ANNALES DE L'I. H. P., SECTION A

ARTHUR E. FISCHER
JERROLD E. MARSDEN
VINCENT MONCRIEF

The structure of the space of solutions of Einstein's equations. I. One Killing field

Annales de l'I. H. P., section A, tome 33, nº 2 (1980), p. 147-194.
<http://www.numdam.org/item?id=AIHPA_1980__33_2_147_0>

Ann. Inst. Henri Poincaré
Vol. XXXIII, n° 2, 1980, p. 147-194.

Section A :
Physique théorique.

The structure of the space of solutions of Einstein's equations. I. One Killing field

by

Arthur E. FISCHER (*), Jerrold E. MARSDEN (**) and Vincent MONCRIEF (***)

(*) Department of Mathematics,
University of California, Santa Cruz, California 95064
(**) Department of Mathematics,
University of California, Berkeley, California 94720
(***) Department of Physics,
Yale University, New Haven, Connecticut 06520

CONTENTS

INTRODUCTION

A vacuum spacetime is a four dimensional Lorentz manifold $(V_4, {}^{(4)}g)$ that satisfies the Einstein empty spacetime equations

$$\mathrm{Ein}({}^{(4)}g) = \mathrm{Ric}({}^{(4)}g) - \frac{1}{2}\,{}^{(4)}g\mathrm{R}({}^{(4)}g) = 0\,,$$

where $\mathrm{Ric}({}^{(4)}g)$ is the Ricci tensor of ${}^{(4)}g$, and $\mathrm{R}({}^{(4)}g)$ is the scalar curvature. We may regard $\mathrm{Ein}({}^{(4)}g)$ as a rational function of degree eight in ${}^{(4)}g$ and its first two derivatives. Therefore one might guess that the space of all solutions $\mathscr{E}$ of $\mathrm{Ein}({}^{(4)}g) = 0$ is a complicated algebraic variety. The main purpose of this paper is to show that the singularities in this variety are precisely conical, i. e., of degree two, and to describe this conical structure explicitly.

The motivation and methods we use come from the perturbation theory of spacetimes and the related concept of linearization stability. A spacetime ${}^{(4)}g_0$ is called *linearization stable* if every solution of the linearized Einstein equations is integrable, i. e., is tangent to a curve of exact solutions of Einstein's equations. For instance, if $\mathscr{E}$ is a smooth manifold in a neighborhood of ${}^{(4)}g_0$, then ${}^{(4)}g_0$ will be linearization stable. (See § 1 for the precise definitions.)

In previous papers the following results have been shown for spacetimes with compact Cauchy surfaces:

1) *For a solution ${}^{(4)}g_0$ of Einstein's equations, the following are equivalent:*

i) *${}^{(4)}g_0$ has no Killing fields,*

ii) *${}^{(4)}g_0$ is linearization stable,*

iii) *the space of solutions $\mathscr{E}$ is a manifold in the neighborhood of ${}^{(4)}g_0$ and has as its tangent space the space of solutions of the linearized equations at ${}^{(4)}g_0$.*

2) *Let ${}^{(4)}g_0$ have Killings fields. If ${}^{(4)}h$ is a solution of the linearized equations at ${}^{(4)}g_0$, then in order for ${}^{(4)}h$ to be integrable certain conserved quantities of Taub quadratic in ${}^{(4)}h$ must vanish identically. There is one such quantity for each linearly independent Killing vector field.* (We refer to these restrictions as the *second order condition.*)

These results are proved in Fischer and Marsden [1973, 1975], Moncrief [1975, 1976] and Arms and Marsden [1979]. They are briefly reviewed in Section 2.

In this paper we prove the converse of 2. This converse can be interpreted in several ways, as follows:

a) If ${}^{(4)}h$ satisfies the linearized equations and the Taub conserved quantities vanish, then there is a curve of exact solutions through ${}^{(4)}g_0$ that is tangent to ${}^{(4)}h$.

b) The set $\mathscr{E}$ has (nontrivial) quadratic, i. e., conical singularities near spacetimes with Killing fields.

c) A perturbation expansion around the background ${}^{(4)}g_0$ may need adjustment at second order, but once this is done, it can always be completed to a convergent expansion.

This paper deals with globally hyperbolic solutions of the vacuum Einstein equations in a neighborhood of spacetimes that have a compact

Cauchy surface of constant mean curvature. (The latter condition, believed to be valid rather generally, is discussed in Marsden and Tipler [1979].) In this part of the paper we deal with solutions which have a single Killing vector field. Part II of the paper will then deal with the case of several Killing fields and general applications to mechanics.

The proofs in this paper rely on the Morse lemma of Tromba [1976], the Ebin-Palais slice theorem (Ebin [1970]), and the authors'earlier work. Strictly speaking, we are only after the structure of the zero set of a function, so the Morse lemma is a bit of overkill and in fact the arguments can be simplified somewhat. However, the stronger conclusions given by the Morse lemma, namely a normal form for the whole function not merely its zero set, may be useful for other purposes.

For many Killing fields discussed in Part II (done jointly with J. M. Arms) we shall use the bifurcation theory developed by Buchner, Schecter and Marsden [1979] and the Kuranishi theory of deformations, in a form inspired by Atiyah, Hitchin and Singer [1978].

There are two interesting directions of generalization possible:

(A) To fields coupled to gravity, such as Yang-Mills fields. As shown by Arms [1978], the Hamiltonian formalism of Fischer and Marsden [1978] is applicable to Yang-Mills fields, and the corresponding generalization of linearization results 1 and 2 have been obtained. It seems that the analysis of the present paper will generalize as well.

(B) The other direction of generalizations is to spacetimes with non-compact Cauchy surfaces, say Cauchy surfaces that are asymptotically flat. Here, under rather general conditions, Choquet-Bruhat, Fischer and Marsden [1978] have shown that $\mathscr{E}$ is a manifold, even near spacetimes with Killing fields. However, the second order conditions emerge at spatial infinity in perturbations of the total energy and momentum; see Moncrief [1978]. This signals not trouble in the solution manifold, but rather the development of cones in the level sets for the total energy and angular momentum. It is this reduced space that will still have singularities and the methods in this paper can be used to analyze them.

Our spacetimes are always assumed to be smooth, oriented and time-oriented with smooth compact spacelike Cauchy surfaces. By Budic *et al.* [1978], any compact smooth spacelike hypersurface will be a Cauchy surface. The spacetime $(V_4, {}^{(4)}g)$ associated with Cauchy data (g, π) on a spacelike hypersurface Σ will always mean the maximal Cauchy development, unique up to diffeomorphisms of the spacetime. The spacetime is topologically $\Sigma \times \mathbb{R}$. Results like these are conveniently available in Hawking and Ellis [1973] and will not be referenced explicitly.

The authors wish to acknowledge a number of helpful remarks provided by J. Arms, Y. Choquet-Bruhat, R. Jantzen, R. Palais, R. Sachs, S. Schecter, I. Singer, F. Tipler, A. Taub and A. Weinstein.

§ 1. PRELIMINARY CALCULATIONS AND DEFINITIONS

Let $(V_4, {}^{(4)}g)$ be a spacetime. We let $\mathrm{Ric}({}^{(4)}g)$ = the Ricci tensor of ${}^{(4)}g$, $\mathrm{R}({}^{(4)}g)$ = trace $[\mathrm{Ric}({}^{(4)}g)]$ = the scalar curvature and

$$\mathrm{Ein}({}^{(4)}g) = \mathrm{Ric}({}^{(4)}g) - \frac{1}{2}\mathrm{R}({}^{(4)}g){}^{(4)}g = \text{the Einstein tensor}.$$

Our sign conventions are those of Misner, Thorne and Wheeler [1973]. These conventions are determined by the Ricci commutation formula

$$X^{\gamma}{}_{;\alpha;\beta} - X^{\gamma}{}_{;\beta;\alpha} = -\mathrm{R}^{\gamma}{}_{\delta\alpha\beta}X^{\delta}$$

where $\alpha, \beta, \ldots = 1, 2, 3, 4$ and ; denotes covariant differentiation with respect to ${}^{(4)}g$, and by the definition

$$\mathrm{R}_{\alpha\beta} = \mathrm{R}^{\gamma}{}_{\alpha\gamma\beta}.$$

Using appropriate functions spaces (see, e. g., Fischer and Marsden [1975]), the functions Ric, R and Ein are C^{∞} in ${}^{(4)}g$. The derivatives of these mappings are given as follows:

$$a)\ \mathrm{D}\,\mathrm{Ric}({}^{(4)}g)\cdot{}^{(4)}h = \frac{1}{2}\{\,\square_{\mathrm{L}}{}^{(4)}h - \alpha_{(4)g}\delta_{(4)g}{}^{(4)}h\text{-Hess tr }({}^{(4)}h)\,\}$$
$$= \frac{1}{2}\left\{\,\square_{\mathrm{L}}{}^{(4)}h - \alpha_{(4)g}\delta_{(4)g}\left({}^{(4)}h - \frac{1}{2}\mathrm{tr}\,({}^{(4)}h){}^{(4)}g\right)\right\},$$

$$b)\ \mathrm{DR}({}^{(4)}g)\cdot{}^{(4)}h = \square\,\mathrm{tr}\,({}^{(4)}h) + \delta_{(4)g}\delta_{(4)g}{}^{(4)}h - {}^{(4)}h\cdot\mathrm{Ric}({}^{(4)}g),$$

$$c)\ \mathrm{D}\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h = \frac{1}{2}\{\,\square_{\mathrm{L}}{}^{(4)}\bar{h} - \alpha_{(4)g}\delta_{(4)g}{}^{(4)}\bar{h} - (\delta_{(4)g}\delta_{(4)g}{}^{(4)}\bar{h}){}^{(4)}g\,\}$$
$$+ \frac{1}{2}\{\,({}^{(4)}\bar{h}\cdot\mathrm{Ric}({}^{(4)}g)){}^{(4)}g - \mathrm{R}({}^{(4)}g){}^{(4)}\bar{h}\,\}.$$

where

$$(\square_{\mathrm{L}}{}^{(4)}h)_{\alpha\beta} = -g^{\mu\nu}h_{\alpha\beta;\mu;\nu} + \mathrm{R}^{\mu}_{\alpha}h_{\mu\beta} + \mathrm{R}^{\mu}_{\beta}h_{\mu\alpha} - 2\mathrm{R}^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}h_{\mu\nu}$$

is the Lichnerowicz Laplacian acting on covariant symmetric two tensors ${}^{(4)}h$,

$$\alpha_{(4)g}({}^{(4)}X) = \mathrm{L}_{(4)X}{}^{(4)}g = X_{\alpha;\beta} + X_{\beta;\alpha},$$
$$\delta_{(4)g}{}^{(4)}h = -{}^{(4)}h_{\alpha}{}^{\beta}{}_{;\beta},$$
$$\mathrm{Hess}(\mathrm{tr}\,{}^{(4)}h) = h^{\mu}{}_{\mu;\alpha;\beta},$$
$$\square\,\mathrm{tr}\,{}^{(4)}h = -g^{\alpha\beta}h^{\mu}{}_{\mu;\alpha;\beta},$$

and where ${}^{(4)}\bar{h} = {}^{(4)}h - \frac{1}{2}(\mathrm{tr}\,{}^{(4)}h){}^{(4)}g$, i. e. $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h^{\mu}{}_{\mu}g_{\alpha\beta}$.

For example, in coordinates

$$[\mathrm{DRic}({}^{(4)}g)\cdot{}^{(4)}h]_{\alpha\beta} = \frac{1}{2}[-g^{\mu\nu}h_{\alpha\beta;\mu;\nu} + h_{\alpha}{}^{\mu}{}_{;\mu;\beta} + h_{\beta}{}^{\mu}{}_{;\mu;\beta}$$
$$+ \mathrm{R}_{\alpha}{}^{\mu}h_{\mu\beta} + \mathrm{R}_{\beta}{}^{\mu}h_{\mu\alpha} - 2\mathrm{R}^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}h_{\mu\nu}]$$

The formulas for $R({}^{(4)}g)$ and $\mathrm{Ein}({}^{(4)}g)$ follow from that for Ric using

$$[D({}^{(4)}g^{-1})\cdot{}^{(4)}h]^{\alpha\beta} = -{}^{(4)}h^{\alpha\beta} = -{}^{(4)}g^{\alpha\mu}{}^{(4)}g^{\beta\nu}{}^{(4)}h_{\mu\nu}$$

We shall now give a number of lemmas which will lead to Taub's conserved quantities. Some of these calculations are « well known » and are included for the reader's convenience.

1.1. LEMMA. — *If* $\mathrm{Ein}({}^{(4)}g) = 0$, *and* ${}^{(4)}h$ *is any symmetric two tensor, then*

$$\delta[D\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h] = 0$$

where $\delta = \delta_{{}^{(4)}g}$ *is the divergence with respect to* ${}^{(4)}g$.

Proof. — The contracted Bianchi identities assert that $\delta\mathrm{Ein}({}^{(4)}g) = 0$. Differentiation gives the identity

$$[D\delta({}^{(4)}g)\cdot{}^{(4)}h]\cdot\mathrm{Ein}({}^{(4)}g) + \delta[D\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h] = 0$$

where $\delta({}^{(4)}g) = \delta_{{}^{(4)}g}$ indicates the functional dependence of δ on ${}^{(4)}g$, and $[D\delta({}^{(4)}g)\cdot{}^{(4)}h]$ is the linearized divergence operator acting on $\mathrm{Ein}({}^{(4)}g)$. The lemma follows since $\mathrm{Ein}({}^{(4)}g) = 0$. ■

1.2. LEMMA. — *Suppose* $\mathrm{Ein}({}^{(4)}g) = 0$ *and* $D\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h = 0$. *Then*

$$\delta[D^2\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h)] = 0.$$

Proof. — This follows by differentiating the contracted Bianchi identities twice to give

$$\begin{aligned}&[D^2\delta({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h)]\cdot\mathrm{Ein}({}^{(4)}g) + [D\delta({}^{(4)}g)\cdot{}^{(4)}h]\cdot(D\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h)\\ &\quad + [D\delta({}^{(4)}g)\cdot{}^{(4)}h]\cdot(D\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h) + \delta[D^2\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h)] = 0\end{aligned}$$

and then using the hypotheses $\mathrm{Ein}({}^{(4)}g) = 0$ and $D\mathrm{Ein}({}^{(4)}g) = 0$. ■

We remark that $D^2\mathrm{Ein}({}^{(4)}g)({}^{(4)}h, {}^{(4)}h)$ can be worked out explicitly using

$$\begin{aligned}&[D^2\mathrm{Ric}({}^{(4)}g)({}^{(4)}h, {}^{(4)}h)]_{\alpha\beta}\\ &\quad = +{}^{(4)}h^{\mu\lambda}[{}^{(4)}h_{\mu\lambda;\alpha\beta} + {}^{(4)}h_{\alpha\beta;\mu\lambda} - {}^{(4)}h_{\alpha\lambda;\beta\mu} - {}^{(4)}h_{\beta\lambda;\alpha\mu}]\\ &\qquad + \left(\frac{1}{2}{}^{(4)}h^{\mu}{}_{\mu}{}^{;\lambda} - {}^{(4)}h^{\mu\lambda}{}_{;\mu}\right)[{}^{(4)}h_{\alpha\lambda;\beta} + {}^{(4)}h_{\beta\lambda;\alpha} - {}^{(4)}h_{\alpha\beta;\lambda}]\\ &\qquad + {}^{(4)}h_{\beta}{}^{\mu;\lambda}({}^{(4)}h_{\mu\alpha;\lambda} - {}^{(4)}h_{\alpha\lambda;\mu}) + \frac{1}{2}{}^{(4)}h_{\mu\lambda;\alpha}{}^{(4)}h^{\mu\lambda}{}_{;\beta},\end{aligned}$$

and $[(D^2({}^{(4)}g^{-1})({}^{(4)}h, {}^{(4)}h)]^{\alpha\beta} = 2{}^{(4)}h^{\alpha\mu}{}^{(4)}h^{\beta}{}_{\mu}$.

1.3. PROPOSITION (Taub [1970]). — *Suppose* $\mathrm{Ein}({}^{(4)}g) = 0$,

$$D\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h = 0$$

and ${}^{(4)}X$ *is a Killing field for* ${}^{(4)}g$. *Then the vector field*

$${}^{(4)}T = {}^{(4)}X\cdot[D^2\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h)]$$

has zero divergence. (Here the first « · » denotes contraction while the second indicates the application of the bilinear map $D^2\mathrm{Ein}({}^{(4)}g)$.)

Proof. — From lemma 1.2, the bracketed quantity has zero divergence. Thus ${}^{(4)}T$ is the contraction of a Killing field and a symmetric divergence-free two tensor field and hence has zero divergence. ■

As a consequence, if Σ_1 and Σ_2 are two compact spacelike hypersurfaces, then

$$\int_{\Sigma_1} {}^{(4)}T \cdot {}^{(4)}Z_{\Sigma_1} d^3\Sigma_1 = \int_{\Sigma_2} {}^{(4)}T \cdot {}^{(4)}Z_{\Sigma_2} d^3\Sigma_2$$

where ${}^{(4)}Z_{\Sigma_i}$, $i = 1, 2$, is the unit forward pointing normal to Σ_i and $d^3\Sigma_i$ is its Riemannian volume element.

If $F : V_4 \to V_4$ is a diffeomorphism, then

$$\mathrm{Ein}(F^{*(4)}g) = F^*(\mathrm{Ein}({}^{(4)}g)),$$

where F^* denotes the pull-back of tensors. This equation is the globalized version of the coordinate covariance of $\mathrm{Ein}({}^{(4)}g)$. The first and second order infinitesimal version of coordinate covariance is the following:

1.4. PROPOSITION. — *Let ${}^{(4)}X$ be any vector field on V_4, and ${}^{(4)}h$ a symmetric two-tensor field. Then*

$$\mathrm{DEin}({}^{(4)}g) \cdot (L_{{}^{(4)}X}{}^{(4)}g) = L_{{}^{(4)}X}(\mathrm{Ein}({}^{(4)}g)),$$

and

$$D^2\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, L_{{}^{(4)}X}{}^{(4)}g) + \mathrm{DEin}({}^{(4)}g) \cdot L_{{}^{(4)}X}{}^{(4)}h = L_{{}^{(4)}X}(\mathrm{DEin}({}^{(4)}g) \cdot {}^{(4)}h),$$

where $L_{{}^{(4)}X}$ denotes Lie differentiation.

Proof. — Let F_λ be the flow of ${}^{(4)}X$, $F_0 = \mathrm{id}_{V_4}$, the identity diffeomorphism on V_4. (Of course, F_λ may be only locally defined.) Thus

$$\mathrm{Ein}(F_\lambda{}^{*(4)}g) = F_\lambda{}^*\mathrm{Ein}({}^{(4)}g).$$

Differentiating this relation in λ gives

$$\mathrm{DEin}(F_\lambda{}^{*(4)}g) \cdot F_\lambda{}^*(L_{{}^{(4)}X}{}^{(4)}g) = F_\lambda{}^*(L_{{}^{(4)}X}\mathrm{Ein}({}^{(4)}g)).$$

Setting $\lambda = 0$ gives the first relation. Then, differentiating this relation with respect to ${}^{(4)}g$ gives the second relation. ■

Remark. — If $\mathrm{Ein}({}^{(4)}g) = 0$, then

$$\mathrm{DEin}({}^{(4)}g) \cdot L_{{}^{(4)}X}{}^{(4)}g = 0 \tag{1}$$

for any vector field ${}^{(4)}X$. Since perturbations of the form $L_{{}^{(4)}X}{}^{(4)}g$ are gauge perturbations, the equation shows that the linearized Einstein operator $\mathrm{DEin}({}^{(4)}g)$ is gauge invariant if ${}^{(4)}g$ is a solution to the empty space equations.

Similarly, if ${}^{(4)}h$ solves the linearized equations

$$\mathrm{DEin}({}^{(4)}g) \cdot {}^{(4)}h = 0,$$

then

$$D^2\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, L_{{}^{(4)}X}{}^{(4)}g) + \mathrm{DEin}({}^{(4)}g) \cdot L_{{}^{(4)}X}{}^{(4)}h = 0 \tag{2}$$

for any ${}^{(4)}X$. We shall need this relationship and the following lemma to obtain the gauge invariance of Taub's conserved quantities ${}^{(4)}T$.

1.5. LEMMA. — *Suppose* $\mathrm{Ein}({}^{(4)}g) = 0$, ${}^{(4)}X$ *is a Killing field of* ${}^{(4)}g$, ${}^{(4)}h$ *is a symmetric two tensor field and* Σ *is a compact spacelike hypersurface. Then*

$$B(\Sigma, h) \equiv \int {}^{(4)}X \cdot [D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h] \cdot {}^{(4)}Z_\Sigma d^3\Sigma = 0 .$$

Proof. — By 1.1, $D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h$ is divergence free, and since ${}^{(4)}X$ is a Killing vector field, ${}^{(4)}X \cdot [D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h]$ is a divergence free vector field. Thus for two spacelike compact hypersurfaces, $B(\Sigma_1, {}^{(4)}h) = B(\Sigma_2, {}^{(4)}h)$. Choose Σ_1 and Σ_2 disjoint and replace ${}^{(4)}h$ by a symmetric two tensor ${}^{(4)}k$ that equals ${}^{(4)}h$ on Σ_1 and vanishes on a neighborhood of Σ_2. Then $B(\Sigma_1, {}^{(4)}h) = B(\Sigma_1, {}^{(4)}k) = B(\Sigma_2, {}^{(4)}k) = 0$. ■

The next proposition establishes the gauge invariance of Taub's conserved quantities ${}^{(4)}T$ when integrated over a hypersurface.

