Hamiltonian Systems with Symmetry, Coadjoint Orbits and Plasma Physics

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Abstract

The symplectic and Poisson structures on reduced phase spaces are reviewed, including the symplectic structure on coadjoint orbits of a Lie group and the Lie-Poisson structure on the dual of a Lie algebra. These results are applied to plasma physics. We show in three steps how the Maxwell-Vlasov equations for a collisionless plasma can be written in Hamiltonian form relative to a certain Poisson bracket. First, the Poisson-Vlasov equations are shown to be in Hamiltonian form relative to the Lie-Poisson bracket on the dual of the (finite dimensional) Lie algebra of infinitesimal canonical transformations. Then we write Maxwell's equations in Hamiltonian form using the canonical symplectic structure on the phase space of the electromagnetic fields, regarded as a gauge theory. In the last step we couple these two systems via the reduction procedure for interacting systems. We also show that two other standard models in plasma physics, ideal MHD and two-fluid electrodynamics, can be written in Hamiltonian form using similar group theoretic techniques.

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Contents

Introduction

This paper describes some recent work of Morrison [1980] and Marsden and Weinstein [1982a] on a Hamiltonian structure for the Maxwell-Vlasov equations of plasma physics, similar structures for two-fluid electrodynamics found by Spencer and Kaufman [1982] and Spencer [1982], and structures for magnetohydrodynamics (MHD) found by Morrison and Greene [1980], Holm and Kupershmidt [1983], and Marsden, Ratiu, and Weinstein [1983]. These different structures are all consistent: one can pass from the Maxwell-Vlasov bracket to the fluid bracket by a Poisson map constructed in a natural way using the momentum map of a group action, and it is expected that one can pass to MHD by a limiting argument consistent with the brackets in the framework of Weinstein [1983].

We shall include some of the necessary backgound material concerning symplectic and Poisson manifolds, symmetry groups, and momentum mappings. For alternative expositions and additional topics, see Abraham and Marsden [1978], Arnold [1978], Guillemin and Sternberg [1980] and Marsden and Weinstein [1982a]. We have not included in this paper discussions of Clebsch or canonical variables. We refer to Holm and Kupershmidt [1983], Marsden and Weinstein [1982b] and Marsden, Ratiu and Weinstein [1983] for this topic. We also have not included a rigorous function space setting for the results. A third topic not included here is incompressible fluids. For this, we refer to Marsden and Weinstein [1982b] and references therein.

Hamiltonian structures for classical systems are useful for several purposes. As in Arnold's original work [1966a, 1966b, 1969] on the rigid body and fluids, these structures can be used for stability calculations. In doing so, one must take into account the symmetry group and associated conserved quantities or constraints. A general context for such calculations is given in Marsden and Weinstein [1974]. There should be several interesting examples of the same type for the systems considered in this paper.

Another use for Hamiltonian structures is in studying perturbations of a given system. The perturbation may introduce chaotic motions as in Holmes and Marsden [1983]. Alternatively, one may wish to study the averaged effects of high frequency interactions such as ponderomotive forces and guiding center motion (see, for example, Kaufman, McDonald and Omohundro [1982]). These topics are currently under investigation.

A third use for Hamiltonian structures is in understanding the classical-quantum relationship. In fact, noncanonical brackets like ours, but for quantum observables, were already introduced in the context of current algebras by Dashen and Sharp [1968]. The corresponding semidirect product groups, as in our work in §9, were identified by Goldin [1971], and their representations have been investigated recently by Goldin, Menikoff, and Sharp [1980]. Dashen and Sharp remark that << Physical theories which are written in terms of variables that are not canonical sometimes lack a mathematical elegance possessed by canonical theories. However, physics, rather than the elegance of canonical variables is the final test.>> We believe that the developments of the last few years display noncanonical variables as being just as elegant as canonical ones.

A Poisson bracket for special relativistic plasmas which agrees with ours in the nonrelativistic limit has been recently obtained by Bialynicki-Birula and Hubbard [1982]. They also point out that the brackets for electrodynamics go back to Pauli [1933] and Born and Infeld [1935], and that Pauli also gives brackets for interacting discrete particles and electromagnetic fields. A canonical formulation of relativistic hydrodynamics was given by Bialynicki-Birula and Iwiński [1973]. In none of these references are the Poisson brackets derived from canonical brackets as we do.

1 Momentum Mappings

Let (P, ω) be a symplectic manifold (possibly infinite dimensional), where ω is closed and weakly nondegenerate, *i.e.,* the map $\omega^b : TP \to T^*P$ defined by $\omega^b(v) \cdot w = \omega(v, w), v, w \in TP$, is one-to-one. Let G be a Lie group (possibly infinite dimensional) which acts on P by maps $\Phi_g : P \to P$ which are symplectomorphisms; *i.e.,* $\Phi_g^* \omega = \omega$ for all $g \in G$. Let $\mathfrak{g}$ be the Lie algebra of G. Each $\xi \in \mathfrak{g}$ induces a one-parameter subgroup $g_t = \exp(t\xi) \in G$. The infinitesimal generator ξ_P of the action Φ, corresponding to $\xi \in \mathfrak{g}$ is defined by

$$\xi_P(x) = \frac{d}{dt} \Phi_{\exp t\xi}(x) \bigg|_{t=0}, \quad x \in P.$$

Because Φ is symplectic Φ_{g_t} is a one-parameter family of canonical transformations, and so the vector field ξ_P is locally Hamiltonian. Suppose in fact that it has a global Hamiltonian function which we denote $\widehat{J}(\xi)$ (defined in this way up to a constant), and we write $\xi_P = X_{\widehat{J}(\xi)}$. We thus obtain a mapping $\widehat{J} : \mathfrak{g} \to C^\infty(P)$, such that

$$X_{\widehat{J}(\xi)}(x) = \xi_P(x) = \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp t\xi}(x)$$

or

$$d(\widehat{J}(\xi)) = i_{\xi_P}\omega.$$

The significance of this definition is the following:

Proposition 1.1. *Let $H : P \to \mathbb{R}$ be G-invariant, that is $H(\Phi_g(x)) = H(x)$ for all $x \in P, g \in G$. Then $\widehat{J}(\xi)$ is a constant of the motion for the dynamics generated by H.*

Proof

$$\begin{aligned}\{H, \widehat{J}(\xi)\}(x) &= dH(x) \cdot X_{\widehat{J}(\xi)}(x) = dH(x) \cdot \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp t\xi}(x) \\ &= \left.\frac{d}{dt}\right|_{t=0} (H(\Phi_{\exp t\xi}(x)) = \left.\frac{d}{dt}\right|_{t=0} H(x) = 0.\end{aligned}$$

QED

Since the condition $d(\widehat{J}(\xi)) = i_{\xi_P}\omega$ is linear in ξ, there is often a map $J : P \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$, such that $J(x) \cdot \xi = \widehat{J}(\xi)(x)$. The map J is called the *momentum mapping* for the action Φ.

Remarks

1. The existence of a symplectic action does not always imply the existence of a momentum mapping; not every locally Hamiltonian vector field is globally Hamiltonian.

2. Care is needed in the topology used to construct $\mathfrak{g}^*$ in the infinite dimensional case. Usually $\mathfrak{g}^*$ is taken to be a convenient function space which is paired with $\mathfrak{g}$, and is not necessarily the strict dual.

If $\omega = d\theta$ is an exact symplectic structure, and the action Φ preserves the one form θ, *i.e.*, $\Phi_g^*\theta = \theta$, for all $g \in G$, then there exists a momentum map J given by $J(x) \cdot \xi = \theta(\xi_P)(x)$. For example, this is the case for *extended point transformations.* Indeed, let G act on a manifold Q by transformations $\Phi_g : Q \to Q$, and define the lifted action to the cotangent bundle $(\Phi_g)_* : T^*Q \to T^*Q$, by pushing forward one forms: $(\Phi_g)_*(\alpha) \cdot v = \alpha(T\Phi_g^{-1} \cdot v)$ where $\alpha \in T_q^*Q$ and $v \in T_{\Phi_g(q)}Q$. The lifted action preserves the canonical one-form on T^*Q and the momentum mapping for this (lifted) action is given by

$$J(\alpha_q) \cdot \xi = \alpha_q \cdot \xi_Q(q), \quad \alpha_q \in T_q^*Q.$$

where ξ_Q is the infinitesimal generator of the action Φ on Q.

A basic fact about momentum mappings follows immediately from Proposition 2.1.

Corollary 1.2. *(Geometric Noether theorem): Let J be a momentum mapping for the symplectic action Φ of G on (P,ω). Then J is a constant of motion for any G-invariant Hamiltonian H; that is, if F_t is the flow of H, then*

$$J(F_t(x)) = J(x), \quad \textit{for all} \quad x \in P.$$

A *Hamiltonian action* is a symplectic action with an Ad*-equivariant momentum map J, that is

$$J(\Phi_g(x)) = \mathrm{Ad}^*_{g^{-1}}(J(x)) \quad \text{for all} \quad x \in P, g \in G,$$

where $\mathrm{Ad}^*_{g^{-1}} = (\mathrm{Ad}_{g^{-1}})^*$ is the coadjoint action of G on $\mathfrak{g}^*$. For connected groups the global condition of equivariance is equivalent to the local condition that the map $\widehat{J} : \mathfrak{g} \to C^\infty(P)$ defined by $\widehat{J}(\xi)(x) = J(x) \cdot \xi$ be a Lie algebra homomorphism with respect to the Poisson bracket on $C^\infty(P)$; that is

$$\widehat{J}([\xi,\eta]) = \{\widehat{J}(\xi), \widehat{J}(\eta)\} \quad \text{for all} \quad \xi, \eta \in \mathfrak{g}.$$

For example, extended point transformations automatically give Hamiltonian actions.

Proposition 1.3. *Let G act transitively on (P,ω) by a Hamiltonian action. Then $J(P) \subset \mathfrak{g}^*$ is a coadjoint orbit and J is a covering map from P onto $J(P)$.*

Proof Since Φ is transitive, for every $x, y \in P$ there is a $g \in G$ such that $\Phi_g(x) = y$. Thus $J(P) = \{J(x) \mid x \in P\} = \{J(\Phi_g(x)) \mid g \in G\}$ and Ad*-equivariance implies that $J(P) = \{\mathrm{Ad}^*_{g^{-1}}(J(x_0)) \mid g \in G\}$ is the coadjoint orbit through $J(x_0)$, where $x_0 \in P$ arbitrary. QED

If G is not transitive, $J(P)$ will be a union of coadjoint orbits. For example if $P = T^*G$ and G acts by left translation, then J is surjective (see §3 below).

Remark The material in this section appears in local form in Lie [1890]. The global results are due to Kostant [1970] and Souriau [1970].

2 Reduction

Let Φ be a Hamiltonian action of G on (P,ω) and fix a point $\mu \in \mathfrak{g}^*$. By Proposition 2.1, $J^{-1}(\mu) = \{x \in P \mid J(x) = \mu\}$ is invariant under the motion of any G-invariant Hamiltonian, *i.e.,* the motion stays in $J^{-1}(\mu)$. For systems with symmetry, therefore, $J^{-1}(\mu) \subset P$ is an invariant set for the dynamics. If $\mu \in \mathfrak{g}^*$ is a regular value of $J : P \to \mathfrak{g}^*$, then $J^{-1}(\mu)$ is a submanifold of P (with codim $J^{-1}(\mu) = \dim \mathfrak{g}$, in finite dimensions).

The phase space can be further reduced if we divide $J^{-1}(\mu)$ by as much of G as leaves $J^{-1}(\mu)$ invariant, that is, by the coadjoint isotropy group $\{g \in G \mid \mathrm{Ad}^*_{g^{-1}}(\mu) = \mu\} = G_\mu$. The quotient space $J^{-1}(\mu)/G_\mu$ is a smooth manifold provided that μ is a regular value of J, and that G_μ acts properly and freely on $J^{-1}(\mu)$. We shall assume this here. The manifold $P_\mu = J^{-1}(\mu)/G_\mu$, called the *reduced phase space,* turns out to be symplectic.

Theorem 2.1. *(Marsden and Weinstein [1974] and Meyer [1973]). There is a unique symplectic structure ω_μ on P_μ consistent with the structure ω on P; that is*

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega,$$

where $i_\mu : J^{-1}(\mu) \to P$ is the inclusion and $\pi_\mu : J^{-1}(\mu) \to P_\mu = J^{-1}(\mu)/G_\mu$ is the projection:

$$\begin{array}{ccc} J^{-1}(\mu) & \xrightarrow{\ i_\mu\ } & (P, \omega) \\ {\scriptstyle \pi_\mu}\big\downarrow & & \\ (P_\mu, \omega_\mu) & & \end{array}$$

The symplectic structure ω_μ on P_μ is constructed in the following way. Let $s \in P_\mu$ and $v, w \in T_s P_\mu$ be two tangent vectors. To calculate $\omega_\mu(v, w)$ we select any $x \in \pi_\mu^{-1}(s) \subset J^{-1}(\mu)$; then there exist tangent vectors $v', w' \in T_x(J^{-1}(\mu))$ such that $v = T_x\pi_\mu(v')$ and $w = T_x\pi_\mu(w')$. We set $\omega_\mu(s)(v, w) = \omega(x)(v', w')$, where $\omega(x)$ is identified with the restriction of $\omega(x)$ to $T_x(J^{-1}(\mu))$. One can check that this gives a well defined symplectic form ω_μ with the desired property.

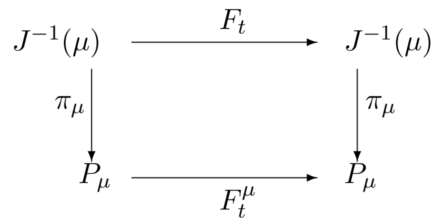
Description of the reduced form in terms of Poisson brackets

Since ω_μ is a symplectic structure on P_μ, it defines a Poisson bracket $\{,\}_\mu$ on the space of C^∞ functions $C^\infty(P_\mu)$. Let $F, G \in C^\infty(P_\mu)$; we want to compute their Poisson bracket $\{F, G\}_\mu$. First define $\widehat{F}, \widehat{G}$ on $J^{-1}(\mu)$, which are constant on G_μ-orbits. Now smoothly extend $\widehat{F}, \widehat{G}$, to functions $\widehat{F}_e, \widehat{G}_e$ on all of P. The Poisson bracket $\{\widehat{F}_e, \widehat{G}_e\} \in C^\infty(P)$ is defined in terms of ω; restrict it to $J^{-1}(\mu)$. One finds that this restriction $\{\widehat{F}_e, \widehat{G}_e\}|_{J^{-1}(\mu)}$ is independent of the extension used, so that it may be written as $\{\widehat{F}, \widehat{G}\}$, and that furthermore as Φ_g is symplectic, $\{\widehat{F}, \widehat{G}\}$ is constant along orbits. This implies that $\{\widehat{F}, \widehat{G}\}$ defines a function $\{F, G\}_\mu$ on the reduced phase space P_μ such that $\{\widehat{F}, \widehat{G}\} = \{F, G\}_\mu \circ \pi_\mu$. This defines a Poisson bracket on P_μ, which may be shown to be that associated with ω_μ.

