Stabilization of Relative Equilibria II

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Abstract

In this paper, we obtain feedback laws to asymptotically stabilize relative equilibria of mechanical systems with symmetry. We use a notion of stability 'modulo the group action' developed by Patrick [1992]. We deal with both *internal* instability and with instability of the rigid motion. The methodology is that of *potential shaping*, but the system is allowed to be internally underactuated, i.e., have fewer internal actuators than the dimension of the shape space.

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1 Introduction

In this paper, we obtain feedback laws to asymptotically stabilize relative equilibria of mechanical systems with symmetry. We use a notion of stability 'modulo the group action' developed by Patrick [1992]. The system is allowed to be internally underactuated, i.e., have fewer internal actuators than the dimension of the shape space.

In our previous work (Jalnapurkar and Marsden [1998]), we showed how to stabilize an internally unstable relative equilibrium using only internal actuators. There, we had assumed that the *rigid* motion of the system was stable, i.e., the system would rotate stably if the internal joints were to be locked. An example of such a system, which we discussed in that paper, is the double spherical pendulum. As we used only internal actuators, the system evolves on a constant momentum surface and we obtain asymptotic stability within level surfaces modulo a group action, but only neutral stability in the momentum directions.

In the present work, we allow actuation in the group variables, with the result that we can change the value of the momentum and therefore we are no longer restricted to constant momentum surfaces. Thus, we can asymptotically stabilize the relative equilibrium in a full phase space neighborhood, modulo a group action. The availability of actuation in the group direction also enables us to handle instability of the rigid motion, that is, the group directions, in addition to the internal instability that we can deal with using the internal actuation.

To give a more precise statement of our objectives we will first discuss relative equilibria of mechanical systems with symmetry, and then describe what exactly is meant by stability and asymptotic stability for these relative equilibria.

Let Q be the configuration manifold of our system, and let G be a compact group that acts freely and properly on Q. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be a G-invariant kinetic energy metric on Q. Let K be the kinetic energy, and let $V : Q \to \mathbb{R}$ be a G-invariant potential function. If $L := K - V$ is the Lagrangian, the equations of motion of the system in local coordinates $q = (q^1, \ldots, q^n)$ on Q, subject to a generalized force τ, are as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t)) - \frac{\partial L}{\partial q^i}(q(t), \dot{q}(t)) = \tau_i(t) \quad i = 1, \ldots, n$$

which we shall write for short as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau \tag{1.1}$$

If $H : T^*Q \to \mathbb{R}$ is the G-invariant Hamiltonian corresponding to the Lagrangian L, then the equivalent Hamiltonian equations on T^*Q, are:

$$\dot{z} = X_H(z) + \begin{bmatrix} 0 \\ \tau \end{bmatrix} \tag{1.2}$$

Here, $\begin{bmatrix} 0 \\ \tau \end{bmatrix}$ denotes the vertically lifted tangent vector in $T_\tau T^*Q$ that corresponds to the cotangent vector τ in $T^*_q Q$.

A point $z_e \in T^*Q$ is said to be a *relative equilibrium* if $X_H(z_e)$ is tangent to the group orbit through z_e, that is, $X_H(z_e) \in \mathfrak{g} \cdot z$, where $\mathfrak{g}$ is the Lie algebra of the group G.

Let $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ be the standard cotangent bundle momentum map corresponding to the group action, and let $\mathbf{J}(z_e) = \mu_e$. Let G_{μ_e} be the coadjoint isotropy subgroup corresponding to μ_e, and let $\mathfrak{g}_{\mu_e}$ be the corresponding Lie algebra. It is a fact (see, for example, Marsden [1992]) that the integral curve $z_e(t)$ of the vector field X_H with initial condition z_e is given by $z_e(t) = \exp(t\xi_e) \cdot z_e \in G_{\mu_e} \cdot z_e \subset G \cdot z_e$, and ξ_e is an element of $\mathfrak{g}_{\mu_e}$.

Now, by the theory of Poisson reduction (see, for example, Marsden and Ratiu [1998]), the Hamiltonian system $\dot{z} = X_H(z)$ drops to a reduced Hamiltonian system on T^*Q/G', where G' can be chosen to be any subgroup of G. In particular, we get the reduced system $\dot{\zeta} = X_{\mathcal{H}}(\zeta)$ on T^*Q/G_{μ_e}, and the reduced system $\dot{\gamma} = X_h(\gamma)$ on T^*Q/G. Here $\mathcal{H}$ and h are the functions obtained on T^*Q/G_{μ_e} and T^*Q/G respectively by dropping the G-invariant function H.

Let $\pi_{\mu_e} : T^*Q \to T^*Q/G_{\mu_e}$ and $\pi : T^*Q \to T^*Q/G$ be the natural projections, and let $\pi_{\mu_e}(z_e) = \zeta_e$, and $\pi(z_e) = \gamma_e$. Then the following statements are equivalent (Marsden [1992]):

1. z_e is a relative equilibrium for the vector field X_H.

2. ζ_e is a fixed point for the reduced vector field $X_{\mathcal{H}}$ on T^*Q/G_{μ_e}.

3. γ_e is a fixed point for the reduced vector field X_h on T^*Q/G.

Thus, in summary, a relative equilibrium is an element z_e of T^*Q such that the integral curve of the Hamiltonian vector field X_H through this element lies in the G-orbit through this element. Furthermore, the projections of z_e onto the reduced spaces T^*Q/G_{μ_e} and T^*Q/G are fixed points of the corresponding reduced Hamiltonian vector fields.

Stability and Asymptotic Stability of Relative Equilibria Now we will turn to what we mean by stability and asymptotic stability of the relative equilibrium z_e. The discussion here uses the work of Patrick [1992].

One possible definition of stability is as follows: Say the equilibrium z_e is stable if an integral curve z_t of the vector field X_H satisfies the following property: z_t stays 'close' to $z_e(t)$ for all time, provided z_0 is sufficiently close to z_e.

The problem with this definition is that if the value of $\mathbf{J}$ at z_0 is different from μ_e, then z_t may eventually drift away from $z_e(t)$, no matter how close z_0 is to z_e. Thus we need a weaker notion of stability.

One possible approach is to use a notion of stability 'modulo the G-action' as follows: Say that the relative equilibrium z_e is 'G-stable' if, for any given G-invariant open neighborhood V of $G \cdot z_e$, there is a G-invariant open neighborhood U of $G \cdot z_e$ such that $z_0 \in U$ implies $z_t \in V \quad$ for all $t \geqslant 0$. It is easy to see that this notion of stability is equivalent to Lyapunov stability of the equilibrium γ_e in the reduced space T^*Q/G.

Following Patrick [1992], we note that one can analogously define a notion of 'G'-stability' for any subgroup G' of G: Say that the relative equilibrium z_e is G'-stable if, for any given G'-invariant open neighborhood V of $G' \cdot z_e$, there is a G'-invariant open neighborhood U of $G \cdot z_e$ such that $z_0 \in U$ implies $z_t \in V \quad$ for all $t \geqslant 0$.

It is straightforward to show that for any subgroup G' of G, G'-stability implies G-stability. Thus G'-stability is a notion of stability that is stronger than G-stability, yet not as strong as the first proposed definition.

For the purposes of this paper, the type of stability we use will be G_{μ_e}-***stability***. The relative equilibrium z_e is thus G_{μ_e}-stable if and only if the equilibrium ζ_e in the reduced space T^*Q/G_{μ_e} is Lyapunov stable. Further we will say that the relative equilibrium z_e is ***asymptotically*** stable if the equilibrium ζ_e in the reduced space T^*Q/G_{μ_e} is asymptotically stable.

Methodology for Stabilization To obtain our feedback laws we will use a combination of the techniques of van der Schaft [1986] on the stabilization of Hamiltonian systems, and the energy-momentum method for stability analysis of relative equilibria due to Simo, Lewis and Marsden [1991].

The methodology is based on *potential shaping*, that is, the selection of controls that, in effect, change the potential energy of the system and add damping to make equilibria asymptotically stable. Using the coupling of the modes as well as the existence of a priori stable directions, one needs only *partial actuation*.

This methodology is to be compared with that of Bloch, Leonard and Marsden [1997, 1998] on controlled Lagrangians, whose methodology involves the reshaping of the *kinetic energy* of the system to stabilize a relative equilibrium. Kinetic shaping is designed for the stabilization of *balance systems* where there might be no internal actuation at all. Typical applications are the stabilization of inverted spherical pendula and underwater vehicles.

Also of interest is the work of Bullo [1998], who considers systems with Abelian symmetry with full actuation along both the group and the internal directions. Under appropriate assumptions, it is shown that it is possible to exponentially stabilize the system to a desired relative equilibrium on a desired level surface of the momentum.

2 Patrick's Stability Result

The set-up considered by Patrick [1992] is as follows: Let z_e be a relative equilibrium for the Hamiltonian vector field X_H on a symplectic manifold (P, Ω), and a regular point for an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. If we assume that G is compact, we know that the Lie algebra $\mathfrak{g}$ of G admits an inner product invariant under the adjoint action of G.

