \beta)$ for $\lambda > \beta$ and $\mu > \beta$.

Let $\tilde{U}^h(t,s)$ be the evolution system associated with

$$A(F_t(x)) + Dg(F_t(x)) + \rho(x,h),$$

and $\tilde{U}(t,s)$ be the evolution system associated with

$$A(F_t(x)) + Dg(F_t(x)).$$

Then, $\lim_{h\to 0} \tilde{U}^h(t,s) = \tilde{U}(t,s)$, as a result of the Trotter-Kato Theorem. Taking the limit of

$$\zeta_t(h) = \tilde{U}^h(t,0)I + \int_s^t \tilde{U}^h(t,s)\tilde{g}(\zeta_s(h))\,ds \tag{3.3.24}$$

as $h \to 0$ produces

$$D_xF_t(x) = \tilde{U}(t,0)I + \int_s^t \tilde{U}(t,s)A'(F_t(x))D_xF_s(x)[F_t(x) + D_xF_s(x)]\,ds. \tag{3.3.25}$$

■

3.6 Main Results

With the preceding preparations we are ready to verify that under the conditions given below, conditions A–I through A–VII from Section 2 are satisfied.

Theorem 3.10. *(Existence and Continuity of Solutions with respect to Initial Data) Let D and E be two Banach spaces, with D continuously and densely included in E. Let $G(u) = A(u)u + g(u)$, where $g(u)$ is a nonlinear map from a neighborhood U of u_0 in D into E, $A(u)$ is a closed linear operator from D into E for each $u \in U$. Assume* B–I, B–II, *and*

B–0 *u_0 is a fixed point of $G(u)$, $g(u_0) = 0$, $Dg(u_0) = 0$, and Dg is continuous at u_0.*

B–III *There are positive numbers ω and β such that $\Sigma(\omega,\beta)$ (defined in Section 3.2) is contained in the resolvent set of the operator $A(u_0)$. Moreover, there exists $C > 0$ such that*

$$\|[A_0 + \lambda I]^{-1}\| \le C[|\lambda| + 1]^{-1},$$

for all $\lambda \in \Sigma(\omega,\beta)$.

Then, there exists a neighborhood U_0 of u_0 and T_0, such that for any $u \in U_0$, the Cauchy problem

$$\frac{dv}{dt} = A(t,v)v + g(t,v),$$
$$v(0) = u \in U_0.$$

has a unique solution $F_t(u) \in D$, with lifetime at least T_0. Furthermore, $F_t(u)$ is continuous in $t \in [0, T_0]$ and $u \in U_0$.

Proof. Recall that the spectrum of a bounded operator on a Banach space is upper-continuous (cf. Lemma 2.5). Since $A(u)$ is continuous in u, we can find a neighborhood U_0 of u_0 and β' such that for any $u \in U_0$, $\Sigma(\omega, \beta')$ is contained in the resolvent set of the operator $A(u)$, and there exists $C' > 0$ such that

$$\|[A(u) + \lambda I]^{-1}\| \leq C'[|\lambda| + 1]^{-1},$$

for all $\lambda \in \Sigma(\omega, \beta')$. The assertion now follows from Theorem 3.2 and Proposition 3.4. ■

Thus, the conditions A–I, A–II and A–III from Section 2 are satisfied.

Theorem 3.11. *(Existence, Norm-continuity in x, and Strong Continuity in t of $D_x F_t(x)$) Under the same notation and conditions as in Theorem 3.10, the solution $F_t(u)$ to the Cauchy problem*

$$\frac{dv}{dt} + A(t, v)v = g(t, v),$$
$$v(0) = u \in U_0.$$

is differentiable with respect to u with

$$D_u F_t(u) = \tilde{U}(t, 0)I + \int_s^t \tilde{U}(t, s)A'(F_t(u))D_u F_s(u)[F_t(u) + D_u F_s(u)]\, ds\ ,$$

where $\tilde{U}(t, s)$ is the evolution system associated with $A(F_t(u)) + Dg(F_t(u))$. Furthermore,

A–IV *$D_u F_t(u)$ is norm-continuous in u for fixed $t \in (0, T_0]$ for some T_0.*

A–V *$D_u F_t(u)$ is strongly continuous in t for fixed u in some neighborhood of u_0.*