Dynamics on the reduced phase space P_μ

Let $H : P \to \mathbb{R}$ be a G-invariant Hamiltonian on P. There is a well defined Hamiltonian $H_\mu : P_\mu \to \mathbb{R}$ on P_μ such that $H_\mu \circ \pi_\mu = H \circ i_\mu$. As real valued functions, H and H_μ define flows F_t and F_t^μ on $J^{-1}(\mu)$ and P_μ, respectively. From this and the construction of ω_μ, we get the following.

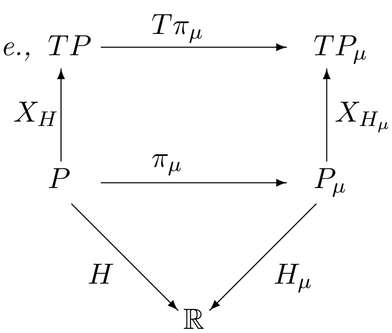
Theorem 2.2. *The projection onto P_μ of the integral curves under H in $J^{-1}(\mu)$ are the integral curves for the dynamics in P_μ determined by H_μ. Thus we have $\pi_\mu \circ F_t = F_t^\mu \circ \pi_\mu$, i.e.,*



commutes.

In terms of the corresponding Hamiltonian vector fields X_H on P and X_{H_μ} on P_μ, this says that X_H and X_{H_μ} are π_μ-related:

$T\pi_\mu \circ X_H = X_{H_\mu} \circ \pi_\mu$, *i.e.*,



Equivalently, in terms of Poisson brackets we can write

$$\{F, G\}_\mu \circ \pi_\mu = \{\widehat{F}_e, \widehat{G}_e\} \circ i_\mu,$$

using the notation above in our discussion of the description of the reduced form in Poisson brackets.

Corollary 2.3. *The equations of motion $\dot{F} = \{F, H\}$ on P reduce to the equations of motion $\dot{F}_\mu = \{F_\mu, H_\mu\}_\mu$ on P_μ.*

3 Coadjoint Orbits

We shall now indicate how coadjoint orbits are a special case of reduced phase spaces; their symplectic structure coincides with the classical Lie-Kirillov-Kostant-Souriau form. The Lie group G acts on itself by left translations $L_g : G \to G, L_g(h) = gh, g, h \in G$. The induced lifted action on T^*G is Hamiltonian and has the momentum map

$$J : T^*G \to \mathfrak{g}^* : \eta_g \mapsto (T_e R_g)^* \eta_g$$

for which each $\mu \in \mathfrak{g}^*$ is a regular value. Because each fiber $T_g^*G \subset T^*G$ is (right) translated onto $\mathfrak{g}^*$ by J, for a given $\mu \in \mathfrak{g}^*$ there will be exactly one point $\eta_g \in T_g^*G$ that maps to μ. Thus

$$J^{-1}(\mu) = \{\eta_g \in T^*G \mid J(\eta_g) = (T_e R_g)^* \eta_g = \mu\}.$$

Then, defining the right-invariant one-form η_μ on G according to $\eta_\mu(g) = \eta_g$, where $\eta_\mu(e) = \eta_e = \mu$, it follows that

$$J^{-1}(\mu) = \{(g, \eta_\mu(g)) \mid g \in G\}$$

is the graph of η_μ. It is easy to see that the isotropy group G_μ of the coadjoint action is

$$G_\mu = \{g \in G \mid L_g^* \eta_\mu = \eta_\mu\},$$

so G_μ acts on $J^{-1}(\mu)$ by left translation of the base points. Now $J^{-1}(\mu) \cong G$ according to $(g, \eta_\mu(g)) \mapsto g^{-1}$. Further, denoting by 0_μ the orbit of μ under the coadjoint action, $G/G_\mu \cong 0_\mu$ via $p_\mu(g) \mapsto \mathrm{Ad}^*_{g^{-1}}(\mu)$ where $p_\mu : G \to G/G_\mu$ is the canonical projection. Hence

$$P_\mu \cong 0_\mu \quad \text{via} \quad \pi_\mu(\eta_\mu(g)) \mapsto \mathrm{Ad}^*_g(\mu)$$

for all $g \in G$. That is, the reduced phase space P_μ can be identified with the coadjoint orbit through μ. Thus, by Theorem 3.1, the coadjoint orbits are symplectic manifolds. This is the statement of the Kirillov-Kostant-Souriau theorem. In order to compute the symplectic structure ω_μ on $P_\mu \cong 0_\mu$, note that tangent vectors to the coadjoint orbit 0_μ at $\beta \in 0_\mu$ are given by $(\mathrm{ad}\ \xi)^*\beta$ for $\xi \in \mathfrak{g}$, where $(\mathrm{ad}\,\xi)^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of the linear map $(\mathrm{ad}\,\xi) : \mathfrak{g} \to \mathfrak{g}$ defined by $(\mathrm{ad}\ \xi)(\eta) = [\xi, \eta], \eta \in \mathfrak{g}$, where $[,]$ is the Lie algebra bracket on $\mathfrak{g}$. If one traces through the definitions, one finds that (note the minus sign)

$$\omega_\mu(\beta)((\mathrm{ad}\,\xi)^*\beta, (\mathrm{ad}\,\eta)^*\beta) = -\langle \beta, [\xi, \eta]\rangle,$$

where $\xi, \eta \in \mathfrak{g}, \beta \in 0_\mu$, and $\langle , \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. This is the canonical Lie-Kirillov-Kostant-Souriau symplectic structure on coadjoint orbits. If one begins with the right action of G on G then one arrives at the same formula with $<< - >>$ replaced by $<< + >>$. Finally, we remark that the reduced space P_μ is naturally identifiable with $J^{-1}(0_\mu)/G$; see Marle [1976].

4 Poisson Brackets on Duals of Lie Algebras

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual space to $\mathfrak{g}$. The pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ is denoted by $\langle , \rangle$. We wish to define a bracket $\{F, G\}$ for functions $F, G : \mathfrak{g}^* \to \mathbb{R}$ which was discovered by Lie [1890] and rediscovered by Berezin [1967] and others. There are three ways to do this:

METHOD 1. (Direct). We just write down the formula $\{F, G\}$.

The formula depends on the notation of $<<$ functional $>>$ derivatives defined as follows. For $F : \mathfrak{g}^* \to \mathbb{R}$, define $\delta F/\delta\mu \in \mathfrak{g}$, where $\mu \in \mathfrak{g}^*$, by

$$DF(\mu) \cdot \nu = \left\langle \nu, \frac{\delta F}{\delta \mu} \right\rangle \quad \text{for all} \quad \nu \in \mathfrak{g}^*,$$

i.e., we formally identify $\mathfrak{g}^{**}$ with $\mathfrak{g}$ so that $DF(\mu) \in \mathfrak{g}^{**}$ becomes an element of $\mathfrak{g}$ (one has to take due precautions in infinite dimensions with the meaning of dual spaces). Then the bracket is defined by

$$\{F, G\}(\mu) = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle,$$

where $[,]$ is the Lie algebra bracket on $\mathfrak{g}$. This bracket, called the $<< - >>$ *Lie-Poisson bracket* defines a *Poisson structure* on $\mathfrak{g}^*$; that is $\{F, G\}$ is bilinear and antisymmetric, satisfies Jacobi's identity, and is a derivation in each argument.

METHOD 2. (Restriction). We can use the Lie-Kirillov-Kostant-Souriau symplectic forms on coadjoint orbits to define the bracket on $\mathfrak{g}^*$. From our discussion on coadjoint orbits, it follows that $\mathfrak{g}^*$ is a disjoint union of symplectic manifolds. For $F, G : \mathfrak{g}^* \to \mathbb{R}$ a Poisson bracket is thus defined by

$$\{F, G\}(\mu) = \{F|0_\mu, G|0_\mu\}_\mu(\mu)$$

where $\mu \in \mathfrak{g}^*, 0_\mu$ is the coadjoint orbit through $\mu, F \mid 0_\mu$ is the *restriction* of F to 0_μ, and $\{,\}_\mu$ is the Poisson bracket on 0_μ defined by ω_μ.

The bracket defined by Method 2 is clearly degenerate, (functions constant on coadjoint orbits have zero bracket with every function); however, it determines a symplectic $<<$ foliation $>>$: on each leaf $\{,\}_\mu$ is nondegenerate. The leaves are just the coadjoint orbits $(0_\mu, \omega_\mu), \mu \in \mathfrak{g}^*$.

METHOD 3. (Extension). Given functions $F, G : \mathfrak{g}^* \to \mathbb{R}$, *extend* them to maps $\widehat{F}, \widehat{G} : T^*G \to \mathbb{R}$ by left invariance, regarding $\mathfrak{g}^*$ as T^*_eG. Then, using the canonical Poisson bracket $\{,\}$ on T^*G, form $\{\widehat{F}, \widehat{G}\}$. Finally restrict $\{\widehat{F}, \widehat{G}\}$ to $T^*_eG = \mathfrak{g}^*$:

$$\{F, G\} = \{\widehat{F}, \widehat{G}\}|_{\mathfrak{g}^*}$$

Theorem 4.1. *The formulas in methods 1, 2 and 3 define the same Poisson structure on* $\mathfrak{g}^*$*.*

This result is implicit in the literature cited in the introduction and may be readily proved by the reader.

If we reduced T^*G using *right* rather than *left* translations, we would have arrived at the $<< + >>$ *Lie Poisson bracket*:

$$\{F, G\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle.$$

If there is danger of confusion, we denote the respective brackets by $\{F, G\}_-$ and $\{F, G\}_+$.

The Lie-Poisson evolution equations for a function $H_- \in C^\infty(\mathfrak{g}^*)$ (for the $<< - >>$ Lie Poisson structure, say) are determined by

$$\dot{F} = \{F, H_-\}_-.$$

These equations are equivalent to the evolution equations

$$\dot{\mu} = X_{H_-}(\mu)$$

on $\mathfrak{g}^*$. To determine X_{H_-}, write

$$\dot{F}(\mu) = DF(\mu) \cdot \dot{\mu} = \left\langle \dot{\mu}, \frac{\delta F}{\delta \mu} \right\rangle$$

and

$$\{F, H_-\}_-(\mu) = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H_-}{\delta \mu} \right] \right\rangle = \left\langle \mu, \mathrm{ad}_{\frac{\delta H_-}{\delta \mu}} \frac{\delta F}{\delta \mu} \right\rangle$$

where $\mathrm{ad}_\xi \eta = [\xi, \eta]$ is the adjoint operator. Thus we get

$$X_{H_-}(\mu) = \mathrm{ad}^*_{\frac{\delta H_-}{\delta \mu}} \mu$$

where $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is the transpose of ad_ξ relative to the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ (depending on the choice of function spaces, transposes may be taken in the sense of unbounded operators, as in Chernoff and Marsden [1974]).

We summarize the calculations just given as follows:

Theorem 4.2. *If $H : T^*G \to \mathbb{R}$ is a left invariant Hamiltonian and $H_- : \mathfrak{g}^* \to \mathbb{R}$ is the induced function on $\mathfrak{g}^*$, then the Hamiltonian evolution equations for H on T^*G induce the Lie-Poisson evolution equations*

$$\dot{F} = \{F, H_-\}_- \quad \text{on} \quad \mathfrak{g}^*$$

which are equivalent to

$$\dot{\mu} = X_{H_-}(\mu) = \mathrm{ad}^*_{\frac{\delta H_-}{\delta \mu}} \mu.$$

The coadjoint orbits in $\mathfrak{g}^*$ are invariant under this evolution. (For right invariant systems, interchange $+$ and $-$).

The notion of momentum map carries over to Poisson manifolds and in particular to duals of Lie algebras as follows. A *Poisson manifold* is a manifold P together with a Lie algebra structure $\{f, g\}$ on $C^\infty(P)$ that is a derivation in each of f and g. If P_1 and P_2 are Poisson manifolds with corresponding brackets $\{,\}_1$ and $\{,\}_2$, a map $\phi : P_1 \to P_2$ is called a *Poisson map* if $\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$ for all $f, g \in C^\infty(P_2)$. Now let a Lie group G act on a Poisson manifold P by Poisson maps. A *Hamiltonian* for this action is a homomorphism $\widehat{J} : \mathfrak{g} \to C^\infty(P)$ such that $X_{\widehat{J}(\xi)} = \xi_P$ for all $\xi \in \mathfrak{g}$. The *momentum map* is $J : P \to \mathfrak{g}^*$, $\langle J(x), \xi \rangle = \widehat{J}(\xi)(x)$. Conservation properties are similar to the symplectic case. The quotient space P/G is the reduced Poisson manifold with brackets defined as in the case of $\mathfrak{g}^* \cong T^*G/G$. Since $P_\mu \approx J^{-1}(0_\mu)/G$, one sees that the symplectic leaves of P/G are the reduced manifolds.

The next result will give a convenient way to generate Poisson maps. In it we distinguish left and right actions and correspondingly $\mathfrak{g}^*$ with the $\pm$ Lie-Poisson bracket by writing $\mathfrak{g}^*_-$ and $\mathfrak{g}^*_+$ in the respective cases. This left-right duality is important and will be elaborated below. For example, the rigid body and heavy top are *left* invariant systems, whereas the continuum systems treated here are *right* invariant. As D. Fried pointed out to us, some interesting systems such as the classical gravitating fluids (of Poincaré, Jacobi and Kirchhoff) have invariance on each side.