1.6. PROPOSITION. — *Let* $\mathrm{Ein}({}^{(4)}g) = 0$, ${}^{(4)}X$ *be a Killing field of* ${}^{(4)}g$, $D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h = 0$ *and* ${}^{(4)}Y$ *an arbitrary vector field. Then for any compact spacelike hypersurface* Σ,

$$\int_\Sigma {}^{(4)}X \cdot [D^2\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g, {}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma$$
$$= \int_\Sigma {}^{(4)}X \cdot [D^2\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma .$$

Proof. — By bilinearity of $D^2\mathrm{Ein}({}^{(4)}g)$, we need only show that

$$\int_\Sigma {}^{(4)}X \cdot [D^2\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}k, L_{{}^{(4)}Y}{}^{(4)}g)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma = 0 ,$$

where by the hypothesis and equation 1 of the remark after 1.4, ${}^{(4)}k = {}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g$ satisfies $D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}k = 0$. But this follows by contracting equation 2 of the remark after 1.4 with ${}^{(4)}X$, integrating over Σ and using 1.5. ■

Next we connect these ideas with linearization stability. If $\mathrm{Ein}({}^{(4)}g) = 0$ and $D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h = 0$, we call ${}^{(4)}h$ an *infinitesimal deformation.* An actual *deformation* is a smooth curve ${}^{(4)}g(\lambda)$ of Lorentz metrics through ${}^{(4)}g_0$ satisfying $\mathrm{Ein}({}^{(4)}g(\lambda)) = 0$. We say ${}^{(4)}h$ is *integrable* if for every compact set $C \subset V_4$ there is an actual deformation ${}^{(4)}g(\lambda)$ defined on C such that ${}^{(4)}g(0) = {}^{(4)}g_0$ on C and

$$\frac{d}{d\lambda} {}^{(4)}g(\lambda) |_{\lambda=0} = {}^{(4)}h \text{ on } C .$$

By the chain rule, every integrable ${}^{(4)}h$ is an infinitesimal deformation. A spacetime is called *linearization stable* if every infinitesimal deformation is integrable.

In the presence of Killing fields any infinitesimal deformation must satisfy a necessary second order condition in order to be integrable, as follows.

1.7. PROPOSITION. – *Suppose* $\mathrm{Ein}({}^{(4)}g_0) = 0$, ${}^{(4)}X$ *is a Killing field of* ${}^{(4)}g_0$ *and* ${}^{(4)}h$ *is integrable. Then the conserved quantity of Taub vanishes identically when integrated over any compact spacelike hypersurface* Σ:

$$\int_\Sigma {}^{(4)}X \cdot [D^2\mathrm{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma = 0 .$$

Proof. — Differentiation of $\mathrm{Ein}({}^{(4)}g(\lambda)) = 0$ twice with respect to λ at $\lambda = 0$ gives the identity

$$D^2\mathrm{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h) + D\mathrm{Ein}({}^{(4)}g_0) \cdot {}^{(4)}k = 0$$

where ${}^{(4)}k = \dfrac{d^2}{d\lambda^2} {}^{(4)}g(\lambda)\,|_{\lambda=0}$. Contracting with ${}^{(4)}X$, integrating over Σ and using lemma 1.5 gives the result.

We shall now summarize some of the formalism of geometrodynamics (ADM formalism) that we shall be using. Consult Misner, Thorne and Wheeler [1973] and Fischer and Marsden [1978] for proofs and additional details.

Given a spacetime $(V_4, {}^{(4)}g)$, a compact three manifold M, and a spacelike embedding $i : M \to V_4$, define

i) $g = i^*({}^{(4)}g)$, a Riemannian metric on M;

ii) k = the second fundamental form of the embedding, a symmetric two tensor on M, with the sign convention

$$k_{ij} = -\,{}^{(4)}Z_{i;j} ,$$

where ${}^{(4)}Z$ is the forward pointing unit formal to $\Sigma = i(M) \subset V_i$,

iii) κ = trace k, the mean curvature of the embedding;

iv) $\pi' = (\kappa g - k)^\#$ where $\#$ denotes the contravariant form of the tensor with respect to g; note that $\mathrm{tr}\,\pi' = 2\kappa$. Similarly $\flat$ denotes the index lowering map.

v) $\pi = \pi' \otimes \mu(g)$ where $\mu(g)$ is the Riemannian volume element of g.

Set $P_i({}^{(4)}g) = (g, \pi)$, the projection of ${}^{(4)}g$ to the Cauchy data (g, π) induced by ${}^{(4)}g$ by the spacelike embedding i. We shall sometimes write P_Σ for P_i.

vi) Set $G_{\alpha\beta} = [\mathrm{Ein}({}^{(4)}g)]_{\alpha\beta}$ and $G^{\perp\perp} = Z^\alpha Z^\beta G_{\alpha\beta}$, the perpendicular-perpendicular projection of $\mathrm{Ein}({}^{(4)}g)$, a scalar function on M;

$(G^\perp{}_{//})_i = -\,Z_\alpha G^\alpha{}_i$, the perpendicular-parallel projection of $\mathrm{Ein}({}^{(4)}g)$, a one-form on M.

vii) $\mathscr{H}(g, \pi) = -\,2G^{\perp\perp}\mu(g) = \left[\pi' \cdot \pi' - \dfrac{1}{2}(\mathrm{tr}\,\pi')^2 - R(g)\right](g)$ a scalar density on M;

$\mathscr{J}(g, \pi) = -\,2G^\perp{}_{//}\mu(g) = 2(\delta\pi)^\flat$, a one form density on M; in

coordinates, $\mathscr{J}_i = -2\pi_i{}^j{}_{|j}$; a vertical bar denoting covariant differentiation on M with respect to g.

Set $\Phi(g, \pi) = (\mathscr{H}(g, \pi), \mathscr{J}(g, \pi))$.

Given a *slicing* of $(V_4, {}^{(4)}g)$; i. e., a curve i_λ in the space $\mathrm{Emb}(M, V_4, {}^{(4)}g)$ of space like embeddings ([1]) of M to V_4 which foliate a neighborhood of $\Sigma_0 = i_0(M)$ in V_4, the Einstein equations $\mathrm{Ein}({}^{(4)}g) = 0$ are equivalent to the following system for which the notation is explained below:

$$\Phi(g, \pi) = 0 \qquad \text{(Constraint Equations)}$$

and

$$\frac{\partial}{\partial\lambda}\begin{pmatrix} g \\ h \end{pmatrix} = -\mathbb{J}\cdot D\Phi(g, \pi)^* \cdot \begin{pmatrix} N \\ X \end{pmatrix} \qquad \text{(Evolution Equations)}$$

Here N and X are the perpendicular and parallel components of ${}^{(4)}X_\lambda = \dfrac{d}{d\lambda} i_\lambda$, the tangent to the curve of embeddings, so ${}^{(4)}X_\lambda : M \to TV_4$ and covers i_λ. To explain the evolution equations, we need some addition notation. Let $\mathscr{M} = \mathscr{M}^{s,p}$ be the space of Riemannian metrics g on M of Sobolov class $W^{s,p}$, $\dfrac{n}{p} + 1 < s \leq \infty$. Then $\mathscr{M}$ is an open convex cone in S_2, the space of symmetric 2-covariant tensor fields h; thus $T\mathscr{M} = \mathscr{M} \times S_2$. Let $S_d{}^2$ be the space of 2-contravariant symmetric tensor densities π on M and let $T^*\mathscr{M} = \mathscr{M} \times S_d{}^2$, the L_2-cotangent bundle of $\mathscr{M}$. The function space topology used is $W^{s,p} \times W^{s-1,p}$ on $T^*\mathscr{M}$.

We can regard $\Phi : T^*\mathscr{M} \to \Lambda_d^0 \times \Lambda_d^1$, where $\Lambda_d^0 \times \Lambda_d^1$ is the space (scalar densities) $\times$ (one form densities). Thus, for $(g, \pi) \in T^*\mathscr{M}$,

$$D\Phi(g, \pi) : S_2 \times S_d{}^2 \to \Lambda_d{}^0 \times \Lambda_d{}^1,$$

mapping the differentiability classes $W^{s,p} \times W^{s-1,p} \to W^{s-2,p} \times W^{s-2,p}$, i. e., $D\Phi(g, \pi)$ is of order (2, 1).

Now $T^*\mathscr{M}$ carries a natural weak symplectic structure Ω given by

$$\Omega_{(g,\pi)}((h_1, \omega_1), (h_2, \omega_2)) = \int_M (\omega_2 \cdot h_1 - \omega_1 \cdot h_2)$$

Also, on $T^*\mathscr{M}$ we define the weak Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ by

$$\langle\langle (h_1, \omega_1), (h_2, \omega_2) \rangle\rangle_{(g,\pi)} = \int_M (h_1 \cdot h_2 + \omega_1' \cdot \omega_2')\mu(g)$$

where $h_1 \cdot h_2$ denote the contraction of h_1 and h_2 to a scalar using g.

([1]) It is an exercise in manifolds of mappings to show that in suitable function spaces $\mathrm{Emb}(M, V_4, {}^{(4)}g_0)$ is a smooth infinite dimensional manifold. See Palais [1968] and Ebin and Marsden [1970] for the relevant techniques.

Associated to Ω and $\langle\langle\,,\rangle\rangle$ is a *complex structure* $\mathbb{J} : T(T^*\mathscr{M}) \to T(T^*\mathscr{M})$ given by

$$\mathbb{J}_{(g,\pi)}(h, \omega) = (-\omega^*, h^*)$$

where * denotes the dualization map $S_2 \to S_d^2$ and $S_d^2 \to S_2$, which depends on g. Explicitly, $h^* = h^\# \mu(g)$ and $\omega^* = (\omega')^\flat$.

Thus Ω, $\langle\langle\,,\rangle\rangle$ and $\mathbb{J}$ stand in the usual relation

$$\langle\langle (h_1, \omega_1), (h_2, \omega_2) \rangle\rangle_{(g,\pi)} = -\Omega_{(g,\pi)}(\mathbb{J}(h_1, \omega_1), (h_2, \omega_2))$$

(see Abraham and Marsden [1978, p. 173]).

The adjoint $D\Phi(g, \pi)^* : \Lambda^0 \times \mathscr{X} \to S_2 \times S_d^2$ (where $\mathscr{X}$ = vector fields on M) is defined by

$$\langle\langle D\Phi(g, \pi)^* \cdot (N, X), (h, \omega) \rangle\rangle = \langle (N, X), D\Phi(g, \pi) \cdot (h, \omega) \rangle$$

where $\langle (N, X), (\alpha, \beta) \rangle = \int_M N\alpha + X \cdot \beta$ is the natural pairing between $\Lambda^0 \times \mathscr{X}$ and $\Lambda_d^0 \times \Lambda_d^1$.

Thus $-\mathbb{J} \circ D\Phi(g, \pi)^* \cdot (N, X) \in S_2 \times S_d^2$ is explained. (For details on the Hamiltonian structure, we refer to Fischer and Marsden [1978] and Marsden [1980] and part II ([2]).

The explicit relationship between ${}^{(4)}g$, g, N and X and the slicing i_λ is given by

$${}^{(4)}g_{\alpha\beta} d_x^\alpha d_x^\beta = -(N^2 - X \cdot X) d\lambda^2 + 2X_i d_x^i d\lambda + g_{ij} d_x^i d\chi^j$$

where coordinates $x^\alpha = (x^i, \lambda)$ of the slicing are used.

We record the explicit formulas for $D\Phi(g, \pi)$ and its adjoint. The proofs are long but straightforward.

$$\begin{aligned} D\Phi(g, \pi) \cdot (h, \omega) &= (D\mathscr{H}(g, \pi) \cdot (h, \omega), D\mathscr{J}(g, \pi) \cdot (h, \omega)) \\ &= ((\mu(g))^{-1} \Big[-\frac{1}{2}\Big(\pi \cdot \pi - \frac{1}{2}(\operatorname{tr} \pi)^2\Big)(\operatorname{tr} h) \\ &\quad + 2\Big(\pi \cdot \omega - \frac{1}{2}(\operatorname{tr} \pi)(\operatorname{tr} \omega)\Big) + 2\Big(\pi \times \pi - \frac{1}{2}(\operatorname{tr} \pi)\pi\Big) \cdot h \Big] \\ &\quad - \mu(g)[\delta\delta h + \Delta(\operatorname{tr} h) - \operatorname{Ein}(g) \cdot h], 2h \cdot \delta\pi + 2g \cdot \delta\omega - 2\pi \cdot (D\Gamma(g) \cdot h)) \end{aligned}$$

([2]) In terms of general field theory, this form $-\mathbb{J} \circ D\Phi(g, \pi)^* \cdot \begin{pmatrix} N \\ X \end{pmatrix}$ is the generator of the momentum mapping association with the gauge variable Emb(M, V_4, ${}^{(4)}g$). In our earlier work (Fischer and Marsden [1979]) we used different conventions for $\mathbb{J}$ and the adjoints, but the present conventions are more useful for the study of general momentum mappings.

and

$$D\Phi(g, \pi)^* \cdot (N, X) = \Bigg(N\Bigg[-\frac{1}{2}\Big(\pi' \cdot \pi' - \frac{1}{2}(\operatorname{tr} \pi')^2\Big)g + 2\Big(\pi' \times \pi' - \frac{1}{2}(\operatorname{tr} \pi')\pi'\Big)^\flat\Bigg] - [\operatorname{Hess} N + (\Delta N)g - N \operatorname{Eng}(g)] - (L_X\pi)^*, 2N\Big[\pi - \frac{1}{2}(\operatorname{tr} \pi)g^\#\Big] + (L_X g)^\# \mu(g)\Bigg).$$

The notation used in these formulas is as follows:

$\cdot$ signifies contraction; e. g., $\pi \cdot h = \pi^{ij}h_{ij}$
tr signifies trace
π' is the tensor part of π
$\pi \times \pi = \pi^{ik}\pi^{kj}(= \operatorname{tr} \pi \times \pi)$
$\Delta N = -N^{|i}{}_{|i}$
$\operatorname{Hess} N = N_{|i|j}$
$\operatorname{Ein}(g) = R_{ij} - \frac{1}{2}Rg_{ij}$
vertical bar = covariant differentiation on M
semi-colon – covariant differentiation on V_4
$h^\#$ = indices raised by g
$\pi^\flat$ = indices lowered by g
$\delta h = -h_{ij}{}^{|j}$
$\delta\delta h = h_{ij}{}^{|i|j}$
$h \cdot \delta\pi = -h_{ik}\pi^{kj}{}_{|j}$
$g \cdot \delta\omega = -\omega_i{}^k{}_{|k}$
$2\pi \cdot (D\Gamma(g) \cdot h) = \pi^{jl}(h_{kj|l} + h_{kl|j} - h_{jl|k})$

The definition of Φ in terms of $\operatorname{Ein}({}^{(4)}g)$ may be phrased as follows: given a hypersurface Σ,

$$\begin{aligned}({}^{(4)}Y \cdot \operatorname{Ein}({}^{(4)}g) \cdot {}^{(4)}Z_\Sigma)\mu(g) &= ([Y_{//} + Y_\perp{}^{(4)}Z_\Sigma] \cdot \operatorname{Ein}({}^{(4)}g) \cdot {}^{(4)}Z_\Sigma)\mu(g)\\ &= (Y_{//} \cdot \operatorname{Ein}({}^{(4)}g) \cdot {}^{(4)}Z_\Sigma + {}^{(4)}Y_\perp{}^{(4)}Z_\Sigma \cdot \operatorname{Ein}({}^{(4)}g) \cdot Z_\Sigma)\mu(g)\\ &= (Y_{//} \cdot G^\perp{}_{//} + Y_\perp G^{\perp\perp})\mu(g)\end{aligned}$$

Therefore,

$$\int_M ({}^{(4)}Y \cdot \operatorname{Ein}({}^{(4)}g) \cdot {}^{(4)}Z_\Sigma)\mu(g) = -\frac{1}{2}\langle (Y_\perp, Y_{//}), \Phi(g, \pi) \rangle$$

where $\langle , \rangle$ denotes the *natural* L_2 pairing as above. Differentiation of this relation yields:

1.8. PROPOSITION. — *Let* $\operatorname{Ein}({}^{(4)}g) = 0$, $i \in \operatorname{Emb}(M, V_4{}^{(4)}g)$ *and* $\Sigma = i(M)$.

If ${}^{(4)}Y$ *is a vector field on* V_4 *with* ${}^{(4)}Y = Y_\perp {}^{(4)}Z_\Sigma + Y_{//}$ *on* Σ, $Y_{//} \cdot {}^{(4)}Z_\Sigma = 0$, *then*

$$-2\,{}^{(4)}Y \cdot [D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h] \cdot {}^{(4)}Z_\Sigma \mu(g) = (Y_\perp, Y_{//}) \cdot [D\Phi(g, \pi) \cdot (h, \omega)]$$

where $(h, \omega) = DP_i({}^{(4)}g) \cdot {}^{(4)}h$, $P_i({}^{(4)}g) = (g, \pi)$. *In particular*

$$-2\int_\Sigma {}^{(4)}Y \cdot [D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h] \cdot {}^{(4)}Z_\Sigma \mu(g) = \langle\langle\, D\Phi(g, \pi)^* \cdot (Y_\perp, Y_{//}), (h, \omega)\,\rangle\rangle$$

The infinitesimal deformations (h, ω) of (g, π) induced by a deformation ${}^{(4)}h$ of ${}^{(4)}g$ are given explicitly by

$$\begin{aligned}(h, \omega) = DP_i({}^{(4)}g) \cdot {}^{(4)}h = \Bigg(h, &\frac{1}{2}\pi \operatorname{tr} {}^{(4)}h \\ &+ \left[g^\#(h \cdot \pi) - \frac{1}{2} g^\#(\operatorname{tr} h)(\operatorname{tr} \pi) + \frac{1}{2} h^\#(\operatorname{tr} \pi) - h^\# \times \pi - \pi \times h^\#\right] \\ &+ \frac{1}{2}\mu(g)(g^{ij}g^{kl} - g^{ik}g^{jl}) \cdot {}^{(4)}Z_\Sigma \mu({}^{(4)}h_{l\mu;k} + {}^{(4)}h_{k\mu;l} - {}^{(4)}h_{kl;\mu})\Bigg)\end{aligned}$$

where $h = h_{ij} = {}^{(4)}h_{//\,//}$, $\operatorname{tr} {}^{(4)}h = {}^{(4)}g^{\alpha\beta(4)}h_{\alpha\beta}$, $g^\# = g^{ij}$, $h^\# = h^{ij}$, $h \cdot \pi = h_{ij}\pi^{ij}$ and $h^\# \times \pi + \pi \times h^\# = h^{ik}\pi j_k + h^{jk}\pi^i k$.

1.9. Corollary. — *If* $\mathrm{Ein}({}^{(4)}g) = 0$ *and* ${}^{(4)}Y$ *is a Killing field of* ${}^{(4)}g$, *then*

$$(Y_\perp, Y_{//}) \in \text{kernel } D\Phi(g, \pi)^*$$

This follows from 1.8 and 1.5. In lemma 2.2 of Section 2, we will prove that Killing fields are in one to one correspondence with elements of ker $D\Phi(g, \pi)$ (Moncrief [1975]).

Differentiation of the identity preceding 1.8 twice with respect to ${}^{(4)}g$ and then setting $\mathrm{Ein}({}^{(4)}g) = 0$ and $D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h = 0$ gives

$$\begin{aligned}&\{\,{}^{(4)}Y \cdot [D^2\mathrm{Ein}({}^{(4)}g)({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma\,\}\,\mu(g) \\ &\quad = -\frac{1}{2}(Y_\perp, Y_{//}) \cdot [D^2\Phi(g, \pi)((h, \omega), (h, \omega))] - \frac{1}{2}(Y_\perp, Y_{//}) \cdot (D\Phi(g, \pi) \cdot (h', \omega'))\end{aligned}$$

where $(h', \omega') = D^2P_i({}^{(4)}g) \cdot ({}^{(4)}h, {}^{(4)}h)$. Integrating this over M and using 1.9 yields the following expression for Taub's quantities.

1.10. Proposition. — *Let* $\mathrm{Ein}({}^{(4)}g) = 0$, $D\mathrm{Ein}({}^{(4)}g) \cdot {}^{(4)}h = 0$ *and* ${}^{(4)}X$ *be a Killing field for* ${}^{(4)}g$. *Then for any* $i \in \mathrm{Emb}(M, V_4, {}^{(4)}g)$

$$\begin{aligned}\int_M {}^{(4)}X \cdot [D^2\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma \mu(g) \\ = -\frac{1}{2}\langle\, (X_\perp, X_{//}), D^2\Phi(g, \pi) \cdot ((h, \omega), (h, \omega))\,\rangle\end{aligned}$$

Just as with $D^2\mathrm{Ein}({}^{(4)}g)$, one can work out $D^2\Phi(g, \pi)$ explicitly. We record the result:

$$
\begin{aligned}
&D^2\Phi(g, \pi)((h, \omega), (h, \omega)) \\
&= \Big\{ (\mu(g))^{-1}\Big[\Big(\frac{1}{4}(\mathrm{tr}\, h)^2 + \frac{1}{2} h\cdot h\Big)\Big(\pi\cdot\pi - \frac{1}{2}(\mathrm{tr}\,\pi)^2\Big) \\
&- 2(\mathrm{tr}\, h)\Big(h\cdot(\pi\times g\times\pi) - \frac{1}{2}(\pi\cdot h)(\mathrm{tr}\,\pi)\Big) - 2(\mathrm{tr}\, h)\Big(\pi\cdot\omega - \frac{1}{2}(\mathrm{tr}\,\pi)(\mathrm{tr}\,\omega)\Big) \\
&+ 2\Big(h\cdot(\pi\times h\times\pi) - \frac{1}{2}(h\cdot\pi)^2\Big) + 2\Big((\omega\cdot\omega) - \frac{1}{2}(\mathrm{tr}\,\omega)^2\Big) \\
&+ 2(4h\cdot(\omega\times\pi) - (h\cdot\omega)(\mathrm{tr}\,\pi) - (h\cdot\pi)(\mathrm{tr}\,\omega))\Big] \\
&- D^2[\mu(g)R(g)](h, h), + 4h\cdot\delta\omega - 4\omega\cdot D\Gamma(g)\cdot h)\Big\}
\end{aligned}
$$

where $\pi\times h\times\pi = \pi^{ij}h_{jk}\pi^{kl}$, $h\cdot\delta\omega = -h_{ij}\omega^{jk}{}_{|k}$,

$$2\omega\cdot(D\Gamma(g)\cdot h) = \omega^{jl}(h_{lk|j} + h_{jk|l} - h_{lj|k})$$

and where

$$
\begin{aligned}
&D^2[\mu(g)R(g)]\cdot(h, h) \\
&= +\mu(g)\Big\{\Big[\frac{1}{4}(\mathrm{tr}\, h)^2 - \frac{1}{2} h\cdot h\Big]R - (\mathrm{tr}\, h)h\cdot\mathrm{Ric}(g) + 2\mathrm{Ric}(g)\cdot(h\times h) \\
&\quad - h^{ij}(h_{ik|j}{}^{|k} + h_{kj|i}{}^{|k}) + (\mathrm{tr}\, h)(\delta\delta h + \Delta(\mathrm{tr}\, h)) \\
&\quad + 2h^{ij}(h_{ij|k}{}^{|k} + (\mathrm{tr}\, h)_{|ij} - h_{ki}{}^{|k}{}_{|j}) + \Big[\frac{1}{2}(\mathrm{tr}\, h)^{|k} - h^{jk}{}_{|j}\Big](2h_{ik}{}^{|i} - (\mathrm{tr}\, h)_{|k}) \\
&\quad + \frac{1}{2} h_{ij|k}h^{ij|k} + h_i{}^{j|k}(h_j{}^i{}_{|k} - h^i{}_{k|j})\Big\}
\end{aligned}
$$

We have already investigated gauge invariance of Ein, DEin, D^2Ein and ${}^{(4)}T$ in 1.4 and 1.6. These results imply corresponding results for the Φ map via 1.8 and 1.10. In obtaining these, the following is used.