Proof. By the same argument as in Theorem 3.10, it follows from Theorem 3.9 that there exists a $t_0 > 0$ such that

$$D_u F_t(u) = \tilde{U}(t, 0)I + \int_s^t \tilde{U}(t, s)A'(F_s(u))D_u F_s(u)[F_s(u) + D_u F_s(u)]\, ds\ ,$$

exists in a neighborhood U_0 of u_0 for $0 \leq t \leq t_0$. By Proposition 3.4, for fixed $t \in [0, t_0]$ $D_u F_t(u)$ is norm-continuous in u. A–V follows because $D_u F_t(u)$ is continuous in t, as a solution to equation (3.3.13). ■

Theorem 3.12. *Under the same notation and conditions as in Theorem 3.10, the solution $F_t(u)$ to the Cauchy problem*

$$\frac{dv}{dt} + A(t,v)v = g(t,v),$$
$$v(0) = u \in U_0.$$

also satisfies the following properties,

A–VI *$DF_t(u_0)$ is norm-continuous in t for $t \in (0, T_0]$, i.e.,*

$$\lim_{t \to t_0} \|DF_t(u_0) - DF_{t_0}(u_0)\| = 0$$

for any $t_0 \in (0, T_0]$.

A–VII *Strong continuity of $D_uF_t(u)$ in t at $t = 0$ is uniformly bounded in u locally at $u = u_0$, that is, given any $x \in E$, there exist $M_x > 0$, $\varepsilon > 0$, and a neighborhood U_x of u_0 such that*

$$\|DF_t(u)x - DF_0(u)x\| \equiv \|DF_t(u)x - x\| \leq M_x,$$

for all $0 \leq t < \varepsilon$ and $u \in U_x$.

A–VIII *the spectrum $\sigma(DF_t(u_0))$ lies uniformly inside the unit circle for $t \in (0, T_0]$.*

Proof. Recall that by Theorem 3.9 there exists a $t_0 > 0$ such that

$$D_uF_t(u) = \tilde{U}(t,0)I + \int_0^t \tilde{U}(s,0)A'(F_s(u))D_uF_s(u)[F_s(u) + D_uF_s(u)]\,ds\ ,$$

in a neighborhood U of u_0 and $\tilde{U}(t,s)$ is the evolution system associated with $A(F_t(u))+Dg(F_t(u))$. Hence, at u_0, $\tilde{U}(t,0) = e^{A(u_0)t}$ with $\|\tilde{U}(t,0)\| \leq e^{-\beta t}$ in view of B-III. Noting that the integrand in the second term is continuous in s, we see that there exist $T_0 > 0$ and $\beta' > 0$ such that the resolvent set of $D_uF_t(u)$ is contained in $\Sigma(\omega, \beta')$, for all $t \in [0, T_0]$ by Lemma 2.5, which is equivalent to A–VIII. To show that A–VI is satisfied, we note first that at u_0,

$$D_uF_t(u_0) = \tilde{U}(t,0) + \int_0^t \tilde{U}(s,0)A'(u_0)D_uF_s(u_0)[u_0 + D_uF_s(u_0)]\,ds\ ,$$

and

$$\frac{\partial \tilde{U}(t,0)}{\partial t} = A(u_0)\tilde{U}(t,0).$$

Pick an arbitrary unit vector $x \in \mathcal{Y}$, and $t_0 \in (0, T_0]$. Then

$$(D_u F_t(u_0) - D_u F_{t_0}(u_0))\, x = (\tilde{U}(t,0) - \tilde{U}(t_0,0))\, x + \left(\int_{t_0}^{t} \tilde{U}(s,0) A'(u_0) D_u F_s(u_0)[u_0 + D_u F_s(u_0)]\, ds \right) x. \quad (3.3.26)$$

By the basic properties of semigroups, $\|(\tilde{U}(t) - \tilde{U}(t_0))x\| \to 0$, when t tends to t_0. Note also that the integrand in second term is bounded in norm and continuous in s. It is obvious that

$$\lim_{t \to t_0} \sup \|(D_u F_t(u_0) - D_u F_{t_0}(u_0))x\| = 0.$$

Let $t_0 = 0$ in (3.3.26). Condition A–VII follows by the same argument. ■

Finally, as a consequence of Theorem 2.8 in Section 1 and the preceding theorems, we have the following result about asymptotic stability and global existence of solutions to (3.3.1) in a neighborhood of a fixed point.