$T_{z_e}[\mathbf{J}^{-1}(\mu_e)]$ is the tangent space to the level set of the momentum at z_e. If $\mathfrak{g}_{\mu_e}$ is the Lie algebra of G_{μ_e}, then the space $\mathfrak{g}_{\mu_e} \cdot z_e$ is contained in $T_{z_e}[\mathbf{J}^{-1}(\mu_e)]$. If $J_{\xi_e} : T^*Q \to \mathbb{R}$ is defined by

$$J_{\xi_e}(z) = \langle \mathbf{J}(z), \xi_e \rangle$$

then it can be shown (see, for example, Marsden[1992]) that z_e is a critical point of the function $(H - J_{\xi_e})$. Let Z be a subspace of $T_{z_e}(T^*Q)$ that is a complement of $\mathfrak{g}_{\mu_e} \cdot z_e$ in $T_{z_e}[\mathbf{J}^{-1}(\mu_e)]$. Then Patrick's theorem says that if $\mathbf{d}^2(H - J_{\xi_e})(z_e)$ is positive definite on Z, then the relative equilibrium z_e of X_H is G_{μ_e}-stable.

We will give an outline of how Patrick proved this result: Since the action of G on Q was assumed to be proper, the action of G_{μ_e} on T^*Q is also proper. This action therefore admits a relatively compact slice at z_e, which means that there is a submanifold S with compact closure containing z_e such that:

1. If $g \in G_{\mu_e}$ and if $gS \cap S \neq \varnothing$, then g is the identity element of G_{μ_e}, and
2. $G_{\mu_e} \cdot S$ is an open neighborhood of $G_{\mu_e} \cdot z_e$ in T^*Q.

Next, a map $\Psi : G_{\mu_e} \cdot S \to \mathfrak{g}$ is constructed, with the following properties:

$$\Psi(gz) = \mathrm{Ad}_g \Psi(z) \quad \text{for all } g \in G_{\mu_e}, \text{ for all } z \in T$$
$$\Psi(z) = \xi_e \ \text{ for all } \ z \in S$$

As a consequence, $\langle \mu_e, \Psi(z) \rangle = \langle \mu_e, \xi_e \rangle$ for all z in $G_{\mu_e} \cdot S$. Next, define a function $f : G_{\mu_e} \cdot S \to \mathbb{R}$ by

$$f(z) = af_1(z) + f_2(z)$$

where a is a positive real number, and f_1 and f_2 are defined as follows:

$$f_1(z) = H(z) - \mathbf{J}(z)\Psi(z)$$
$$f_2(z) = |\mathbf{J}(z) - \mu_e|^2$$

Note that f_1 is a G_{μ_e}-invariant function which equals $(H - J_{\xi_e})$ on S. The norm used in defining f_2 is the G-invariant norm on $\mathfrak{g}^*$ induced by the G-invariant inner product on $\mathfrak{g}$. Thus the function f is a G_{μ_e}-invariant function on $G_{\mu_e} \cdot S$. It can be shown that for an appropriately chosen value of a, $\mathbf{d}^2 f(z_e)$, the second derivative of f at z_e, is positive definite on $T_{z_e}S$. Note that G_{μ_e}-invariance of f means that for all $g \in G_{\mu_e}$, $\mathbf{d}^2 f(gz_e)$ is positive definite on $T_{gz_e}gS$

Now, let z_t be an integral curve of the vector field X_H. Then G_{μ_e}-stability can be proved by showing that for any given $\epsilon > 0$, $f(z_t) < \epsilon$ for all $t > 0$, provided $f(z_0)$ is sufficiently small. For the details, refer to Patrick [1992].

3 Description of Forces

In this section we shall describe the forces on the system, and outline our strategy for achieving asymptotic stability.

As we stated in section 1, the equations of motion for the system are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

where τ is the generalized force acting on the system. Since Q is a principle G-bundle, locally, Q is diffeomorphic to $G \times S$, where $S := Q/G$ is the shape space.

The generalized force τ is regarded as an element of T_q^*Q, which can be identified with $T_g^*G \times T_x^*S$. Thus, we can use the local trivialization of Q to express τ in the form (τ_g, τ_x), where $\tau_x \in T_g^*G$ is called the ***internal force***, and represents the mutual interactions between the components of the system, whereas $\tau_g \in T_x^*S$ is the ***external force*** acting on the system.

Strictly speaking, one should proceed more intrinsically by picking a connection, such as the mechanical connection on the bundle $Q \to Q/G$ and dividing τ into its horizontal and vertical components. That is, we consider the restriction of τ to the horizontal space and vertical space, respectively, of the connection. In a local trivialization, these components differ from the naive decomposition by terms involving the connection coefficients. In addition, one should use the intrinsic breakup of the Euler-Lagrange equations into vertical and horizontal components (the reduction of these equations are the Lagrange-Poincaré equations, which is a reasonably lengthy story—see Cendra, Marsden and Ratiu [1998]). These modification does not affect the main techniques of the present paper in a substantive way, so for simplicity we will work with the naive decomposition (τ_g, τ_x) associated to a given local trivialization. We will comment on the intrinsic version of the constructions as we proceed.

We will assume that τ_x is given by

$$\tau_x = \mathbf{d}F_1(q)u_1 + \ldots + \mathbf{d}F_m(q)u_m,$$

where $F_1, \ldots, F_m$ are G-invariant functions on Q (or, equivalently, functions on $S = Q/G$). The u_i are our control inputs. Here m could be less than $\dim S$, i.e., we could have only *partial* internal actuation. Note that this class of forces makes intrinsic sense as horizontal one forms τ_{hor} on Q (that is, they annihilate vertical vectors), independent of the local trivialization.

We will assume that we have full freedom to set the value of the external force τ_g, which in general will change the value of the momentum, μ. The precise relationship is given by the following lemma, which is readily verified.

Lemma 3.1 $\dot{\mu} = T^*R_g\tau_g$

Here R_g denotes right translation on the group G by the group element g. If one wanted to do this intrinsically, one would choose the mechanical connection, let τ_{ver} be a vertical one form (vanishing on horizontal vectors) and then the equation above would read

$$\dot{\mathbf{J}}(v_q) = \sigma_q^* \cdot \tau_{\mathrm{ver}}(v_q),$$

where $\sigma_q : \mathfrak{g} \to T_qQ$ is the infinitesimal generator map; $\xi \mapsto \xi_Q(q)$ and where we now use the same symbol $\mathbf{J} : TQ \to \mathfrak{g}^*$ for the momentum map on the tangent bundle.

Recall that the Euler-Lagrange equations with generalized force vector τ correspond to the following equations on T^*Q:

$$\dot{z} = X_H(z) + \begin{bmatrix} 0 \\ \tau \end{bmatrix}$$

Here, $\begin{bmatrix} 0 \\ \tau \end{bmatrix}$ denotes the vertically lifted tangent vector in $T_\tau T^*Q$ that corresponds to the cotangent vector τ in T_q^*Q.

Alternatively, assuming that the Legendre transformation is invertible and letting H be the corresponding Hamiltonian on T^*Q, in cotangent bundle coordinates (q,p) on T^*Q, the equations can be written as:

$$\begin{aligned} \dot{q} &= \partial H/\partial p \\ \dot{p} &= -\partial H/\partial q + \tau \end{aligned}$$

We have the following equation for the time derivative of the Hamiltonian:

$$\frac{dH}{dt} = \langle \tau, \dot{q} \rangle = \langle \tau_g, \dot{g} \rangle + \langle \tau_x, \dot{x} \rangle \tag{3.1}$$

Now consider the feedback:

$$\begin{aligned} &u_i = -k_i \dot{F}_i, \quad \text{where } k_i,\ i = 1, \ldots, m \text{ are positive constants, and} \\ &\tau_g = T^* R_{g^{-1}}[-k(\mu - \mu_e)] \quad \text{where } k \text{ is a positive constant.} \end{aligned}$$

With this feedback, $\tau_x = -\sum_i k_i \mathbf{d}F_i \, \dot{F}_i$. Thus,

$$\begin{aligned} \langle \tau_x, \dot{x} \rangle &= -\sum_i k_i \dot{F}_i \, \mathbf{d}F_i \, \dot{x} \\ &= -\sum_i k_i (\dot{F}_i)^2 \leqslant 0 \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned} \langle \tau_g, \dot{g} \rangle &= \langle T^* R_{g^{-1}}[-k(\mu - \mu_e)], \dot{g} \rangle \\ &= \langle -k(\mu - \mu_e), \dot{g} g^{-1} \rangle \\ &= -k \langle (\mu - \mu_e), \nu \rangle, \quad \text{where } \nu = \dot{g} g^{-1} . \end{aligned} \tag{3.3}$$

Since $\dot{\mu} = T^* R_g \tau_g$, the feedback for τ_g implies

$$\dot{\mu} = -k(\mu - \mu_e) . \tag{3.4}$$

Intrisically, the feedback $\tau_g = T^* R_{g^{-1}}[-k(\mu - \mu_e)]$ can be expressed as:

$$\tau_{\text{ver}}(v_q) = A_q^* \cdot [-k(\mathbf{J}(v_q) - \mu_e)]$$

Here $A_q : T_q Q \to \mathfrak{g}$ is the connection one form.

Strategy for showing Asymptotic Stability At first we will proceed under the assumption that $\mathbf{d}^2(H - J_{\xi_e})(z_e)$ is positive definite on the subspace Z. Thus, by Patrick's result, the relative equilibrium z_e of the vector field X_H is G_{μ_e}-stable. We want to show that the feedback $u_i = -k_i \dot{F}_i$, $\tau_g = T^* R_{g^{-1}}[-k(\mu - \mu_e)]$ makes this relative equilibrium asymptotically stable. In the following sections, we will be using equations (3.1) through (3.4) to investigate the behaviour of the trajectory $z_t \in T^*Q$ of the feedback system.