1.11. LEMMA. — *Let* $\mathrm{Ein}({}^{(4)}g) = 0$ *and* $i\in\mathrm{Emb}(M, V_4, {}^{(4)}g)$. *Let* ${}^{(4)}Y$ *be a vector field on* V_4 *and* ${}^{(4)}h_1 = L_{{}^{(4)}Y}{}^{(4)}g$. *Then*

$$(h_1, \omega_1) \equiv DP_i({}^{(4)}g)\cdot{}^{(4)}h_1 = -\mathbb{J}\circ D\Phi(g, \pi)^*\cdot(Y_\perp, Y_{//})$$

Proof. — Let i_λ be the slicing of V_4 determined by the flow of ${}^{(4)}Y$; i. e., $i_\lambda = F_\lambda\circ i$, where F_λ is the flow of ${}^{(4)}Y$. Strictly speaking, this need not be a slicing of V_4 because ${}^{(4)}Y$ may have zeros, but this does not affect the validity of the evolution equations as they hold for any curve in $\mathrm{Emb}(M, V_4, {}^{(4)}g)$. Therefore

$$-\mathbb{J}\circ D\Phi(g, \pi)^*\begin{pmatrix} Y_\perp \\ Y_{//}\end{pmatrix} = \frac{\partial}{\partial\lambda} P_{i_\lambda}({}^{(4)}g)\,|_{\lambda=0}$$

Now for any diffeomorphism F of V_4, one has the identity

$$P_{F\circ i}({}^{(4)}g) = P_i(F^*{}^{(4)}g)$$

Thus

$$\begin{aligned}\frac{\partial}{\partial\lambda}\mathrm{P}_{i_\lambda}({}^{(4)}g) &= \frac{\partial}{\partial\lambda}\mathrm{P}_{\mathrm{F}_\lambda\circ i}({}^{(4)}g)\,|_{\lambda=0}\\ &= \frac{\partial}{\partial\lambda}\mathrm{P}_i(\mathrm{F}_\lambda{}^{*(4)}g)\,|_{\lambda=0}\\ &= \mathrm{DP}_i({}^{(4)}g)\cdot \mathrm{L}_{({}^{(4)}\mathrm{Y})}{}^{(4)}g\end{aligned}$$

by the chain rule. ■

The *linearized evolution equations* are obtained by differentiating the evolution equations $\frac{\partial}{\partial\lambda}\begin{pmatrix} g \\ \pi\end{pmatrix} = -\,\mathbb{J}\circ \mathrm{D}\Phi(g,\pi)^*\cdot(\mathrm{X}_\perp, \mathrm{X}_{//})$ with respect to $(g, \pi, \mathrm{X}_\perp, \mathrm{X}_{//})$ for a *fixed* slicing:

$$\frac{\partial}{\partial\lambda}\begin{pmatrix} h \\ \omega\end{pmatrix} = -\,\mathbb{J}\circ \mathrm{D}\left[\mathrm{D}\Phi(g,\pi)^*\begin{pmatrix}\mathrm{X}_\perp \\ \mathrm{X}_{//}\end{pmatrix}\right]\cdot\begin{pmatrix} h \\ \omega\end{pmatrix} - \mathbb{J}\circ \mathrm{D}\Phi(g,\pi)^*\cdot\begin{pmatrix}\mathrm{Y}_\perp \\ \mathrm{Y}_{//}\end{pmatrix}$$

Likewise, differentiation of $\Phi(g, \pi) = 0$ with respect to (g, π) yields the *linearized constraint equations:*

$$\mathrm{D}\Phi(g,\pi)(h,\omega) = 0$$

(See Fischer and Marsden [1978] for more information on the linearized Einstein system.)

Combining 1.6, 1.10, and 1.11 gives

1.12. Proposition. — *Let* $\mathrm{Ein}({}^{(4)}g) = 0$ *and* $i \in \mathrm{Emb}(\mathrm{M}, \mathrm{V}_4, {}^{(4)}g)$. *Let* ${}^{(4)}\mathrm{Y}$ *be a vector field on* V_4 *and* ${}^{(4)}h_1 = \mathrm{L}_{({}^{(4)}\mathrm{Y})}{}^{(4)}g$. *Then*

i) $\mathbb{J}\circ \mathrm{D}\Phi(g,\pi)^*\cdot\begin{pmatrix}\mathrm{Y}_\perp \\ \mathrm{Y}_{//}\end{pmatrix} \in \ker \mathrm{D}\Phi(g,\pi)$

ii) If $\mathrm{DEin}({}^{(4)}g)\cdot{}^{(4)}h = 0$ *and* ${}^{(4)}\mathrm{X}$ *is a Killing field for* ${}^{(4)}g$ *then*

$$\langle (\mathrm{X}_\perp, \mathrm{X}_{//}), \mathrm{D}^2\Phi(g,\pi)((h,\omega),(h_1,\omega_1))\rangle = 0$$

and

$$\begin{aligned}&\langle (\mathrm{X}_\perp, \mathrm{X}_{//}), \mathrm{D}^2\Phi(g,\pi)((h,\omega)+(h_1,\omega_1),(h,\omega)+(h_1,\omega_1))\rangle\\ &\qquad = \langle (\mathrm{X}_\perp, \mathrm{X}_{//}), \mathrm{D}^2\Phi(g,\pi)\cdot((h,\omega),(h,\omega))\,.\end{aligned}$$

For later use, we record here explicit formulas for the various projections of $\mathrm{L}_{({}^{(4)}\mathrm{Y})}{}^{(4)}g$ onto Σ:

$$(\mathrm{L}_{({}^{(4)}\mathrm{Y})}{}^{(4)}g)_{////} = \frac{2\mathrm{Y}_\perp}{\mu(g)}\left(\pi^\flat - \frac{1}{2}g(\mathrm{tr}\,\pi)\right) + \mathrm{L}_{\mathrm{Y}_{//}}g$$

$$(\mathrm{L}_{({}^{(4)}\mathrm{Y})}{}^{(4)}g)^{\perp\perp} = -\,2{}^{(4)}\mathrm{Z}_\Sigma\cdot\nabla\mathrm{Y}_\perp - \frac{2}{\mathrm{N}}\mathrm{Y}_{//}\cdot\nabla\mathrm{N}$$

$$(\mathrm{L}_{({}^{(4)}\mathrm{Y})}{}^{(4)}g)^{\perp}{}_{//} = ({}^{(4)}\mathrm{Z}_\Sigma\cdot\mathrm{L}_{({}^{(4)}\mathrm{Y})}{}^{(4)}g)_{//} = -\,\mathrm{N}\nabla\left(\frac{\mathrm{Y}_\perp}{\mathrm{N}} + \frac{1}{\mathrm{N}}g\cdot\frac{\partial\mathrm{Y}_{//}}{\partial t} + \frac{1}{\mathrm{N}}g\cdot\mathrm{L}_{\mathrm{Y}_{//}}\mathrm{X}\right)$$

where ${}^{(4)}\mathrm{Z}_\Sigma\cdot\nabla\mathrm{Y}_\perp = {}^{(4)}\mathrm{Z}_\Sigma{}^\mu\frac{\partial}{\partial x^\mu}\mathrm{Y}_\perp$, $\mathrm{Y}_{//}\cdot\nabla\mathrm{N} = \mathrm{Y}_{//}{}^i\frac{\partial\mathrm{N}}{\partial x^i}$, and (N, X) are the

lapse function and shift vector field of the chosen slicing. We recall that in terms of these quantities

$$ {}^{(4)}Z_\Sigma = \frac{1}{N}\frac{\partial}{\partial t} - \frac{X^i}{N}\frac{\partial}{\partial^{*i}}, \quad {}^{(4)}g^{00} = \frac{-1}{N^2}, $$
$$ {}^{(4)}g_{0i} = X_i \, {}^{(4)}g_{ij} = g_{ij}. $$

§ 2. LINEARIZATION STABILITY AND KILLING FIELDS

This section reviews the known connections between linearization stability and Killing vector fields. We supply those proofs that are either new or are needed in later sections.

Let $(V, {}^{(4)}g)$ be a vacuum spacetime and, as introduced in section 1, $\Sigma = i(M)$ is a compact embedded spacelike smooth hypersurface. Recall that our spacetimes are globally hyperbolic, so that Σ is a Cauchy hypersurface.

2.1. THEOREM. — *If* $(V, {}^{(4)}g)$ *has no Killing fields, then it is linearization stable.*

This theorem is proved by combining the results of Fischer and Marsden [1973, 1975] with those of Moncrief [1975]. We begin with the following lemma and give a simplified proof, inspired by Coll [1976].

2.2. LEMMA. — *Let* $(V_4, {}^{(4)}g)$ *be fixed. The space of Killing fields of* ${}^{(4)}g$ *is isomorphic to* $\ker D\Phi(g, \pi)^*$ *by the map* $\mathscr{V}_\Sigma {}^{(4)}X \mapsto (X_\perp, X_{//})$, *the perpendicular and parallel projections of* ${}^{(4)}X$ *on* Σ.

Proof. — By 1.9, the range of $\mathscr{V}_\Sigma$ lies in the kernel of $D\Phi(g, \pi)^*$. To prove the lemma, we construct an inverse for $\mathscr{V}_\Sigma$, i. e., a map from elements $(X_\perp, X_{//}) \in \ker D\Phi(g, \pi)^*$ to Killing fields ${}^{(4)}X$ whose projections are $(X_\perp, X_{//})$. Let ${}^{(4)}X$ be the unique global solution on $({}^{(4)}V, {}^{(4)}g)$ of the linear hyperbolic equation

$$ \Box {}^{(4)}X = 0 $$
$$ \nabla_{{}^{(4)}Z} \, {}^{(4)}X = - {}^{(4)}Z_\Sigma \cdot \mathscr{P}(\nabla {}^{(4)}X) = - Z_\Sigma{}^\mu \mathscr{P}^{\alpha\beta} X_{\mu\beta} \text{ on } \Sigma $$

where $\mathscr{P}^{\alpha\beta} = g^{\alpha\beta} + Z_\Sigma{}^\alpha Z_\Sigma{}^\beta$ is the projection tensor. This choice of Cauchy data and the projection formulas following 1.12 imply that

$$ {}^{(4)}h^{\perp\perp} = 0 \quad \text{and} \quad {}^{(4)}h^{\perp}{}_{//} = 0 \text{ on } \Sigma $$

where ${}^{(4)}h = L_{{}^{(4)}X} {}^{(4)}g$. By the Ricci commutation formulas we get

$$ {}^{(4)}\delta({}^{(4)}\hat{h}) = \Box {}^{(4)}X = 0 $$

where ${}^{(4)}\hat{h} = {}^{(4)}h - \frac{1}{2}\operatorname{tr}({}^{(4)}h){}^{(4)}g$. By gauge invariance 1.4,

$$0 = \mathrm{DEin}({}^{(4)}g)\cdot{}^{(4)}h = \Box_{\mathrm{L}}{}^{(4)}\hat{h}.$$

We shall demonstrate that ${}^{(4)}h = 0$ and $\nabla_{{}^{(4)}Z_\Sigma}{}^{(4)}h = 0$ on Σ. This implies the Caucy data of ${}^{(4)}h$ is zero, so ${}^{(4)}h$ and hence ${}^{(4)}h$ vanishes.

STEP 1. — ${}^{(4)}h^{\perp\perp}$ and ${}^{(4)}h^{\perp}{}_{//} = 0$ on Σ by the choice of Cauchy data for ${}^{(4)}\mathrm{X}$.

STEP 2. — ${}^{(4)}h = h = 0$ on Σ by 1.11 and the fact that $\mathrm{D}\Phi(g, \pi)^*\cdot(\mathrm{X}_\perp, \mathrm{X}_{//}) = 0$ (see 1.9).

STEP 3. — In a slicing i_λ near Σ, (h, ω) induced by ${}^{(4)}h$ satisfy the linearized evolution equations (see the remark following 1.11), with ${}^{(4)}\mathrm{Y} = {}^{(4)}\mathrm{X}$. Since (h, ω) themselves and ${}^{(4)}h^{\perp\perp}$ and ${}^{(4)}h^{\perp}{}_{//}$ vanish, so does $\frac{\partial}{\partial\lambda}\begin{pmatrix} h \\ \omega \end{pmatrix}$. In particular, $\frac{\partial h}{\partial\lambda} = 0$ and ${}^{(4)}h = 0$ on Σ implies $(\nabla_{{}^{(4)}Z_\Sigma}{}^{(4)}h)_{////} = 0$ on Σ.

STEP 4. — $(\nabla_{{}^{(4)}Z_\Sigma}{}^{(4)}h)^{\perp\perp}$ vanishes on Σ since it equals

$$2{}^{(4)}Z_\Sigma{}^\alpha\mathscr{P}^{\beta\gamma(4)}h_{\alpha\beta;\gamma} + 2{}^{(4)}Z_\Sigma\cdot(\Box{}^{(4)}\mathrm{X}) - 2{}^{(4)}Z_\Sigma\cdot\mathrm{Ric}({}^{(4)}g)\cdot{}^{(4)}\mathrm{X} + {}^{(4)}h\cdot(\nabla_{{}^{(4)}Z_\Sigma}\mathscr{P}) - \nabla_{{}^{(4)}Z_\Sigma}\operatorname{tr}({}^{(4)}h_{////})$$

and $\Box{}^{(4)}\mathrm{X} = 0$, $\mathrm{Ric}({}^{(4)}g) = 0$, ${}^{(4)}h|_\Sigma = 0$, $(\nabla_{Z_\Sigma}{}^{(4)}h)_{////}|_\Sigma = 0$ and $\mathscr{P}^{\beta\gamma(4)}h_{\alpha\beta;\gamma}$ involves only derivatives tangent to Σ.

STEP 5. — $(\nabla_{{}^{(4)}Z_\Sigma}{}^{(4)}h)^{\perp}{}_{//}$ vanishes on Σ since it equals

$$\mathscr{P}^{\alpha\sigma}\left\{\mathscr{P}^{\beta\gamma(4)}h_{\alpha\beta;\gamma} + (\Box{}^{(4)}\mathrm{X})_\alpha - {}^{(4)}\mathrm{X}^\delta\mathrm{R}_{\delta\alpha} - \frac{1}{2}(\operatorname{tr}{}^{(4)}h)_{;\alpha}\right\}. \qquad \blacksquare$$

Proof of 2.1 (Sketch). — On a compact spacelike hypersurface Σ, $\ker \mathrm{D}\Phi^*(g, \pi) = \{0\}$ by lemma 2.2. A computation shows that $\mathrm{D}\Phi^*(g, \pi)$ is an elliptic operator in the sense of Douglis and Nirenberg; cf. Fischer and Marsden [1978]. Thus in the appropriate Sobolev spaces, $\mathrm{D}\Phi(g, \pi)$ is a surjective operator whose kernel splits. By the implicit function theorem, $\Phi^{-1}(0)$ is a smooth manifold near (g, π) and in particular the constraint equations $\Phi(g, \pi) = 0$ are linearization stable. This stability is propagated to the spacetime using the evolution equations (see Fischer and Marsden [1978] and [1979]). $\blacksquare$

It follows that the condition $\ker \mathrm{D}\Phi(g, \pi)^* = \{0\}$ which guarantees linearization stability, is hypersurface independent. If Σ is a hypersurface of constant mean curvature, then $\ker \mathrm{D}\Phi(g, \pi)^*$ can be worked out explicitly. In fact, from the formula for $\mathrm{D}\Phi(g, \pi)^*$, one obtains the following (Fischer and Marsden [1973 or 1978]).

2.3. PROPOSITION. — *Let Σ be a smooth spacelike compact hypersurface*

of constant mean curvature with induced metric and conjugate momentum (g_0, π_0). *Then*

i) *if* $\pi_0 \neq 0$ *or* g_0 *is not flat,*

$$\ker D\Phi(g_0, \pi_0)^* = \{ (0, X) \mid L_X g_0 = 0 \quad \textit{and} \quad L_X \pi_0 = 0 \}$$

ii) *if* $\pi_0 = 0$ *and* g_0 *is flat,*

$$\ker D\Phi(g_0, 0)^* = \{ (0, X) \mid L_X g_0 = 0 \} \cup \{ (N, O)\ N \textit{ is constant} \}.$$

In particular, if $\Sigma = i(M)$ *has constant mean curvature and* $\pi_0 \neq 0$ *or* g_0 *is not flat, then any Killing field on* V_4 *must be tangent to* Σ.

(The last remark of this proposition also follows from the uniqueness of hypersurfaces of constant mean curvature; see Marsden and Tipler [1978].) Cases *i*) and *ii*) correspond to the presence of spacelike and timelike Killing fields on V_4. The analysis in later sections shall treat these cases separately. The converse of 2.1 is the following:

2.4. THEOREM. — *If* $(V, {}^{(4)}g)$ *is linearization stable then it has no Killing fields.*

The idea of the proof, suggested by the work of Fischer and Marsden [1975] and Moncrief [1976], is to show that the second order conditions 1.7 (and 1.10) are non-vacuous, i. e., if ${}^{(4)}X \neq 0$ is a Killing field, there exist $(h, \omega \neq 0$ satisfying the linearized equations such that the Taub quantity in 1.10 does not vanish. The hypersurface invariance of the Taub quantities, the explicit expression for $D^2\Phi(g, \pi)$ and underdetermined elliptic systems (Bourguignon, Ebin and Marsden [1976]) play an important role in the proof. For details, see Arms and Marsden [1978].

As we have seen, one of the crucial features of the proofs of 2.1 and 2.4 is the analysis of the Φ map and the associated adjoint formulation of the evolution equations. Another application of the Φ-formalism that we shall need is a decomposition of tensors (h, ω) due to Moncrief [1975 *b*].

As was remarked above, $D\Phi(g, \pi)^*$ and hence $\mathbb{J} \circ D\Phi(g, \pi)^*$ are elliptic. Thus we get the following two splittings by the Fredholm alternative:

$$T_{(g,\pi)}(T^*\mathscr{M}) = \text{range}(D\Phi(g, \pi)^*) \oplus \ker (D\Phi(g, \pi))$$

and

$$T_{(g,\pi)}(T^*\mathscr{M}) = \text{range}\,(\mathbb{J} \circ D\Phi(g, \pi)^*) \oplus \ker (D\Phi(g, \pi) \circ \mathbb{J})$$

The summand $\ker D\Phi(g, \pi)$ represents the infinitesimal deformations (h, ω) of (g, π) that maintain $\Phi(g, \pi)$ and range $(D\Phi(g, \pi)^*)$ represents the infinitesimal deformations which change $\Phi(g, \pi)$. Thus, if $\Phi(g, \pi) = 0$, $\ker D\Phi(g, \pi)$ represents those infinitesimal deformations that conserve the constraints. From the adjoint form of the evolution equations and the fact that they preserve the constraints, we have

$$\text{range}\ \mathbb{J} \circ D\Phi(g, \pi)^* \subset \ker D\Phi(g, \pi).$$

Thus these two splittings can be intersected to give Moncrief's splitting:

2.5. THEOREM. — *If* $\Phi(g, \pi) = 0$, *then the tangent space*

$$T_{(g,\pi)}(T^*\mathscr{M}) \approx S_2 \times S_d^2$$

splits L_2*-orthogonally as*

$$T_{(g,\pi)}(T^*\mathscr{M}) = \text{range } D\Phi(g, \pi)^* \oplus \text{range } (\mathbb{J} \circ D\Phi(g, \pi)^*) \oplus \ker (D\Phi(g, \pi) \circ \mathbb{J}) \cap \ker (D\Phi(g, \pi))$$

(The differentiability classes are the obvious ones.)

The two summands in the splitting

$$\ker D\Phi(g, \pi) = \text{range } (\mathbb{J} \circ D\Phi(g, \pi)^*) \oplus \ker (D\Phi(g, \pi) \circ \mathbb{J}) \cap \ker (D\Phi(g, \pi))$$

can be interpreted as follows. Elements of the first summand infinitesimally deform (g, π) to Cauchy data that generate isometric solutions to the Einstein equations, and elements of the second summand infinitesimally deform (g, π) in the direction of new Cauchy data that generate nonisometric solutions.