Proposition 4.3. *Let $J_L : P \to \mathfrak{g}^*$ be a momentum map for a left action of G on P. Then*

$$J_L : P \to \mathfrak{g}^*_+ \text{ is a Poisson map.}$$

Likewise, if J_R is the momentum map for a right action, then

$$J_R : P \to \mathfrak{g}^*_- \text{ is a Poisson map.}$$

Proof By definition of the Lie-Poisson bracket,

$$\begin{aligned} \{F, G\}_+(\mu) &= \left\langle J(x), \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle \\ &= \widehat{J}\left(\left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right)(x), \end{aligned}$$

where $\mu = J(x)$. Since $\widehat{J}$ is a Lie algebra homomorphism,

$$\widehat{J}\left(\left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right)(x) = \left\{\widehat{J}\left(\frac{\delta F}{\delta \mu}\right), \widehat{J}\left(\frac{\delta G}{\delta \mu}\right)\right\}(x).$$

The proof will be complete if we can show that

$$d\left(\widehat{J}\left(\frac{\delta F}{\delta \mu}\right)\right) = d(F \circ J)$$

where $\delta F/\delta \mu$ is regarded as a constant element of $\mathfrak{g}$ evaluated at $\mu = J(x)$. Indeed, we have

$$\begin{aligned} d(F \circ J)(x) \cdot v_x &= dF(\mu) \cdot (dJ(x) \cdot v_x) \\ &= \left\langle dJ(x) \cdot v_x, \frac{\delta F}{\delta \mu}\right\rangle \end{aligned}$$

for $v_x \in T_x P$. Also,

$$\begin{aligned} d\left(\widehat{J}\left(\frac{\delta F}{\delta \mu}\right)\right) \cdot v_x &= d\left(\left\langle J(x), \frac{\delta F}{\delta \mu}\right\rangle\right) \cdot v_x \\ &= \left\langle dJ(x) \cdot v_x, \frac{\delta F}{\delta \mu}\right\rangle \end{aligned}$$

since $\delta F/\delta \mu$ is regarded as a constant element of $\mathfrak{g}$. QED

Note that Ad*-equivariant momentum maps in the usual sense for actions on symplectic manifolds, are also momentum maps in the Poisson sense. In what follows, most of the momentum maps we consider are standard ones from symplectic geometry (see Abraham and Marsden [1978], §4.2).

A consequence of the formula

$$d\left(\widehat{J}\left(\frac{\delta F}{\delta \mu}\right)\right) = d(F \circ J)$$

proved above is the following fact about collective Hamiltonians. (cf. Marle [1976] and Guillemin and Sternberg [1980]). (This holds for momentum maps associated with either left or right actions).

Corollary 4.4. *Let $F \in C^\infty(\mathfrak{g}^*)$. Then*

$$X_{F\circ J}(x) = \left(\frac{\delta F}{\delta \mu}\right)_P (x)$$

where $\delta F/\delta\mu$ is evaluated at $\mu = J(x)$.

It is convenient for later purpose to be more explicit about the map by which the reduced space is identified with $\mathfrak{g}^*$, and the corresponding left-right duality. If L_g and R_g denote left and right translation by g in G, these actions can be lifted to left and right actions on T^*G as follows. Define

$$L : G \times T^*G \to T^*G, L(g, \alpha_h) = (T_{gh}L_{g^{-1}})^*\alpha_h$$

and

$$R : T^*G \times G \to T^*G, R(\alpha_h, g) = (T_{hg}R_{g^{-1}})^*\alpha_h.$$

These two commuting actions have the (Ad*-equivariant) momentum maps

$$J_L : T^*G \to \mathfrak{g}^*, J_L(\alpha_g) = (T_e R_g)^*(\alpha_g) \;\text{ for }\; L$$

and

$$J_R : T^*G \to \mathfrak{g}^*, J_R(\alpha_g) = (T_e L_g)^*(\alpha_g) \;\text{ for }\; R.$$

Thus by 5.3,

$$J_L : T^*G \to \mathfrak{g}^*_+$$

and

$$J_R : T^*G \to \mathfrak{g}^*_-$$

are Poisson maps. Moreover J_L is R-invariant and J_R is L-invariant so that these maps induce Poisson manifold diffeomorphisms on the corresponding quotient spaces

$$\bar{J}_L : R\backslash T^*G \to \mathfrak{g}^*_+$$
$$\bar{J}_R : T^*G/L \to \mathfrak{g}^*_-$$

with inverses $J_L^{-1}(\mu) = [\mu]_R, J_R^{-1}(\mu) = [\mu]_L$. ($[\alpha_g]_R$ and $[\alpha_g]_L$ denote equivalence classes in $R\backslash T^*G, T^*G/L$). Denote the canonical projections by $\pi_L : T^*G/L$, and $\pi_R : T^*G \to R\backslash T^*G$.

The symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits. In $\mathfrak{g}^*_\pm$ the orbit $0^\pm_\mu$ through $\mu \in \mathfrak{g}^*$ has the Lie-Berezin-Kirillov-Kostant-Souriau symplectic form

$$\omega_\mu(\nu)((\operatorname{ad}\xi)^*\nu, (\operatorname{ad}\eta)^*\nu) = \pm\langle \nu, [\xi, \eta]\rangle$$

where $\nu \in 0^\pm_\mu$ and $\xi, \eta \in \mathfrak{g}$. Hence the symplectic leaves of T^*G/L are of the form $\bar{J}_R^{-1}(0^-_\mu)$, and those of $R\backslash T^*G$ are of the form $\bar{J}_L^{-1}(0^+_\mu)$. But it is easily seen that

$$\bar{J}_R^{-1}(0^-_\mu) = \pi_L J_L^{-1}(0^+_\mu), \quad \bar{J}_L^{-1}(0^+_\mu) = \pi_R J_R^{-1}(0^-_\mu)$$

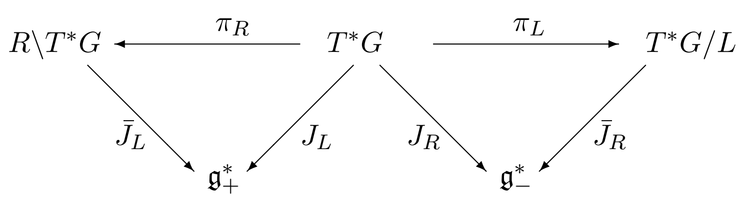
and hence the symplectic leaves in T^*G/L and $R\backslash T^*G$ are the reduced manifolds $\pi_L J_L^{-1}(0^+_\mu)$ and $\pi_R J_R^{-1}(0^-_\mu)$, where $0^\pm_\mu$ has the + or − Lie-Poisson symplectic form. We summarize these results in the following theorem.

Theorem 4.5. *The Ad*-equivariant momentum maps J_L and J_R for the actions L and R of G on T^*G induce Poisson manifold diffeomorphisms*

$$\bar{J}_L : R\backslash T^*G \to \mathfrak{g}^*_+ \ \ \textit{and} \ \ \bar{J}_R : T^*G/L \to \mathfrak{g}^*_-$$

where $\mathfrak{g}^*_\pm$ is endowed with the $\pm$ Lie-Poisson bracket. The symplectic leaves in the quotient spaces T^*G/L and $R\backslash T^*G$ are the reduced manifolds $J_L^{-1}(0^+_\mu)/L$, and $R\backslash J_R^{-1}(0^-_\mu)$, where $\mu \in \mathfrak{g}^*$ and $0^\pm_\mu$ is the coadjoint orbit of G in $\mathfrak{g}^*$ through μ with the $\pm$ Lie-Poisson symplectic form.

In particular, note that the actions L and R form a *dual pair* in the sense that the reduced spaces for one action are coadjoint orbits for the other, and we get the diagram



We can reformulate Theorem 5.2 in these terms as follows. Let $H : T^*G \to \mathbb{R}$ be a Hamiltonian invariant under the lifted action L on T^*G of the left translation on G. Then H induces a smooth mapping $H_L : T^*G/L \to \mathbb{R}$, and hence the function $H_- = H_L \circ \bar{J}_R^{-1} : \mathfrak{g}^*_- \to \mathbb{R}$ defines Lie-Poisson equations on $\mathfrak{g}^*$. Since Lie-Poisson equations have trajectories which remain in the coadjoint orbit of their initial conditions, H_L when restricted to the reduced manifolds $J_L^{-1}(0^+_\mu)/L$ where $\mu \in \mathfrak{g}^*$, define Hamiltonian systems on these manifolds. Moreover, if F_t denotes the flow of X_H and G^-_t is the flow of the Lie-Poisson equation defined by H_-, we have $G^-_t \circ \bar{J}_R \circ \pi_L = \bar{J}_R \circ \pi_L \circ F_t$. We summarize these results in the following theorem.

Theorem 4.6. *A left invariant Hamiltonian $H : T^*G \to \mathbb{R}$ canonically induces Lie-Poisson equations on $\mathfrak{g}^*_-$ defined by the Hamiltonian function $H_- = H_L \circ \bar{J}_R^{-1}$*

where $H_L \circ \pi_L = H$. If F_t denotes the flow of X_H and G_t^- is the flow of the Lie-Poisson equations for H_- on $\mathfrak{g}_-^*$, then $G_t^- \circ \bar{J}_R \circ \pi_L = \bar{J}_R \circ \pi_L \circ F_t$. The Hamiltonian $H_L | J_L^{-1}(0_\mu^+)/L$ where $\mu \in \mathfrak{g}^*$, induces a Hamiltonian system on the reduced manifold $J_L^{-1}(0_\mu^+)/L$. The same result holds if $<< left >>$ and $<< right >>$, and $<< - >>$ and $<< + >>$ are interchanged.

5 Plasma Physics

In this section we apply the theory of Hamiltonian systems with symmetry to plasma physics. In subsequent sections, we shall derive a Hamiltonian formulation, relative to certain Poisson brackets, for the equations of motion in three different models for plasma physics: 1) the Maxwell-Vlasov equations, 2) compressible fluids and MHD (magnetohydrodynamics) and 3) two-fluid plasma dynamics.

To motivate our derivation of the Hamiltonian structure for the equations of plasma physics, we first discuss these equations here and provide a heuristic derivation rather than a detailed treatment. The reader may wish to consult a text on the subject, such as Chen [1978] or Davidson [1972].

As a gas of electrically neutral molecules is heated, the molecules begin to dissociate into atoms when the thermal energy exceeds the molecular binding energy. Further heating produces dissociation of the atoms themselves, called ionization, when even the binding energy of electrons to nuclei is exceeded by particle thermal energies. The end product is called a (completely ionized) *plasma*. Thus a plasma is a collection of various species, labeled by s, of charged particles with, say, mass m_s and charge q_s.

According to the Lorentz force law of electrodynamics, the position x_i and velocity v_i of the i-th charged particle of species s, are determined by

$$\begin{aligned} \dot{x}_i &= v_i \\ m_s \ddot{x}_i &= q_s(E + v_i \times B) \end{aligned}$$

where E and B are the electric and magnetic fields due to the positions and motions of the other particles, possibly augmented by external fields. The particle motions both determine and are determined by the electromagnetic fields. There are equations for each of the enormous number of particles in the plasma, and the system is coupled through the Maxwell equations.

For notational simplicity only, we consider a plasma consisting only of one species of particles with charge q and mass m moving in Euclidean space $\mathbb{R}^3$ with positions x and velocities v. The plasma particle's motion evolves with time in phase space (x, v), which we denote by Ω. Since the number of particles is large, it is useful to approximate their positions and velocities by a density on phase space which may be a smooth function. Let $f(x, v, t)$ be the plasma density at time t and denote by $u = (\dot{x}, \dot{v})$ the phase $<<$ velocity $>>$ field of the plasma. Let Ω_0 be a fixed subregion of Ω, with boundary $\partial\Omega_0$, and let $m(\Omega_0, t)$ be the mass in Ω_0. Then conservation of mass (and sufficient smoothness) leads in standard fashion to the *equation of continuity*:

$$\frac{\partial f}{\partial t}(x,v,t)+\mathrm{div}_{(x,v)}(fu)=0.$$

Rewriting this with $u=(\dot{x},\dot{v})$, we get

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x}(f\dot{x})+\frac{\partial}{\partial v}(f\dot{v})=0$$

or

$$\frac{\partial f}{\partial t}+\dot{x}\frac{\partial f}{\partial x}+\dot{v}\frac{\partial f}{\partial v}+f\left(\frac{\partial \dot{x}}{\partial x}+\frac{\partial \dot{v}}{\partial v}\right)=0$$

i.e.,

$$\frac{\partial f}{\partial t}+\dot{x}\frac{\partial f}{\partial x}+\dot{v}\frac{\partial f}{\partial v}+f\cdot\mathrm{div}_{(x,v)}u=0.$$

Assuming the plasma to be << incompressible >> in phase space *i.e.*, $\mathrm{div}_{(x,v)}u=0$, we obtain in the famous *Boltzmann equation* of kinetic theory:

$$\frac{\partial f}{\partial t}(x,v,t)+\dot{x}\frac{\partial f}{\partial x}(x,v,t)+\dot{v}\frac{\partial f}{\partial v}(x,v,t)=0.$$

The incompressibility condition $\mathrm{div}_{(x,v)}u=0$ is just the Liouville theorem which states that phase space volumes are preserved and which applies to particles moving under the influence of an electromagnetic field. Thus we find that it is a condition appropriate for a *collisionless* plasma. Inserting the Lorentz force law

$$m\dot{v}=q(E+v\times B),$$

we get the (collisionless) *Vlasov equation*

$$\frac{\partial f}{\partial t}(x,v,t)+v\frac{\partial f}{\partial x}(x,v,t)+\frac{q}{m}(E+v\times B)\frac{\partial f}{\partial v}(x,v,t)=0.$$

For the Maxwell-Vlasov system, the dynamical variables are $f(x,v,t),E(x,t)$ and $B(x,t)$, which satisfy a system of coupled nonlinear evolution equations, with initial conditions on f,E and B. In appropriate units, the *Maxwell-Vlasov* equations are:

$$\frac{\partial f}{\partial t}+v\cdot\frac{\partial f}{\partial x}+\frac{q}{m}(E+v\times B)\cdot\frac{\partial f}{\partial v}=0 \tag{5.1}$$

$$\frac{\partial B}{\partial t}=-\mathrm{curl}\,E \tag{5.2}$$

$$\frac{\partial E}{\partial t}=\mathrm{curl}\,B-J_f,\quad \text{where the current is } J_f=q\int vf(x,v,t)\,dv \tag{5.3}$$

$$\mathrm{div}E=\rho_f,\quad \text{where the charge density is } \rho_f=q\int f(x,v,t)\,dv \tag{5.4}$$

$$\mathrm{div} B = 0. \tag{5.5}$$

We think of this system as an initial value problem for f, E and B.

The restriction of the Maxwell-Vlasov equations to the Coulomb or electrostatic case, in which $B = 0$ (or the velocity of light $\to \infty$) is also important, and leads to the *Poisson Vlasov* equations

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{q}{m}\frac{\partial \phi_f}{\partial x} \cdot \frac{\partial f}{\partial v} = 0 \tag{5.6}$$

$$\Delta\varphi_f = -\rho_f, \quad \text{where } \Delta = \nabla^2 \text{ is the Laplacian, and } \varphi_f \text{ the scalar potential.} \tag{5.7}$$

6 Hamiltonian Formulation of the Poisson-Vlasov System

We first exhibit the Poisson-Vlasov equation as a Hamiltonian system on an appropriate Lie group by using the Lie-Poisson structure discussed in §4. To guess the right group, consider the single particle energy, given by

$$\mathcal{H}_f = \frac{1}{2}mv^2 + q\varphi_f.$$

This is called the *self-consistent Hamiltonian*. It is a direct verification that the Poisson-Vlasov equation can be written as

$$\frac{\partial f}{\partial t} = -\{f, \mathcal{H}_f\}$$

where $\{,\}$ is the standard Poisson bracket on (x, mv)-phase space. From this, we see that f evolves by means of an infinitesimal canonical transformation at each time t. Thus, the evolution of f can be described by

$$f_t = \eta_t^* f_0$$

where f_0 is the initial value of f, f_t is its value at time t, and η_t is a canonical transformation. This motivates a guess that the appropriate group for the Poisson-Vlasov equation is $\mathcal{S}$, the group of canonical transformations on phase space $\mathbb{R}^6$. $\mathcal{S}$ is an infinite dimensional Lie group in the sense of diffeomorphism groups and we shall not deal here with the delicate functional analytic issues needed to make precise all the infinite dimensional geometry. We expect that this gap can be filled by using techniques of Ebin-Marsden [1970] and Ratiu-Schmid [1981].