Theorem 3.13. *Let D and E be two Banach spaces, with D continuously and densely included in E. Let $G(u) = A(u)u + g(u)$, where $g(u)$ is a nonlinear map from a neighborhood U of u_0 in D into E, $A(u)$ is a closed linear operator from D into E for each $u \in U$. Assume*

B–0 *u_0 is a fixed point of $G(u)$, $g(u_0) = 0$, $Dg(u_0) = 0$, and Dg is continuous at u_0;*

B–I *$(A'(u) \cdot v)x$ is Lipschitz continuous in u, i.e.,*

$$\|(A'(u_1) - A'(u_2)) \cdot v)x\| \le C\, \|u_1 - u_2\| \cdot \|v\| \cdot \|x\|_D,$$

where $u_1, u_2 \in U$;

B–II *for all $u_1, u_2 \in U$, we have*

$$\|g(u_1) - g(u_2)\| \le C \|u_1 - u_2\|;$$

B–III *There are positive ω and β such that $\Sigma(\omega, \beta)$ is contained in the resolvent set of the operator $A(u_0)$, and there exists $C > 0$ such that*

$$\|[A_0 + \lambda I]^{-1}\| \le C[|\lambda| + 1]^{-1},$$

for all $\lambda \in \Sigma(\omega, \beta)$.

Then, there exists a neighborhood U of u_0 such that $G(u) = A(u)u + g(u)$ generates a semiflow $F_t(u)$ in U, and

I *Global Existence in U: each $u \in U$ has infinite lifetime;*

II *Asymptotic Stability at u_0:*

$$\lim_{t\to\infty} \|F_t(u) - u_0\| = 0 \qquad \text{for all } u \in U.$$

Future Directions

It is our belief that the present context also allows one to prove invariant manifold theorems. Some progress in this direction was already made by, for example, Renardy [1992]. For example, it would be interesting to be able to apply some of the work on dissipation induced instabilities of Bloch, Marsden, Krishnaprasad, and Marsden [1994, 1995] to the present context. This should also allow one to prove theorems on, for example, the Hopf bifurcation for quasilinear pde's of the sort that occur in nonlinear elasticity. Remarkably little has been done in this area despite all of the activity in infinite dimensional dynamical systems.

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REFERENCES

Abraham, R., J. E. Marsden, and T. S. Ratiu [1988] *Manifolds, Tensor Analysis, and Applications.* Second Edition, Appl. Math. Sci. **75**, Springer, New York

Bloch, A. M., P. S. Krishnaprasad, J. E. Marsden, and T. S. Ratiu [1994] Dissipation Induced Instabilities, *Ann. Inst. H. Poincaré, Analyse Nonlineare* **11**, 37–90.

Bloch, A. M., P. S. Krishnaprasad, J. E. Marsden, and T. S. Ratiu [1995] The Euler-Poincaré equations and double bracket dissipation. *Comm. Math. Phys.* (to appear).

Browne, R. C. [1978], Dynamical Stability of One-dimensional Nonlinearly Viscoelastic Bodies, *Arch. Rat. Mech. Anal.*, **68**, 257–282.

Chernoff, P. R. and J. E. Marsden [1972] On continuity and smoothness of group actions, *Bull. Am. Math. Soc.* **76**, 1044–1049.

Dunford, N. and Schwartz, J. [1963], *Linear Operators, Part I*, John Wiley & Sons, Inc.

Hughes, T. J. R., T. Kato, and J. E. Marsden [1977] Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rat. Mech. Anal.* **90**, 545–561.

Kato, T. [1977] *Perturbation Theory for Linear Operators.* Springer-Verlag, Second Edition, 1984.

Marsden, J. E. and T. J. R. Hughes [1983] *Mathematical Foundations of Elasticity.* Prentice Hall, reprinted by Dover Publications, N.Y., 1994.

Marsden, J. E. and M. McCracken [1976] *The Hopf Bifurcation and its Applications*. Appl. Math. Sci. **19**, Springer, New York.

Potier–Ferry, M. [1981], Linearization Principle for Stability in Quasilinear Parabolic Equations I, *Arch. Rat. Mech. Anal.*, **77**, 301–320.

Potier–Ferry, M. [1982] On the mathematical foundations of elastic stability theory. *Arch. Rat. Mech. Anal.* **78**, 55–72.

Renardy, M. [1992] A centre manifold theorem for hyperbolic PDEs'. *Proc. Roy. Soc. Edinburgh* **122A**, 363–377.

Simo, J. C., J. E. Marsden, and P. S. Krishnaprasad [1988] The Hamiltonian structure of nonlinear elasticity: The material, spatial, and convective representations of solids, rods, and plates, *Arch. Rat. Mech. Anal.* **104**, 125–183.

Simo, J. C., T. A. Posbergh, and J. E. Marsden [1990] Stability of coupled rigid body and geometrically exact rods: block diagonalization and the energy-momentum method, *Physics Reports* **193**, 280–360.

Sobolevskii, P. E. [1966], Equations of Parabolic Type in Banach Space, *Amer. Math. Soc. Transl.*, **49**, 1–62.