Consistent with our work in Section 1, we call elements of range $\mathbb{J} \circ D\Phi(g, \pi)^*$ *gauge transformations.* We shall sometimes denote the component of (h, ω) in $\ker (D\Phi(g, \pi) \circ \mathbb{J}) \cap \ker D\Phi(g, \pi)$ by (h^{TT}, ω^{TT}) and call it the *transverse-traceless part of* (h, ω). This terminology arises from the original decompositions of Arnowitt, Deser and Misner (circa 1960) for non-compact spacelike sections and g flat, $\pi = 0$; there the « TT » part signifies transverse traceless: $\delta h = 0$ and $\operatorname{tr} h = 0$. Compact spacelike sections (specifically $T^3 \times \mathbb{R}$) were studied by Brill and Deser (1973), where $\pi = 0$ and g is flat, entailing that (h, ω) satisfy $\delta h = 0$, $\delta\omega = 0$ and $\operatorname{tr} h =$ constant, $\operatorname{tr} \omega =$ constant.

§ 3. THE MAIN THEOREM AND IDEA OF PROOF

We make the following hypotheses on the vacuum spacetime $(V_4, {}^{(4)}g_0)$:

i) it has a compact Cauchy surface,

ii) it has a compact spacelike hypersurface Σ of constant mean curvature (it follows that Σ is a Cauchy surface; cf. Budic *et al.* [1978]), and

iii) its space of Killing fields is one dimensional, spanned by ${}^{(4)}X$.

3.1. THEOREM. — *If* ${}^{(4)}h$ *satisfies the linearized equations*

$$D\text{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0$$

and the necessary second order condition (*see* 1.7)

$$\int_\Sigma {}^{(4)}X \cdot [D^2\text{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma = 0,$$

then ${}^{(4)}h$ *is integrable.*

Moreover, curves tangent to such ${}^{(4)}h$*'s in* $\mathscr{E}$ *generate all the solutions to* $\mathrm{Ein}({}^{(4)}g) = 0$ *near* ${}^{(4)}g_0$.

Our proof will show that the singularities in $\mathscr{E}$ are conical and that the second order conditions define the tangent directions to the conical singularity.

To prove the theorem, it is sufficient to show that the induced solution of the linearized constraint equations (h, ω), which then satisfies

$$\int_M (X_\perp, X_{//}) \cdot (D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) = 0$$

(see (1.10)), is tangent to a curve of solutions of $\Phi = 0$. Corresponding to $\mathscr{E}$, we shall show that the space of solutions $\mathscr{C}$ of $\Phi = 0$ has a conical singularity at (g_0, π_0).

The projection map $P_i : {}^{(4)}g \to (g, \pi); \mathscr{E} \to \mathscr{C}$ makes $\mathscr{E}$ into a principal bundle over $\mathscr{C}$ with structure group the diffeomorphisms $\phi : V_4 \to V_4$ such that $T\phi \mid T\Sigma =$ identity; this group acts freely on the Lorentz metrics on V_4. Thus, singularities in $\mathscr{E}$ are in one to one correspondence with singularities in $\mathscr{C}$.

The singularities in $\mathscr{E}/{}^{(4)}\mathscr{D}$ are probably wrose than those in $\mathscr{E}$; $\mathscr{E}/{}^{(4)}\mathscr{D}$ may be a stratified set, whose strata are « $\forall$-manifolds ». Here ${}^{(4)}\mathscr{D}$ is the group of diffeomorphisms of V_4.

In broad outline, the plan of the proof to show that $\mathscr{C}$ has conical singularities proceeds as follows:

STEP 1. — Use the Liapunov-Schmidt procedure from bifurcation theory to reduce the problem to the study of the zero set of a real valued function. We carry out this step below.

STEP 2. — Determine the degeneracy in the second derivative of f at (g_0, π_0). This part is of separate interest and is studied in the next section. This direction of degeneracy is due to nearby (g, π) that come from a spacetime with one Killing field.

STEP 3. — Carefully fix gauges and freeze out the additional degeneracies studied in step 2 so the problem is further reduced to that of studying the zero set of a real valued function $\tilde{f}$ with a non-degenerate critical manifold.

This step represents the heart of the proof and is carried out in Sections 5 and 6, treating the spacelike and timelike cases separately.

STEP 4. — Apply a suitable version of the Morse lemma (or a theorem on the structure of zero sets of maps) to $\tilde{f}$ to show that its zero set is (a cone) $\times$ (a manifold) ([3]).

([3]) As was pointed out by J. Arms, in the spacelike case this step can also be done directly using the ideas in Atiyah, Hitchin and Singer [1978]. See part II.

STEP 5. — Remove the gauges to show that the zero set of Φ is a cone $\times$ manifold with the correct cone directions as determined by the second order conditions.

The proof is then completed by invoking properties of the *initial value problem* as was explained above.

We now carry out step 1. The space $\Lambda_d{}^0 \times \Lambda_d{}^1$, the target of Φ, has the following L_2 orthogonal decomposition (in $W^{s-2,p} \times W^{s-2,p}$):

$$\text{range } D\Phi(g_0, \pi_0) \oplus \ker D\Phi(g_0, \pi_0)^*$$

Let P denote the orthogonal projection onto the first factor. As we proved in § 2, the second factor is spanned by $(X_\perp, X_{//})$, the perpendicular and parallel projections of the Killing field ${}^{(4)}X$. We can assume $(X_\perp, X_{//})$ has L_2 length 1, and identify $\ker D\Phi(g_0, \pi_0)$ with the real line $\mathbb{R}$.

Thus the projection $(I - P)$ is given by

$$(I - P)\mathscr{H}, \mathscr{J}) = \int_M \mathscr{H} X_\perp + \mathscr{J} \cdot X_{//}$$

Let

$$\mathscr{C}_P = \{ (g, \pi) \mid P\Phi(g, \pi) = 0 \}.$$

3.2. PROPOSITION. — *Near* (g_0, π_0), *in the* $W^{s,p} \times W^{s-1,p}$ *topology,* $\mathscr{C}_P$ *is a smooth manifold with tangent space the space of solutions to the linearized constraint equations*

$$T_{(g_0,\pi_0)}\mathscr{C}_P = \ker D\Phi(g_0, \pi_0).$$

Proof. — The map Φ is clearly transverse to $\ker D\Phi(g_0, \pi_0)^*$ and its kernel splits. ■

Let $f : \mathscr{C}_P \to \mathbb{R}$ be defined by

$$\begin{aligned} f(g, \pi) &= (I - P)\Phi(g, \pi) \\ &= \int_M (X_\perp, X_{//}) \cdot \Phi(g, \pi) \\ &= \int_M \mathscr{H}(g, \pi) X_\perp + \mathscr{J}(g, \pi) \cdot X_{//}. \end{aligned}$$

Clearly, the constraint set $\mathscr{C} = \Phi^{-1}(0)$ is given by

$$\mathscr{C} = f^{-1}(0).$$

Thus, our problem is to analyze the zero set of f near (g_0, π_0).

(In bifurcation theory, the construction just carried out is called the Liapunov-Schmidt procedure; cf. Marsden [1978].)

3.3. PROPOSITION. — *The point* (g_0, π_0) *is a critical point of* f *and its Hessian is given by*

$$\begin{aligned} &d^2 f(g_0, \pi_0) \cdot ((h_1, \omega_1), (h_2, \omega_2)) \\ &\qquad = \int_M \langle (X_\perp, X_{//}), D^2\Phi(g_0, \pi_0) \cdot ((h_1, \omega_1), (h_2, \omega_2)) \rangle \end{aligned}$$

Proof. — $df(g, \pi)(h, \omega) = \int_M \langle (X, X), \ D\Phi(g, \pi) \cdot (h, \omega) \rangle$ since the contraction is natural. This vanishes at (g_0, π_0) since $(X_\perp, X_{//})$ lies in the kernel of $D\Phi(g_0, \pi_0)^*$. Since we are at a critical point, the second derivative of f may be computed in the ambient space $T^*\mathscr{M}$ and be restricted to $\mathscr{C}_P$. This remark makes the formula for d^2f clear. ■

By gauge invariance (1.12 *ii*)) any element of the range of $\mathbb{J} \circ D\Phi(g_0, \pi_0)^*$ is in the *degeneracy space* for $d^2f(g_0, \pi_0)$, i. e. in the space

$$\mathscr{S}(g_0, \pi_0) = \{ (h_1, \omega_1) \mid d^2f(g_0, \pi_0) \cdot ((h, \omega), (h_1, \omega_1)) = 0 \quad \text{for all } (h, \omega) \in \ker D\Phi(g_0, \pi_0) \}.$$

The next section will determine the degeneracy space completely and discuss its spacetime significance.

§ 4. DEGENERACY SPACES

The degeneracy space for a solution (g_0, π_0) of the constraint equations was defined at the end of § 3. A detailed analysis of this space is crucial for the proof of our main theorem, but it is also of independent interest. We treat these spaces from both the spacetime and dynamic points of view, and both from a finite and an infinitesimal perspective. Our definitions below will be shown to be equivalent to those already given.

For a vacuum spacetime $(V_4, {}^{(4)}g)$ (with a compact Cauchy surface, as above), let

$$\mathscr{S}({}^{(4)}g) = \{ {}^{(4)}X \mid {}^{(4)}X \text{ is a Killing field of } {}^{(4)}g \},$$

and let the *degeneracy manifold* of ${}^{(4)}g_0$ be defined by

$$S({}^{(4)}g_0) = \{ {}^{(4)}g \mid \mathrm{Ein}({}^{(4)}g) = 0 \quad \text{and} \quad \dim \mathscr{S}({}^{(4)}g) = \dim \mathscr{S}({}^{(4)}g_0) \},$$

i. e., $S({}^{(4)}g_0)$ is the set of vacuum spacetimes with the same number of Killing fields as ${}^{(4)}g_0$. Eventually we will prove that $S({}^{(4)}g_0)$ is a smooth manifold; its tangent space may be formally computed by linearizing its defining condition $L_{{}^{(4)}g}{}^{(4)}X = 0$. This leads us to define the *infinitesimal degeneracy space* of ${}^{(4)}g_0$ to be

$$\mathscr{S}({}^{(4)}g_0) = \{ {}^{(4)}h \mid D\mathrm{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0 \quad \text{and for each } {}^{(4)}X \ {}^{(4)}\mathscr{S}({}^{(4)}g_0) \text{ there is a vector field } {}^{(4)}Y \text{ such that } L_{{}^{(4)}X}{}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g_0 = 0 \}.$$

Here ${}^{(4)}Y$ represents the variation in the Killing fields ${}^{(4)}X$ with respect to ${}^{(4)}g$.

Given a compact spacelike hypersurface* $\Sigma \subset V_4$, we have the map $P_\Sigma : {}^{(4)}g \mapsto (g, \pi)$ defined in Section 1, and its derivative

$$DP_\Sigma({}^{(4)}g_0) \cdot {}^{(4)}h = (h, \omega);$$

we have suppressed the embedding $i : M \mapsto \Sigma$ for simplicity. We also let $\mathscr{V}_\Sigma$ denote the projection ${}^{(4)}X \mapsto (X_\perp, X_{//})$. In 2.2 we proved that

$$\mathscr{V}_\Sigma(\mathscr{S}({}^{(4)}g_0)) \equiv \mathscr{S}(g_0, \pi_0, \Sigma) = \{ (X_\perp, X_{//}) \in \ker D\Phi(g_0, \pi_0)^* \}.$$

$\mathscr{V}_\Sigma$ of course depends also on ${}^{(4)}g$; when we wish to make this explicit, we write $\mathscr{V}_\Sigma({}^{(4)}X, {}^{(4)}g) = (X_\perp, X_{//})$.

Let

$$P_\Sigma(S({}^{(4)}g_0)) = S(g_0, \pi_0, \Sigma),$$

and let

$$DP_\Sigma({}^{(4)}g_0) \cdot \mathscr{S}({}^{(4)}g_0) = \mathscr{S}(g_0, \pi_0, \Sigma).$$

We expect $\mathscr{S}(g_0, \pi_0, \Sigma)$ to be the formal tangent space to $S(g_0, \pi_0, \Sigma)$. The next proposition verifies this.

4.1. PROPOSITION. — *The following equality holds:*

$\mathscr{S}(g_0, \pi_0, \Sigma) = \{ (h, \omega) \mid D\Phi(g_0, \pi_0) \cdot (h, \omega) = 0$
and for each $(X_\perp, X_{//}) \in \ker D\Phi(g_0, \pi_0)^*$, there is a $(X_\perp', X_{//}')$ such that
$D_{(g,\pi)}[\mathbb{J} \circ D\Phi(g, \pi)^*(X_\perp, X_{//})]|_{(g_0,\pi_0)} \cdot (h, \omega) + \mathbb{J} \circ D\Phi(g_0, \pi_0)^*(X_\perp', X_{//}') = 0 \}$.

Remark. — A curve $(g(\lambda), \pi(\lambda))$ in $S(g_0, \pi_0, \Sigma)$ satisfies

$$D\Phi(g(\lambda), \pi(\lambda))^* \cdot (X_\perp(\lambda), X_{//}(\lambda)) = 0$$

for some $(X_\perp(\lambda), X_{//}(\lambda))$. Differentiating in λ and evaluating at $\lambda = 0$ yields

$$D_{(g,\pi)}(D\Phi(g, \pi)^*(X_\perp, X_{//})|_{(g_0,\pi_0)}) \cdot (h, \omega) + D\Phi(g_0, \pi_0)^* \cdot (X_\perp', X_{//}') = 0$$

where $(h, \omega) = \dfrac{d}{d\lambda}(g(\lambda), \pi(\lambda))|_{\lambda=0}$ and $(X_\perp', X_{//}') = \dfrac{d}{d\lambda} X_\perp((\lambda), X_{//}(\lambda))$

Inserting a factor of $\mathbb{J}$ to parallel the definition of $\mathscr{S}$ yields the formula for $\mathscr{S}(g_0, \pi_0, \Sigma)$ in a formal way.

Proof of 4.1. — Let $(h, \omega) \in \mathscr{S}(g_0, \pi_0, \Sigma)$, so by definition

$$(h, \omega) = DP_\Sigma({}^{(4)}g_0) \cdot {}^{(4)}h$$

for some ${}^{(4)}h \in \mathscr{S}({}^{(4)}g_0)$. By 1.11 and its derivative in the ${}^{(4)}g$ variable, we get

$$DP_\Sigma({}^{(4)}g_0) \cdot (L_{(4)Y}{}^{(4)}g_0) = -\mathbb{J} \circ D\Phi(g, \pi)^*(Y_\perp, Y_{//})$$

and

$$\begin{aligned} DP_\Sigma({}^{(4)}g_0) \cdot (L_{(4)X}{}^{(4)}h) = &- D_{(g,\pi)}[\mathbb{J} \circ D\Phi(g, \pi)^*(X_\perp, X_{//})]|_{(g_0,\pi_0)}(h, \omega) \\ &- \mathbb{J} \circ D\Phi(g, \pi)^* \cdot (D_{(4)g}\mathscr{V}_\Sigma({}^{(4)}X, {}^{(4)}g_0) \cdot {}^{(4)}h). \end{aligned}$$

(4) If Σ is spacelike for ${}^{(4)}g_0$, it will also be spacelike for nearby ${}^{(4)}g$, which is all that concerns us here.

Let

$$(X_\perp', X_{//}') = D\mathscr{V}_\Sigma({}^{(4)}X, {}^{(4)}g)\cdot({}^{(4)}Y, {}^{(4)}h)$$
$$= (Y_\perp, Y_{//}) + D_{{}^{(4)}g}P_\Sigma({}^{(4)}X, {}^{(4)}g)\cdot{}^{(4)}h\,.$$

Explicitly,

$$X_\perp' = Y_\perp + \frac{X_\perp}{N}N',\ X_{//}' = Y_{//} + \frac{X_\perp}{N}X',$$

where $(N', X') = D_{{}^{(4)}g}\mathscr{V}_\Sigma({}^{(4)}X, {}^{(4)}g_0)\cdot{}^{(4)}h$; compare the linearized Einstein equations.

Adding the above expressions gives

$$DP_\Sigma({}^{(4)}g_0)\cdot(L_{{}^{(4)}X}{}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g_0)$$
$$= -D_{(g,\pi)}[\mathbb{J}\circ D\Phi(g,\pi)^*\cdot(X_\perp, X_{//})]\,|_{(g_0,\pi_0)}\cdot(h,\omega)$$
$$-\mathbb{J}\circ D\Phi(g_0,\pi_0)^*\cdot(D_{({}^{(4)}X,{}^{(4)}g)}\mathscr{V}_\Sigma({}^{(4)}X, {}^{(4)}g)\cdot({}^{(4)}Y, {}^{(4)}h))\,.$$

This shows that (h, ω) belongs to the right hand side of the formula for $\mathscr{S}(g_0, \pi_0, \Sigma)$.

Conversely, let (h, ω) belong to the right hand side of the stated formula for $\mathscr{S}(g_0, \pi_0, \Sigma)$. Let ${}^{(4)}h$ satisfy the linearized equations and induce (h, ω). We will show that ${}^{(4)}h \in \mathscr{S}({}^{(4)}g_0)$ by a procedure parallel to the proof of 2.2. This requires the following steps.

4.2. LEMMA. — *Let* $\mathrm{Ric}({}^{(4)}g_0) = 0$, $D\mathrm{Ric}({}^{(4)}g_0)\cdot{}^{(4)}h = 0$, *and* $L_{{}^{(4)}X}{}^{(4)}g = 0$. *For a vector field* ${}^{(4)}Y$, *define*

$${}^{(4)}\tilde{h} = L_{{}^{(4)}X}{}^{(4)}h + L_{{}^{(4)}Y}{}^{(4)}g_0\,.$$

Then

$$D\mathrm{Ric}({}^{(4)}g_0)\cdot{}^{(4)}\tilde{h} = 0\,.$$

Proof. — By 1.4, $D\mathrm{Ric}({}^{(4)}g_0)\cdot(L_{{}^{(4)}Y}{}^{(4)}g_0) = 0$. We see that

$$D\mathrm{Ric}({}^{(4)}g_0)\cdot L_{{}^{(4)}X}{}^{(4)}h = 0$$

by letting F_λ denote the flow of ${}^{(4)}X$, differentiating the covariance relation

$$D\mathrm{Ric}(F_\lambda{}^{*(4)}g_0)\cdot(F_\lambda{}^{*(4)}h) = 0$$

in λ and using $L_{{}^{(4)}X}{}^{(4)}g_0 = 0$. ■

We will now determine a vector field ${}^{(4)}Y$ such that ${}^{(4)}\tilde{h} = 0$. Define ${}^{(4)}Y$ to be the solution of the differential equation

$$\delta\left[{}^{(4)}\tilde{h} - \frac{1}{2}{}^{(4)}g_0\,\mathrm{tr}\,{}^{(4)}\tilde{h}\right] = 0\,,$$

which is equivalent to the hyperbolic equation

$$\Box{}^{(4)}Y + \delta\left[L_{{}^{(4)}X}{}^{(4)}h - \frac{1}{2}{}^{(4)}g_0\,\mathrm{tr}\,(L_{{}^{(4)}X}{}^{(4)}h)\right] = 0$$

(since $\mathrm{Ric}({}^{(4)}g_0) = 0$) with Cauchy data

$${}^{(4)}Y\,|_\Sigma = (Y_\perp, Y_{//}) = (X_\perp', X_{//}') - D_{{}^{(4)}g}\mathscr{V}_\Sigma({}^{(4)}X, {}^{(4)}g)\cdot{}^{(4)}h$$

and

$$(\nabla_{(4)Z}\,{}^{(4)}Y)^{\alpha} = -\frac{1}{2}\,{}^{(4)}Z_{\Sigma}{}^{\beta}(L_{(4)X}{}^{(4)}h)^{\alpha}{}_{\beta} - {}^{(4)}Z_{\Sigma}{}^{\gamma}\mathscr{P}^{\alpha\beta}\left[{}^{(4)}Y_{\gamma;\beta} + \frac{1}{2}(L_{(4)X}{}^{(4)}h)_{\gamma\beta}\right].$$

(The right hand side of the last equation only involves tangential derivatives of ${}^{(4)}Y$ and so is expressible in terms of $(Y_{\perp}, Y_{//})$.