The first step is to show that the relative equilibrium remains G_{μ_e}-stable with this feedback. Since the feedback $\tau_g = T^*R_{g^{-1}}[-k(\mu - \mu_e)]$ could result in energy being pumped into the system, we will need to proceed with some care. This will be done in section 4.

Once G_{μ_e}-stability has been shown, we will be able to restrict our attention to the open neighborhood $G_{\mu_e} \cdot S$ of $G_{\mu_e} \cdot z_e$. The compactness of G implies that G_{μ_e} is compact, which implies that the set $G_{\mu_e} \cdot S$ has compact closure. In section 5 we will use an argument that is similar in spirit to the proof of LaSalle's theorem to show that the system converges to a certain invariant set $M \subset G_{\mu_e} \cdot S$.

Then, in section 6, we will study the properties of this set M. We will show that, subject to a certain rank condition on the derivatives of the functions F_i and their repeated Poisson brackets with the Hamiltonian H, the projection of this set M onto T^*Q/G_{μ_e} is simply ζ_e. This will enable us to show asymptotic stability of the relative equilibrium, which we have defined as asymptotic stability of ζ_e in T^*Q/G_{μ_e}.

Finally, in section 7 we will show that we can sometimes relax the assumption that $\mathbf{d}^2(H - J_{\xi_e})(z_e)$ is positive definite on the subspace Z. We will show how, under certain conditions, we can use the inputs to modify the potential in a such way that the modified Hamiltonian satisfies the required positivity condition.

4 G_{μ_e}-Stability of the Feedback System

Assume, as in section 2, that $\mathbf{d}^2(H - J_{\xi_e})(z_e)$ is positive definite on Z. Then we know that there exists a G_{μ_e}-invariant function f on $G_{\mu_e} \cdot S$ such that $\mathbf{d}^2 f(gz_e)$ is positive definite on $T_{gz_e}(gS)$ for all $g \in G_{\mu_e}$.

Let z_t be a trajectory of the feedback system with initial condition z_0. Note that

$$f(z_t) = f(z_0) + (f(z_t) - f(z_0))$$

If we can find an upper bound for $f(z_t) - f(z_0)$, we will have an upper bound for $f(z_t)$. Having an upper bound on $f(z_t)$ will ensure that z_t lies in a G_{μ_e}-invariant open neighborhood of $G_{\mu_e} \cdot z_e$.

Note that

$$\begin{aligned} f(z_t) - f(z_0) &= (af_1(z_t) - af_1(z_0)) + (f_2(z_t) - f_2(z_0)) \\ &= a[H(z_t) - H(z_0)] \\ &\quad - a[\mathbf{J}(z_t)\Psi(z_t) - \mathbf{J}(z_0)\Psi(z_0)] \\ &\quad + (|\mathbf{J}(z_t) - \mu_e|^2 - |\mathbf{J}(z_0) - \mu_e|^2) \end{aligned}$$

Let is consider the three terms in this expression. Define

$$\begin{aligned} T1 &= a[H(z_t) - H(z_0)] \\ T2 &= -a[\mathbf{J}(z_t)\Psi(z_t) - \mathbf{J}(z_0)\Psi(z_0)] \\ T3 &= |\mathbf{J}(z_t) - \mu_e|^2 - |\mathbf{J}(z_0) - \mu_e|^2 \end{aligned}$$

Let us examine the term $T3$. With the feedback $\tau_g = T^* R_{g^{-1}}[-k(\mu - \mu_e)]$, the momentum μ satisfies the differential equation

$$\dot{\mu} = -k(\mu - \mu_e).$$

Thus $|\mathbf{J}(z_t) - \mu_e| \leqslant |\mathbf{J}(z_0) - \mu_e|$ and therefore $T3 \leqslant 0$.

Next, we look at $T2$:

$$\begin{aligned} T2 &= a[\mathbf{J}(z_0)\Psi(z_0) - \mathbf{J}(z_t)\Psi(z_t)] \\ &= a[\langle \mathbf{J}(z_0) - \mu_e, \Psi(z_0)\rangle - \langle \mathbf{J}(z_t) - \mu_e, \Psi(z_t)\rangle] \\ &\leqslant a[|\mathbf{J}(z_0) - \mu_e||\Psi(z_0)| + |\mathbf{J}(z_t) - \mu_e||\Psi(z_t)|] \\ &\leqslant a[|\mathbf{J}(z_0) - \mu_e||\Psi(z_0)| + |\mathbf{J}(z_0) - \mu_e||\Psi(z_t)|]. \end{aligned}$$

If $z \in G_{\mu_e} \cdot S$, $z = gz'$, where $z' \in S$ and g is some element of G_{μ_e}. Thus $|\Psi(z)| = |\Psi(gz')| = |\mathrm{Ad}_g \Psi(z')| = |\Psi(z')|$, since the norm is Ad_g-invariant. But $\Psi(z') = \Psi(z_e) = \xi_e$ for all $z' \in S$. Thus $|\Psi(z)| = |\xi_e|$ for all $z \in G_{\mu_e} \cdot S$. Thus

$$T2 \leqslant 2a|\mathbf{J}(z_0) - \mu_e||\xi_e|.$$

Now we shall obtain an upper bound for the $T1$. We know that

$$\dot{H} = \langle \tau, \dot{q}\rangle = \langle \tau_g, \dot{g}\rangle + \langle \tau_x, \dot{x}\rangle.$$

Thus,

$$H(z_t) - H(z_0) = \int_0^t \langle \tau_g, \dot{g}\rangle\, ds + \int_0^t \langle \tau_x, \dot{x}\rangle\, ds\,.$$

We have already seen that $\langle \tau_x, \dot{x}\rangle \leqslant 0$. Thus

$$H(z_t) - H(z_0) \leqslant \int_0^t \langle \tau_g, \dot{g}\rangle\, ds\,.$$

We have seen that for the feedback system, $\langle \tau_g, \dot{g}\rangle = -k\langle \mu - \mu_e, \nu\rangle$ where $\nu = \dot{g}g^{-1}$. At this point we will assume that the trajectory z_t stays inside the set $G_{\mu_e} \cdot S$. We will later justify this assumption. Since $G_{\mu_e} \cdot S$ has compact closure, $|\nu|$ remains bounded on this set. Let m be an upper bound for $|\nu|$. Thus,

$$|\langle \mu - \mu_e, \nu\rangle| \leqslant |\mu - \mu_e||\nu| \leqslant m|\mu - \mu_e|.$$

Also, since $\dot{\mu} = -k(\mu - \mu_e)$,

$$|\mu(z_t) - \mu_e|\,|_{t=s} = e^{-ks}|\mu_0 - \mu_e|.$$

Thus,

$$|\langle \mu - \mu_e, \nu\rangle| \leqslant m|\mu_0 - \mu_e|e^{-ks}.$$

Thus,

$$\begin{aligned} H(z_t) - H(z_0) &\leqslant -k \int_0^t \langle \mu - \mu_e, \nu \rangle \, ds \\ &\leqslant k \int_0^t |\langle \mu - \mu_e, \nu \rangle| \, ds \\ &\leqslant km|\mu_0 - \mu_e| \int_0^t e^{-ks} \, ds \\ &= m|\mu_0 - \mu_e|. \end{aligned}$$

Thus,

$$T1 \leqslant am|\mu_0 - \mu_e|.$$

Using the above inequalities on $T1$, $T2$, $T3$, we get

$$f(z_t) - f(z_0) \leqslant a|\mu_0 - \mu_e|(2|\xi_e| + m),$$

and thus

$$f(z_t) \leqslant f(z_0) + a|\mu_0 - \mu_e|(2|\xi_e| + m).$$

Thus if $f(z_0) < \epsilon/2$ and if $|\mu_0 - \mu_e| < \epsilon/(2a(2|\xi_e| + m))$, then $f(z_t) < \epsilon$.

Let $\epsilon > 0$ be chosen such that the closure of the G_{μ_e}-invariant open neighborhood $T_\epsilon := \{z \mid f(z) < \epsilon\}$ is contained inside $G_{\mu_e} \cdot S$. Now choose an initial condition z_0 such that $f(z_0) < \epsilon/2$ and $|\mu_0 - \mu_e| < \epsilon/(2a(2|\xi_e| + m))$. Then, *if we assume that z_t remains inside the set $G_{\mu_e} \cdot S$ for all $t > 0$*, we can conclude that z_t remains inside the set T_ϵ. But since we have assumed that T_ϵ is contained inside $G_{\mu_e} \cdot S$, the conclusion is that z_t must remain inside $\{z \mid f(z) < \epsilon\}$ for all $t \geqslant 0$.

Given any G_{μ_e}-invariant open neighborhood V of $G_{\mu_e} \cdot z_e$ we can always find an $\epsilon > 0$ such that $T_\epsilon \subset V$, $T_\epsilon \subset G_{\mu_e} \cdot S$. Then, if we set U to be the G_{μ_e}-invariant neighborhood $T_{\epsilon/2} \cap \{z \mid |\mu_0 - \mu_e| < \epsilon/(2a(2|\xi_e| + m))\}$, then $z_0 \in U$ implies $z_t \in V$ for all $t > 0$. Thus we conclude that the relative equilibrium z_e remains G_{μ_e}-stable even with the feedback.

5 Showing convergence of z_t to an invariant set M

Since the trajectory z_t stays inside the set $G_{\mu_e} \cdot S$, which has compact closure, we conclude that the limit set $\mathcal{L}$ of z_t must be compact and non-empty. Also, $z_t \to \mathcal{L}$ (using an appropriately chosen metric d on T^*Q) as $t \to \infty$, and $\mathcal{L}$ is an invariant set. See, for example, Vidyasagar [1993] for the proofs of these statements.