The Lie algebra s of $\mathcal{S}$ consists of all Hamiltonian vector fields on $\mathbb{R}^6$. We can identify elements of s with their generating functions, so that s consists of the C^∞ functions on $\mathbb{R}^6$ and the (left) Lie algebra structure is given by $\{f, g\}$, the usual Poisson bracket on phase space[1]. The dual space s^* consists of the linear functionals on s which can be identified with the distribution densities on phase space.

[1] Strictly speaking, $\mathcal{S}$ is the group of diffeomorphisms preserving the contact form $\Sigma p_i dq^i + d\tau$ on $\mathbb{R}^6 \times \mathbb{R}$.

For $f\,dx\,dv \in s_+^*$, $h \in s_+$, the natural pairing is

$$\langle f, h\rangle = \int f h\,dx\,dv.$$

Applying the general definition of the Lie-Poisson bracket on the dual of a Lie algebra to s^*, we get for $F, G : s^* \to \mathbb{R}$

$$\{F, G\}(f) = \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx\,dv.$$

(This is the bracket appropriate to right invariant systems).

We now verify that the Poisson-Vlasov equation is a Hamiltonian evolution equation with the Hamiltonian equal to the total energy:

$$H(f) = \frac{1}{2}\int mv^2 f(x, v, t)\,dx\,dv + \frac{1}{2}\int \varphi_f \rho_f\,dx.$$

The second term can be written as $1/2 \int |\nabla \varphi_f|^2\,dx$, from which it follows that $\delta H/\delta f = \mathcal{H}_f$. By integration by parts one gets the following identity

$$\int g\{h, k\}\,dx\,dv = -\int k\{h, g\}\,dx\,dv;\ i.e.,\quad \langle g, \{h, k\}\rangle = \langle \{g, h\}, k\rangle,$$

for $g, h, k : \mathbb{R}^6 \to \mathbb{R}$.

Proposition 6.1. *The Poisson-Vlasov equation can be written in Lie-Poisson form*

$$\dot{f} = -\mathrm{ad}^*_{\frac{\delta H}{\delta f}} f$$

which is equivalent to $\dot{f} = \{f, H\}$ (by Theorem 4.1).

Proof We have

$$\begin{aligned} \langle \mathrm{ad}^*_g f, h\rangle &= \langle f, \mathrm{ad}_g h\rangle \\ &= \langle f, \{g, h\}\rangle \\ &= \int f\{g, h\}\,dx\,dv \\ &= \int \{f, g\} h\,dx\,dv. \end{aligned}$$

Thus $\mathrm{ad}^*_g f = \{f, g\}$: Hence

$$-\mathrm{ad}^*_{\frac{\delta H}{\delta f}} f = \left\{ \frac{\delta H}{\delta f}, f \right\} = -\{f, \mathcal{H}_f\}.$$

Thus $\dot{f} = -\mathrm{ad}^*_{\frac{\delta H}{\delta f}} f$ is equivalent to $\dot{f} = -\{f, \mathcal{H}_f\}$ which, as remarked above, is equivalent to the Poisson-Vlasov equation. QED

7 Maxwell's Equations and Reduction

We will describe Maxwell's equations with a given charge density as a Hamiltonian system and construct, in a natural way, a Poisson bracket for functions of the field variables E and B. Two of Maxwell's equations, namely $\dot{E} = \text{curl } B$ and $\dot{B} = -\text{curl } E$ will be Hamilton's equations relative to this bracket, while the remaining Maxwell equations, namely, div $E = \rho$ and div $B = 0$, will be associated with gauge invariance and the reduction procedure.

The basic configuration variables we begin with are the vector potentials on $\mathbb{R}^3$, *i.e.,* the configuration space is $\mathcal{A} = \{A : \mathbb{R}^3 \to \mathbb{R}^3\}$. The corresponding phase space is the cotangent bundle $T^*\mathcal{A}$ with its canonical symplectic structure. Elements in $T^*\mathcal{A}$ may be identified with pairs (A, Y), where Y is a vector field density on $\mathbb{R}^3$ (we do not distinguish Y and $Y dx$), *i.e.,* $T^*\mathcal{A} \cong \mathcal{A} \times \mathcal{A}^*$. We have the L^2 pairing between $A \in \mathcal{A}$ and $Y \in \mathcal{A}^*$ given by

$$\langle A, Y \rangle = \int A(x) \cdot Y(x)\, dx,$$

so that the canonical symplectic structure ω on $T^*\mathcal{A}$ is given by

$$\omega((A_1, Y_1), (A_2, Y_2)) = \int (Y_2 \cdot A_1 - Y_1 \cdot A_2)\, dx.$$

The associated Poisson bracket for $F, G : T^*\mathcal{A} \to \mathbb{R}$ is given as follows

$$\{F, G\}(A, Y) = \int \left(\frac{\delta F}{\delta A} \cdot \frac{\delta G}{\delta Y} - \frac{\delta F}{\delta Y} \cdot \frac{\delta G}{\delta A} \right) dx.$$

With the Hamiltonian

$$H(A, Y) = \frac{1}{2} \int (|\text{curl}\, A|^2 + |Y|^2)\, dx.$$

Hamilton's equations are easily computed to be

$$\frac{\partial A}{\partial t} = \frac{\delta H}{\delta Y} = Y \text{ and } \frac{\partial Y}{\partial t} = -\frac{\delta H}{\delta A} - \text{curl}\,\text{curl}\, A.$$

By setting $B = \text{curl } A$ and $E = -Y$, the Hamiltonian becomes the usual field energy

$$H(E, B) = \frac{1}{2} \int (|B|^2 + |E|^2)\, dx$$

and Hamilton's equations imply Maxwell's equations

$$\frac{\partial B}{\partial t} = -\text{curl}\, E \text{ and } \frac{\partial E}{\partial t} = \text{curl}\, B.$$

The remaining two Maxwell equations will appear as a consequence of gauge invariance.

Since $H(A, Y)$ depends on curl A, it is invariant under the gauge transformations $A \mapsto A + \nabla\varphi$ for any $\varphi : \mathbb{R}^3 \to \mathbb{R}$. Consider the gauge group $G = \{\varphi : \mathbb{R}^3 \to \mathbb{R}\}$, the group operation being addition; G acts on $\mathcal{A}$ by $\Phi_\varphi(A) = A + \nabla\varphi$. This translation of A extends in the usual way to a canonical transformation (extended point transformation) of $T^*\mathcal{A}$, given by

$$\tilde{\Phi}_\varphi(A, Y) = (A + \nabla\varphi, Y).$$

This action is Hamiltonian and has a momentum map $J : T^*\mathcal{A} \to \mathfrak{g}^*$, where $\mathfrak{g}$ is identified with G, the real valued functions on $\mathbb{R}^3$. The momentum map is given by the formula:

$$\langle J(A, Y), \psi \rangle = \langle Y, \psi_{\mathcal{A}}(A) \rangle,$$

where $\psi \in \mathfrak{g}$ and $\psi_{\mathcal{A}}$ is the corresponding infinitesimal generator of the action Φ_φ on $\mathcal{A}$. One computes that $\psi_{\mathcal{A}}(A) = \nabla\psi$, which leads to

$$\langle J(A, Y), \psi \rangle = \int (Y \cdot \nabla\psi)\, dx = -\int (\operatorname{div} Y) \cdot \psi\, dx.$$

Thus we may write

$$J(A, Y) = -\operatorname{div} Y.$$

If ρ is an element of $\mathfrak{g}^*$ (the densities on $\mathbb{R}^3$), $J^{-1}(\rho) = \{(A, Y) \in T^*\mathcal{A} \mid \operatorname{div} Y = -\rho\}$. In terms of E, the condition div $Y = -\rho$ becomes the Maxwell equation div $E = \rho$, so we may interpret the elements of $\mathfrak{g}^*$ as charge densities.

Since the Hamiltonian is gauge invariant, *i.e.,* $H(\tilde{\Phi}_\varphi(A, Y)) = H(A, Y)$, we can pass to the reduced phase space $J^{-1}(\rho)/G$, which we know by the general theory is a symplectic manifold into which the motion can be projected.

Proposition 7.1. *The reduced phase space $J^{-1}(\rho)/G$ can be identified with $\mathfrak{M} = \{(E, B) \mid \operatorname{div} E = \rho, \operatorname{div} B = 0\}$, and the Poisson bracket on $\mathfrak{M}$ is given in terms of E and B by*

$$\{F, G\}(E, B) = \int \left(\frac{\delta F}{\delta E} \cdot curl\, \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} \cdot curl\, \frac{\delta F}{\delta B} \right) dx.$$

Maxwell's equations with an ambient charge density ρ are Hamilton's equations for

$$H(E, B) = \frac{1}{2} \int (|E|^2 + |B|^2)\, dx$$

on the space $\mathfrak{M}$.

Proof To each $(A, Y) \in J^{-1}(\rho)$, we associate the pair $(B, E) = (\operatorname{curl} A, -Y) \in \mathfrak{M}$. Since two vector fields on $\mathbb{R}^3$ have the same curl if and only if they differ by a gradient, and every divergence-free B is a curl, this association gives a 1-1 correspondence between $J^{-1}(\rho)/G$ and $\mathfrak{M}$.

Now let $F, G : \mathfrak{M} \to \mathbb{R}$. To compute their Poisson bracket $\{F, G\}$, we pull them back to $J^{-1}(\rho)$, extend them to $T^*\mathcal{A}$, take their canonical Poisson bracket in $T^*\mathcal{A}$, restrict this to $J^{-1}(\rho)$, and finally << push-down >> the resulting G-invariant function to $\mathfrak{M}$. The result does not depend upon the choice of extension made, and in fact we can do the computation without mentioning the extension again. Given $F(B,E)$ we define the pull back $\widehat{F}(A,Y)$ to $J^{-1}(\rho)$ by

$$\widehat{F}(A,Y) = F(\operatorname{curl} A, -Y).$$

Notice that the chain rule gives

$$D_A \widehat{F}(A,Y) \cdot A' = D_B F(B,E) \cdot \operatorname{curl} A'$$

i.e.,

$$\begin{aligned} \int \frac{\delta \widehat{F}}{\delta A} \cdot A' \, dx &= \int \frac{\delta F}{\delta B} \cdot (\operatorname{curl} A') \, dx \\ &= \int \left(\operatorname{curl} \frac{\delta F}{\delta B} \right) \cdot A' \, dx \end{aligned}$$

(integrating by parts). Thus

$$\frac{\delta \widehat{F}}{\delta A} = \operatorname{curl} \left(\frac{\delta F}{\delta B} \right).$$

Similarly

$$\frac{\delta \widehat{F}}{\delta Y} = -\frac{\delta F}{\delta E}.$$

Using the canonical bracket on $T^*\mathcal{A}$, we thus have

$$\begin{aligned} \{F,G\}(B,E) = \{\widehat{F},\widehat{G}\}(A,Y) &= \int \left(\frac{\delta \widehat{F}}{\delta A} \cdot \frac{\delta \widehat{G}}{\delta Y} - \frac{\delta \widehat{G}}{\delta A} \cdot \frac{\delta \widehat{F}}{\delta Y} \right) dx \\ &= \int \left(\frac{\delta}{\delta A} F(\operatorname{curl} A, -Y) \cdot \frac{\delta}{\delta Y} G(\operatorname{curl} A, -Y) \right. \\ &\quad \left. - \frac{\delta}{\delta A} G(\operatorname{curl} A, -Y) \cdot \frac{\delta}{\delta Y} F(\operatorname{curl} A, -Y) \right) dx \\ &= -\int \left(\operatorname{curl} \frac{\delta F}{\delta B} \cdot \frac{\delta G}{\delta E} - \operatorname{curl} \frac{\delta G}{\delta B} \cdot \frac{\delta F}{\delta E} \right) dx \\ &= \int \left(\frac{\delta F}{\delta E} \cdot \operatorname{curl} \frac{\delta G}{\delta B} - \frac{\delta G}{\delta B} \cdot \operatorname{curl} \frac{\delta F}{\delta E} \right) dx. \end{aligned}$$

The rest of the proposition follows from the general theory of reduction. QED

The Maxwell-Vlasov equations in terms of potentials

We will now combine our results for the Maxwell equations and the Poisson-Vlasov system to obtain the Poisson structure for the Maxwell-Vlasov system. The Hamiltonian will be given by the total energy, which is the sum of the particle kinetic energy and the field energy. For simplicity, we set $m = 1, q = 1$:

$$\begin{aligned} H(f,E,B) &= \frac{1}{2} \int |v|^2 f(x,v,t) \, dx \, dv \\ &\quad + \frac{1}{2} \int (|E(x,t)^2 + |B(x,t)^2) \, dx. \end{aligned}$$

The calculation is simpler if we choose as our variables $(A, Y) \in T^*\mathcal{A}$ and densities f_{mom} on (x, p)-space, where p is the particle canonical momentum, rather than densities f on (x, v)-space. The relation $p = v + A$ leads to $f_{\text{mom}}(x, p) = f_{\text{mom}}(x, v + A) = f(x, v)$ and of course, replacing f by f_{mom} and v by p in the Lie-Poisson bracket on s^* still defines a Poisson structure on s^*.

The Poisson bracket on $s^* \times T^*\mathcal{A}$ is just the sum of those on s^* and $T^*\mathcal{A}$. For functionals $\bar{F}, \bar{G} : s^* \times T^*\mathcal{A} \to \mathbb{R}$ set

$$\begin{aligned} \{\bar{F}, \bar{G}\}(f_{\text{mom}}, A, Y) &= \int f_{\text{mom}} \left\{ \frac{\delta \bar{F}}{\delta f_{\text{mom}}}, \frac{\delta \bar{G}}{\delta f_{\text{mom}}} \right\} dx\, dp \\ &\quad + \int \left(\frac{\delta \bar{F}}{\delta A} \frac{\delta \bar{G}}{\delta Y} - \frac{\delta \bar{G}}{\delta A} \frac{\delta \bar{F}}{\delta Y} \right) dx \end{aligned} \tag{7.1}$$

and the Hamiltonian expressed in these variables becomes

$$\begin{aligned} \bar{H}(f_{\text{mom}}, A, Y) &= \frac{1}{2} \int |p - A(x)|^2 f_{\text{mom}}(x, p)\, dx\, dp \\ &\quad + \frac{1}{2} \int (|Y|^2 + |\text{curl}\, A|^2)\, dx = H_{\text{matter}} + H_{\text{field}}. \end{aligned} \tag{7.2}$$

Notice that at this stage (pre-reduction), there is no coupling of particles and fields in the Poisson bracket, but there is coupling in the first term of $\bar{H}$.