Lemma 4.2 and the defining equation for ${}^{(4)}Y$, namely

$$\delta\left[{}^{(4)}\tilde{h} - \frac{1}{2}\,{}^{(4)}g_0 \operatorname{tr} {}^{(4)}\tilde{h}\right] = 0$$

show that

$$\Box_L\left[{}^{(4)}\tilde{h} - \frac{1}{2}\,{}^{(4)}g_0 \operatorname{tr} {}^{(4)}\tilde{h}\right] = 0.$$

Thus ${}^{(4)}\tilde{h}$ will vanish if its Cauchy data does. Clearly ${}^{(4)}\tilde{h}^{\perp\perp} = 0$ and ${}^{(4)}\tilde{h}^{\perp}{}_{//} = 0$ since ${}^{(4)}Y$ projects to $Y_{\perp}$, $Y_{//}$. The remaining projection ${}^{(4)}\tilde{h}_{////}$ is the first slot of

$$DP_{\Sigma}({}^{(4)}g_0)\cdot{}^{(4)}\tilde{h} = -D[\mathbb{J}\circ D\Phi(g_0, \pi_0)^*(X_{\perp}, X_{//})]\cdot(h, \omega) - \mathbb{J}\circ D\Phi(g_0, \pi_0)^*\cdot(X_{\perp}', X_{//}')$$

which vanishes by assumption. Thus ${}^{(4)}\tilde{h}\,|_{\Sigma} = 0$. The second slot vanishing then implies that

$$\nabla_{(4)Z_{\Sigma}}{}^{(4)}\tilde{h}_{////}\,|_{\Sigma} = 0.$$

The identity

$$\begin{aligned}{}^{(4)}Z_{\Sigma}\cdot(\nabla_{(4)Z_{\Sigma}}{}^{(4)}\tilde{h})\cdot{}^{(4)}Z_{\Sigma} = {}^{(4)}Z_{\Sigma}{}^{\alpha}\{\, & 2\mathscr{P}^{\beta\gamma}{}^{(4)}\tilde{h}_{\alpha\beta;\gamma} - 2{}^{(4)}Y^{\beta}(\mathrm{Ric}({}^{(4)}g))_{\alpha\beta} \\ & + 2\left[\Box{}^{(4)}Y + \delta\cdot\left(L_{(4)X}{}^{(4)}h - \frac{1}{2}\,{}^{(4)}g \operatorname{tr}(L_{(4)X}{}^{(4)}h)\right)\right]_{\alpha} \\ & - [\mathscr{P}^{\beta\gamma}{}^{(4)}\tilde{h}_{\beta\gamma}]_{;\alpha} + \mathscr{P}^{\beta\gamma}{}_{;\alpha}{}^{(4)}\tilde{h}_{\beta\gamma}\,\}\end{aligned}$$

expresses $(\nabla_{(4)Z_{\Sigma}}{}^{(4)}\tilde{h})^{\perp\perp}$ in terms of ${}^{(4)}\tilde{h}\,|_{\Sigma}$ and the derivatives of ${}^{(4)}\tilde{h}$ tangential to Σ and other quantities which vanish by virtue of $\mathrm{Ric}({}^{(4)}g) = 0$, and the chosen evolution equation for ${}^{(4)}Y$. Thus we have

$$(\nabla_{(4)Z_{\Sigma}}{}^{(4)}\tilde{h})^{\perp\perp}\,|_{\Sigma} = 0.$$

Finally the identity

$$\begin{aligned}\mathscr{P}^{\alpha\sigma}\,{}^{(4)}Z_{\Sigma}{}^{\beta}(\nabla_{(4)Z_{\Sigma}}{}^{(4)}\tilde{h})_{\alpha\beta} = \mathscr{P}^{\alpha\sigma}\Big\{\, & \mathscr{P}^{\beta\gamma}({}^{(4)}\tilde{h}_{\alpha\beta;\gamma}) - {}^{(4)}Y^{\beta}(\mathrm{Ric}({}^{(4)}g))_{\alpha\beta} \\ & + \left[\Box{}^{(4)}Y + \delta\cdot\left[L_{(4)X}{}^{(4)}h - \frac{1}{2}\,{}^{(4)}g \operatorname{tr}(L_{(4)X}{}^{(4)}h)\right]\right]_{\alpha} - \frac{1}{2}(\operatorname{tr}{}^{(4)}\tilde{h})_{;\alpha}\Big\}\end{aligned}$$

expresses $(\nabla_{(4)Z_{\Sigma}}{}^{(4)}\tilde{h})^{\perp}{}_{//}$ in terms of ${}^{(4)}\tilde{h}\,|_{\Sigma}$, the tangential derivatives of ${}^{(4)}\tilde{h}$ at Σ and terms which also vanish by virtue of $\mathrm{Ric}({}^{(4)}g) = 0$ and the evolution equation for ${}^{(4)}Y$.

Thus $\tilde{h}$ and $\nabla_{(4)Z_{\Sigma}}{}^{(4)}\tilde{h}$ vanish on Σ and hence ${}^{(4)}\tilde{h}$ vanishes on all of V_4. ■

Next we show that $\mathscr{S}(g_0, \pi_0, \Sigma)$ is the degeneracy space in the sense defined at the end of the previous section.

4.3. PROPOSITION. — $\mathscr{S}(g_0, \pi_0, \Sigma) = \{ (h, \omega) \in \ker D\Phi(g_0, \pi_0) \mid$ for each $(X_\perp, X_{//}) \in \ker D\Phi(g_0, \pi_0)^*$, $(h_1, \omega_1) \in \ker D\Phi(g_0, \pi_0)$,

$$\int_M \langle (X_\perp, X_{//}), D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h_1, \omega_1)) \rangle = 0 \Big\}.$$

Proof. — From 4.1, $(h, \omega) \in \mathscr{S}(g_0, \pi_0, \Sigma)$ if and only if

$$D_{(g,\pi)} [D\Phi(g, \pi)^*(X_\perp, X_{//})] |_{(g_0,\pi_0)} \cdot (h, \omega) \in \text{range } D\Phi(g_0, \pi_0)^*,$$

i. e., the left hand side is L_2 orthogonal to $\ker D\Phi(g_0, \pi_0)$,

$$\int_M \langle (h_1, \omega_1), D_{(g,\pi)} [D\Phi(g, \pi)^* \cdot (X_\perp, X_{//})]_{(g_0,\pi_0)} \cdot (h, \omega) \rangle = 0$$

for all $(h_1, \omega_1) \in \ker D\Phi(g_0, \pi_0)$. Differentiating the identity

$$\langle\langle (h_1, \omega_1), D\Phi(g, \pi)^* \cdot (X_\perp, X_{//}) \rangle\rangle = \langle\langle (X_\perp, X_{//}), D\Phi(g, \pi)(h_1, \omega_1) \rangle\rangle$$

in (g, π), we see that the preceding condition is equivalent to the one stated in the proposition. ■

For later purposes we shall need to know what the degeneracy spaces are in case Σ is a hypersurface of constant mean curvature (see Proposition 2.3).

4.4. PROPOSITION. — *If Σ has constant mean curvature, we have the L_2 orthogonal decomposition*

$$\mathscr{S}(g_0, \pi_0, \Sigma) = \mathscr{D} \oplus \text{range } \mathbb{J} \circ D\Phi(g_0, \pi_0)^*$$

where

$$\mathscr{D} = \{ (h, \omega) \in \ker (D\Phi(g_0, \pi_0) \circ \mathbb{J}) \cap \ker D\Phi(g_0, \pi_0) \mid L_X h = 0 \text{ and } L_X \omega = 0 \text{ for each } X \text{ such that } L_X g = L_X \pi = 0 \},$$ *if $\pi_0 \neq 0$ or g_0 is not flat*

and

$$\mathscr{D} = \{ (h, 0) \mid \nabla h = 0 \text{; i. e., } h \text{ is covariant constant and } L_X g_0 = 0 \text{ implies } L_X h = 0 \},$$ *if $\pi_0 = 0$ and g_0 is flat.*

Proof. — Let us begin with the following observation:

4.5. LEMMA. — $\mathscr{S}(g_0, \pi_0, \Sigma) \supset \text{range } \mathbb{J} \circ D\Phi(g_0, \pi_0)^*$.

Proof. — This may be verified by a computation but the following proof is easier. Note that

$$\mathscr{S}({}^{(4)}g_0) \supset \{ L_{{}^{(4)}Y} {}^{(4)}g_0 \mid {}^{(4)}Y \text{ is a vector field on } V_4 \}$$

by its definition and 1.4. But $\mathscr{S}(g_0, \pi_0, \Sigma)$ is the image of $\mathscr{S}({}^{(4)}g_0)$ under

$DP_\Sigma({}^{(4)}g_0)$ and by 1.11, range $[\mathbb{J} \circ D\Phi(g_0, \pi_0)^*]$ is the image of the right hand side. ■

To prove 4.4 let us first consider the case in which $\pi_0 \neq 0$ or g_0 is not flat. By 2.3, $\ker D\Phi(g_0, \pi_0)^* = \{ (0, X) \mid L_X g_0 = 0 \text{ and } L_X \pi_0 = 0 \}$. By 2.5 we have

$$\ker D\Phi(g_0, \pi_0) = \operatorname{range} \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \oplus (\ker (D\Phi(g_0, \pi_0) \circ \mathbb{J}) \cap \ker D(\Phi(g_0, \pi_0))$$

A straightforward calculation gives

$$- D_{(g,\pi)} [\mathbb{J} \circ D\Phi(g, \pi)^*(0, X)] \cdot (h, \omega) = (L_X h, L_X \omega)$$

By 4.5 and 4.1, this maps range $\mathbb{J} \circ D\Phi(g_0, \pi_0)^*$ to itself. On the other hand, this maps the summand $(\ker (D\Phi(g_0, \pi_0) \circ \mathbb{J})) \cap \ker D\Phi(g_0, \pi_0)$ orthogonal to range $[\mathbb{J} \circ D\Phi(g_0, \pi_0)^*]$. Indeed, since $L_X g = 0$,

$$\int_M (h_1, \omega_1) \cdot (L_X h, L_X \omega) = - \int_M (L_X h_1, L_X \omega_1) \cdot (h, \omega)$$

Let $(h_1, \omega_1) \in \operatorname{range} (\mathbb{J} \circ D\Phi(g_0, \pi_0)^*) \subset \mathscr{S}(g_0, \pi_0, \Sigma)$ so

$$0 = (L_X h_1, L_X \omega_1) - \mathbb{J} \circ D\Phi(g_0, \omega_0)^* \cdot (Y_\perp, Y_{//})$$

for some $(Y_\perp, Y_{//})$ by 4.1, and let $(h, \omega) \in (\ker D\Phi(g_0, \pi_0) \circ \mathbb{J}) \cap \ker D\Phi(g_0, \pi_0)$. Then we get

$$\int_M (h_1, \omega_1) \cdot (L_X h, L_X \omega) = - \int_M [\mathbb{J} \circ D\Phi(g_0, \pi_0)^*(Y_\perp, Y_{//})] \cdot (h, \omega)$$
$$= \int_M (Y_\perp, Y_{//}) \cdot (D\Phi(g_0, \pi_0) \circ \mathbb{J})(h, \omega) = 0 .$$

Thus the only elements of the second summand of $\ker D\Phi(g_0, \pi_0)$ that lie in $\mathscr{S}(g_0, \pi_0, \Sigma)$ are those for which $(L_X h, L_X \omega) = 0$ by virtue of 4.1. This proves the first part of 4.4.

Now suppose that $\pi_0 = 0$ and g_0 is flat. Then $(h, \omega) \in \mathscr{S}(g_0, \pi_0, \Sigma)$ must satisfy the conditions just derived, plus the condition of (4.1) corresponding to the elements $(N, O) \in \ker D\Phi(g_0, \pi_0)^*$ (see 2.3 *ii*)). We can assume $(h, \omega) \in$ the second summand of $\ker D\Phi(g_0, \pi_0)$. But for $\pi_0 = 0$ and g_0 flat, this reduces to the Brill-Deser [1972] « TT » condition (cf. Moncrief [1976]); i. e.,

$$h = h^{TT} + \alpha g_0, \ \omega = \omega^{TT} + \beta g_0{}^\# \mu(g_0)$$

where α, β are constant and $\delta(h^{TT}) = 0$, $\operatorname{tr}(h^{TT}) = 0$. The condition

$$O = D_{(g,\pi)} [\mathbb{J} \circ D(g, \pi)^* \cdot (X_\perp, X_{//})] |_{(g_0, \omega_0)} \cdot (h, \omega) + \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (X_\perp', X_{//}')$$

of 4.1 for g_0 flat, $\pi_0 = 0$, $X_\perp = 1$ and $X_{//} = 0$, evaluated on

$$(h, \omega) = (h^{TT} + \alpha g_0, \omega^{TT} + \beta g_0{}^\# \mu g_0)$$

becomes

$$\begin{cases} 2(\omega^{TT} - \frac{1}{2}\beta g_0{}^{\#}\mu(g_0)) = -(L_{X'_{//}}g_0)^{\#}\mu(g_0) \\ \frac{1}{2}(h^{TT})^{ij|}{}_{k}{}^{|k} \qquad = -X_\perp{}'^{|i|j} + g^{ij}X_\perp{}'_{|k}{}^{|k} \end{cases}$$

However the left and right hand sides of each equation are L_2 orthogonal, and so vanish separately. Moreover ω^{TT} and $\beta g_0{}^{-1}\mu(g_0)$ are orthogonal, so both vanish. Thus $\omega = 0$. Also $(h^{TT})_{ij|k}{}^{|k} = 0$ implies h^{TT} is covariant constant. This proves the second part of 4.4. ■

In 1.7 and 1.10 we developed necessary second order conditions that must be satisfied by an integrable element $(h, \omega) \in \ker D\Phi(g_0, \pi_0)$. Our main result may be phrased by saying that there cannot be any higher order conditions. One manifestation of this is the following result that can be checked by a completely different technique from the proof of our main result.

4.6. PROPOSITION. — a) *If* $(h, \omega) \in \ker D\Phi(g_0, \pi_0)$ *is integrable,* $(X_\perp, X_{//}) \in \ker D\Phi(g_0, \pi_0)^*$, $(h, \omega) \in \mathscr{S}(g_0, \pi_0, \Sigma)$, *and* $(X_\perp', X_{//}')$ *is as in* 4.1, *then the following third order condition holds:*

$$3\int_M \langle (X_\perp', X_{//}'), D^2\Phi(g_0, \omega_0)\cdot((h, \omega), (h, \omega)) \rangle + \int_M \langle (X_\perp, X_{//}), (D^3\Phi(g_0, \pi_0)\cdot((h, \omega), (h, \omega)(h, \omega)) \rangle = 0$$

b) *The third order condition in* a) *is vacuous in the sense that it is an identity that is satisfied automatically for any* $(h, \omega) \in \mathscr{S}(g_0, \pi_0, \Sigma)$ *satisfying the first and second order conditions. (We assume, as always, that there is at least one compact spacelike hypersurface of constant mean curvature somewhere in the spacetime.)*

This situation is in contrast to the second order conditions that are always a non-vacuous condition on $(h, \omega) \in \ker D\Phi(g_0, \pi_0)$ if $\ker D\Phi(g_0, \pi_0)^* \neq \{0\}$.

Sketch of Proof of 4.6. — *a)* Let $(h^{(k)}, \omega^{(k)}) = \frac{d^k}{d\lambda^k}(g(\lambda), \pi(\lambda))|_{\lambda=0}$.

Differentiating $\Phi(g(\lambda), \pi(\lambda)) = 0$ three times and setting $\lambda = 0$ gives

$$D\Phi(g_0, \pi_0)\cdot(h^{(3)}, \omega^{(3)}) + 3D^2\Phi(g_0, \omega_0)\cdot((h, \omega), (h^{(2)}, \omega^{(2)})) + D^3\Phi(g_0, \pi_0)\cdot((h, \omega), (h, \omega), (h, \omega)) = 0.$$

Contracting with $(X_\perp, X_{//})$ and integrating gives

$$0 = 3\int_M \langle (X_\perp, X_{//}), (D^2\Phi(g_0, \pi_0)\cdot((h, \omega), (h^{(2)}, \omega^{(2)}))) \rangle + \int_M \langle (X_\perp, X_{//}), D^3\Phi(g_0, \pi_0)\cdot((h, \omega), (h, \omega), (h, \omega))) \rangle,$$

i. e.,

$$0 = 3\int_M \langle (h^{(2)}, \omega^{(2)}), (D_{(g_0,\pi_0)}[D\Phi(g, \pi)^* \cdot (X_\perp, X_{//})]_{(g_0,\pi_0)} \cdot (h, \omega)) \rangle + \int_M \langle (X_\perp, X_{//}), (D^3\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega), (h, \omega))) \rangle .$$

Substituting from 4.1 (with the $\mathbb{J}$'s removed),

$$0 = -3\int_M \langle (Y_\perp, Y_{//}), (D\Phi(g_0, \pi_0) \cdot (h^{(2)}, \omega^{(2)})) \rangle + \int_M \langle (X_\perp, X_{//}), (D^3\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega), (h, \omega))) \rangle$$

Finally, substitution of the second order equations

$$0 = D\Phi(g_0, \pi_0) \cdot (h^{(2)}, \omega^{(2)}) + D^2\Phi(g_0, \omega_0) \cdot ((h, \omega), (h, \omega))$$

yields *a*).

The outline of the proof of *b*) is as follows.

STEP 1. — *Show that the third order conditions are hypersurface and gauge invariant.*

This step follows the pattern used in establishing the second order conditions. For example one needs to differentiate the contracted Bianchi identities $\mathrm{Ein}({}^{(4)}g) = 0$ twice rather than just once. The second order conditions imply that there exist $(h^{(2)}, \omega^{(2)})$ such that

$$D\Phi(g_0, \pi_0) \cdot (h^{(2)}, \omega^{(2)}) + D^2\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega)) = 0 ,$$

and the integrability conditions

$$\delta[D^2\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, {}^{(4)}h)] = 0$$

enable one to propagate these to a solution of

$$D\mathrm{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h^{(2)} + D^2\mathrm{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h) = 0 .$$

This shows that

$$\int_\Sigma {}^{(4)}X \cdot [3D^2\mathrm{Ein}({}^{(4)}g) \cdot ({}^{(4)}h, {}^{(4)}h^{(2)}) + D^3\mathrm{Ein}({}^{(4)}g)({}^{(4)}h, {}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma d^3\Sigma$$

is hypersurface invariant and reduces to $-\frac{1}{2}$ our third order integral in terms of Φ. Gauge invariance then follows, as for the second order conditions. This enables one to pass to a hypersurface Σ of constant mean curvature and to omit the first summand in the decomposition of $\ker D\Phi(g_0, \pi_0)$. On this hypersurface there are the two cases *i*) and *ii*) of 2.3. Proposition 4.4 characterizes $\mathscr{S}(g_0, \pi_0, \Sigma)$ explicitly in these two cases.

STEP 2. — *Verification of the third order condition for* $\pi_0 = 0$, *g flat and* $(X_\perp, X_{//}) = (N, O)$.

By gauge invariance of the third order condition and using 4.4, it is enough to take $(h, \omega) = (h, 0)$ where h is covariant constant. Then the condition on $(X_\perp', X_{//}')$ in 4.1 reduces to $(X_\perp', X_{//}') \in \ker D\Phi(g, 0)^*$ since

$$- D_{(g,\pi)}[\mathbb{J} \circ D\Phi(g, \pi)^*(N, O)]_{(g_0,0)}(h, \omega) = 0$$

as a straightforward consequence of $\pi_0 = \omega = 0$, $\nabla N = 0$, $\nabla h = 0$, and $\mathrm{Ric}(g_0) = 0$. The third order condition then reduces to

$$0 = \int_M \langle (1, 0), D^3\Phi(g, 0) \cdot (((h, 0), (h, 0), (h, 0))) \rangle .$$

Writing this out explicitly, we see that each term vanishes as a consequence of $\mathrm{Ric}(g_0) = 0$, $h = 0$ or $= 0$.

STEP 3. — *Verification of the third order condition for* $(X_\perp, X_{//}) = (O, X)$ *where* $L_X g = 0$, $L_X \pi = 0$.

By gauge invariance we can assume that

$$(h, \omega) \in (\ker D\Phi(g_0, \pi_0) \circ \mathbb{J}^* \cap \ker D\Phi(g_0, \pi_0) .$$

Then if $(X_\perp', X_{//}')$ satisfies the condition of 4.1, each side must separately vanish, as they are orthogonal, as was shown in the proof of 4.4. Therefore

$$(L_X h, L_X \omega) = 0 ,$$

and

$$(X_\perp', X_{//}') \in \ker D\Phi(g_0, \pi_0)^* .$$

Thus the third order condition reduces to

$$0 = \int_M \langle (0, X), (D^3\Phi(g_0, \pi_0) \cdot ((h, \omega), (h, \omega), (h, \omega)) \rangle .$$

But this is true since $D^3\mathscr{J}(g, \pi) \equiv 0$ as $\mathscr{J}$ is quadratic in (g, π). ■

§ 5. A SLICE IN $T^*\mathscr{M}$ FOR THE ACTION OF $\mathscr{D}^3$

The proof of our results depends on a carefully constructed slice for the action of the three dimensional diffeomorphism group $\mathscr{D}^3$ on $T^*\mathscr{M}$. For the action of $\mathscr{D}^3$ on $\mathscr{M}$, the Ebin-Palais slice theorem (see Ebin [1970]) asserts the existence of a slice. We shall find one in $T^*\mathscr{M}$ by utilizing the Φ-map. Our slice will be an affine one, L_2 orthogonal to the orbit of $\mathscr{D}^3$. A « more covariant » slice, like the one constructed by Ebin would technically complicate our proof; affine slices have all the desired properties needed here and are of the type originally proposed (and unpublished) by Palais. The analogue of Ebin's slice on $T^*\mathscr{M}$ is discussed in Fischer and Marsden [1977].

We recall that a *slice* S_{x_0} at x_0 in a manifold M relative to the action of a Lie group G on M is a submanifold containing x_0 such that

i) if $g \in I_{x_0}$, the isotropy group of x_0, then

$$g \cdot S_{x_0} = S_{x_0};$$

ii) If $g \in G$ and $g \cdot S_{x_0} \cap S_{x_0} \neq \phi$, then $g \in I_{x_0}$; and
iii) there is a local cross-section

$$\chi : G/I_{x_0} \to G$$

defined in a neighborhood U of the identity coset such that the map $(\phi, x) \mapsto \chi(\phi) \cdot x$ is a homeomorphism of $U \times S_{x_0}$ onto a neighborhood of x_0 in M. In particular, the slice S_{x_0} sweeps out a neighborhood of x_0 under the group action; see Palais [1957].

In effect, if G is interpreted as a gauge group, a slice may be thought of as the local choice of a gauge.

Let M be a smooth compact manifold and $\mathscr{D}$ the group of $W^{s+1,p}\left(s > \frac{n}{p} + 1\right)$ diffeomorphisms of M and $\mathscr{M}$ the space of $W^{s,p}$ Riemannian metrics, with $\mathscr{D}$ acting on $\mathscr{M}$ by pull-back (a « right » action). The orbit of a $W^{s+r,p}$ element $g_0 \in \mathscr{M}$ is a C^r closed submanifold $\mathscr{O}_{g_0} \subset \mathscr{M}$ with tangent space

$$T_{g_0}\mathscr{O}_{g_0} = \{ L_X g_0 \mid X \text{ is a } W^{s+1,p} \text{ vector field} \}.$$

This is closed since the operator $X \mapsto L_X g_0$ is elliptic; its L_2 orthogonal complement is $\ker \delta$. In fact, a neighborhood of g_0 in the affine space $g_0 + \ker \delta$ may be chosen as the slice; the neighborhood is a ball in a Sobolev norm that is invariant under the isometry group of g_0. These facts follow from Palais [1957] and Ebin [1970].