We have seen that $\langle \tau_x, \dot{x} \rangle = -\sum_i k_i(\dot{F}_i)^2 \leqslant 0$. Let us assume for now that $\langle \tau_x, \dot{x} \rangle \to 0$ as $t \to \infty$. Then we have $\dot{F}_i \to 0$ as $t \to \infty$ for $i = 1, \dots, m$, or, in short, $\dot{F} \to 0$ as $t \to \infty$, where $F = (F_1, \dots, F_m) : Q \to \mathbb{R}^m$. Thus if we define the set $S1 := \{z \in G_{\mu_e} \cdot S \mid \dot{F}(z) = 0\}$, then , by continuity of the function $\dot{F}$, it must be that $\mathcal{L} \subset S1$.

We know that $\mu \to \mu_e$ exponentially as $t \to \infty$. Thus if we define the set $S2 := \{z \in G_{\mu_e} \cdot S \mid \mathbf{J}(z) = \mu_e\}$, we also have $\mathcal{L} \subset S2$.

Thus we know that $\mathcal{L}$ is an invariant set contained in $S1 \cap S2$, and that $z_t \to \mathcal{L}$ as $t \to \infty$. Let us define the set M as the largest invariant subset contained in $S1 \cap S2$. Then $\mathcal{L} \subset M$, and therefore $z_t \to M$ as $t \to \infty$. In the next section, we will show that the projection of this set M onto T^*Q/G_{μ_e} is the point ζ_e, which will enable us to show asymptotic stability.

Now let us go back and prove that $\langle \tau_x, \dot{x} \rangle \to 0$ as $t \to \infty$. We have seen that

$$\langle \tau_g, \dot{g} \rangle = -k \langle (\mu - \mu_e), \nu \rangle \,.$$

We know that $(\mu - \mu_e) \to 0$ exponentially as $t \to \infty$. Further, $|\nu|$ remains bounded on $G_{\mu_e} \cdot S$. Thus the integral

$$\int_0^t \langle \tau_g, \dot{g} \rangle \, ds$$

converges to some real number K as $t \to \infty$. Define $\varphi(t)$ by

$$\varphi(t) := \int_0^t \langle \tau_x, \dot{x} \rangle (z_s) \, ds \,.$$

and note that

$$H(z_t) = H(z_0) + \int_0^t \dot{H}(z_s) \, ds.$$

Since $\dot{H} = \langle \tau_g, \dot{g} \rangle + \langle \tau_x, \dot{x} \rangle$,

$$H(z_t) - \varphi(t) = H(z_0) + \int_0^t \langle \tau_g, \dot{g} \rangle \, ds \to H(z_0) + K \quad \text{as}\, t \to \infty \,. \tag{5.1}$$

Since $\langle \tau_x, \dot{x} \rangle \leqslant 0$, we know that $\varphi(t)$ is a non-increasing function of t. The set $G_{\mu_e} \cdot S$ has compact closure and thus $H(z_t)$ remains bounded. Since $H(z_t) - \varphi(t)$ converges to a constant, we conclude that $\varphi(t)$ must also remain bounded. Thus we can conclude that there exists a constant C such that $\varphi(t) \to C$ as $t \to \infty$. Thus, from equation (5.1), we conclude that $H(z_t) \to H(z_0) + C + K$ as $t \to \infty$.

If y is in the limit set $\mathcal{L}$, by continuity of H, we can conclude that $H(y) = H(z_0) + C + K$. Since $\mathcal{L}$ is an invariant set and H is constant on $\mathcal{L}$, it must be that $\dot{H}(y) = 0$ for all $y \in \mathcal{L}$.

Lemma 5.1 *$\dot{H}(z_t) \to 0$ as $t \to \infty$.*

Proof: Choose any $\epsilon > 0$. By continuity of $\dot{H}$, for each $y \in \mathcal{L}$, there exists an open ball B_y such that $z \in B_y$ implies $|\dot{H}(z)| < \epsilon$. Since $\mathcal{L}$ is compact, we can cover $\mathcal{L}$ with a finite number of such balls. The complement of these balls is a closed set which we shall call $\mathcal{S}$. Let $\delta = d(\mathcal{S}, \mathcal{L})$ be the distance between $\mathcal{L}$ and $\mathcal{S}$. Thus, if $d(z, \mathcal{L}) < \delta$, then $|\dot{H}(z)| < \epsilon$. But, we know that $z_t \to \mathcal{L}$ as $t \to \infty$. Thus for any $\epsilon > 0$, $|\dot{H}(z_t)| < \epsilon$ for t large enough. Thus $\dot{H}(z_t) \to 0$ as $t \to \infty$. ■

Now,

$$\dot{H}(z_t) = \langle \tau_g, \dot{g} \rangle + \langle \tau_x, \dot{x} \rangle \,.$$

Since $\langle \tau_g, \dot{g} \rangle = -k \langle \mu - \mu_e, \nu \rangle$, and $(\mu - \mu_e) \to 0$ exponentially and $|\nu|$ remains bounded, we know that $\langle \tau_g, \dot{g} \rangle \to 0$. Since we have shown that $\dot{H}(z_t) \to 0$, we conclude that $\langle \tau_x, \dot{x} \rangle \to 0$ as $t \to \infty$.

6 Characterization of the invariant set M

Let $y(t)$ be a trajectory in M. Since $M \subset S1 \cap S2$, we know that $\dot{F}(y(t)) = 0$, and $\mathbf{J}(y(t)) = \mu_e$. Since the feedback inputs, which are given by $u_i = -k_i \dot{F}_i$, and $\tau_g = T^* R_{g^{-1}}[-k(\mu - \mu_e)]$ must be zero along such a trajectory, the trajectory $y(t)$ is also a trajectory of the Hamiltonian system, $\dot{z} = X_H(z)$.

The equation $\dot{F}(y(t)) = 0$ implies that the functions $F_1, \ldots, F_m$ are constant along the trajectory $y(t)$. Thus the functions $\dot{F}_i, \ddot{F}_i, \dddot{F}_i, \ldots$ $i = 1, \ldots, m$ are all constant along the trajectory. Since this trajectory is an integral curve of the system $\dot{z} = X_H(z)$,

$$\begin{aligned} \dot{F}_i &= \{F_i, H\}, \\ \ddot{F}_i &= \{\{F_i, H\}, H\}, \\ \dddot{F}_i &= \{\{\{F_i, H\}, H\}, H\}, \quad \text{and so on.} \end{aligned}$$

Thus, if we define a set of functions

$$\mathcal{C} = \operatorname{span}\{F_i, \{F_i, H\}, \{\{F_i, H\}, H\}, \{\{\{F_i, H\}, H\}, H\}, \ldots\}, \quad i = 1, \ldots, m\,, \tag{6.1}$$

and if we define the codistribution $\mathbf{d}\mathcal{C}$

$$\mathbf{d}\mathcal{C}(z) := \operatorname{span}\{\mathbf{d}g(z) \mid g \in \mathcal{C}\}\,,$$

then it is easy to see that the tangent vector $\dot{y}(t)$ must be annihilated by this codistribution, i.e., $\dot{y} \in \ker \mathbf{d}\mathcal{C}(y)$. All the functions in $\mathcal{C}$ are G-invariant. Thus we know that $\ker \mathbf{d}\mathcal{C}(y)$ must include the vertical space at y, namely, $\mathfrak{g} \cdot y$. We shall make the assumption

$$\ker \mathbf{d}\mathcal{C}(z) = \mathfrak{g} \cdot z \quad \text{for all } z \text{ in } G_{\mu_e} \cdot S \tag{6.2}$$

This assumption implies that the dimension of the codistribution $\mathbf{d}\mathcal{C}$ is as large as it possibly can be. We will refer to this assumption as the 'rank assumption'. We will make some comments about this assumption in section 8. With this assumption we conclude that $\dot{y} \in \mathfrak{g} \cdot y$. We also know that $M \subset \mathbf{J}^{-1}(\mu_e)$. Thus, $\dot{y} \in T_y[\mathbf{J}^{-1}(\mu_e)]$. It is a fact (see Marsden [1992]) that

$$\mathfrak{g} \cdot y \cap T_y[\mathbf{J}^{-1}(\mu_e)] = \mathfrak{g}_{\mu_e} \cdot y\,.$$

Thus we have $\dot{y} \in \mathfrak{g}_{\mu_e} \cdot y$. If $\pi_{\mu_e} : T^*Q \to T^*Q/G_{\mu_e}$ is the projection, then we have $T\pi_{\mu_e} \cdot \dot{y} = 0$. Thus if $y(t)$ is a trajectory in M, then $\pi_{\mu_e}(y(t)) =$ constant. Thus $\pi_{\mu_e}(M)$ must consist of a collection of equilibrium points of the vector field $X_{\mathcal{H}}$ in T^*Q/G_{μ_e}.