Theorem 7.2. *The evolution equations $\dot{\bar{F}} = \{\bar{F}, \bar{H}\}$ for a function $\bar{F}$ on $s^* \times T^*\mathcal{A}$ with $\bar{H}$ and $\{,\}$ given above, are equations 5.1, 5.2 and 5.3 of the Maxwell-Vlasov equations.*

The proof of this theorem is a straightforward verification. The constraints 5.4 and 5.5 of the Maxwell-Vlasov equations are subsidiary equations which are consistent with the evolution equations. Equation 5.4 holds since $B = \text{curl } A$ and equation 5.5 expresses the fact that we are on the zero level of the momentum map generated by the gauge transformations. (If an external charge density is present, we replace $J^{-1}(0)$ by $J^{-1}(\rho_{\text{ext}})$; see §10). The corresponding reduced space decouples the energy, while coupling the Poisson structures. We turn to this in the next section.

8 The Poisson Structure for the Maxwell-Vlasov Equations

The final form of the Poisson structure for the Maxwell-Vlasov system will now be obtained by applying the reduction procedure to the formulation of the previous sections.

First we must specify an action of the gauge group $G = \{\varphi : \mathbb{R}^3 \to \mathbb{R}\}$ on s^* in such a way, that when combined with the action of G on $T^*\mathcal{A}$, $\tilde{\Phi}_\varphi(A, Y) = (A + \nabla\varphi, Y)$ it will leave the Hamiltonian invariant. Furthermore, it must preserve the Poisson structure on $s^* \times T^*\mathcal{A}$. A natural choice for this extended action, which satisfies the first requirement is:

$$\Phi_\psi(f_{\mathrm{mom}}, A, Y) = (f_{\mathrm{mom}} \circ \tau_{-d\psi}, A + \nabla\psi, Y)$$

where $\tau_{-d\psi} : \mathbb{R}^6 \to \mathbb{R}^6$ is the momentum translation map defined by

$$\tau_{-d\psi}(x,p) = (x, p - d\psi(x)).$$

Proposition 8.1. *The above action of G on $s^* \times T^*\mathcal{A}$ has a momentum map $J : s^* \times T^*\mathcal{A} \to \mathfrak{g}^*$ given by*

$$J(f_{\mathrm{mom}}, A, Y) = -\int f_{\mathrm{mom}}(x,p)dp - div\, Y.$$

Proof The second term is the momentum map for the action of $T^*\mathcal{A}$, so it suffices to calculate the momentum map for the action of G on s^*. This can be done by observing that the action on s^* is the restriction of an extended point transformation on $T^*\mathcal{S}$ to the fiber at the identity $T^*_e\mathcal{S} \cong s^*$, or by the following direct computation. The infinitesimal generator $\psi_{s^*} : s^* \to s^*$ of the action $(\psi, f_{\mathrm{mom}}) \mapsto f_{\mathrm{mom}} \circ \tau_{-d\psi}$ is given by

$$\begin{aligned}
\psi_{s^*}(f)(x,p) &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\psi)} f(x,p) \\
&= \left.\frac{d}{dt}\right|_{t=0} (f \circ \tau_{-td\psi})(x,p) = \left.\frac{d}{dt}\right|_{t=0} f(x, p - td\psi),
\end{aligned}$$

hence $\psi_{s^*}(f) = -d_p f \cdot d_x \psi = \{f, \psi\}$.

Let $J_{s^*}(f_{\mathrm{mom}}) = -\int f_{\mathrm{mom}}(x,p)dp$, so $J_{s^*} : s^* \to \mathfrak{g}^*$. We need to verify that

$$X_{\langle J^*_s, \psi\rangle} = \psi_{s^*}.$$

But $\langle J_{s^*}, \psi\rangle(f_{\mathrm{mom}}) = -\int f_{\mathrm{mom}}(x,p)\psi(x)\, dx\, dp$ and by Theorem 4.2,

$$-X_{\langle J_{s^*}, \psi\rangle}(f_{\mathrm{mom}}) = \mathrm{ad}^*_{\frac{\delta\langle J_{s^*},\psi\rangle}{\delta f_{\mathrm{mom}}}}(f) = \mathrm{ad}^*_{-\psi}(f) = -\mathrm{ad}^*_\psi(f).$$

But $\mathrm{ad}^*_\psi(f) = \{f, \psi\}$ since

$$\begin{aligned}
\langle \mathrm{ad}^*_\psi(f), h\rangle &= \langle f, \mathrm{ad}_{-\psi} h\rangle \\
&= \langle f, \{\psi, h\}\rangle \\
&= \langle \{f, \psi\}, h\rangle.
\end{aligned}$$

Thus $X_{\langle J_{s^*}, \psi\rangle} = \psi_{s^*}$ and the result follows. QED

An alternative proof of this proposition, suggested by S. Sternberg, proceeds as follows. We use the following.

Lemma 8.2. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and $\alpha : \mathfrak{g} \to \mathfrak{h}$ a Lie algebra homomorphism. Regarding $\mathfrak{h} \subset C^\infty(\mathfrak{h}^*)$ as linear functions, then $\alpha : \mathfrak{g} \to C^\infty(\mathfrak{h}^*)$ is a Lie algebra homomorphism i.e., a Hamiltonian action with $\mathfrak{g}$ acting on $\mathfrak{h}^*$ by $-\mathrm{ad}^*_{\alpha(\xi)}$. The associated momentum map is the dual $\alpha^* : \mathfrak{h}^* \to \mathfrak{g}^*$.*

Proof If η_1 and $\eta_2 \in \mathfrak{h}$ are regarded as linear functions on $\mathfrak{h}^*$, their (+) Lie-Poisson bracket is

$$\begin{aligned} \{\eta_1, \eta_2\}(\nu) &= \left\langle \nu, \left[\frac{\delta\eta_1}{\delta\nu}, \frac{\delta\eta_2}{\delta\nu}\right]\right\rangle \\ &= \langle \nu, [\eta_1, \eta_2]\rangle. \end{aligned}$$

So $\{\eta_1, \eta_2\} = [\eta_1, \eta_2]$. Thus $\alpha : \mathfrak{g} \to C^\infty(\mathfrak{h}^*)$ is a Lie algebra homomorphism. To check that α^* is the momentum map, note that $\langle \alpha^*(\nu), \xi\rangle = \langle \nu, \alpha(\xi)\rangle$. Thus we must verify that for $\xi \in \mathfrak{g}$,

$$X_{\alpha(\xi)} = \xi_{\mathfrak{h}^*}.$$

But $\xi_* = -\mathrm{ad}^*_{\alpha(\xi)}$ by assumption and $X_{\alpha(\xi)} = -\mathrm{ad}^*_{\alpha(\xi)}$ by Theorem 4.2. QED

This proof also shows that the dual of a linear map $\alpha : \mathfrak{g} \to \mathfrak{h}$ is a Poisson map if and only if α is a Lie algebra homomorphism.

To get the proposition, let $\mathfrak{g} = C^\infty(\mathbb{R}^3)$ and $\mathfrak{h} = s = C^\infty(T^*\mathbb{R}^3)$ with $\alpha : \mathfrak{g} \to \mathfrak{h}$ given by mapping the function ψ of x to $-\psi$ regarded as a function of (x, p). The dual of α is minus integration over the fiber, so we just need to check that α gives the correct action of $\mathfrak{g}$ on s^*, namely

$$(\psi, f) \mapsto \mathrm{ad}^*_\psi(f).$$

Now

$$\begin{aligned} \mathrm{ad}^*_\psi(f)(h) &= \langle f, \mathrm{ad}_{-\psi} h\rangle \\ &= \langle f, \{\psi, h\}\rangle \\ &= \langle \{f, \psi\}, h\rangle \end{aligned}$$

so $\mathrm{ad}^*_{-\psi}(f) = \{f, \psi\}$.

On the other hand, the infinitesimal generator of the action, computed above is

$$\psi_{s^*}(f) = -d_p f \cdot d_x \psi = \{f, \psi\}.$$

QED

We may describe the reduced Poisson manifold in terms of densities $f(x, v)$ on position-velocity space. We reduce at an element $\mu \in \mathfrak{g}^*$.

Proposition 8.3. *The reduced manifold $(s^* \times T^*\mathcal{A})_\mu = J^{-1}(\mu)/G$ may be identified with the Maxwell-Vlasov phase space.*

$$MV = \left\{ (f, B, E) \mid \mathrm{div} B = 0, \mathrm{div} E = \mu + \int f(x, v)\, dv \right\}.$$

Proof To each $(f_{\rm mom}, A, Y)$ in $J^{-1}(\mu)$ we associate the triple (f, B, E) in MV where

$$f(x,v) = f_{\rm mom}(x, v + A(x)), B = \text{ curl } A \text{ and } E = -Y.$$

Then the proposition follows from the momentum map construction above, and a simple verification that two elements of $J^{-1}(\mu)$ are associated to the same triple in MV if and only if they are related by a gauge transformation. (Note that $\mu \in \mathfrak{g}^*$ can be interpreted as an external charge density). QED

By a general theory of reduction, MV inherits a Poisson bracket from the one on $s^* \times T^*\mathcal{A}$. Since the Hamiltonian H is invariant under gauge transformations, it follows from Theorem 2.2 that the Maxwell-Vlasov equations 5.1, 5.2, and 5.3 are a Hamiltonian system on MV with respect to this Poisson structure. It remains now to compute the explicit form of the Poisson bracket in terms of the reduced variables (f, B, E).

Theorem 8.4. *For two functions F, G of the field variables (f, B, E), i.e., $F, G : MV \to \mathbb{R}$, the Poisson bracket is given by :*

$$\begin{aligned}
\{F, G\}(f, B, E) \quad = \quad & \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx\, dv \\
& + \int \left(\frac{\delta F}{\delta E} \cdot \operatorname{curl} \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} \cdot \operatorname{curl} \frac{\delta F}{\delta B} \right) dx \\
& + \int \left(\frac{\delta F}{\delta E} \cdot \frac{\partial f}{\partial v} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta E} \cdot \frac{\partial f}{\partial v} \frac{\delta F}{\delta f} \right) dx\, dv \\
& + \int f B \cdot \left(\frac{\partial}{\partial v} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial v} \frac{\delta G}{\delta f} \right) dx\, dv. \qquad (8.1)
\end{aligned}$$

Proof Given $F(f, B, E)$ on MV, define $\bar{F}(f_{\rm mom}, A, Y)$ by

$$\bar{F}(f_{\rm mom}, A, Y) = F(f, B, E),$$

where the relation between the two sets of variables is as in the previous proposition. Then $\{F, G\}(f, B, E)$ is given by computing $\{\bar{F}, \bar{G}\}(f_{\rm mom}, A, Y)$ according to the last section and expressing the result in terms of the variables (f, B, E) on MV.

By definition of the functional derivative we see that

$$\frac{\delta \bar{F}}{\delta f_{\rm mom}}(x, p) = \frac{\delta F}{\delta f}(x, p - A).$$

The first term of 8.1 therefore becomes

$$\int f(x, p - A) \left\{ \frac{\delta F}{\delta f}(x, p - A), \frac{\delta G}{\delta f}(x, p - A) \right\} dx\, dp. \qquad (8.2)$$

Using the formula for the Poisson bracket under a change of variables and $B = \operatorname{curl} A$, we have for two functions $g(x, v), h(x, v)$:

$$\{g(x, p - A), h(x, p - A)\}_{(x,p)} = \{g(x, v), h(x, v)\}_{(x,v)} + B \cdot \left(\frac{\partial g}{\partial v} \times \frac{\partial h}{\partial v} \right).$$

Letting $g(x,v) = \delta F/\delta f(x,v)$ and $h(x,v) = \delta G/\delta f(x,v)$, and making the substitution $v = p - A$, 8.2 becomes

$$\int f(x,v)\left\{\frac{\delta F}{\delta f}(x,v), \frac{\delta G}{\delta f}(x,v)\right\}_{(x,y)} dx\, dv$$
$$+ \int f(x,v)\cdot B\left(\frac{\partial}{\partial v}\frac{\delta F}{\delta f}(x,v) \times \frac{\partial}{\partial v}\frac{\delta G}{\delta f}(x,v)\right) dx\, dv. \tag{8.3}$$

This is the first and the last term in 8.1. For the second term in 8.1 we use

Lemma 8.5.

$$\frac{\delta \bar{F}}{\delta A} = \operatorname{curl}\frac{\delta F}{\delta B} + \frac{\delta F}{\delta f}\frac{\partial f}{\partial v} \textit{ and } \frac{\delta \bar{F}}{\delta Y} = -\frac{\delta F}{\delta E}.$$

Proof By the chain rule and definition of the functional derivative,

$$\begin{aligned}\int \frac{\delta \bar{F}}{\delta A} A' dx &= D_A\bar{F}(f_{\mathrm{mom}}, A, Y)\cdot A' \\ &= D_f F(f,B,E)\cdot D_A f\cdot A' + D_B F(f,B,E)\cdot \operatorname{curl} A'.\end{aligned}$$

But

$$D_A f\cdot A'(x,p) = \frac{\partial f_{\mathrm{mom}}}{\partial p}(x, v+A)\cdot A' = \frac{\partial f}{\partial v}(x,v)\cdot A'.$$

Thus,

$$\begin{aligned}\int \frac{\delta \bar{F}}{\delta A}\cdot A'\, dx &= \int \frac{\delta F}{\delta f}\frac{\partial f}{\partial v}A' dx + \int \frac{\delta F}{\delta B}\operatorname{curl} A'\, dx \\ &= \int \frac{\delta F}{\delta f}\frac{\partial f}{\partial v}A' + \int \operatorname{curl}\frac{\delta F}{\delta B}A'\, dx.\end{aligned}$$

QED

Substituting this into the second term of 7.1 yields the second and third term of 8.1. Thus Theorem 8.4 is proved. QED

Observe that the first term of 8.1 is the bracket for the Poisson-Vlasov system and involves only f, the second term is the bracket for the vacuum Maxwell equations and involves only the electromagnetic fields, and the third and fourth terms provide the coupling between the Vlasov system and the Maxwell system. This Poisson bracket automatically satisfies the Jacobi identity, by the way it is constructed from general methods of symplectic geometry using the Lie-Poisson bracket and reduction.