To construct a slice in $T^*\mathscr{M}$ we proceed according to the following steps.

5.1. LEMMA. — *Let* (g_0, π_0) *be of class* $W^{s+r+2,p} \times W^{s+r+1,p}$, $s > n/p + 1$ *and* $\mathscr{O}_{(g_0,\pi_0)}$ *its orbit under the action of* $\mathscr{D}^3$ *diffeomorphisms of class* $W^{s+1,p}$ *on* $T^*\mathscr{M}$ *under pull-back*, i. e.,

$$\mathscr{O}_{(g_0,\pi_0)} = \{ (\eta^* g_0, \eta^* \pi_0) \mid \eta \in \mathscr{D}^3 \}.$$

Then $\mathscr{O}_{(g_0,\pi_0)}$ *is a closed* C^r *submanifold of* $T^*\mathscr{M}$ *with tangent space at* (g_0, π_0) *given by*

$$\begin{aligned} T_{(g_0,\pi_0)}\mathscr{O}_{(g_0,\pi_0)} &= \{ -\mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (0, X) \mid \textit{is a } W^{s+1,p} \textit{ vector field on } M \} \\ &= \{ (L_X g_0, L_X \pi_0) \mid X \textit{ is a } W^{s+1,p} \textit{ vector field on } M \}. \end{aligned}$$

Proof. — Let $\Psi : \mathscr{D}^3 \to T^*\mathscr{M}$; $\eta \mapsto (\eta^* g_0, \eta^* \pi_0)$. By standard composition properties of Sobolev spaces, Ψ is a C^r map; see Ebin [1970]. We have

$$T_\eta \Psi(X) = (\eta^* L_X g_0, \eta^* L_X \pi_0).$$

Since $X \mapsto L_X g_0$ is elliptic, $T_\eta \Psi$ has closed range and finite dimensional kernel. By the arguments of Ebin-Marsden [1970, Appendix B], $\ker T_\eta \Psi$ is a C^r subbundle of $T\mathscr{D}^3$. It follows from the implicit function theorem

that the range of Ψ is an immersed submanifold. From Ebin [1970, Prop. 6.13], it follows that Ψ is an open map onto its range and that the range is closed. The lemma then follows. ■

5.2. LEMMA. — *There is a unique linear operator* $C \mapsto Y(C)$ *from* $W^{s+2,p}$ *functions to* $W^{s+1,p}$ *vector fields on* M *such that*

$$-\mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (C, Y(C))$$

is L_2 *orthogonal to* $T_{(g_0,\pi_0)}\mathcal{O}_{(g_0,\pi_0)}$.

Proof. — The condition is

$$\begin{aligned} 0 &= \langle\langle -\mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (O, X), -\mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (C, Y(C)) \rangle\rangle \\ &= \langle\langle D\Phi(g_0, \pi_0)^* \cdot (O, X), D\Phi(g_0, \pi_0)^* \cdot (C, Y(C)) \rangle\rangle \\ &= \langle\langle (O, X), D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* \cdot (C, Y(C)) \rangle\rangle \end{aligned}$$

i. e. $P_2 \cdot (D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* \cdot (C, Y(C))) = 0$,

where P_2 is the projection onto the second factor. Thus, Y is to be solved for from the equation

$$\begin{aligned} &P_2 \cdot (D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* \cdot (O, Y)) \\ &\qquad = -P_2 \cdot (D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* \cdot ((C, O))). \end{aligned}$$

The proof will be complete if we can show

i) the left side is an elliptic operator in Y,
ii) the left side has kernel $\{ Y \mid L_Y g_0 = 0, L_Y \pi_0 = 0 \}$, and
iii) the right side is L_2 orthogonal to this kernel.

Proof of i). — Let $\Sigma_{(g_0,\pi_0)}(Y)$ be the left side of the above equation. Then $\Sigma_{(g_0,\pi_0)}$ is a second order operator whose leading terms are

$$\begin{aligned} &-2\mu(g_0)(Y_i{}^{|k}{}_{|k} + Y^k{}_{|i|k}) \\ &+ 2\pi^{jl}(Y^m{}_{|ml}\pi'_{ij} - Y_{i|ml}\pi'^m{}_j - Y_{j|ml}\pi'_i{}^m) \\ &- \pi^{jl}(Y^m{}_{|mi}\pi'_{jl} - Y_{j|mi}\pi'^m{}_l - Y_{l|mi}\pi'_j{}^m). \end{aligned}$$

The symbol $\sigma_\xi(\Sigma_{(g_0,\pi_0)})$ of the operator $\Sigma_{(g_0,\pi_0)}$

$$\begin{aligned} \sigma_\xi(\Sigma_{(g_0,\pi_0)}) \cdot Y = &-2\mu(g_0)(\xi^k\xi_k Y_i + \xi_i\xi_k Y^k) \\ &+ 2\pi^{jl}(\xi_m\xi_l Y^m\pi'_{ij} - \xi_m\xi_l Y_i\pi'^m{}_j - \xi_m\xi_l Y_j\pi'_i{}^m) \\ &- \pi^{jl}(\xi_m\xi_i[Y^m\pi'_{jl} - Y_j\pi'^m{}_l - Y_l\pi'_j{}^m]). \end{aligned}$$

An algebraic computation shows that

$$\sigma_\xi(\Sigma_{(g_0,\pi_0)}) = -\mu(g_0)\sigma_\xi(\alpha_{g_0}) \cdot \sigma_\xi(\alpha_{g_0}) - \frac{1}{\mu(g_0)}\sigma_\xi(\alpha_{\pi_0}) \cdot \sigma_\xi(\alpha_{\pi_0})$$

where $\alpha_{g_0}(Y) = L_Y g_0$ and $\alpha_{\pi_0}(Y) = L_Y \pi_0$. Therefore, $Y \cdot {}_\xi(\Sigma_{(g_0,\pi_0)}) \cdot Y = 0$ implies $Y \cdot {}_\xi(\alpha_{g_0}) \cdot Y = 0$ and $Y \cdot {}_\xi(\alpha_{\pi_0}) \cdot Y = 0$. From the first, we get $Y = 0$ if $\xi \neq 0$. Thus $\Sigma_{(g_0,\pi_0)}$ is elliptic.

Proof of ii). — Let $\Sigma_{(g_0,\pi_0)} \cdot Y = 0$. Then

$$\begin{aligned} 0 &= \langle Y, P_2 \cdot (D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* \cdot (0, Y)) \rangle \\ &= \langle (O, Y), D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* \cdot (0, Y) \rangle \\ &= \| D\Phi(g_0, \pi_0)^* \cdot (0, Y) \|^2 \end{aligned}$$

and so $D\Phi(g_0, \pi_0)^* \cdot (O, Y) = (- L_Y\pi_0, L_Y g_0) = 0$.

Proof of iii). — Let Y be in the kernel of $\Sigma_{(g_0,\pi_0)}$. Then

$$\begin{aligned} &\langle Y, P_2 D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* \cdot (C, O) \rangle \\ &\qquad = \langle (O, Y), D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* \cdot (C, O) \rangle \\ &\qquad = \langle D\Phi(g_0, \pi_0)^* \cdot (O, Y), D\Phi(g_0, \pi_0)^* \cdot (C, O) \rangle \end{aligned}$$

vanishes since $(0, Y) \in \ker D\Phi(g_0, \pi_0)^*$. ■

The isotropy group of (g_0, π_0) is $I_{(g_0,\pi_0)} = \{ \phi \in \mathscr{D}^3 \mid \phi^* g_0 = g_0$ and $\phi^*\pi_0 = \pi_0 \}$.

5.3. COROLLARY. — *For $\phi \in I_{(g_0,\pi_0)}$, we have the covariance relation*

$$\phi^* [Y(C)] = Y(\phi^* C).$$

Indeed, Y is a covariant operator, so this follows from uniqueness.

5.4. LEMMA. — *If $\Phi(g_0, \pi_0) = 0$, then the L_2 orthogonal complement of* $T_{(g_0,\pi_0)}\mathscr{O}_{(g_0,\pi_0)}$ is given by

$$\begin{aligned} [T_{(g_0,\pi_0)}\mathscr{O}_{(g_0,\pi_0)}]^\perp = \{ (h, \omega) \mid (h, \omega) &= (h, \omega)^{TT} + D\Phi(g_0, \pi_0)^* \cdot (\tilde{C}, \tilde{Y}) \\ &\quad - \mathbb{J} \circ D\Phi(g_0, \pi_0)^*(C, Y(C)) \text{ for some } \tilde{C}, \tilde{Y}, C \}. \end{aligned}$$

Proof. — From 5.2 we have the L_2-orthogonal splitting

$$\begin{aligned} \operatorname{range}(- \mathbb{J} \circ D\Phi(g_0, \pi_0)^*) &= T_{(g_0,\pi_0)}\mathscr{O}_{(g_0,\pi_0)} \\ &\quad \oplus \{ - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (C, Y(C)) \text{ for some } C \} \end{aligned}$$

by

$$\begin{aligned} - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (C, X) &= - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (O, X - Y(C)) \\ &\quad - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (C, Y(C)). \end{aligned}$$

The result therefore follows by the general splitting theorem 2.5. ■

Now let U be an ε-ball, for ε sufficiently small in $[T_{(g_0,\pi_0)}\mathscr{O}_{(g_0,\pi_0)}]^\perp$ in a $W^{s,p}$ norm that is invariant under $I_{(g_0,\pi_0)}$; e. g., the $W^{s,p}$ norm constructed using the first s covariant derivatives with respect to the metric g_0. Let

$$S_{(g_0,\pi_0)} = \{ (g_0, \pi_0) \} + U.$$

5.5. THEOREM. — *If $\Phi(g_0, \pi_0) = 0$ and (g_0, π_0) is $W^{s+r,p}$, $r \geq 1$, then for ε sufficiently small, $S_{(g_0,\pi_0)}$ is a slice in $T^*\mathscr{M}$ for the action of $\mathscr{D}^3$ on $T^*\mathscr{M}$.*

Proof (sketch). — Covariance of Φ implies that for $\phi \in I_{(g_0,\pi_0)}$, ϕ preserves the orthogonal splitting (see for example 5.3), and hence maps $S_{(g_0,\pi_0)}$ to itself. This gives property *i)* of the slice.

By construction, the set of $S_{\phi^*(g_0,\pi_0)} = \phi(S_{(g_0,\pi_0)})$ forms a smooth bundle over $\mathcal{O}_{(g_0,\pi_0)}$. Therefore, by the implicit function theorem, for ε small enough, this bundle forms a tubular neighborhood of $\mathcal{O}_{(g_0,\pi_0)}$. In particular, if any two $S_{\phi^*(g_0,\pi_0)}$ intersect, they must coincide and their base points be equal. Therefore property *ii)* of a slice holds. This also gives *iii)* using the canonical homeomorphism between $\mathcal{O}_{(g_0,\pi_0)}$ and $\mathcal{D}^3/I_{(g_0,\pi_0)}$ and a local right inverse for the projection $\pi : \mathcal{D}^3 \to \mathcal{D}^3/I_{(g_0,\pi_0)}$. ■

The usual consequences of the slice theorem follow. For example, in a neighborhood of (g_0, π_0), any $I_{(g,\pi)}$ is conjugate to a subgroup of $I_{(g_0,\pi_0)}$. The $I_{(g,\pi)}$'s are, of course, compact finite dimensional Lie groups.

§ 6. PROOF OF THE TIMELIKE CASE

Let Σ_0 by a hypersurface of constant mean curvature. Our assumption that the Killing fields form a one dimensional space is equivalent to $\dim \ker [D\Phi(g_0, \pi_0)^*] = 1$, where g_0, π_0, Φ are the geometrodynamical objects defined previously in conjunction with $\Sigma_0 = i_0(M)$. By 2.3, $\ker [D\Phi(g_0, \pi_0)^*]$ is spanned by either $(0, X)$, where $L_X g_0 = 0$ and $L_X \pi_C = 0$ if $\pi_0 \neq 0$ or g_0 is not flat, or by $(1, 0)$ if $\pi_0 = 0$ and g_0 is flat. The latter possibility is the *timelike case* and is dealt with in this section.

Steps one and two in the outline in Section 3 have already been carried out. It has been shown that the degeneracy space for the map

$$f : \mathcal{C}_P \to \mathbb{R}, (g, \pi) \mapsto \int_M \mathcal{H}(g, \pi)$$

at the critical point $(g_0, 0)$ is given by

$$\mathcal{S}(g_0, \pi_0, \Sigma_0) = \mathcal{F} \oplus \text{range } [\mathbb{J} \circ \text{range } [\mathbb{J} \circ D\Phi(g_0, 0)^*]$$

where $\mathcal{F} = \{ (h^{cc}, 0) \mid \nabla h^{cc} = 0 \}$;
see Proposition 4.4. First, let us construct an affine space whose tangent space is $\mathcal{F}$.

6.1. LEMMA. — *Let* $F = \{ (g_0 + h^{cc}, 0) \in T^*\mathcal{M} \mid h^{cc}$ is covariant constant with respect to $g_0 \}$. *Then* $T_{g_0}F = \mathcal{F}$ *and each* $g = g_0 + h^{cc}$ *is flat.*

Proof. — In a normal coordinate chart for g_0, the components of g are constants. ■

Remark. — The first component of $\mathcal{S}$ corresponds to the splitting of the tangent space of the space of all flat metrics into covariant constants plus gauges. This construction is discussed in Fischer and Marsden [1975], Theorem 6, p. 530.

Now consider the slice $S_{(g_0,0)}$ at $(g_0, 0)$ defined in the previous section. This slice consists of a piece of an affine space whose linear part contains

the three summands described in lemma 5.4. For g_0 flat and $\pi_0 = 0$, the ^{TT} summand has the following form (Moncrief [1975]):

$$(h, \omega)^{TT} = \left(h^{tr\,tr} + \frac{1}{3}\alpha g_0,\ \omega^{tr,tr} + \frac{1}{3}\beta g_0{}^{\#}\mu(g_0)\right)$$

where ^{tr,tr} literally means zero divergence (transverse) and zero trace and where α, β are constants.

6.2. LEMMA. — *$F \subset S_{(g_0,0)}$ and F is a finite dimensional affine submanifold of $S_{(g_0,0)}$.*

Proof. — Let h^{cc} satisfy $\nabla h^{cc} = 0$. Write

$$h^{cc} = \left[h^{cc} - \frac{1}{3}g_0(\operatorname{tr} h^{cc})\right] + \frac{1}{3}g_0(\operatorname{tr} h^{cc})$$

and observe that the right hand side has the form $h^{tr,tr} + \frac{1}{3}g_0\alpha$. ■

Restriction to $S_{(g_0,0)}$ imposes a spatial gauge. For the timelike gauge, we shall impose the condition that $\operatorname{tr}\pi'$ is constant; we also wish to coordinate these gauge choices with the manifold $\mathscr{C}_P$ introduced in Section 3 (see step 1 of the proof). Let $\mathscr{C}_{tr} = \{(g, \pi) \in T^*\mathscr{M} \mid \nabla \operatorname{tr}\pi = 0\}$.

6.3. LEMMA. — *$\mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0,0)}$ is a smooth submanifold of $S_{(g_0,0)}$ in a neighborhood of $(g_0, 0)$ with tangent space at $(g_0, 0)$ given by the ^{TT} tensors $(h, \omega)^{TT}$.*

Proof. — Define $\Gamma : S_{(g_0,0)} \to (\Lambda_d^0/\mathbb{R}) \times \Lambda_d^1 \times (\Lambda_d^0/\mathbb{R})$, where

$$\Lambda_d^0/\mathbb{R} = \left\{\omega \in \Lambda_d^0 \mid \int_M \omega = 0\right\},$$

by

$$\Gamma(g, \pi) = (P\Phi(g, \pi), \Delta(\operatorname{tr}\pi)),$$

where

$$P\Phi(g, \pi) = L_2 \text{ orthogonal projection of } \Phi(g, \pi) \text{ to range } D\Phi(g_0, 0).$$

Note that

$$\begin{aligned}\text{range } D\Phi(g_0, 0) &= L_2 \text{ orthogonal complement of } (1, 0)\\ &= \Lambda_d^0/\mathbb{R} \times \Lambda_d^1\end{aligned}$$

since $(1, 0)$ spans $\ker D\Phi(g_0, 0)^*$ by assumption.

Now

$$D\Gamma(g_0, 0)\cdot(h, \omega) = (D\Phi(g_0, 0)\cdot(h, \omega), D(\Delta \operatorname{tr}\pi)|_{(g_0,0)}\cdot(h, \omega))$$

since P = identity on range $D\Phi(g_0, 0)$. By definition of $S_{(g_0,0)}$, we get

$$D\Phi(g_0, 0)\cdot(h, \omega) = D\Phi(g_0, 0)\circ D\Phi(g_0, 0)^*\cdot(\tilde{C}, \tilde{Y})$$

and one computes that

$$D(\Delta \operatorname{tr} \pi)|_{(g_0,0)} \cdot (h, \omega) = \Delta(\operatorname{tr} \omega)$$

and

$$D\Delta(\operatorname{tr} \pi)|_{(g_0,0)} \cdot [-\mathbb{J} \circ D\Phi(g_0, 0)^*(C, O)] = -2\mu(g_0) \cdot \Delta(\Delta C).$$

These two maps are surjective from different components of $S_{(g_0,0)}$ and so $D\Gamma(g_0, 0)$ is surjective. Its kernel splits from the splitting theorems and so $\Gamma^{-1}(0)$ is a submanifold of $S_{(g_0,0)}$. The proof shows that

$$T_{(g_0,0)}[\mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0,0)}]$$
$$= \{ (h, \omega) = (h, \omega)^{TT} + \mathbb{J} \circ D\Phi(g_0, 0)^* \cdot (C, O) \mid D(\Delta \operatorname{tr} \pi)|_{(g_0,0)} \cdot (h, \omega) = 0 \},$$

where we have used the fact that $Y(C) = 0$. But from the above formulae for $D(\Delta \operatorname{tr} \pi)|_{(g_0,0)} \cdot (h, \omega)$ we have

$$D(\Delta \operatorname{tr} \pi)|_{(g_0,0)} \cdot (h, \omega)^{TT} = 0$$

and so

$$-2\mu(g_0)\Delta\Delta C = 0$$

and thus C is constant. Thus the tangent space reduces to the ^{TT} tensors. ■

Not only does $F \subset S_{(g_0,0)}$ (see (5.3)) but

$$F \subset \mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0,0)}.$$

Indeed, any $g = g_0 + h^{cc}$, $\pi = 0$ satisfies $\nabla \operatorname{tr} \pi = 0$ and $\Phi(g, \pi) = 0$ automatically.

Now let $\tilde{f} : \mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0,0)} \to \mathbb{R}$ be the restriction of f, i. e.,

$$\tilde{f}(g, \pi) = \int_M \mathscr{H}(g, \pi) = \int_M (1, 0) \cdot \Phi(g, \pi).$$

Clearly $D\tilde{f}(g_0, 0) = 0$. Thus $D^2\tilde{f}(g_0, 0)$ is well-defined.

6.4. LEMMA. — *The degeneracy space of* $D^2\tilde{f}(g_0, 0)$ *is exactly*

$$T_{(g_0,0)_F} = \{ (h^{cc}, 0) \mid \nabla h^{cc} = 0 \}.$$

Also $F \subset \mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0,0)}$ *is a manifold of critical points of* $\tilde{f}$, *i. e.,* F *is a weakly non-degenerate critical manifold for* $\tilde{f}$. (« Weakly » is explained below.)

Proof. — We have $D\tilde{f}(g, \pi) \cdot (h, \omega) = \int_M (1, 0) \cdot [D\Phi(g, \pi) \cdot (h, \omega)$ and so

$$D^2\tilde{f}(g_0, 0) \cdot ((h_1, \omega_1), (h_2, \omega_2)) = \int_M (1, 0) \cdot [D^2\Phi(g_0, 0) \cdot ((h_1, \omega_1), (h_2, \omega_2))].$$

By gauge invariance and the formula

$$(h, \omega)^{TT} + \mathbb{J} \circ D\Phi(g_0, 0)^*(C, O)$$

for elements of the tangent space to $\mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0,0)}$ at $(g_0, 0)$, we get

$$D^2\tilde{f}(g_0, 0)\cdot((h_1, \omega_1), (h_2, \omega_2)) = \int_M (1, 0)\cdot [D^2\Phi(g_0, 0)\cdot((h_1, \omega_1)^{TT}, (h_2, \omega_2)^{TT})].$$

The following statement is readily verified: *if* B *is a bilinear form on a Banach space* $\mathbb{E}$ *with degeneracy space* D *and if* $\mathbb{F} \subset \mathbb{E}$ *satisfies* $\mathbb{F} + \mathbb{F}_1 = \mathbb{E}$ *where* $B(\mathbb{F}, \mathbb{F}_1) = 0$, *then the degeneracy space of* B *restricted to* $\mathbb{F}$ *is* $\mathbb{F} \cap D$. Now from 6.3 and the splitting theorems,

$$T_{(g_0,0)}(\mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0,0)}) + \text{range}\,[\mathbb{J} \circ D\Phi(g_0, 0)^*] = \ker D\Phi(g_0, 0)$$

From this remark, gauge invariance of Taub's conserved quantities, and Propositions 4.4 and 6.3, the degeneracy space of $D^2\tilde{f}(g_0, 0)$ is

$$[\mathscr{F} \oplus \text{range}\,\mathbb{J} \circ D\Phi(g_0, 0)^*] \cap T_{(g_0,0)}[\mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0,0)}] = \mathscr{F}.$$

Since all members of F are of the form $(g_{\text{flat}}, 0)$, each point of F is a critical point for $\tilde{f}$, with the same degeneracy space $\mathscr{F}$. ■

A continuous symmetric bilinear form $B : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ on a Banach space $\mathbb{E}$ is called *weakly nondegenerate* if $B(x, y) = 0$ for all y implies $x = 0$, i. e., the induced map of $\mathbb{E}$ to $\mathbb{E}^*$ is injective; *strongly nondegenerate* means that the induced map of $\mathbb{E}$ to $\mathbb{E}^*$ is an isomorphism.