We know that $z_t \to \mathcal{L}$ as $t \to \infty$. Choose any $\epsilon > 0$. By the continuity of the map π_{μ_e}, for each $y \in \mathcal{L}$, there exists an open ball B_y such that $z \in B_y$ implies $d(\pi_{\mu_e}(z), \pi_{\mu_e}(y)) < \epsilon$, which in turn implies $d(\pi_{\mu_e}(z), \pi_{\mu_e}(\mathcal{L})) < \epsilon$. Now $\mathcal{L}$ is compact, and so it can be covered by a finite number of such open balls. Just as in the proof of Lemma 5.1, we can find a $\delta > 0$ such that $d(z, \mathcal{L}) < \delta$ implies

$d(\pi_{\mu_e}(z), \pi_{\mu_e}(\mathcal{L})) < \epsilon$. Thus $\pi_{\mu_e}(z_t) \to \pi_{\mu_e}(\mathcal{L}) \subset \pi_{\mu_e}(M)$ as $t \to \infty$. Thus $\pi_{\mu_e}(z_t)$ must converge to an equilibrium point of the vector field $X_{\mathcal{H}}$ in T^*Q/G_{μ_e}.

The next step is to prove that $\pi_{\mu_e}(z_t)$ converges to the equilibrium $\pi_{\mu_e}(z_e) = \zeta_e$. We will show this by proving that the set $\pi_{\mu_e}(M)$ consists of just one element, which is the equilibrium ζ_e. Recall that $M \subset \mathbf{J}^{-1}(\mu_e)$. Thus $\pi_{\mu_e}(M)$ can be thought of as a subset of $\mathbf{J}^{-1}(\mu_e)/G_{\mu_e}$. Note that ζ_e is also an equilibrium of the vector field $X_{H_{\mu_e}}$ on $\mathbf{J}^{-1}(\mu_e)/G_{\mu_e}$. Here H_{μ_e} is the function on $\mathbf{J}^{-1}(\mu_e)/G_{\mu_e}$ obtained by restricting H to $\mathbf{J}^{-1}(\mu_e)$ and then using the G-invariance of H to drop this restriction to the quotient $\mathbf{J}^{-1}(\mu_e)/G_{\mu_e}$.

We have assumed (in section 4) that $\mathbf{d}^2(H - J_{\xi_e})(z_e)$ is positive definite on the space Z. It is easy to check (using the fact that J_{ξ_e} is constant in the set $\mathbf{J}^{-1}(\mu_e)$) that this is equivalent to the condition that $\mathbf{d}^2 H_{\mu_e}(\zeta_e)$ is a positive definite 2-form on $T_{\zeta_e}[\mathbf{J}^{-1}(\mu_e)/G_{\mu_e}]$.

The positive definiteness of $\mathbf{d}^2 H_{\mu_e}(\zeta_e)$ implies that there is a neighborhood W of ζ_e in $\mathbf{J}^{-1}(\mu_e)/G_{\mu_e}$ that does not contain any other equilibria of $X_{H_{\mu_e}}$ besides ζ_e. By choosing the set S to be sufficiently small, we can arrange that

$$\pi_{\mu_e}[\mathbf{J}^{-1}(\mu_e) \cap G_{\mu_e} \cdot S] \subset W$$

Thus $\pi_{\mu_e}(M) \subset W$. Now since $\pi_{\mu_e}(M)$ consists only of equilibria, $\pi_{\mu_e}(M) = \{\zeta_e\}$ and therefore $\pi_{\mu_e}(z_t)$ converges to ζ_e.

We shall summarize this discussion in the following theorem.

Theorem 6.1 *Let Q be a configuration manifold, and let G be a compact group that acts freely and properly on Q. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be a G-invariant kinetic energy metric on Q. Let $K(v_q) := \frac{1}{2}\langle\langle v_q, v_q \rangle\rangle$ be the kinetic energy, and let $V : Q \to \mathbb{R}$ be a G-invariant potential function. Let $L := K - V$ be the G-invariant the Lagrangian, and let $H : T^*Q \to \mathbb{R}$ be the corresponding G-invariant Hamiltonian. Let $z_e \in T^*Q$ be a relative equilibrium for the vector field X_H. Let $\tau = (\tau_g, \tau_x)$ be the vector of generalized forces acting on the system. We assume that*

$$\tau_x = \mathbf{d}F_1(q)u_1 + \ldots + \mathbf{d}F_m(q)u_m,$$

where $F_1, \ldots, F_m$ are G-invariant functions on Q (or, equivalently, functions on $S = Q/G$). We choose τ_g and $u_1, \ldots, u_m$ to be our control inputs. Let the codistribution $\mathbf{d}\mathcal{C}$, defined in equation (6.1), satisfy the property that for all z in a neighborhood of z_e, $\ker \mathbf{d}\mathcal{C}(z)$ is equal to the tangent space to the G orbit through z. Let Z be a subspace of $T_{z_e}(T^*Q)$ that is a complement of $\mathfrak{g}_{\mu_e} \cdot z_e$ in $T_{z_e}[\mathbf{J}^{-1}(\mu_e)]$, and let $\mathbf{d}^2(H - J_{\xi_e})(z_e)$ be positive definite on Z. Then the feedback

$$u_i = -k_i \dot{F}_i, \quad \tau_g = T^* R_{g^{-1}}[-k(\mu - \mu_e)]$$

asymptotically stabilizes the relative equilibrium z_e.

7 Relaxing the positivity condition on $\mathbf{d}^2(H - \mathbf{J}_{\xi_e})(z_e)$.

The previous theorem required that $\mathbf{d}^2(H - \mathbf{J}_{\xi_e})(z_e)$ be positive definite on Z, where Z is a complement of $\mathfrak{g}_{\mu_e} \cdot z_e$ in $T_{z_e}\mathbf{J}^{-1}(\mu_e)$, which is equivalent to the positive definiteness of $\mathbf{d}^2 H_{\mu_e}(\zeta_e)$ on $T_{\zeta_e}[\mathbf{J}^{-1}(\mu_e)/G_{\mu_e}]$. If this condition fails to hold then we can still achieve asymptotic stability, but we have to do it in a two step process:

1. We first use the inputs to modify the potential in such a way that the modified Hamiltonian, $\tilde{H}$, satisfies $\mathbf{d}^2(\tilde{H} - \mathbf{J}_{\xi_e})(z_e) > 0$ on Z.

2. We then apply the previously developed techniques to get asymptotic stability.

7.1 Stability Analysis at a Relative Equilibrium

We will now briefly describe some of the basics of the Energy-Momentum method for analyzing the stability of relative equilibria. For a complete description, see Marsden [1992]. A relative equilibrium $z_e \in T^*_{q_e}Q$ has the form $\alpha_{\mu_e}(q_e)$, where $\alpha_{\mu_e}(q_e)$ is defined by:

$$\langle \alpha_{\mu_e}(q_e), v_{q_e} \rangle = \langle \mu_e, A(v_{q_e}) \rangle .$$

Here $A : TQ \to \mathfrak{g}$ is the *mechanical connection*, and q_e has to be a critical point of the *amended potential* V_{μ_e}, which is defined by:

$$V_{\mu_e}(q) = V(q) + \frac{1}{2}\langle \mu_e, \mathbb{I}^{-1}(q)\mu_e \rangle,$$

where $\mathbb{I}$ is the *locked inertia tensor*. By the Energy-Momentum method, it is possible to choose a basis of $T_{\zeta_e}[\mathbf{J}^{-1}(\mu_e)/G_{\mu_e}]$ such that $\mathbf{d}^2 H_{\mu_e}(\zeta_e)$ has a convenient block-diagonal form. To describe this form we will need some more definitions and constructions.

Let $\mathfrak{g}_{\mu_e}$ be the Lie algebra of the isotropy subgroup G_{μ_e}. The subspace $\mathcal{V} \subset T_{q_e}Q$ is defined as the orthogonal complement of the tangent space to the G_{μ_e}-orbit through q_e. The metric on Q is used for defining the orthogonal complement. Thus $\mathcal{V} = (\mathfrak{g}_{\mu_e} \cdot q_e)^{\perp}$. Let $\mathcal{V}_{\text{RIG}} := (\mathfrak{g}_{\mu_e})^{\perp} \cdot q_e \subset \mathcal{V}$, where the orthogonal complement of $\mathfrak{g}_{\mu_e}$ is computed using the inner product on $\mathfrak{g}$ defined by the locked inertia tensor at q_e, $\mathbb{I}_{q_e}$. Since q_e is a critical point of V_{μ_e}, the second derivative $\mathbf{d}^2 V_{\mu_e}(q_e)$ is a symmetric 2-form on $T_{q_e}Q$. Now let $\mathcal{V}_{\text{INT}}$ be a complement of $\mathcal{V}_{\text{RIG}}$ in $\mathcal{V}$, chosen in such a way that the restriction of $\mathbf{d}^2 V_{\mu_e}(q_e)$ to $\mathcal{V}$ block-diagonalizes with respect to the splitting $\mathcal{V} = \mathcal{V}_{\text{RIG}} \oplus \mathcal{V}_{\text{INT}}$.

Thus, with respect to a basis of $\mathcal{V}$ that is the union of a basis of $\mathcal{V}_{\text{RIG}}$ and a basis of $\mathcal{V}_{\text{INT}}$, the matrix representation of $\mathbf{d}^2 V_{\mu_e}(q_e)|\mathcal{V}$ has the form:

$$\begin{bmatrix} A_{\mu_e} & 0 \\ 0 & B_{\mu_e} \end{bmatrix}, \tag{7.1}$$

where $A_{\mu_e} = \mathbf{d}^2 V_{\mu}(q_e)|\mathcal{V}_{\text{RIG}}$ and $B_{\mu_e} = \mathbf{d}^2 V_{\mu}(q_e)|\mathcal{V}_{\text{INT}}$. The Energy-Momentum method tells us that with respect to an appropriately chosen basis, the matrix of $\mathbf{d}^2 H_{\mu_e}(\zeta_e)$ is:

$$\mathbf{d}^2 H_{\mu_e}(\zeta_e) = \begin{bmatrix} A_{\mu_e} & 0 & 0 \\ 0 & B_{\mu_e} & 0 \\ 0 & 0 & K_{\mu_e} \end{bmatrix} \tag{7.2}$$

The matrices A_{μ_e}, B_{μ_e}, have been defined earlier; and the matrix K_{μ_e} is a matrix of size $\dim S \times \dim S$ that depends on the kinetic energy metric only and is known to be positive definite.