The final ingredient in the Hamiltonian structure is the Hamiltonian expressed in the reduced manifold variables:

$$\begin{aligned}H(f,B,E) &= \frac{1}{2}\int |v^2| f(x,v)\, dx\, dy \\ &\quad + \frac{1}{2}\int |B(x)|^2\, dx + \frac{1}{2}\int |E(x)|^2\, dx.\end{aligned}$$

Theorem 8.6. *The Maxwell-Vlasov equations of motion 5.1, 5.2, and 5.3 may be written as*

$$\dot{F} = \{F, H\},$$

where $\{,\}$ is given by 8.1.

This follows from Theorem 7.2 by reduction, and can also be checked directly by a straightforward calculation.

There is an analogous bracket to 8.1 in (f_{mom}, E, B) variables. For an interpretation of this in terms of semidirect products and additional results, see Marsden, Ratiu and Weinstein [1983].

9 Semi-Direct Products, Fluids and Magnetohydrodynamics

In the preceding sections we saw that the Vlasov variable f is regarded as an element of s^* the dual of the Lie algebra of the group of canonical transformations and that the bracket on $C^\infty(s^*)$ is the Lie-Poisson bracket for this group. One can view s^* as arising from $T^*\mathcal{S}$ by reduction by the action of $\mathcal{S}$ by right translations.

Now we consider reducing T^*G not by G but by a subgroup. In doing so we shall arrive not at $\mathfrak{g}^*$ but at the dual of the Lie algebra of a semi-direct product. We shall then apply this result to compressible fluids and to MHD and indicate briefly the relationship of the structures obtained to the Lie-Poisson bracket on $C^\infty(s^*)$ and the Maxwell-Vlasov bracket.

The results described below are due to Marsden, Ratiu and Weinstein [1983], which should be consulted for detailed proofs. They are a consolidation and extension of earlier results of Vinogradov and Kupershmidt [1977], Ratiu [1980], Guillemin and Sternberg [1980], Morrison and Greene [1980], Dzyalozhinskii and Volovick [1980], Holm and Kupershmidt [1983], Holmes and Marsden [1983] and Kupershmidt [1982]. These references also consider the heavy top, elasticity and other systems which are not discussed here.

We first apply the results on reduction of T^*G by left and right translations to the case of semidirect products. Let $\rho : G \to A\text{ut}(V)$ denote a left Lie group representation of G in the vector space V and $\rho' : \mathfrak{g} \to \text{End }(V)$ the induced Lie algebra representation. Denote by $S = G \overset{\times}{\rho} V$ the semidirect product group of G with V by ρ with multiplication.

$$(g_1, u_1)(g_2, u_2) = (g_1, g_2, u_1 + \rho(g_1)u_2).$$

Let $s = \mathfrak{g} \overset{\rho}{\times} V$ be the Lie algebra of S. The Lie bracket on s is given by

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \rho'(\xi_1)v_2 - \rho'(\xi_2)v_1).$$

The adjoint and coadjoint actions of S on s and s^* are given by

$$\text{Ad}_{(g,u)}(\xi, v) = (\text{Ad}_g\xi, \rho(g)v - \rho'(Ad_g\xi)u)$$

and

$$[\mathrm{Ad}_{(g,u)^{-1}}]^*(\nu,a) \equiv \mathrm{Ad}^*_{(g,u)^{-1}}(\nu,a)$$
$$= (\mathrm{Ad}^*_{g^{-1}}\nu + (\rho'_u)^*(\rho_*(g))a, \rho_*(g)a)$$

where $g \in G, u, v \in V, \nu \in \mathfrak{g}^*$, and $a \in V^*; \rho'_u : \mathfrak{g} \to V$ is given by $\rho'_u(\xi) = \rho'(\xi)u$ and $(g,u)^{-1} = (g^{-1}, -\rho(g^{-1})u)$. The $\pm$ Lie-Poisson bracket of $F, G : s^* \to \mathbb{R}$ is

$$\begin{aligned} \{F,G\}_\pm(\mu,a) \quad &= \quad \pm\left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right]\right\rangle \pm \left\langle a, \rho'\left(\frac{\delta F}{\delta\mu}\right)\cdot\frac{\delta G}{\delta a}\right\rangle \\ &\quad \mp \left\langle a, \rho'\left(\frac{\delta G}{\delta\mu}\right)\cdot\frac{\delta F}{\delta a}\right\rangle \end{aligned} \tag{9.1}$$

where $\delta F/\delta\mu \in \mathfrak{g}$ and $\delta F/\delta a \in V$. Also from Theorem 4.2, we compute the Hamiltonian vector field of $H : s^* \to \mathbb{R}$ to be

$$X_H(\mu,a) = \mp\left(\mathrm{ad}^*_{\frac{\delta H}{\delta\mu}}\mu - \rho'_{\frac{\delta H}{\delta a}}{}^*a, \rho'_{\frac{\delta H}{\delta\mu}}{}^*a\right) \tag{9.2}$$

where $\rho'_{\delta H/\delta a} : \mathfrak{g} \to V$ is given by $\rho'_{\delta H/\delta a}(\xi) = \rho'(\xi)\cdot\delta H/\delta a$, and $\rho'_{\delta H/\delta a}{}^*$ is its adjoint.

We shall explicitly compute the left and right actions L and R of S on T^*S. Since

$$L_{(g,u)}(h,v) = (gh, u+\rho(g)v),$$

we have

$$T_{(h,v)}L_{(g,\mu)}(v_h,v,w) = (T_hL_g(v_h), u+\rho(g)v, \rho(g)w)$$

for $(v_h,v,w) \in T_{(h,v)}(G\times V)$. Thus

$$T_{(g,u)(h,v)}L_{(g,u)^{-1}}(v_{gh}, u+\rho(g)v, w)$$
$$= (T_{gh}L_{g^{-1}}(v_{gh}), v, \rho(g^{-1})w)$$

for $(v_{gh}, u+\rho(g)v, w) \in T_{(g,u)(h,v)}(G\times V)$, and hence

$$\begin{aligned} L((g,u),(\alpha_h,v,a)) \quad &= \quad (T_{(g,u)(h,v)}L_{(g,u)^{-1}})^*(\alpha_h,v,a) \\ &= \quad ((T_{gh}L_{g^{-1}})^*\alpha_h, u+\rho(g)v, \rho_*(g)a) \end{aligned}$$

for $(\alpha_h,v,a) \in T^*_{(h,v)}(G\times V)$. Since $R_{(g,u)}(h,v) = (hg, v+\rho(h)u)$, we have

$$T_{(h,v)}R_{(g,u)}(v_h,v,w) = (T_hR_g(v_h), v+\rho(h)u, w+T_h\rho(v_h)\cdot u)$$

for $(v_h,v,w) \in T_{(h,v)}(G\times V)$. Thus

$$T_{(h,v)(g,u)}R_{(g,u)^{-1}}(v_{hg}, v+\rho(h)u, w)$$
$$= (T_{hg}R_{g^{-1}}(v_{hg}), u, w - T_{hg}\rho(v_{hg})\cdot\rho(g^{-1})u)$$

for $(v_{hg}, v+\rho(g)u, w) \in T_{(h,v)(g,u)}(G\times V)$ and hence

$$\begin{aligned} R((g,u),(\alpha_h,v,a)) \quad &= \quad (T_{(h,v)(g,u)}R_{(g,u)^{-1}})^*(\alpha_h,v,a) \\ &= \quad ((T_{hg}R_{g^{-1}})^*\alpha_h - df^a_{\rho(g^{-1})u}(hg), v+\rho(h)u, a) \end{aligned}$$

where $f_u^a(g)$ is the << matrix elements >> $\langle a, \rho(g)u\rangle$. The last equality is obtained by applying the left hand side to $(v_{hg}, v + \rho(g)u, w) \in T_{(h,v)(g,u)}(G \times V)$ and using the easily verifiable formula

$$\langle a, T_k\rho(v_k)z\rangle = df_z^a(k)v_k$$

for $a \in V^*, k \in G, z \in V, v_k \in T_kG$.

The corresponding momentum mappings are

$$\begin{aligned} J_L : T^*S \to s_+^*, J_L(\alpha_g, v, a) &= (T_{(e,0)}R_{(g,v)})^*(\alpha_g, v, a) \\ &= ((T_eR_g)^*\alpha_g + (\rho_v')^*a, a) \end{aligned} \tag{9.3}$$

and

$$\begin{aligned} J_R : T^*S \to s_-^*, J_R(\alpha_g, v, a) &= (T_{(e,0)}L_{(g,v)})^*(\alpha_g, v, a) \\ &= ((T_eL_g)^*\alpha_g, \rho^*(g)a) \end{aligned} \tag{9.4}$$

Theorem 4.5 applies to this situation to give:

Proposition 9.1. *Let J_L and J_R be given by 9.3 and 9.4. Then*

1. *J_L and J_R are Poisson maps.*
2. *J_L (resp. J_R) induces a Poisson diffeomorphism of $R\backslash T^*S$ (resp. T^*S/L) with s_+^* (resp. s_-^*).*
3. *The reduced spaces for the L (resp. R) action are coadjoint orbits in s_-^* (resp. s_+^*).*

In many physical examples a Hamiltonian system on T^*G is given whose Hamiltonian function H_a depends smoothly on a parameter $a \in V^*$. In addition, H_a is left invariant under the stabilizer $G_a = \{g \in G \mid \rho_*(g)a = a\}$ where $\rho_*(g) = [\rho(g^{-1})]^*$, the dual of the linear transformation $\rho(g^{-1})$. The Lie algebra of G_a is $\mathfrak{g}_a = \{\xi \in \mathfrak{g} \mid \rho'(\xi)^*a = 0\}$. Denote by T^*G/G_a the orbit space of the lift to T^*G of left translation of G_a on G. We wish to study the motion on the Poisson manifolds T^*G/G_a for all $a \in V^*$. We shall prove below that this is equivalent to the study of the motion on s_-^* where $s = \mathfrak{g} \overset{\rho}{\times} V$.

For fixed $a \in V^*$, the lift to T^*G of the left translation of G_a on G has the Ad^*-equivariant momentum map $J_L^a : T^*G \to \mathfrak{g}_a^*, J_L^a(\alpha_g) = (T_eR_g)^*\alpha_g \mid \mathfrak{g}_a$. The mapping $i_L^a : T^*G \to T^*S, i_L^a(\alpha_g) = (\alpha_g, 0, a)$ is an *embedding* of Poisson manifolds inducing a Poisson embedding of the quotients $\bar{i}_L^a : T^*G/G_a \to T^*S/L$. Thus the symplectic leaves in T^*S/L pull back to symplectic leaves in T^*G/G_a. The natural candidates for these leaves are the reduced symplectic manifolds $(J_L^a)^{-1}(0_{\mu_a}^+)/G_a$, where $\mu \in \mathfrak{g}^*, \mu_a = \mu|\mathfrak{g}_a$ and $0_{\mu_a}^+ \subset \mathfrak{g}_a^*$ denotes the coadjoint orbit of G_a in $\mathfrak{g}_a^*$ with the << + >> Lie-Poisson symplectic structure.

For Hamiltonians that are right invariant under G_a one proceeds exactly as above, interchanging << left >> and << right >>, and << − >> and << + >>, and replacing the mapping i_L^a by $i_R^a(\alpha_g) = (\alpha_g, 0, \rho_*(g)a)$. The mappings i_L^a and i_R^a

are necessarily different; if one would use i_L^a for the right actions, $\bar{i}_L^a$ would fail to be injective. Summarizing: *we have the following sequence of Poisson embeddings, the last arrow being a diffeomorphism.*

$$(J_L^a)^{-1}(0_{\mu_a}^+)/G_a \hookrightarrow T^*G/G_a \hookrightarrow T^*S/L \to s_-^*$$
$$G_a\backslash(J_R^a)^{-1}(0_{\mu_a}^-) \hookrightarrow G_a\backslash T^*G \hookrightarrow R\backslash T^*S \to s_+^*.$$

Moreover

$$\begin{aligned}(\bar{J}_R \circ \bar{i}_L^a)((J_L^a)^{-1}(0_{\mu_a}^+)/G_a) &= (J_R \circ i_L^a)((J_L^a)^{-1}(0_{\mu_a}^+)) \\ &= \{(v,b) \in s^* | \text{ there exists } g \in G_a \text{ such that } \rho_*(g)a \\ &= b, \mathrm{Ad}_g^* v \in 0_{\mu_a}\} \\ &= \bigcup_{\chi|\mathfrak{g}_a=\mu_a} S\cdot(\chi,a)_-\end{aligned}$$

as a simple verification shows; here $S\cdot(\chi,a)_- \subset s_-^*$ denotes the coadjoint orbit of S through (χ,a) with the << − >> Lie-poisson structure. But

$$\{(\rho_u')^*a \mid u \in V\} = \{\mu \in \mathfrak{g}^* | \mu|\mathfrak{g}_a \equiv 0\}$$

implies that $S\cdot(\chi,a) = S\cdot(\mu,a)$ for all $\chi \in \mathfrak{g}^*$ such that $\chi \mid \mathfrak{g}_a = \mu_a$ and so we have proved the following theorem.

Theorem 9.2.

1. *$\bar{J}_R \circ \bar{i}_L^a$ maps the reduced space $(J_L^a)^{-1}(0_{\mu_a}^+)/G_a$ in a symplectically diffeomorphic way to the coadjoint orbit $S\cdot(\mu,a)_-$ in the dual of the semidirect product s_-^*. Similarly,*

2. *$\bar{J}_L \circ \bar{i}_R^a$ maps the reduced space $G_a\backslash(J_R^a)^{-1}(0_{\mu_a}^-)$ in a symplectically diffeomorphic way to the coadjoint orbit $S\cdot(\mu,a)_+$ in the dual of the semidirect product s_+^*.*

Next we shall explain how to use Theorem 9.2 in examples. This will allow one to begin with the standard phase space T^*G and then to reduce a Hamiltonian system on T^*G to a Lie-Poisson system on s^*.

Let $H_a : T^*G \to \mathbb{R}$ be a Hamiltonian depending smoothly on the parameter $a \in V^*$ and assume that H_a is invariant under the lift to T^*G of the left translations of G_a on G. Thus H_a induces a Hamiltonian on T^*G/G_a. Via the symplectic diffeomorphism $\bar{J}_R \circ \bar{i}_L^a$, we may regard it as being defined on $S\cdot(\mu,a)_-\mu, \in \mathfrak{g}^*$, and now varying $a \in V^*$ we obtain a smooth Hamiltonian H on s_-^* ; thus $H \circ J_R \circ i_L^a = H_a$, *i.e.*, $H((T_eL_g)^*\alpha_g, \rho^*(g)a) = H_a(\alpha_g)$. Thus the family of Hamiltonians H_a on T^*G induces a single Hamiltonian on s_-^*, and the original problem has been embedded in a larger one which yields Lie-Poisson equations on s_-^*. For right invariant Hamiltonians, interchange << left >> and << right >>, and << − >> and << + >>. However, since the maps i_L^a and i_R^a are different, we have $H((T_eR_g)^*\alpha_g, \rho_*(g)a) = H_a(\alpha_g)$.