A *weakly non-degenerate critical manifold* for a smooth map $f : M \to \mathbb{R}$ on a Banach manifold M is a submanifold $N \subset M$ such that each $x \in N$ is a critical point for f (so f is constant, say zero, on N) and, restricted to a complement of T N, $D^2f(x)$ is weakly non-degenerate.

If $f : M \to \mathbb{R}$ is a smooth map such that $N \subset M$ is a manifold of critical points, and if the degeneracy space of $D^2f(x)$ at each $x \in N$ coincides with T_xN, then it is clear that N is a weakly nondegenerate critical manifold for f. This completes the explanation of the terminology in 6.4.

We have now completed Step 3 for the timelike case.

For Step 4 we shall invoke the Morse lemma. The usual version (see, e. g., Palais [1969]) assumes that the critical point (or manifold) is strongly nondegenerate, which is not the case here. There is, however, a version due to Tromba [1976] that will apply to our case. We state the result we need in the following form.

6.5. LEMMA. — *Let* M *be a Banach manifold and* $f : M \to \mathbb{R}$ *a* C^2 *function. Let* $N \subset M$ *be a submanifold of critical points of* f *and suppose that* f *vanishes on* N. *Suppose that*

i) N *is a weakly nondegenerate critical manifold;*

ii) *there is a weak Riemannian (or pseudo-Riemannian) structure* $\langle\,,\rangle_x$, *on* M *that has a smooth connection* (cf. Ebin [1970] and Ebin-Marsden [1970]);

iii) f *has a* C^2 $\langle\, ,\rangle_x$ *gradient* $Y(x)$, *where* $Y : M \to TM$ (*so* $Y(x) = 0$ *if* $x \in N$); *and*

iv) *if* $T_xM = T_xN \oplus E_x$ *is a* $\langle\, ,\rangle_x$*-orthogonal splitting* (*so* $DY(x) : E_x \to E_x$ *if* $x \in N$), *then* $DY(x) : E_x \to E_x$ *is an isomorphism.*

Then in a neighborhood of $x_0 \in N$, *there is a* C^2 *local chart in* M *in which the local representative of* f *takes the normal form*

$$f(x_1, x_2) = \frac{1}{2} D^2 f(x_1, 0) \cdot ((0, x_2), (0, x_2))$$

where x_1 *is a coordinate in* N *and* x_2 *is a complementary coordinate* (*for* E_x); *with* $x_2 = 0$ *defining* N.

The special case in which N is a point is proved in Tromba [1976]; see Choquet-Bruhat, Fischer and Marsden [1979] for an alternative proof. The lemma here is proved in the same way noting that all the constructions depend smoothly on the parameter x, a coordinate for N.

Remarks 1. — If one just wants the structure of the zero set of f, the proof can be simplified. In Buchner, Marsden and Schecter [1979], this is done by a « blowing up » method. Their results generalize to the case of $f : M \to \mathbb{R}^k$ that will be needed in part II of this paper.

2. — If we let C denote the cone of solutions of $D^2 f(x_0) \cdot (v, v) = 0$ in a complement E_{x_0} of $T_{x_0}N$, then the lemma implies that $f^{-1}(0)$ is diffeomorphic to $C \times N$ in a neighborhood of x_0, where « diffeomorphic » means one set is mapped to the other by means of the restriction of a local diffeomorphism. Thus the set of zeros of f is a cone $\times$ manifold, the manifold being the critical manifold.

3. — As a particular case of remark 2, a vector $v \in T_{x_0}M$ which satisfies the linearized equations $Df(x_0) \cdot v = 0$ is tangent to a curve of solutions of $f(x) = 0$, i. e., v will be integrable, if and only if $D^2 f(x_0) \cdot (v, v) = 0$, i. e., if and only if the second order conditions hold.

6.6. LEMMA. — *The previous lemma applies to the map*

$$\tilde{f} : \mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0, 0)} \to \mathbb{R}$$

with $N = F$.

Proof. — We have already checked *i)* of 6.5 in 6.4, namely that F is a weakly nondegenerate critical manifold for $\tilde{f}$. Let $\langle\quad\rangle_{(g,\pi)}$ be the $H^1 \times L^2$ metric on $\mathscr{C}_{tr} \cap \mathscr{C}_P \cap S_{(g_0, 0)}$ defined by

$$\langle (h_1, \omega_1), (h_2, \omega_2) \rangle_{(g,\pi)} = \int_M (h_1 \cdot h_2 + \nabla h_1 \cdot \nabla h_2) \mu(g) + \int_M \omega_1' \cdot \omega_2' \mu(g)$$

where the contraction is with respect to g. By the methods of Ebin [1970] one sees that this metric considered on $T^*\mathscr{M}$ has a smooth connection. On the other hand, from the methods of our splitting theorems and those

in Appendix A of Ebin and Marsden [1970], one sees that the $\langle\ \ \rangle$-orthogonal projection

$$\mathscr{P} : \mathrm{T}(\mathrm{T}^*\mathscr{M})\,|_{\mathscr{C}_{\mathrm{tr}} \cap \mathscr{C}_{\mathrm{P}} \cap \mathrm{S}_{(g_0,\pi)}} \to \mathrm{T}(\mathscr{C}_{\mathrm{tr}} \cap \mathscr{C}_{\mathrm{P}} \cap \mathrm{S}_{(g_0,0)})$$

is a smooth map, using $\mathscr{C}$ the $\mathrm{W}^{s+1,p} \times \mathrm{W}^{s,p}$ topology for (g, π), $s > \dfrac{n}{p}$. It follows that $\langle\, , \rangle$ has a smooth connection on the $\mathrm{W}^{s+1,p} \times \mathrm{W}^{s,p}$ completion of $\mathscr{C}_{\mathrm{tr}} \cap \mathscr{C}_{\mathrm{P}} \cap \mathrm{S}_{(g_0,0)}$. This establishes *ii*) of 6.5.

The $\mathrm{L}_2 \times \mathrm{L}_2$ gradient of $\tilde{f}$ is given by

$$(\mathrm{grad}_{\mathrm{L}_2 \times \mathrm{L}_2} \tilde{f})(g, \pi) = \mathrm{D}\Phi(g, \pi)^* \cdot (1, 0)$$

and so the $\mathrm{H}^1 \times \mathrm{L}_2$ gradient is given by

$$\mathrm{Y}(g, \pi) = \mathrm{grad}_{\mathrm{H}^1 \times \mathrm{L}_2} \tilde{f}(g, \pi) = \mathscr{P} \circ [(\mathrm{Id} + \Delta) \times \mathrm{Id}]^{-1} \cdot (\mathrm{D}\Phi(g, \pi)^* \cdot (1, 0))$$

where Id is the identity operator. From the explicit form of $\mathrm{D}\Phi(g, \pi)^*$ we see that it is second order in g and zero[5] order in π; thus by elliptic theory $\mathrm{Y}(g, \pi)$ is a smooth map of $\mathscr{C}_{\mathrm{tr}} \cap \mathscr{C}_{\mathrm{P}} \cap \mathrm{S}_{(g_0,0)}$ to its tangent bundle. This proves *iii*).

To prove *iv*) we note that the kernel of $\mathrm{DY}(g_0, 0)$ is exactly the degeneracy space. Thus on a complement, $\mathrm{DY}(g_0, 0)$ is one-to-one. But $\mathrm{DY}(g_0, 0)$ is bounded, self-adjoint, and elliptic, so it is an isomorphism on this complement. ■

We have now completed step four of the outline in Section 3. We have proved that the set of zeros of Φ within the space $\mathscr{C}_{\mathrm{tr}} \cap \mathrm{S}_{(g_0,0)}$ is a (cone) × (manifold). It remains to remove these gauge conditions. Removing $\mathrm{S}_{(g_0,0)}$ may be done by properties of a slice and the fact that Φ is covariant:

$$\eta^*\Phi(g, \pi) = \Phi(\eta^* g, \eta^* \pi),$$

i. e., zeros of Φ are propagated by the action of $\mathscr{D}_\Sigma$. Thus the zeros of Φ within $\mathscr{C}_{\mathrm{tr}}$ form a (cone) × (manifold); it is readily checked that the directions of integrability are still defined by the second order conditions*.

We have so far shown that

$$\{\, (g, \pi) \mid \Phi(g, \pi) = 0 \text{ and } \mathrm{tr}\, \pi' = \text{constant} \,\}$$

has the structure of a cone × manifold in a neighborhood of (g_0, π_0). We can use this to prove that any [4]h satisfying the linearized equations and the second order conditions is integrable. We need:

---

(5) In the Ebin-Palais slice theorem some derivatives may be lost by using the slice map χ, i. e., the map χ is smooth only if we drop our differentiability class. Thus the zeros of Φ within $\mathrm{W}^{s+k+1,p} \times \mathrm{W}^{s+k,p}$ regarded as a cone in $\mathrm{W}^{s+1,p} \times \mathrm{W}^{s,p}$ will be a « C^k cone », i. e., a literal cone after a C^k coordinate change.

6.7. LEMMA. — *Let Σ_0 be a hypersurface of constant mean curvature for (g_0, π_0), and ${}^{(4)}h$ any solution to the linearized equations. Then in a suitable linearized gauge, its induced (h, ω) satisfies*

$$D_{(g,\pi)}[\operatorname{tr} \pi'] \cdot (h, \omega) = \text{constant}.$$

Proof. — The linearized gauge transformations are range $\mathbb{J} \circ D\Phi(g_0, \pi_0)^*$. If $\operatorname{tr} \pi_0' = \text{const.}$ and $\Phi(g_0, \pi_0) = 0$, we have

$$D(\operatorname{tr} \pi') \cdot (-\mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (C, Y)) = -2C^{|i}{}_{|i} + \frac{2C}{[\mu(g_0)]^2}\left[\pi_0{}^{ij}\pi_{0ij} - \frac{1}{4}(\operatorname{tr} \pi_0)^2\right].$$

If $\pi_0 \neq 0$ then $\pi_0{}^{ij}\pi_{0ij} - \frac{1}{4}(\operatorname{tr} \pi_0)^2 = \left(\pi - \frac{g \operatorname{tr} \pi}{2}\right)^2 > 0$ and so the above operator is an isomorphism in C. If $\pi_0 = 0$ then the range of $C \mapsto 2\Delta C$ is the L_2 orthogonal complement of the constants. The result then follows. ■

Thus any ${}^{(4)}h$ satisfying the linearized equations induces $(h, \omega) \in T_{(g_0,0)}\mathscr{C}_{tr}$. Therefore, if the (gauge-invariant) second order conditions also hold, (h, ω) will be tangent to the cone defined by $\Phi = 0$ within $\mathscr{C}_{tr}$ and hence (h, ω) is tangent to a curve of solutions of the constraint equations, and therefore ${}^{(4)}h$ is tangent to a curve of solutions of $\operatorname{Ein}({}^{(4)}g) = 0$ (see Fischer and Marsden [1978 *a*] for details of the latter statement).

It remains to prove that all the solutions of $\operatorname{Ein}({}^{(4)}g) = 0$ in a neighborhood of ${}^{(4)}g_0$ are obtained by this process. We have shown that all the solutions admitting hypersurfaces of constant mean curvature are so obtained. That this is all solutions follows from the perturbation theory of constant mean curvature hypersurfaces, namely:

6.8. LEMMA. — *Let Σ_0 be a hypersurface on which $\pi_0 = 0$ and g_0 is flat. There is a smooth map ψ from a neighborhood of ${}^{(4)}g_0$ in the space of all Lorentz metrics on V_4 to hypersurfaces such that $\psi({}^{(4)}g)$ is a compact spacelike hypersurface of constant* (6) *mean curvature for ${}^{(4)}g$.*

This is proved by the methods in Choquet-Bruhat, Fischer and Marsden [1979] and Marsden and Tipler [1979]. Namely, consider Gaussian normal coordinates in a neighborhood of Σ_0. For a real valued function N on Σ_0, Let $\Gamma(N)$ be its graph and let $P({}^{(4)}g, N)$ be the density $\operatorname{tr} \pi$ induced by ${}^{(4)}g$ on $\Gamma(N)$. Then since $D_N P({}^{(4)}g_0, 0) \cdot N' = 2\Delta N'$, one sees that P is transversal to the constants. The result then follows.

This completes the proof of the timelike case of our main theorem.

(6) If we had attempted to use the gauge condition $\operatorname{tr} \pi = 0$, then solutions would be isolated, by Fischer and Marsden [1975 *a*].

§ 7. PROOF OF THE SPACELIKE CASE

In the spacelike case we are concerned with a compact hypersurface of constant mean curvature $\Sigma_0 = i_0(M)$ on which $\ker D\Phi(g_0, \pi_0)^*$ is spanned by the vector (O, X) where $L_X g_0 = 0$ and $L_X \pi_0 = 0$. By the results of section three, the set of zeros of Φ near (g_0, π_0) equals the set of zeros of

$$f : \mathscr{C}_P \to \mathbb{R}; (g, \pi) \mapsto \int_M X \cdot \mathscr{J}(g, \pi)$$

near (g_0, π_0). This function vanishes at (g_0, π_0), has a critical point there and by 4.4 the degeneracy space of $D^2 f(g_0, \pi_0)$ is given by

$$\mathscr{S}(g_0, \pi_0, \Sigma_0) = \mathscr{D} \oplus \text{range } [- \mathbb{J} \circ D\Phi(g_0, \pi_0)^*]$$

where

$$\mathscr{D} = \{ (h, \omega) \in \ker (D\Phi(g_0, \pi_0) \circ \mathbb{J}) \cap \ker D\Phi(g_0, \pi_0) \mid L_X h = 0 \text{ and } L_x \omega = 0 \}.$$

We now proceed to freeze out the spatial gauges by passing to a slice $S_{(g_0, \pi_0)}$ (see 5.21) and the temporal gauges by imposing a condition on tr π. Then within this space we shall obtain a manifold of critical points of the restriction $\tilde{f}$ of f that is tangent to the degeneracy space. This will carry out step 3 of the proof outlined in § 3.

Motivated by the above degeneracy space for $d^2 f(g_0, \pi_0)$, we define the affine space

$$\begin{aligned} \mathscr{A}_X = \{ (g_0 + \mathring{h}, \pi_0 + \mathring{\omega}) \in T^* \mathscr{M} \mid (\mathring{h}, \mathring{\omega}) \\ = (\mathring{h}, \mathring{\omega})^{TT} + D\Phi(g_0, \pi_0)^* \cdot (\mathring{\tilde{C}}, \mathring{\tilde{Y}}) - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\mathring{C}, \mathring{Y}) \end{aligned}$$

where $L_X \mathring{h}^{TT} = 0 = L_X \mathring{\omega}^{TT}$, $L_X \mathring{\tilde{C}} = 0 = L_X \mathring{\tilde{Y}}$ and $L_X \mathring{C} = 0 = L_X \mathring{Y} \}$.

7.1. LEMMA

$$\mathscr{A}_X = \{ (g_0 + \mathring{h}, \pi_0 + \mathring{\omega}) \in T^* \mathscr{M} \mid L_X \mathring{h} = 0 = L_X \mathring{\omega} \}.$$

Proof. — Let $(g_0 + \mathring{h}, \pi_0 + \mathring{\omega}) \in \mathscr{A}_X$. Since $L_X g_0 = 0$ and $L_X \pi_0 = 0$, covariance of Φ implies that $L_X \mathring{h} = 0$, $L_X \mathring{\omega} = 0$. Conversely, if $L_X \mathring{h} = 0$ and $L_X \mathring{\omega} = 0$, write

$$(\mathring{h}, \mathring{\omega}) = (\mathring{h}, \mathring{\omega})^{TT} + D\Phi(g_0, \pi_0)^* \cdot (\mathring{\tilde{C}}, Y) - J \circ D\Phi(g_0, \pi_0)^* \cdot (\mathring{C}, \mathring{Y})$$

by the splitting theorem. We can choose

$$(\mathring{\tilde{C}}, \mathring{\tilde{Y}}) = [D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^*]^{-1} \cdot [D\Phi(g_0, \pi_0) \cdot (\mathring{h}, \mathring{\omega})]$$

from the proof of the theorem, with a similar formula for $(\mathring{C}, \mathring{Y})$. Again, by

covariance of Φ, if (g_0, π_0) and $(\overset{\circ}{h}, \overset{\circ}{\omega})$ are X-invariant, si is $(\overset{\circ}{\tilde{C}}, \overset{\circ}{\tilde{Y}})$ and $(\overset{\circ}{C}, \overset{\circ}{Y})$. ■

Now define

$$\mathscr{B}_X = \mathscr{A}_X \cap S_{(g_0,\pi_0)}$$

where the slice $S_{(g_0,\pi_0)}$ is given by 5.4, so $\mathscr{B}_X$ is the affine space

$$\mathscr{B}_X = \{ (g_0 + \overset{\circ}{h}, \pi_0 + \overset{\circ}{\omega}) \in T^*\mathscr{M} \mid (\overset{\circ}{h}, \overset{\circ}{\omega})$$
$$= (\overset{\circ}{h}, \overset{\circ}{\omega})^{TT} + D\Phi(g_0, \pi_0)^* \cdot (\tilde{C}, \tilde{Y}) - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{C}, Y(\overset{\circ}{C}))$$

where $\quad L_X\overset{\circ}{h} = 0 = L_X\overset{\circ}{\omega}, \quad L_X\overset{\circ}{\tilde{C}} = 0 = L_X\overset{\circ}{\tilde{Y}} \quad$ and $\quad L_X\overset{\circ}{C} = 0 \}$.

Note that covariance of the operator $C \mapsto Y(C)$ implies that $L_X[Y(\overset{\circ}{C})] = 0$; indeed, if F_t is the flow of X, $F_t \in I_{(g_0,\pi_0)}$ so $F_t^*[Y(C)] = Y(F_t^*C)$.

Let $\kappa_0 = \operatorname{tr} k_0$ be the mean curvature of the hypersurface Σ_0 in the metric ${}^{(4)}g_0$. Let

$$\mathscr{C}_{\kappa_0} = \{ (g, \pi) \in T^*\mathscr{M} \mid \operatorname{tr} k = \kappa_0 \}.$$

We have assumed that κ_0 is a constant. $\mathscr{C}_{\kappa_0}$ is a smooth submanifold of $T^*\mathscr{M}$. Indeed, the map $(g, \pi) \mapsto \operatorname{tr} k$ from $T^*\mathscr{M}$ to Λ^0 has a surjective derivative even as a function of k alone. Of more interest is the intersection of the manifolds $\mathscr{B}_X \cap \mathscr{C}_{\kappa_0} \cap \mathscr{C}_P$. In order to deal with this intersection, shall have to make one more simplification.

7.2. LEMMA. — *Under our assumption that g_0 is not flat or that $\pi_0 \neq 0$, there is a compact hypersurface Σ_0' near Σ_0 in V_4 that has constant mean curvature $\kappa_0' \neq 0$.*

This lemma is easy if $k_0 \neq 0$ and is more delicate if $k_0 = 0$ (and so g_0 is not flat). The proof is given in lemma 5 of Marsden and Tipler [1979] and is based on the idea of Choquet-Bruhat, Fischer and Marsden [1979]. Note that if $\pi_0 = 0$, the assumption that g_0 is not flat is equivalent to ${}^{(4)}g_0$ not being flat.

This lemma means that *we can, and will assume that our constant mean curvature hypersurface satisfies $\kappa_0 \neq 0$.* Note that by 2.3, $\ker D\Phi(g_0, \pi_0)^*$ is still spanned by (O, X). With this assumption, we have

7.3. LEMMA. — *a) $\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0,\pi_0)}$ is a smooth manifold in the neighborhood of (g_0, π_0) with tangent space at (g_0, π_0) given by*

$$\{ (h, \omega) = (h, \omega)^{TT} - \mathbb{J} \circ D\Phi(g_0, \pi_0)^*(C, Y(C)) \mid D \operatorname{tr} k(g_0, \pi_0) \cdot (h, \omega) = 0 \}.$$

b) $\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap \mathscr{B}_X$ is a smooth submanifold of $\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0,\pi_0)}$ with tangent space at (g_0, π_0) given by

$$\{ (\overset{\circ}{h}, \overset{\circ}{\omega}) = (\overset{\circ}{h}, \overset{\circ}{\omega})^{TT} - \mathbb{J} \circ D\Phi(g_0, \pi_0)^*(\overset{\circ}{C}, Y(\overset{\circ}{C})) \mid L_X\overset{\circ}{C} = 0,$$
$$D \operatorname{tr} k(g_0, \pi_0) \cdot (\overset{\circ}{h}, \overset{\circ}{\omega}) = 0 \text{ and } L_X\overset{\circ}{h} = 0 = L_X\overset{\circ}{\omega} \}.$$

Proof. — Parts *a*) and *b*) are similar. We shall prove *b*). For *a*), replace $\mathscr{B}_X$ by $S_{(g_0,\pi_0)}$ and drop the conditions $L_X(\cdot) = 0$ in what follows.