For positivity of $\mathbf{d}^2 H_{\mu_e}(\zeta_e)$ we thus need both A_{μ_e} and B_{μ_e} to be positive definite. If A_{μ_e} is positive definite, it means that the *rigid motion* is stable. In other words, if we were to lock up the internal joints of the system, then the system will rotate stably if A_{μ_e} is positive definite.

We will now proceed to describe the modifications that we need to make to our feedback if $\mathbf{d}^2 H_{\mu_e}(\zeta_e)$ fails to be positive definite. We will first consider the case in which A_{μ_e} is positive definite.

7.2 The Case $A_{\mu_e} > 0$

Recall that the inputs available to us are $\tau_g \in T_g^* G$ and $u_1, \ldots, u_m$. Thus the equations of motion of the system are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau_g + \sum_{i=1}^{m} \mathbf{d}F_i \, u_i$$

This form of the equations (i.e. Euler-Lagrange equations with force terms) is convenient for showing how we can modify the potential (and therefore the Hamiltonian) using the inputs. Once we have the modified potential, we will use it to calculate the block-diagonal form for the second derivative of the modified reduced Hamiltonian, as in equation (7.2). The positive definiteness of this second derivative is one of the conditions we need to apply Theorem 6.1. Recall that the setting for the proof of Theorem 6.1 is T^*Q. Our use of the Euler-Lagrange equations here (which give first order dynamics on TQ) is only for calculating the modified potential.

We will now proceed to show how we can (under appropriate conditions) use the internal or shape space inputs $u_1, \ldots, u_m$ to make the required modifications in the Hamiltonian.

Consider the feedback $u_i = -c_i F_i(q) + v_i$. With this feedback, we have

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} &= -\sum_{i=1}^{m} c_i \mathbf{d}F_i \, F_i + \tau_g + \sum_{i=1}^{m} \mathbf{d}F_i \, v_i \\ &= -\frac{\partial}{\partial q}\left(\frac{1}{2}\sum_{i=1}^{m} c_i F_i^2(q)\right) + \tau_g + \sum_{i=1}^{m} \mathbf{d}F_i \, v_i \end{aligned}$$

Now define a modified potential $\tilde{V}(q) := V(q) + \frac{1}{2}\sum_{i=1}^{m} c_i F_i^2(q)$. We have a corresponding modified amended potential $\tilde{V}_{\mu_e}(q) := V_{\mu_e}(q) + \frac{1}{2}\sum_{i=1}^{m} c_i F_i^2(q)$. Note that since the functions F_i are zero at q_e, q_e is a critical point of $\tilde{V}_{\mu_e}$ as well, and thus $z_e = \alpha_{\mu_e}(q_e)$ remains a relative equilibrium of the modified system. If we let $\tilde{L} := K - \tilde{V}$ be the corresponding modified Lagrangian, then the new equations of motion are:

$$\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q} = \tau_g + \sum_{i=1}^{m} \mathbf{d}F_i \, v_i$$

We shall now compute $\mathbf{d}^2\tilde{V}_{\mu_e}(q_e)|\mathcal{V}$:

$$\begin{aligned}\mathbf{d}^2\tilde{V}_{\mu_e}(q_e)|\mathcal{V} &= \mathbf{d}^2V_{\mu_e}(q_e)|\mathcal{V} + \mathbf{d}^2\left(\frac{1}{2}\sum_{i=1}^{m} c_iF_i^2(q)\right)(q_e)|\mathcal{V} \\ &= \begin{bmatrix} A_{\mu_e} & 0 \\ 0 & B_{\mu_e} \end{bmatrix} + \mathbf{d}^2\left(\frac{1}{2}\sum_{i=1}^{m} c_iF_i^2(q)\right)(q_e)|\mathcal{V}\end{aligned}$$

Since the functions F_i are G-invariant, it is easy to verify that

$$\mathbf{d}^2\left(\frac{1}{2}\sum_{i=1}^{m} c_iF_i^2\right)(q_e)|\mathcal{V} = \begin{bmatrix} 0 & 0 \\ 0 & (\mathbf{d}F(q_e))^TC(\mathbf{d}F(q_e)) \end{bmatrix}$$

Here $\mathbf{d}F(q_e)$ is the matrix of the restriction of the derivative of $F = (F_1, \ldots, F_m)$ to $\mathcal{V}_{\mathrm{INT}}$, and $C := \mathrm{diag}\{c_1, \ldots, c_m\}$.

Let $\tilde{H}$ be the modified Hamiltonian corresponding to $\tilde{L}$. Since we have changed only the potential energy, the $(3,3)$ block in the block diagonal form for $\mathbf{d}^2\tilde{H}_{\mu_e}(\zeta_e)$, which depend only on the kinetic energy, remains unchanged. Thus we have

$$\mathbf{d}^2\tilde{H}_{\mu_e}(\zeta_e) = \begin{bmatrix} A_{\mu_e} & 0 & 0 \\ 0 & B_{\mu_e} + \mathbf{d}F(q_e)^TC\mathbf{d}F(q_e) & 0 \\ 0 & 0 & K_{\mu_e} \end{bmatrix}. \tag{7.3}$$

We would like $\mathbf{d}^2\tilde{H}_{\mu_e}(\zeta_e)$ to be positive definite. We have assumed that A_{μ_e} is positive definite, and K_{μ_e} is known to be positive definite. Thus we need $B_{\mu_e} + \mathbf{d}F(q_e)^TC\mathbf{d}F(q_e)$ to be positive definite. It is a fact from linear algebra that we can choose the coefficients $c_1, \ldots, c_m$ so as to make this matrix positive definite if and only if B_{μ_e} is positive definite on $\ker \mathbf{d}F(q_e)$.

Thus we conclude that if $\mathbf{d}^2V_{\mu_e}(q_e)$ is positive definite on $\ker \mathbf{d}F(q_e)$, then we can choose coefficients $c_1, \ldots, c_m$ so that the feedback

$$u_i = -c_iF_i(q) + v_i$$

yields a modified system with Hamiltonian $\tilde{H}$ such that $\mathbf{d}^2\tilde{H}_{\mu_e}(\zeta_e)$ is positive definite. For the modified system, we define a collection of functions $\tilde{\mathcal{C}}$, which is analogous to the set $\mathcal{C}$ defined earlier.

$$\tilde{\mathcal{C}} := \mathrm{span}\{F_i, \{F_i, \tilde{H}\}, \{\{F_i, \tilde{H}\}, \tilde{H}\}, \{\{\{F_i, \tilde{H}\}, \tilde{H}\}, \tilde{H}\}, \ldots\}, \quad i = 1, \ldots, m.$$

As in section 6, we require that this modified system satisfy the rank condition:

$$\ker \mathbf{d}\tilde{\mathcal{C}}(z) = \mathfrak{g}\cdot z \quad \text{for all } z \text{ in } G_{\mu_e}\cdot S. \tag{7.4}$$

Thus the modified system satisfies the conditions of Theorem 6.1. If we were to apply the asymptotically stabilizing feedback of Theorem 6.1 to the modified system, then the overall feedback applied to the system is:

$$\begin{aligned} u_i &= -c_iF_i(q) - k_i\dot{F}_i(q),\ i = 1, \ldots, m \\ \tau_g &= T^*R_{g^{-1}}[-k(\mu - \mu_e)] \end{aligned}$$

7.3 The Case $A_{\mu_e} \not> 0$.

We have defined A_{μ_e} to be equal to $\mathbf{d}^2 V(q_e)|\mathcal{V}_{\mathrm{RIG}}$, where $\mathcal{V}_{\mathrm{RIG}}$ is a subspace of the tangent space to the group orbit at $q_e \in Q$. We have also seen above that if we try to modify the potential using the internal inputs alone, A_{μ_e} does not change. Thus we cannot use the stabilization technique in the previous section if A_{μ_e} is not positive definite. In this section we will give an alternate sufficient condition that may hold even when A_{μ_e} is not positive definite.

In order to get around the fact that A_{μ_e} is not positive definite, we will need to use the inputs in the group directions also to modify the potential. But if we do this, our modified system will no longer be invariant under the action of the symmetry group G, and thus the momentum $\mathbf{J}$ will no longer be conserved. However, the arguments we have used in the earlier sections of this paper do require that momentum be conserved by the system to which we are applying the feedback described in section 3.