It is of interest to investigate the evolution of $a \in V^*$. Let $c_a(t) \in T^*G$ denote an integral curve of H_a and let $g_a(t)$ be its projection on G. Then the curve $t \mapsto (J_R \circ i_L^a)c_a(t)$ is an integral curve of X_H on s_-^* and thus $t \mapsto \rho^*(g_a(t))$ is the evolution of the initial condition a in s_-^*. We summarize these results in the following theorem.

Theorem 9.3. *Let $H_a : T^*G \to \mathbb{R}$ be a Hamiltonian depending smoothly on $a \in V^*$ and left invariant under the action on T^*G of the stabilizer G_a. The family of Hamiltonians $\{H_a \mid a \in V^*\}$ induces a Hamiltonian function H on s_-^*, defined by $H((T_eL_g)^*\alpha_g, \rho^*(g)a) = H_a(\alpha_g)$ thus yielding Lie-Poisson equations on s_-^*. The curve $t \mapsto c_a(t) \in T^*G$ is a solution for Hamilton's equations defined by H_a on T^*G iff $t \mapsto (J_R \circ i_L^a)c_a(t)$ is a solution of the Hamiltonian system X_H on s_-^*. In particular, the evolution of $a \in V^*$ is given by $t \mapsto \rho^*(g_a(t))a$, where $g_a(t)$ is the projection of $c_a(t)$ on G. For right invariant systems, interchange everywhere << left >> and << right >>, << − >> and << + >>, and replace the formula for H by $H((T_eR_g)^*\alpha_g, \rho_*(g)a) = H_a(\alpha_g)$.*

This theorem is sometimes not applied directly, since ρ and V may not be given but need to be discovered in the course of analyzing the system. In the process of finding V one also discovers whether left or right actions of G are involved. The determination of ρ and V usually is done by means of the evolution of $a \in V^*$. We shall elaborate on this remark in the context of the examples. Our first example is ideal compressible flow and the second is MHD.

Ideal compressible isentropic fluids[2]. Let Ω be a compact submanifold of $\mathbb{R}^3$ with smooth boundary, filled with a moving fluid free of exterior forces. Denote by $x(t) = \eta_t(X) = \eta(X,t)$ where $X \in \Omega$, the trajectory of a fluid particle which at time $t = 0$ is at X. As is customary in continuum mechanics, capital letters will denote entities in the reference configuration, *i.e.,* in << body >> coordinates; lower case letters denote spatial entities (see Marsden and Hughes [1983]).

Given $\eta_t : \Omega \to \Omega$, a time dependent diffeomorphism of Ω, denote by $v_t(x) = v(x,t)$ the spatial velocity field of the fluid, *i.e.,*

$$\frac{\partial \eta(X,t)}{\partial t} = v(\eta(X,t),t);$$

v_t is thus a time dependent vector field with flow η_t. Let $\rho_t(x) = \rho(x,t)$, denote the mass density of the fluid at time t and ρ_0 the mass density in the reference configuration. Thus the physical problem of fluid motion has as configuration space the group $\mathcal{D}$ of diffeomorphisms of Ω, and ρ_t is determined by the configuration when ρ_0 is known.

The equations of motion are derived from three fundamental principles: conservation of mass, momentum, and energy. It will be useful to recall these well-known derivations, as they are relevant to understanding how Theorem 9.3 is applied.

[2]For expository reasons, technical details on function spaces are omitted. See Ebin and Marsden [1970] for what is needed. The velocity fields are at least C^1. For incompressible fluids, see Marsden and Weinstein [1982b].

(a) The principle of conservation of mass stipulates that mass can be neither created or destroyed, *i.e.,*

$$\int_{\eta_t(\mathcal{W})} \rho_t(x)\, dx = \int_{\mathcal{W}} \rho_0(X)\, dX$$

for all compact $\mathcal{W}$ with nonempty interior having smooth boundary. Changing variables, this becomes

$$\int_{\mathcal{W}} \eta_t^*(\rho_t(x)dx) = \int_{\mathcal{W}} \rho_0(X)\, dX \tag{9.5}$$

which is equivalent to

$$(\eta_t^* \rho_t) J(\eta_t) = \rho_0 \tag{9.6}$$

where $J(\eta_t) = dx/dX$ is the Jacobian of η_t. Using the relation between Lie derivatives and flows, this is equivalent to the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho_t v) = 0. \tag{9.7}$$

The present derivation of conservation of mass shows that the physical entity to be dealt with is the density $\rho\, dx$ rather than the function ρ and that 9.5 is more convenient than the standard fluid mechanics formulation 9.7. This observation will be crucial later on.

(b) The balance of momentum is described by Newton's second law: the rate of change of momentum of a portion of the fluid equals the total force applied to it. Since we assume that no external forces are present, the only forces acting on the fluid are forces of stress. The assumption of an *ideal fluid* means that the force of stress per unit area exerted across a surface element at x, with outward unit normal n at time t, is $-p(x,t)n$ for some function $p(x,t)$ called the pressure. With this hypothesis, the balance of momentum becomes *Euler's equations of motion*

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p \tag{9.8}$$

with the boundary condition $v || \partial\Omega$ (no friction exists between fluid and boundary) and the initial condition $v(x,0) = v_0(x)$ on Ω.

(c) The kinetic energy of the fluid is $1/2 \int_\Omega \rho ||v||^2\, dx$. The assumption of an *isentropic* fluid means that the internal energy of the fluid is $\int_\Omega \rho w(\rho)\, dx$ and $p = \rho^2 w'(\rho)$ for some real valued function w of ρ. These hypotheses imply that the total energy, which should be the Hamiltonian of the system, is conserved.

The configuration space of this problem is $\mathcal{D}$, and so the corresponding phase space is $T^*\mathcal{D}$. For $\eta \in \mathcal{D}$ we have $T_\eta \mathcal{D} = \{V_\eta : \Omega \to T\Omega \mid V_\eta(X) \in T_{\eta(X)}\Omega\}$ and

$T^*_\eta\mathcal{D} = \{\alpha_\eta : \Omega \to T^*\Omega \otimes \Lambda^3(\Omega) \mid \alpha_\eta(X) \in T^*_{\eta(X)}\Omega \otimes \Lambda^3_X(\Omega)\}$; the pairing between the velocity V in $T_\eta\mathcal{D}$ and the momentum density α_η in the dual space $T^*_\eta\mathcal{D}$ is

$$\langle \alpha_\eta, V_\eta \rangle = \int_\Omega \alpha_\eta(X) \cdot V_\eta(X).$$

For later use, we shall express the energy on $T^*\mathcal{D}$ by passing to material coordinates. Let $V_t(X) = \partial\eta(X,t)/\partial t$ be the material velocity. Then $V_t = v_t \circ \eta_t$, showing that V_t is not a vector field but an element of $T_{\eta_t}\mathcal{D}$. The metric on Ω and the density $\rho_0(X)dX$ establish an isomorphism of $T_\eta\mathcal{D}$ with $T^*_\eta\mathcal{D}$ given by $V_\eta(X) \mapsto \alpha_\eta(X) = \rho_0(X)V^b_\eta(X)dX$ where $b : T\Omega \to T^*\Omega$ is the bundle isomorphism induced by the metric on Ω. Finally, $\mathcal{D}$ has a smooth metric defined by

$$\langle\langle V_\eta, W_\eta \rangle\rangle_\eta = \int_\Omega V_\eta(X) \cdot \mathcal{W}_\eta(X)\rho_0(X)\, dX$$

which determines a length function on $T^*\mathcal{D}$, denoted by $||\ ||$.

To apply Theorem 9.3 directly, use a simple change of variables to express the energy on $T^*\mathcal{D}$ as

$$H_{\rho_0}(\alpha_\eta) = \frac{1}{2}||\alpha_\eta||^2 + \int_\Omega \rho_0(X) w(\rho_0(X) J_\eta^{-1}(X))\, dX. \tag{9.9}$$

H_{ρ_0} clearly depends smoothly on ρ_0, and again the change of variables formula shows that is it *right* invariant under the action of the subgroup

$$\mathcal{D}_{\rho_0} = \{\varphi \in \mathcal{D} \mid \rho_0 = (\rho_0 \circ \varphi^{-1}) J_{\varphi^{-1}}\}$$

where J_φ denotes the Jacobian of φ. This shows that we should work with the density $\rho_0(X)dX$ and not with the function ρ_0, and consider the representation of $\mathcal{D}$ on $\mathcal{F}(\Omega) = C^\infty(\Omega)$ to be the push-forward, *i.e.,* $(\eta, f) \mapsto \eta_* f$. Then the induced left representation on $\mathcal{F}(\Omega)^* =$ densities on Ω, is again push-forward and

$$\mathcal{D}_{\rho_0} = \{\varphi \in \mathcal{D} \mid \varphi_* = (\rho_0(X)dX) = \rho_0(X)dX\}.$$

Let $S = \mathcal{D} \times F(\Omega)$ be the semidirect product of $\mathcal{D}$ with $F(\Omega)$ by the push-forward representation and $s = \mathfrak{X} \times F(\Omega)$ its Lie algebra. By Theorem 9.3 for right invariant systems the family of Hamiltonians H_{ρ_0} determines a unique Hamiltonian system on s^*_+. If $\bar{M} = (T_e R_\eta)^* \alpha_\eta \in T^*_e\mathcal{D} = \mathfrak{X}^* =$ one-form densities, for every $\mathcal{W} \in \mathfrak{X}$

$$\begin{aligned} \langle \bar{M}, \mathcal{W} \rangle &= \int_\Omega \alpha_\eta(X) \cdot \mathcal{W}(\eta(X))\, dX \\ &= \int_\Omega \rho_0(X) V^b_\eta(X) \cdot \mathcal{W}(\eta(X))\, dX = \int_\Omega \rho(x) v(x) \cdot \mathcal{W}(x)\, dx, \end{aligned}$$

i.e., $\bar{M} = \rho(x) v^b(x)\, dx$. Thus the Hamiltonian H on s^*_+ has the expression $H(\bar{M}, \rho(x)dx) = H((T_e R_\eta)^* \alpha_\eta, \eta_*(\rho_0(X)dX)) = H_{\rho_0}(\alpha_\eta) = 1/2 \int_\Omega \rho(x)||v(x)||^2 dx + \int \rho(x) w(\rho)(x)\, dx$

which is the physical energy function. Thus, identifying $\bar{M}$ with the momentum density $M(x) = \rho(x)v(x)$ and $\rho(x)dx$ with $\rho(x)$, the physical energy function

$$H(M,\rho) = \frac{1}{2}\int_\Omega \frac{1}{\rho(x)}||M(x)||^2 dx + \int_\Omega \rho(x)w(\rho)(x)dx$$

defines Lie-Poisson equations on $\mathfrak{X}^* \times F(\Omega)^*_+$; we shall determine them later on.

We now apply Theorem 9.3 backwards. Start with the configuration space $\mathcal{D}$, the physical energy function $H(M,\rho)$, conservation of mass 9.7, balance of momentum 9.8, and $p = \rho^2 w'(\rho)$. Then notice that 9.7 is equivalent to $L_v(\rho(x)dx) = 0$, *i.e.,* $\eta_t^*(\rho(x)dx) = \rho_0(X)dX$, for ρ_0 the initial mass density. Hence the dual of the representation space is the space of densities, *i.e.,* the representation space is $F(\Omega)$. Moreover, $\eta_{t^*}(\rho_0(X)dX) = \rho(x)dx$ shows that the induced *left* representation on $F(\Omega)^*$ is push-forward, so that by the general formula on the evolution of the parameter $a \in V^*$ in Theorem 9.3, the representation of $\mathcal{D}$ on $F(\Omega)$ is also push-forward. We now identify M with $\bar{M}$ and $\rho(x)$ with $\rho(x)dx$, so that the physical energy function $H(\bar{M},\rho(x)dx)$ is defined on $\mathfrak{X}^* \times F(\Omega)^*$. Since $H(\bar{M},\rho(x)dx) = H_{\rho_0}(\alpha_\eta)$ for $(T_eR_\eta)^*\alpha_\eta = \bar{M}, \eta_*(\rho_0(X)dX) = \rho(x)dx$, and H_{ρ_0} is *right* invariant under $\mathcal{D}_{\rho_0}$, Theorem 9.3 for right invariant systems can be applied yielding Lie-Poisson equations on s^*_+.

Having seen how Theorem 9.3 was applied backwards let us make some general remarks. In many examples, one is given the phase space T^*G, but it is not obvious *a priori* what V and ρ should be. The phase space T^*G is often interpreted as "material" or "Lagrangian" coordinates, while the equations of motion may be partially or wholly derived in "spatial" or "Eulerian" coordinates. This means that the Hamiltonian might be given directly on a space of the form $\mathfrak{g}^* \times V^*$, where the evolution of the V^* variable is by "dragging along" or "Lie transport" *i.e.,* it is of the form $t \mapsto \rho^*(g(t))a$ for left invariant systems (or $t \mapsto \rho_*(g(t))a$ for right invariant ones), where $a \in V^*$ and $g(t)$ is the solution curve in the configuration space G. This then determines the representation ρ and shows whether one should work with left or right actions. The relation between H and H_a in the theorem uniquely determines H_a, which is automatically G_a-invariant, and 9.1 and 9.2 give the corresponding Lie-Poisson bracket and equations of motion. The parameter $a \in V^*$ often appears in the form of an initial condition on some physical variable of the given problem.