Define the map

$$\Gamma : \mathscr{B}_X \to (\Lambda_d^0 \times \Lambda_d^1)_{\mathbb{P},X} \times \Lambda_X^0$$
$$(g, \pi) \mapsto (\mathbb{P}\Phi(g, \pi), \operatorname{tr} k)$$

where $(\Lambda_d^0 \times \Lambda_d^1)_{\mathbb{P},X}$ stands for the X-invariant members of the orthogonal complement of (O, X) i. e., $\left\{(\mu, \alpha) \,\middle|\, \int_M \alpha \cdot X = 0 \text{ and } L_X\alpha = 0,\ L_X\mu = 0\right\}$, and $\Lambda_X^0 = \{ f \in \Lambda^0 \mid L_X f = 0 \}$. Recall that $\mathbb{P}$ is the orthogonal projection onto the range of $D\Phi(g_0, \pi_0)$. As in 7.1, the map $\mathbb{P}$ commutes with Lie differentiation by X and so it is clear that Γ takes values in the stated spaces.

The derivative of Γ at (g_0, π_0) in the direction of a tangent vector $(\overset{\circ}{h}, \overset{\circ}{\omega}) = (\overset{\circ}{h}, \overset{\circ}{\omega})^{TT} + D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{\tilde{C}}, \overset{\circ}{\tilde{Y}}) - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{C}, Y(\overset{\circ}{C}))$ to $\mathscr{B}_X$ is given by

$$\begin{aligned} D\Gamma(g_0, \pi_0) \cdot (\overset{\circ}{h}, \overset{\circ}{\omega}) = (&D\Phi(g_0, \pi_0) \cdot (D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{\tilde{C}}, \overset{\circ}{\tilde{Y}})), \\ &D \operatorname{tr} k(g_0, \pi_0) \cdot [- \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{C}, Y(\overset{\circ}{C}))] \\ &+ D \operatorname{tr} k(g_0, \pi_0) \cdot ((\overset{\circ}{h}, \overset{\circ}{\omega})^{TT} + D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{\tilde{C}}, \overset{\circ}{\tilde{Y}}))). \end{aligned}$$

To show this is surjective, let $((\mu, \alpha), f)$ lie in the range space. Since $D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^*$ is invertible, we can let

$$(\overset{\circ}{\tilde{C}}, \overset{\circ}{\tilde{Y}}) = [D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^*]^{-1}(\mu, \alpha)$$

Again, X-invariance of (μ, α) implies X-invariance for $(\tilde{C}, \overset{\circ}{\tilde{Y}})$.

Write

$$\begin{aligned} D \operatorname{tr} k(g_0, \pi_0) \cdot (- \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{C}, Y(\overset{\circ}{C})) \\ = D \operatorname{tr} k(g_0, \pi_0) \cdot (- \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{C}, O)) \\ + D \operatorname{tr} k(g_0, \pi_0) \cdot (- \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (0, Y(\overset{\circ}{C})). \end{aligned}$$

The second term equals

$$D \operatorname{tr} k(g_0, \pi_0) \cdot (L_{Y(\overset{\circ}{C})} g_0, L_{Y(\overset{\circ}{C})} \pi_0) = L_{Y(\overset{\circ}{C})} \operatorname{tr} k(g_0, \pi_0) = 0 .$$

Note that this vanishes in case *a*) as well since $k_0 =$ constant.

Next, let

$$R(\overset{\circ}{C}) = D \operatorname{tr} k(g_0, \pi_0) \cdot (- \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{C}, O))$$

This operator is computed (for example, in Choquet-Bruhat, Fischer and Marsden [1979]) to be

$$R(\overset{\circ}{C}) = (\Delta + k_0 \cdot k_0)\overset{\circ}{C}$$

Our assumption that $\kappa_0 \neq 0$ implies that $k_0 \cdot k_0 \neq 0$ and so R is invertible.

Thus we can choose

$$\overset{\circ}{C} = R^{-1}[f - D \operatorname{tr} k(g_0, \pi_0) \cdot (\overset{\circ}{h}, \overset{\circ}{\omega})^{TT} + D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{\tilde{C}}, \overset{\circ}{\tilde{Y}})]$$

and again $\overset{\circ}{C}$ is X-invariant by covariance. Thus Γ is a submersion and so

$$\Gamma^{-1}((0, 0), \kappa_0) = \mathscr{B}_X \cap \mathscr{C}_{\kappa_0} \cap \mathscr{C}_P$$

is a smooth manifold. Its tangent space at (g_0, π_0) is the kernel of $D\Gamma(g_0, \pi_0)$; i. e.

$$\begin{aligned}\{ (\overset{\circ}{h}, \overset{\circ}{\omega}) \mid (\overset{\circ}{h}, \overset{\circ}{\omega}) &= (\overset{\circ}{h}, \overset{\circ}{\omega})^{TT} + D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{\tilde{C}}, \overset{\circ}{\tilde{Y}}) \\ &\quad - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\overset{\circ}{C}, Y(\overset{\circ}{C})), L_X \overset{\circ}{h} = 0, L_X \overset{\circ}{\omega} = 0, \\ &\quad (\overset{\circ}{h}, \overset{\circ}{\omega}) \in \ker D\Phi(g_0, \pi_0) \text{ and } D \operatorname{tr} k(g_0, \pi_0) \cdot (\overset{\circ}{h}, \overset{\circ}{\omega}) = 0 \}.\end{aligned}$$

Since $(\overset{\circ}{h}, \overset{\circ}{\omega}) \in \ker D\Phi(g_0, \pi_0)$, the term $D\Phi(g_0, \pi_0{}^* \cdot (\overset{\circ}{\tilde{C}}, \overset{\circ}{\tilde{Y}})$ vanishes. The lemma then follows. ■

Now let $\tilde{f} : \mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0, \pi_0)} \to \mathbb{R}$ be the restriction of f; i. e.,

$$\tilde{f}(g, \pi) = \int_M (O, X) \cdot \Phi(g, \pi) = \int_M X \cdot \mathscr{J}(g, \pi)$$

It is clear that $\tilde{f}(g_0, \pi_0) = 0$ and that $D\tilde{f}(g_0, \pi_0) = 0$.

7.4. LEMMA. — *$\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap \mathscr{B}_X$ is a (weakly) non-degenerate critical manifold for $\tilde{f}$ (for the explanation of these terms see the text following* 6.4*).*

Proof. — From the definition

$$\tilde{f}(g, \pi) = \int_M X \cdot \mathscr{J}(g, \pi) = \int_M (L_X g) \cdot \pi .$$

But $L_X g = 0$ for each $(g, \pi) \in \mathscr{B}_X$, so $\tilde{f}$ vanishes on $\mathscr{B}_X$. Also, for $(h, \omega) \in T_{(g,\pi)}(\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0,\pi_0)})$

$$\begin{aligned} D\tilde{f}(g, \pi) \cdot (h, \omega) &= \int_M \langle (O, X) D\Phi(g, \pi)(h, \omega) \rangle \\ &= \int_M \langle D\Phi(g, \pi)^* \cdot (O, X)(h, \omega) \rangle . \end{aligned}$$

Now if $(g, \pi) \in \mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap \mathscr{B}_X$, then Σ_0 has constant mean curvature for (g, π) and so by (2.3), $L_X g = 0 = L_X \pi$ implies $(O, X) \in \ker D\Phi(g, \pi)^*$. Thus Df vanishes at points of $\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap \mathscr{B}_X$ so the latter is a critical manifold for $\tilde{f}$.

From the splitting theorems and 7.3,

$$T_{(g_0,\pi_0)}(\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0,\pi_0)}) + \operatorname{range} [- \mathbb{J} \circ D\Phi(g_0, \pi_0)^*] = \ker D\Phi(g_0, \pi_0) .$$

Since $D^2 f(g_0, \pi_0)$ is gauge invariant, the remark on the degeneracy space of a restriction in 6.9 then applies, so the degeneracy space of $D^2 \tilde{f}(g_0, \pi_0)$ is, by 4.4 and 7.3

$$\begin{aligned}
&\mathscr{S}(g_0, \pi_0, \Sigma_0) \cap T_{(g_0,\pi_0)}(\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0,\pi_0)}) \\
&\quad = (\mathscr{D} \oplus \text{range}\,[-\mathbb{J} \circ D\Phi(g_0, \pi_0)^*]) \cap \{ (h, \omega) = (h, \omega)^{TT} \\
&\qquad - \mathbb{J} \circ D\Phi(g_0, \pi_0)^*(C, (Y(C)) \mid D\,\mathrm{tr}\, k(g_0, \pi_0) \cdot (h, \omega) = 0 \} \\
&\quad = \{ (\mathring{h}, \mathring{\omega}) = (\mathring{h}, \mathring{\omega})^{TT} - \mathbb{J} \circ D\Phi(g_0, \pi_0)^* \cdot (\mathring{C}, Y(\mathring{C})) \mid L_X \mathring{C} = 0, \\
&\qquad D\,\mathrm{tr}\, k(g_0, \pi_0) \cdot (\mathring{h}, \mathring{\omega}) = 0 \text{ and } L_X \mathring{h} = 0 = L_X \mathring{\omega} \} \\
&\quad = T_{(g_0,\pi_0)}(\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap \mathscr{B}_X). \qquad \blacksquare
\end{aligned}$$

We have now completed step 3 for the spacelike case.

For step 4, we need to verify the conditions of lemma 6.5 for the function $\tilde{f}$, with $M = \mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0,\pi_0)}$ and $N = \mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap \mathscr{B}_X$. We have already verified condition *i*). For condition *ii*) we choose this time not an $H^1 \times L^2$ norm but an $H^{1/2} \times H^{1/2}$ norm:

$$\langle (h_1, \omega_1), (h_2, \omega_2) \rangle_{(g,\pi)} = \int_M ((I + \Delta)^{1/2} h_1) \cdot h_2 \mu(g) + \int_M ((I + \Delta)^{1/2} \omega_1') \cdot \omega_2' \mu(g)$$

Here $(I + \Delta)^{1/2}$ is the square root of the positive self-adjoint operator $I + \Delta$. We have chosen the $H^{1/2} \times H^{1/2}$ norm so the $\langle\,,\rangle$ gradient is zeroth order. We have

$$f(g, \pi) = \int (L_X g) \cdot \pi = -\int g \cdot L_X \pi$$

and so

$$Df(g, \pi) \cdot (\mathring{h}, \mathring{\omega}) = \int (L_X g) \cdot \mathring{\omega} - \int \mathring{h} \cdot L_X \pi .$$

Thus the $L_2 \times L_2$ gradient of f is

$$\mathrm{grad}_{L_2 \times L_2} f(g, \pi) = (-L_X \pi, L_X g)$$

and so the $\langle\,,\rangle$ gradient of $\tilde{f}$ is

$$Y(g, \pi) = \mathscr{P}(-(I + \Delta)^{-1/2} L_X \pi, (I + \Delta)^{-1/2} L_X g)$$

where $\mathscr{P}$ is the $\langle\,,\rangle$-orthogonal projection of $T(T^*\mathscr{M})$ onto

$$T(\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0,\pi_0)})$$

at points of $\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap S_{(g_0,\pi_0)}$.

As in 6.6, $\langle\,,\rangle$ has a smooth connection and $\mathscr{P}$ is a smooth bundle map; it is clear that Y is a smooth vector field. Note that

$$Y(g, \pi) = \mathscr{P} \circ [(I + \Delta)^{-1/2} \times (I + \Delta)^{-1/2}] \cdot D\Phi(g, \pi)^* \cdot (O, Y).$$

As in 5.7, $DY(g_9, \pi_0)$ is an isomorphism on a complement to $T(\mathscr{C}_{\kappa_0} \cap \mathscr{C}_P \cap \mathscr{B}_X)$. Thus the conditions of 6.5 hold, and step 4 of the spacelike case is completed.

Finally, removal of the gauges proceeds as before. Since $\kappa_0 \neq 0$, lemma 6.8 now simplifies as follows (we can prescribe $\operatorname{tr} k = \kappa_0$ now rather than a « floating » constant).

6.5. LEMMA. — *Let $\Sigma_0 \subset V_4$ be a hypersurface of constant mean curvature $\kappa_0 \neq 0$ for ${}^{(4)}g_0$. There is a unique map Ψ from a neighborhood of ${}^{(4)}g$ in the space of Lorentz metrics on V_4 to hypersurfaces such that $\Psi({}^{(4)}g)$ is a compact spacelike hypersurface of constant mean curvature κ_0 for ${}^{(4)}g$.*

The proof is found in Choquet-Bruhat, Fischer and Marsden [1979]. This, together with the same arguments involving 6.7, completes the proof of the spacelike case.

§ 8. DISCUSSION

We have established that the space of solutions of Einstein's equations for spacetimes with compact Cauchy surfaces of constant mean curvature is a smooth manifold near spacetimes with no Killing fields and has a conical singularity near spacetimes with one Killing field. These singularities reflect the fact that in order to complete a perturbation expansion from the linearized solution to second and higher order it is necessary and sufficient that the conserved integrals of Taub vanish identically. In part II of this paper we shall extend the analysis from one to many Killing fields, and to the study of general momentum mappings in mechanics.

Although generically (on an open dense subset), the space of solutions is a smooth manifold, the singularities occur at precisely the spacetimes the most studied and of the most interest, i. e., those with symmetries. Not only must one take care with perturbation theory near such spacetimes, but other physical phenomena such as quantization will be affected by these singularities as well; see Moncrief [1978]. Indeed, some path integral techniques and the WKB method implicitly assume that the solution space has a reasonable Banach space structure, at least locally; our analysis indicates that this question is a good deal more subtle near spacetimes with symmetries.

All explicitly known spacetimes with compact Cauchy surfaces have Killing fields and therefore are singular points in the space of solutions. For example, the Taub universe which has four Killing fields, is a singular point in the space of solutions, as are the other Bianchi IX models which usually have three « spacelike » Killing fields. In addition, there are the Gowdy and Kasner universes. We do not know any *explicit* solution with precisely one spacelike Killing field. However, non-flat 3-metrics

with one Killing field and zero scalar curvature would provide examples, taking $\pi_0 = 0$.

Our analysis has shown that near a metric with one Killing field, all nearby solutions with one Killing field form a smooth manifold pointing in the degeneracy direction. This is also true for more Killing fields. *Within a given symmetry class, breaking one extra symmetry can be dealt with by the methods already presented here.* One obtains a cone of solutions within the given symmetry class. This is exactly what happens for the Taub universe within the Bianchi IX type class, as has been demonstrated explicitly by Jantzen [1978].

For time-like Killing fields, most examples have extra spatial symmetries as well, the flat universe $T^3 \times \mathbb{R}$, originally studied by Brill and Deser [1972], being an example. For one Killing field that is time-like, we have seen in 2.3 that this forces one to a situation where Σ_0 is flat with $\pi_0 = 0$. While our main thrust in § 6 has been to set the stage for the many Killing field cases, one can ask if there are any examples with only one Killing field. We now give some information on this question, and in particular, show that there *are* some examples. First of all, one cannot find such a spacetime with $T^3 \times \mathbb{R}$ topology:

8.1. PROPOSITION. — *Any flat metric on* T^n *has n commuting Killing fields.*

Proof. — From Wolf [1974, p. 123], any flat metric on T^n is induced from the Euclidean metric on $\mathbb{R}^n$ by taking the quotient of $\mathbb{R}^n$ with a discrete subgroup Γ of $\mathbb{R}^n$ generated by a lattice of vectors $v_1, \ldots, v_n$. The corresponding translations give the isometries. ■

If (M, g) is a Ricci-flat manifold and X is a vector field on it, it follows readily from the formula

$$\delta_g L_X g = \Delta X + d\delta X - 2\mathrm{Ric}(g) \cdot X$$

that X is a Killing field iff X is harmonic (iff X is covariant constant). Thus, from De Rham's theorem, the dimension of the space of Killing fields $= b_1(M)$, the first Betti number of M.

8.2. PROPOSITION. — *There exist flat compact three manifolds with no Killing fields.*

Proof. — From Wolf [1974, p. 122] there is a class $\mathscr{G}_6$ with $b_1(M) = 0$. This gives a 3-parameter family of compact connected orientable flat 3-manifolds with no Killing fields by our above remarks. (Curiously all non-orientable ones have a Killing field.) ■

The classes $\mathscr{G}_2$, $\mathscr{G}_3$, $\mathscr{G}_4$ and $\mathscr{G}_5$ have one Killing field. These classes give examples of flat spacetimes $M \times \mathbb{R}$ with precisely two Killing fields. Spacetimes with more than one Killing vector field will be considered in part II.

REFERENCES

R. ABRAHAM and J. MARSDEN, *Foundations of Mechanics*, 2nd edition, Benjamin/Cummings, Reading, Massachusetts, 1978.

J. ARMS, Linearization stability of the Einstein-Maxwell system. *J. Math. Phys.*, t. **18**, 1977, p. 830-833.

J. ARMS, On coupled gravitational and gauge fields. *J. Math. Phys.*, t. **20**, 1979, p. 443-453.

J. ARMS and J. MARSDEN, The absence of Killing vector fields is necessary for linearization stability of Einstein's equations. *Indiana Univ. Math. J.*, t. **28**, 1979, p. 119-125.

M. BERGER and D. EBIN, Some decompositions of the space of symmetric tensors on a Riemannian manifold. *J. Diff. Geom.*, t. **3**, 1969, p. 379-392.

J. BOURGNION, D. EBIN and J. MARSDEN, Sur le noyau des opérateurs pseudo-différentiels a symbole surjective et non-injective. *C. R. Acad. Sci.*, t. **282**, 1976, p. 867-870.

D. BRILL and S. DESER, Instability of closed spaces in general relativity. *Comm. Math. Phys.*, t. **32**, 1973, p. 291-304.

M. BUCHNER, J. MARSDEN and S. SCHECHTER, *Differential topology and singularity theory in the solution of nonlinear equations* (to appear), 1979.

R. BUDIC, J. ISENBERG, L. LINDBLOM and P. YASSKIN, On the determination of Cauchy surfaces from intrinsic properties. *Comm. Math. Phys.*, t. **61**, 1977, p. 87-93.

Y. CHOQUET-BRUHAT, A. FISCHER and J. MARSDEN, *Maximal hypersurfaces and positivity of mass*, in *Proceedings of the International School of Physics « Enrico Fermi »*, Course LXVII, *Isolated Gravitating Systems in General Relativity*, ed. J. Ehlers, North Holland Publishing Company, Amsterdam, New York.

B. COLL, Sur la stabilité linéaire des équations d'Einstein du vide. *C. R. Acad. Sci. Paris*, t. **282**, 1976, p. 247-250.

D. EBIN, On the space of Riemannian metrics. *Symp. Pure Math. Amer. Math. Soc.*, t. **15**, 1970, p. 11-40.

D. EBIN and J. MARSDEN, Groups of diffeomorphisms and the motion of an uncompressible fluid. *Ann. Math.*, t. **42**, 1970, p. 102-163.

A. FISCHER and J. MARSDEN, Linearization stability of the Einstein equations. *Bull. Am. Math. Soc.*, t. **79**, 1973, p. 997-1003.

A. FISCHER and J. MARSDEN, Linearization stability of the nonlinear partial differential equations. *Proc. Symp. Pure Math. A. M. S.*, t. **27**, 1975 *a*, part 2, p. 219-263.

A. FISCHER and J. MARSDEN, Deformations of the scalar curvature. *Duke Math. J.*, t. **42**, 1975 *b*, p. 519-547.

A. FISCHER and J. MARSDEN, *The Initial Value Problem and the Canonical Formalism of General Relativity*, in *Gravitational Theories since Einstein*, ed. S. W. Hawking and W. Israel, Cambridge University Press, Cambridge, 1979 *a*.

A. FISCHER and J. MARSDEN, *Topics in the dynamics of general relativity*, in *Proceedings of the International School of Physics « Enrico Fermi »*, Course LXVII, *Isolated Gravitating Systems in General Relativity*, ed. J. Ehlers, North Holland Publishing Company, Amsterdam, New York, 1979 *b*.

S. W. HAWKINGS and G. F. R. ELLIS, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge, 1973.

R. T. JANTZEN, The dynamical degrees of freedom in spatially homogeneous cosmology. *Comm. Math. Phys.*, t. **64**, 1979, p. 211-232.

J. MARSDEN, Qualitative methods in bifurcation theory. *Bull. Am. Math. Soc.*, t. **84**, 1978, 1125-1148.

J. MARSDEN, *Geometric methods in Mathematical Physics*, CBMS regional conference series, American Mathematical Society, 1980.

J. MARSDEN and F. TIPLER, *Maximal hypersurfaces and foliations of constant mean curvature in general relativity* (to appear), 1978.

C. W. MISNER, R. S. THORNE and J. A. WHEELER, *Gravitation*, W. H. Freeman and Company, San Francisco, 1974.

V. MONCRIEF, Spacetime symmetries and linearization stability of the Einstein equations I. *J. Math. Phys.*, t. **16**, 1975 *a*, p. 493-498.

V. MONCRIEF, Decompositions of gravitational perturbations. *J. Math. Phys.*, t. **16**, 1975 *b*, p. 1556-1560.

V. MONCRIEF, Spacetime symmetries and linearization stability of the Einstein equations II. *J. Math. Phys.*, t. **17**, 1976, p. 1893-1902.

V. MONCRIEF, Invariant states and quantized gravitational perturbations. *Phys. Rev.*, t. **D 18**, 1978, p. 983-989.

R. PALAIS, Global formulation of Lie theory of transformation groups, in *Memoris A. M. S.*, t. **22**, 1957.

R. PALAIS, The Morse lemma on Banach spaces. *Bull. Am. Math. Soc.*, t. **75**, 1969, p. 968-971.

A. H. TAUB, Approximate stress energy tensor for gravitational fields. *J. Math. Phys.*, t. **2**, 1961, p. 787-793.

A. H. TAUB, *Variational principles in general relativity*, ed. C. Cattanes (*Lectures at the Centro Internazionale Matematico Estiud*, Bressanone), Edizioni Cremonese, Rome, 1970, p. 206-300.

A. J. TROMBA, Almost Riemannian structures on Banach manifolds, the More lemma, and the Darboux theorem. *Can. J. Math.*, t. **28**, 1976, p. 640-652.

J. A. WOLF, *Spaces of Constant Curvature*, 3rd edition, J. A. Wolf, Berkeley, California, 1974.

(Manuscrit reçu le 22 avril 1980)