To overcome this difficulty we will proceed using the G_{μ_e} action on Q rather than the G action. Note that Q is locally diffeomorphic to $G \times S$, and that G is locally diffeomorphic to $G_{\mu_e} \times \mathcal{O}$, where $\mathcal{O}$ is the coadjoint orbit through μ_e. Thus Q is locally diffeomorphic to $G_{\mu_e} \times \mathcal{O} \times S$. We will now regard $Q/G_{\mu_e} \approx \mathcal{O} \times S$ as the new shape space, and we will use inputs along this shape space to modify the potential. We will modify the potential in such a way that z_e remains a relative equilibrium. Since we will not use any actuation along the G_{μ_e} direction to modify the potential, the momentum map $\hat{\mathbf{J}}$ corresponding to the G_{μ_e} action will be conserved. Let $\hat{\mu_e} = \hat{\mathbf{J}}(z_e)$. The amended potential is now called $V_{\hat{\mu}_e}$, and is defined by:

$$V_{\hat{\mu}_e}(q) := V(q) + \frac{1}{2}\langle \hat{\mu}_e, \hat{\mathbb{I}}^{-1}(q)\hat{\mu}_e \rangle$$

$\hat{\mathbb{I}}$ is the locked inertia tensor associated with the G_{μ_e} action. Let $\hat{Z}$ be a complement of $\mathfrak{g}_{\mu_e}$ in $\hat{\mathbf{J}}^{-1}(\hat{\mu_e})$. If we can show that $\mathbf{d}^2(\tilde{H} - \hat{\mathbf{J}}\xi_e)$ is positive definite on $\hat{Z}$, where $\tilde{H}$ is the modified Hamiltonian, or equivalently, $\mathbf{d}^2 \tilde{H}_{\hat{\mu_e}}(\zeta_e)$ is a positive definite two-form on $T_{\zeta_e}[\hat{\mathbf{J}}^{-1}(\hat{\mu}_e)/G_{\mu_e}]$, then we will be able to use the previously developed theory to show asymptotic stability.

Recall that the open loop equations of motion for the (unmodified) system are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau_g + \sum_{i=1}^{m} \mathbf{d}F_i\, u_i$$

Now we let $\tau_g = \tau_1 + \tau_2$, where τ_1 is the force along the G_{μ_e} direction, and τ_2 is the force along the $\mathcal{O}$ direction.

Let $\omega^1, \ldots, \omega^r$ be local coordinates on $\mathcal{O}$. Define the functions $\psi^1, \ldots, \psi^{r+m}$ as follows:

$$\psi_i = \begin{cases} (\omega^i - \omega^i(z_e)), & i = 1, \ldots, r \\ F_{i-r}, & i = r+1, \ldots, r+m \end{cases}$$

Let $\tau_2 = \sum_{i=1}^{r} \mathbf{d}\psi_i \, v_i$. Thus the force can be written as

$$\tau = \tau_1 + \sum_{i=1}^{r+m} \mathbf{d}\psi_i \, v_i \, .$$

where we let $(v_{r+1}, \ldots, v_{r+m}) = (u_1, \ldots, u_m)$. We will now regard τ_1 and $v_1, \ldots, v_{r+m}$ as the inputs to the system. If we set $v_i = -c_i \, \psi_i + w_i$, we get

$$\begin{aligned} \tau &= -\sum_{i=1}^{m+r} c_i \mathbf{d}\psi_i \, \psi_i + \tau_1 + \sum_{i=1}^{m+r} \mathbf{d}\psi_i \, w_i \\ &= -\frac{\partial}{\partial q} \left(\frac{1}{2} \sum_{i=1}^{m+r} c_i \psi_i^2(q) \right) + \tau_1 + \sum_{i=1}^{m+r} \mathbf{d}\psi_i \, w_i \, . \end{aligned}$$

Now, as in section 7.2, we define a modified potential $\tilde{V}(q) := V(q) + \frac{1}{2} \sum_{i=1}^{m+r} c_i \psi_i^2(q)$. We get a corresponding modified amended potential, $\tilde{V}_{\hat{\mu}_e}(q) := V_{\hat{\mu}_e}(q) + \frac{1}{2} \sum_{i=1}^{m+r} c_i \psi_i^2(q)$. Note that since the functions ψ_i are zero at q_e, q_e is a critical point of $\tilde{V}_{\hat{\mu}_e}$ as well, and thus $z_e = \alpha_{\mu_e}(q_e)$ remains a relative equilibrium of the modified system. If we let $\tilde{L} := K - \tilde{V}$ be the corresponding modified Lagrangian, then the new equations of motion are:

$$\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q} = \tau_1 + \sum_{i=1}^{m+r} \mathbf{d}\psi_i \, w_i$$

Now, if $\tilde{H}$ is the modified Hamiltonian, let us see what $\mathbf{d}^2 \tilde{H}_{\hat{\mu}_e}$ will look like. Recall that now we are considering the G_{μ_e} action. Let $\hat{\mathcal{V}}_{\mathrm{RIG}}$ and $\hat{\mathcal{V}}_{\mathrm{INT}}$ be the subspaces that correspond to the subspaces $\mathcal{V}_{\mathrm{RIG}}$ and $\mathcal{V}_{\mathrm{INT}}$ in section 7.1.

Since $(G_{\mu_e})_{\hat{\mu}_e} = G_{\mu_e}$, we conclude that $\hat{\mathcal{V}}_{\mathrm{RIG}} = \{0\}$, and thus $\hat{\mathcal{V}}_{\mathrm{INT}} = \mathcal{V} = (\mathfrak{g}_{\mu_e} \cdot q_e)^{\perp}$. Thus, as in equation (7.3), we can express $\mathbf{d}^2 \tilde{H}_{\hat{\mu}_e}(\zeta_e)$ in block diagonal form as follows:

$$\mathbf{d}^2 \tilde{H}_{\hat{\mu}_e}(\zeta_e) = \begin{bmatrix} B_{\hat{\mu}_e} + (\mathbf{d}\psi(q_e))^T C \mathbf{d}\psi(q_e) & 0 \\ 0 & K_{\hat{\mu}_e} \end{bmatrix}$$

Note that since $\hat{\mathcal{V}}_{\mathrm{RIG}} = \{0\}$, there is no $(1,1)$ block. In the above equation, $\psi = (\psi_1, \ldots, \psi_m)$, and $\mathbf{d}\psi(q_e) : \mathcal{V} = \mathcal{V}_{\mathrm{RIG}} \oplus \mathcal{V}_{\mathrm{INT}} \to \mathbb{R}^{r+m}$ is a $(r+m) \times (r + \dim S)$ matrix of the form:

$$\mathbf{d}\psi(q_e) = \begin{bmatrix} I & 0 \\ 0 & \mathbf{d}F(q_e) \end{bmatrix}$$

$B_{\hat{\mu}_e}$ is the second derivative of $V_{\hat{\mu}_e}$ restricted to the space $\mathcal{V}$. We know that $K_{\hat{\mu}_e}$ is always positive definite. Thus $\mathbf{d}^2 \tilde{H}_{\hat{\mu}_e}(\zeta_e)$ is positive definite if and only if

$$B_{\hat{\mu}_e} + (\mathbf{d}\psi(q_e))^T C \mathbf{d}\psi(q_e)$$

is positive definite. We can find a diagonal matrix C such that

$$B_{\hat{\mu}_e} + (\mathbf{d}\psi(q_e))^T C \mathbf{d}\psi(q_e)$$

is positive definite if and only if $B_{\hat{\mu}_e}$ is positive definite on $\ker \mathbf{d}\psi(q_e)$. But $\ker \mathbf{d}\psi(q_e) = \ker \mathbf{d}F(q_e) \cap \mathcal{V}_{\mathrm{INT}}$. Thus we need $B_{\hat{\mu}_e}$ to be positive definite on $\ker \mathbf{d}F(q_e) \cap \mathcal{V}_{\mathrm{INT}}$. Assume that this condition is satisfied and that C is chosen such that

$$B_{\hat{\mu}_e} + (\mathbf{d}\psi(q_e))^T C \mathbf{d}\psi(q_e)$$

is positive definite. In addition assume that the modified system satisfies the rank condition. In this case the rank condition can be expressed as:

$$\ker \mathbf{d}\tilde{\mathcal{C}}(z) = \mathfrak{g}_{\mu_e} \cdot z \quad \text{for all } z \text{ in } G_{\mu_e} \cdot S \tag{7.5}$$

where $\tilde{\mathcal{C}}$ is defined as

$$\tilde{\mathcal{C}} := \operatorname{span}\{\psi_i, \{\psi_i, \tilde{H}\}, \{\{\psi_i, \tilde{H}\}, \tilde{H}\}, \{\{\{\psi_i, \tilde{H}\}, \tilde{H}\}, \tilde{H}\}, \ldots\}, \quad i = 1, \ldots, r+m \tag{7.6}$$

To the modified system we can apply the asymptotically stabilizing feedback of Theorem 6.1. The overall feedback applied to the system will then be:

$$\begin{aligned} \tau_1 &= T^* R_{g^{-1}}[-k(\hat{\mu} - \hat{\mu}_e)] \\ v_i &= -c_i \psi_i - k_i \dot{\psi}_i, \quad i = 1, \ldots, r+m \end{aligned}$$

Alternatively, in terms of $\tau_g = (\tau_1, \tau_2)$ and u_i, we can write the feedback as

$$\begin{aligned} \tau_1 &= T^* R_{g^{-1}}[-k(\hat{\mu} - \hat{\mu}_e)] \\ \tau_2 &= \sum_{i=1}^{r} \mathbf{d}\psi_i(-c_i \psi_i - k_i \dot{\psi}_i) \\ u_i &= -c_{r+i} F_i - k_{r+i} \dot{F}_i, \quad i = 1, \ldots, m \end{aligned}$$

We will now summarize the results we have obtained in this section in the following theorem:

Theorem 7.1 *Let Q be a configuration manifold, and let G be a compact group that acts freely and properly on Q. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be a G-invariant kinetic energy metric on Q. Let $K(v_q) := \frac{1}{2}\langle\langle v_q, v_q \rangle\rangle$ be the kinetic energy, and let $V : Q \to \mathbb{R}$ be a G-invariant potential function. Let $L := K - V$ be the G-invariant the Lagrangian, and let $H : T^*Q \to \mathbb{R}$ be the corresponding G-invariant Hamiltonian. Let $z_e \in T^*Q$ be a relative equilibrium for the vector field X_H. Let $\tau = (\tau_g, \tau_x)$ be the vector of generalized forces acting on the system. We assume that*

$$\tau_x = \mathbf{d}F_1(q)u_1 + \ldots + \mathbf{d}F_m(q)u_m,$$

where $F_1, \ldots, F_m$ are G-invariant functions on Q (or, equivalently, functions on $S = Q/G$). Let τ_g and $u_1, \ldots, u_m$ be our control inputs. We then have the following sufficient conditions under which we can find an asymptotically stabilizing feedback:

1. *Assume that $\mathbf{d}^2 V_{\mu_e}(q_e)|\mathcal{V}_{\mathrm{RIG}}$ is positive definite and let $\mathbf{d}^2 V_{\mu}(q_e)|\mathcal{V}_{\mathrm{INT}}$ be positive definite on $\ker \mathbf{d}F(q_e)$. Further, assume that the rank condition (7.4) is*

satisfied. Then we can find constants c_i, k_i*, and a positive constant* k *such that the feedback*

$$u_i = -c_i F_i(q) - k_i \dot{F}_i(q),\ i = 1, \ldots, m \quad \textit{and} \quad \tau_g = T^* R_{g^{-1}}[-k(\mu - \mu_e)]$$

asymptotically stabilizes the relative equilibrium.

2. *Assume that* $\mathbf{d}^2 V_{\hat{\mu}_e}$ *is positive definite on ker* $\mathbf{d}F(q_e) \cap \mathcal{V}_{INT}$*. Assume that the rank condition (7.5) is satisfied. Then we can find constants* $c_i, k_i,\ i = 1, \ldots, r+m$ *and a positive constant* k *such that the relative equilibrium is asymptotically stabilized by the following feedback:*

$$u_i = -c_{r+i} F_i - k_{r+i} \dot{F}_i, \quad i = 1, \ldots, m$$

and where $\tau_g = (\tau_1, \tau_2)$ *is given by*

$$\tau_1 = T^* R_{g^{-1}}[-k(\hat{\mu} - \hat{\mu}_e)]$$
$$\tau_2 = \sum_{i=1}^{r} \mathbf{d}\psi_i(-c_i \psi_i - k_i \dot{\psi}_i)$$

8 On the rank assumption on the codistribution $\mathbf{d}\tilde{\mathcal{C}}$.

In sections 7.2 and 7.3 we had assumed that the codistribution $\mathbf{d}\tilde{\mathcal{C}}$ had maximum possible dimension. This assumption is needed to apply Theorem 6.1, where we had used a rank condition on the codistribution $\mathbf{d}\mathcal{C}$, defined in equation 6.2. This rank condition is not a stringent one, as we will show by the following discussion. Our discussion here is in the context of section 7.2, but it can be applied with appropriate modifications to section 7.3 also.

We have defined $\mathbf{d}\tilde{\mathcal{C}}$ by:

$$\mathbf{d}\tilde{\mathcal{C}} = \operatorname{span}\{\mathbf{d}F_i, \mathbf{d}\{F_i, \tilde{H}\}, \mathbf{d}\{\{F_i, \tilde{H}\}, \tilde{H}\}, \ldots\}$$

The rank condition says that the $\ker \mathbf{d}\tilde{\mathcal{C}}(z) = \mathfrak{g} \cdot z$, or equivalently, $\dim \mathbf{d}\tilde{\mathcal{C}} = 2\dim Q - \dim G$. All the functions in $\tilde{\mathcal{C}}$ are G-invariant functions on T^*Q. Let f_i be the function on T^*Q/G defined by $F_i = f_i \circ \pi$, where $\pi : T^*Q \to T^*Q/G$ is the projection. Similarly, let $\tilde{h}$ be the function on T^*Q/G induced by $\tilde{H}$. It is a fact (see Marsden and Ratiu [1998]) that Poisson bracket of two G-invariant functions A, B on T^*Q is G-invariant and the function it induces on T^*Q/G is the Poisson bracket of the functions a, b induced on T^*Q/G by A, B respectively. Thus the collection of functions $\tilde{\mathcal{C}}$ induces a collection of functions $\tilde{\mathcal{C}_1}$ on T^*Q/G, defined as:

$$\tilde{\mathcal{C}_1} := \operatorname{span}\{f_i, \{f_i, \tilde{h}\}, \{\{f_i, \tilde{h}\}, \tilde{h}\}, \{\{\{f_i, \tilde{h}\}, \tilde{h}\}, \tilde{h}\}, \ldots\}, \quad i = 1, \ldots, m$$

We can define a codistribution $\mathbf{d}\tilde{\mathcal{C}_1}$ as follows:

$$\mathbf{d}\tilde{\mathcal{C}_1} := \operatorname{span}\{\mathbf{d}f \mid f \in \tilde{\mathcal{C}_1}\}$$

It is fairly easy to check that $\ker \mathbf{d}\tilde{\mathcal{C}}(z) = \mathfrak{g} \cdot z$ if and only if $\ker \mathbf{d}\tilde{\mathcal{C}}_1(\gamma) = \{0\}$, where γ is the projection of z.

Now using the fact that $B(\mathbf{d}f, \cdot) = i_{\mathbf{d}f} B = X_f$, where f is any function on T^*Q/G, and B is the Poisson tensor on T^*Q/G, it is easy to conclude that the condition $\ker \mathbf{d}\tilde{\mathcal{C}}_1(\gamma) = \{0\}$ is satisfied iff

$$\operatorname{span}\{X_{f_i}(\gamma), X_{\{\tilde{h}, f_i\}}(\gamma), X_{\{\tilde{h},\{\tilde{h}, f_i\}\}}(\gamma), \ldots\} = T_\gamma(T^*Q/G) \tag{8.1}$$

which is equivalent to:

$$\operatorname{span}\{X_{f_i}(\gamma), [X_{\tilde{f}}, X_{f_i}](\gamma), [X_{\tilde{h}}, [X_{\tilde{h}}, X_{f_i}]](\gamma), \ldots\} = T_\gamma(T^*Q/G) \tag{8.2}$$

This condition is reminiscent of the condition for local accessibility of a control system (see Nijmeijer and van der Schaft [1990]). Comparing the above condition with the condition for local accessibility we see that the above condition is more stringent than local accessibility.

Now we know that γ_e is an equilibrium of $X_{\tilde{h}}$. Let A be the linearization of $X_{\tilde{h}}$ at γ_e, and let $b_i := X_{f_i}(\gamma_e)$, and let $B = [b_1 \ldots b_m]$. An easy calculation shows that

$$[X_{\tilde{h}}, \ldots [X_{\tilde{h}}, X_{f_i}] \ldots](\gamma_e)$$

(k times repeated bracket) is equal to $(-1)^k A^k b_i$. Thus we can conclude that the condition (8.2) is satisfied iff

$$\operatorname{rank}[B\; AB\; A^2B \ldots A^{n-1}B] = n,$$

where n is the dimension of T^*Q/G, i.e., iff the linearization of the reduced system on T^*Q/G is controllable.

Amongst the space of pairs of matrices A, B of appropriate dimension the pairs which are controllable forms an open dense subset. This suggests that the condition that we need for asymptotic stabilizability is not a very stringent one. Indeed, van der Schaft [1986] notes that in general just one dissipation term $-k_i\dot{F}_i$ is enough to assure asymptotic stability. (See also Jonckheere [1981].)

9 Concluding Remarks

In this paper, we have derived feedback laws that asymptotically stabilize relative equilibria of a mechanical system with symmetry. The inputs corresponding to forces in the internal or shape directions are of the proportional-derivative (P-D) form: $u_i = -c_iF_i - k_i\dot{F}_i$. The functions F_i depend only on the internal configuration of the system. (This statement refers to the case in which A_{μ_e} is positive definite. The situation for the case $A_{\mu_e} \not> 0$ is analogous, the only difference being that G_{μ_e} is regarded as the group and Q/G_{μ_e} as the shape space.) The proportional terms $(-c_iF_i)$ modify the potential and convert the equilibrium to a minimum of the (modified) reduced Hamiltonian, thereby stabilizing the equilibrium in the sense of Lyapunov. The derivative terms $(-k_i\dot{F}_i)$, which are linear functions of the velocities

$\dot{q}$, and the forces in the group directions are used to introduce an effective dissipation in the system.

Note that in order to achieve asymptotic stability, we need two conditions: The first involves checking the positive definiteness of the second derivative of the amended potential on an appropriately defined space, and the second is a rank condition on the codistribution $\mathbf{d}\tilde{\mathcal{C}}$. The positive definiteness condition ensures that we can modify the potential using the proportional terms in the feedback law so as to get Lyapunov stability. The rank condition on $\mathbf{d}\tilde{\mathcal{C}}$ ensures that despite the fact that we have only partial actuation, the dissipation introduced by the derivative terms spreads throughout the system, and makes the system asymptotically stable. It is easy to see that if these conditions are satisfied, then they continue to be satisfied even if the system parameters are perturbed. Thus, our asymptotic stabilization scheme is robust with respect to perturbation of system parameters.

Lastly, we recall that the rank condition on $\mathbf{d}\tilde{\mathcal{C}}$ is not stringent, and is likely to be true in most examples, as explained in section 8.

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