Let us now return to ideal compressible isentropic fluids. To write the bracket and the Lie-Poisson equations explicitly, remark first that the induced Lie algebra representation $\rho' : \mathfrak{X} \to \mathrm{End}\,(F(\Omega))$ is given by $\rho'(v)f = -L_v f$, where L_v is the Lie derivative. To see this, denote by η_t the flow of v and obtain

$$\begin{aligned}
\rho'(v)f &= \left.\frac{d}{dt}\right|_{t=0} \rho(\eta_t)f = \left.\frac{d}{dt}\right|_{t=0} (\eta_t)_* f \\
&= \left.\frac{d}{dt}\right|_{t=0} f \circ \eta_{-t} = -df(v) = -L_v f.
\end{aligned}$$

The bracket is hence given by 9.1:

$$\begin{aligned}\{F,G\}_+(M,\rho) &= \int_\Omega M\cdot\left[\frac{\delta F}{\delta M},\frac{\delta G}{\delta M}\right]dx\\ &\quad -\int_\Omega \rho\left(L_{\frac{\delta F}{\delta M}}\frac{\delta G}{\delta\rho}-L_{\frac{\delta G}{\delta M}}\frac{\delta F}{\delta\rho}\right)dx.\end{aligned}$$

But the left Lie bracket on $\mathfrak{X}$ which shows up in the first term is *minus* the standard Lie bracket of vector fields (Abraham and Marsden [1978], Example 4.1 G), so this becomes explicitly

$$\begin{aligned}\{F,G\}_+(M,\rho) &= \int_\Omega M\cdot\left(\left(\frac{\delta G}{\delta M}\cdot\nabla\right)\frac{\delta F}{\delta M}-\left(\frac{\delta F}{\delta M}\cdot\nabla\right)\frac{\delta G}{\delta M}\right)dx\\ &\quad +\int_\Omega \rho\left(\frac{\delta G}{\delta M}\cdot\left(\nabla\frac{\delta F}{\delta\rho}\right)-\frac{\delta F}{\delta M}\cdot\left(\nabla\frac{\delta G}{\delta\rho}\right)\right)dx. \qquad (9.10)\end{aligned}$$

This bracket agrees with that of Morrison and Greene [1980]. From the Lie-Poisson equations $\dot{F}=\{F,H\}_+$ we can obtain the equations of motion directly by choosing $F=\int M_1\,dx, \int M_2\,dx, \int M_3\,dx, \int \rho\,dx$; the last one represents the equation of continuity and the first three the balance of momentum. We can also get the same result by using 10.2 directly, but with more computations. For example, the term $\rho'_{\delta H/\delta\mu^*}a$ corresponds to div M, as an easy integration by parts argument shows, so that the second component of X_H on s^*_+ yields $\rho=-\operatorname{div}M$, which is the continuity equation.

Remarks

1. In the same manner, one can treat the case of an incompressible inhomogeneous fluid. The semidirect product in question is now $\mathcal{D}_{\rm vol}\times F(\Omega)$, where $\mathcal{D}_{\rm vol}$ denotes the volume preserving diffeomorphisms of Ω. For the technical details regarding the correct choices of function spaces, see Marsden [1976].

2. We can also allow σ, the entropy per unit volume, to be variable. The thermodynamics equation of entropy advection

$$\frac{\partial\sigma}{\partial t}+\operatorname{div}\left(\frac{\sigma}{\rho}M\right)=0 \qquad (9.11)$$

has to be added to the compressible fluid equations. In addition, the internal energy is $w=w(\rho,\sigma)$ and the pressure is

$$p=\rho^2\left(\frac{\partial w}{\partial\rho}+\frac{\sigma}{\rho}\frac{\delta w}{\delta\sigma}\right).$$

Thus our system of partial differential equations is 9.7, 9.8, 9.11, with Hamiltonian 9.9 for $w=w(\rho,\sigma)$.

In our framework, this system is Hamiltonian on the dual of the Lie algebra of $\mathcal{D} \times F(\Omega) \times F(\Omega)$, where $\mathcal{D}$ acts on $F(\Omega) \times F(\Omega)$ by push-forward in each factor. Hence the bracket is given by, 9.10) to which the term

$$\int \sigma \left(\left(\frac{\delta G}{\delta M} \cdot \nabla \right) \frac{\delta F}{\delta \sigma} - \left(\frac{\delta F}{\delta M} \cdot \nabla \right) \frac{\delta G}{\delta \sigma} \right) dx \tag{9.12}$$

has been added. In this way $\dot{F} = \{F, H\}_+$ for $F = \int M_1 dx, \int M_2 dx, \int M_3 dx, \int \rho dx, \int \sigma dx$ becomes the system 9.7, 9.8, 9.11. One can also see the form of the Lie-Poisson equation directly from 9.2 by remarking that the term $\rho'_{\delta H/\delta \mu^*} a$ corresponds to $(\mathrm{div} M, \mathrm{div}((\sigma/\rho)M))$.

To finish this example, let us describe the relationship between the Lie-Poisson bracket on s^*

$$\{F, G\} = \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx\, dv$$

and that on $(\mathfrak{X} \times F)^*$ given by 9.10.

Proposition 9.4. *The map*

$$f \mapsto \left(M = \int v f(x, v)\, dv, \quad \rho = \int f(x, v)\, dv \right)$$

is a Poisson map from s^*_+ to $(\mathfrak{X} \times F)^*_+$.

Proof One may use a direct calculation, but it is more in context to observe that $\mathcal{D} \times F$ acts on s^* by the left action

$$(\eta, h) \cdot f = f \circ \eta^* \circ \tau_{dh}$$

where $\eta^* : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$ is the induced pull-back canonical transformation on $T^*\mathbb{R}^3 : \eta^*(x, p) = (\eta^{-1}(x), p \cdot T\eta(x))$ and τ_{dh} is fiber translation by dh, as in 8.1. (This action corresponds to regarding $\mathcal{D} \times F$ as a subgroup of $\mathcal{S}$ by $(\eta, h) \mapsto \tau_{-dh} \circ \eta_*$). The momentum map may be computed to be the given map $f \mapsto (M, \rho)$ as in 8.1. The result now follows from Proposition 4.3. QED

In summary, the map from the plasma variable f to the fluid variables (M, ρ) is the momentum map for a group action naturally occurring in the problem and therefore this map collapses the bracket for the Poisson-Vlasov equation to that for the Euler equations.

Our second example which involves semi-direct products is MHD.

Magnetohydrodynamics of an ideal compressible perfectly conducting fluid. We keep the same hypotheses and notations as above but in addition we assume that the fluid consists of charged particles in a quasi-neutral state. The configuration space remains $\mathcal{D}$, and conservation of mass is unchanged. In the balance of momentum law, one must add the net Lorentz force of the magnetic field created by the fluid in motion. In addition, the hypothesis of infinite conductivity leads one to the conclusion that magnetic lines are frozen in the fluid, *i.e.,* that they are transported along the particle paths. If ρ is the mass density, v the spatial velocity, $M = \rho v$ the momentum density,, and $B \in \Lambda^2(\Omega)$ the magnetic field regarded as a 2-form, then the equations of motion are

$$\begin{aligned} \dot{M}_i &= -\sum_j \frac{\partial}{\partial x_j}\left(\frac{M_i M_j}{\rho} + \delta_{ij}\left(p - \frac{1}{4}TrB^2\right) - \sum_k B_{ik}B_{kj}\right) \\ &\dot{\rho} + \operatorname{div} M = 0 \; i.e., \rho dx + L_v(\rho(x)dx) = 0 \end{aligned}$$

and

$$\dot{B} + L_v B = 0$$

where $\rho = \rho^2 w'(\rho)$ is the pressure and L_v the Lie derivative. The last equation just says that B is << frozen >> in the fluid. As before, the initial mass density ρ_0, is given. In addition, the initial magnetic field B_0 must now be specified. The energy of this fluid is given by

$$\begin{aligned} H(M,\rho,B) &= \frac{1}{2}\int_\Omega \frac{||M(x)||^2}{\rho(x)}\,dx \\ &+ \int_\Omega \rho(x)w(\rho)(x)\,dx + \frac{1}{2}\int_\Omega ||B(x)||^2\,dx \end{aligned} \tag{9.13}$$

where $||B||^2 = B_{12}^2 + B_{13}^2 + B_{23}^2$. Since the last two equations of motion are Lie transport equations, the pattern of the previous example shows that the relevant semidirect product is $\mathcal{S} = \mathcal{D} \times F(\Omega) \times \Lambda^1(\Omega)$, where $\mathcal{D}$ acts on $F(\Omega) \times \Lambda^1(\Omega)$ by push-forward on each factor. The Lie algebra is $s = \mathfrak{X} \times F(\Omega) \times \Lambda^1(\Omega)$, and its dual is $s^* = \mathfrak{X}^* \times F(\Omega)^* \times \Lambda^2(\Omega)$, the pairing between $\alpha \in \Lambda^1(\Omega)$ and $\beta \in \Lambda^2(\Omega)$ being $\langle \alpha, \beta \rangle = \int_\Omega \alpha \wedge \beta$.

To $H(M,\rho,B)$ there corresponds the Hamiltonian

$$\begin{aligned} H_{\rho_0,B_0}(\alpha_\eta) &= \frac{1}{2}\langle\langle \alpha_\eta, \alpha_\eta \rangle\rangle + \int_\Omega \rho_0(X)w(\rho_0(X)J_\eta^{-1}(X))\,dX \\ &+\frac{1}{2}\int_\Omega ||(\eta_* B_0 \circ \eta)(X)||^2 J_\eta(X)\,dX \end{aligned}$$

on $T^*\mathcal{D}$ which is right invariant under the action of

$$\mathcal{D}_{\rho_0,B_0} = \{\varphi \in \mathcal{D} \mid \rho_0 = (\rho_0 \circ \varphi^{-1})J_{\varphi^{-1}}, \varphi_* B_0 = B_0\}.$$

Thus Theorem 9.3 for right actions applies, and we conclude that $H(M,\rho,B)$ defines Lie-Poisson equations in $s_+^* = (\mathfrak{X}^* \times F(\Omega)^* \times \Lambda^2(\Omega))_+$. Recall again that $\mathfrak{X}$ has the *left* Lie bracket which is *minus* the usual one. The semidirect product Lie group is $S = \mathcal{D} \times F(\Omega) \times \Lambda^1(\Omega)$ where $\mathcal{D}$ acts on $F(\Omega) \times \Lambda^1(\Omega)$ by push forward. As in the previous example, $\mathfrak{X}$ act on $F(\Omega) \times \Lambda^1(\Omega)$ by minus the Lie derivative on each factor. The variables $(\bar{M}, \rho(x)\,dx, B) \in s_+^*$ are of the following geometric type: $\bar{M}(x) = \rho(x)v^\flat(x)\,dx \in \mathfrak{X}^*$ is a one-form density, $\rho(x)\,dx \in F(\Omega)^*$ is a density, and $B \in \Lambda^1(\Omega)^* = \Lambda^2(\Omega)$ is a two-form.

To write out the Lie-Poisson bracket 10.1 of two functions $F, G : s_+^* \to \mathbb{R}$, we again identify $\bar{M}(x)$ with $M(x) = \rho(x)v(x), \rho(x)\, dx$ with $\rho(x)$, and proceed as in the previous example. We get

$$\begin{aligned}
\{F,G\}_+(M,\rho,B) &= \int_\Omega M \cdot \left[\frac{\delta F}{\delta M}, \frac{\delta G}{\delta M}\right] dx \\
&\quad - \int_\Omega \rho \left(L_{\frac{\delta F}{\delta M}} \frac{\delta F}{\delta \rho} - L_{\frac{\delta G}{\delta M}} \frac{\delta F}{\delta \rho} \right) dx \\
&\quad - \int_\Omega B \wedge \left(L_{\frac{\delta F}{\delta M}} \frac{\delta G}{\delta B} - L_{\frac{\delta G}{\delta M}} \frac{\delta F}{\delta B} \right).
\end{aligned}$$

The first two terms coincide with 10.10. To bring the third term into a more familiar form, identify $B \in \Lambda^2(\Omega)$ with the vector field $\mathbf{B} = (B_1, B_2, B_3)$ by $B_1 = B_{23}, B_2 = B_{31}, B_3 = B_{12}$, and identify any one form $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3$ with the vector field $(\alpha_1, \alpha_2, \alpha_3)$. A straight-forward computation shows that

$$\begin{aligned}
B \wedge L_{\frac{\delta F}{\delta M}} \frac{\delta G}{\delta B} &= B_{12} \left[\left(\frac{\delta F}{\delta M}\right)_i \frac{\partial}{\partial x_i} \left(\frac{\delta G}{\delta B}\right)_3 \right. \\
&\quad \left. + \left(\frac{\delta G}{\delta B}\right)_i \frac{\partial}{\partial x_3} \left(\frac{\delta F}{\delta M}\right)_i \right] dx_1 \wedge dx_2 \wedge dx_3 \\
&\quad + B_{31} \left[\left(\frac{\delta F}{\delta M}\right)_i \frac{\partial}{\partial x_i} \left(\frac{\delta G}{\delta B}\right)_2 \right. \\
&\quad \left. + \left(\frac{\delta G}{\delta B}\right)_i \frac{\partial}{\partial x_2} \left(\frac{\delta F}{\delta M}\right)_i \right] dx_1 \wedge dx_2 \wedge dx_3 \\
&\quad + B_{23} \left[\left(\frac{\delta F}{\delta M}\right)_i \frac{\partial}{\partial x_i} \left(\frac{\delta G}{\delta B}\right)_1 \right. \\
&\quad \left. + \left(\frac{\delta G}{\delta B}\right)_i \frac{\partial}{\partial x_1} \left(\frac{\delta F}{\delta M}\right)_i \right] dx_1 \wedge dx_2 \wedge dx_3 \\
&= \mathbf{B} \cdot \left[\left(\frac{\delta F}{\delta M} \cdot \nabla\right) \frac{\delta G}{\delta B} + \left(\nabla \frac{\delta F}{\delta M}\right) \cdot \frac{\delta G}{\delta B} \right] dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}$$

by identifying the one-form $\delta G/\delta B$ with the corresponding vector field with components

$$\left(\frac{\delta G}{\delta B}\right)_1, \left(\frac{\delta G}{\delta B}\right)_2, \left(\frac{\delta G}{\delta B}\right)_3.$$

The Lie-Poisson bracket thus becomes

$$\begin{aligned}
\{F,G\}_+(M,\rho,B) &= \int_\Omega M \cdot \left[\left(\frac{\delta G}{\delta M} \cdot \nabla\right) \frac{\delta F}{\delta M} - \left(\frac{\delta F}{\delta M} \cdot \nabla\right) \frac{\delta G}{\delta M} \right] dx \\
&\quad + \int_\Omega \rho \left[\frac{\delta G}{\delta M} \cdot \left(\nabla \frac{\delta F}{\delta \rho}\right) - \frac{\delta F}{\delta M} \cdot \left(\nabla \frac{\delta G}{\delta \rho}\right) \right] dx \\
&\quad + \int_\Omega \mathbf{B} \cdot \left[\left(\frac{\delta G}{\delta M} \cdot \nabla\right) \frac{\delta F}{\delta B} - \left(\frac{\delta F}{\delta M} \cdot \nabla\right) \frac{\delta G}{\delta B} \right] dx \\
&\quad + \int_\Omega \mathbf{B} \cdot \left[\left(\nabla \frac{\delta G}{\delta M}\right) \cdot \frac{\delta F}{\delta B} - \left(\nabla \frac{\delta F}{\delta M}\right) \cdot \frac{\delta G}{\delta B} \right] dx. \quad (9.14)
\end{aligned}$$

This bracket coincides with the one derived by Morrison and Greene [1980], Holm and Kupershmidt [1983], and Morrison [1982]. With respect to this bracket, the equations are in Lie-Poisson form $\dot{F} = \{F, H\}_+$. The equations of motion are obtained by putting $F = \int M_i dx, \int \rho\, dx, \int B_i dx, i = 1, 2, 3$.

If entropy is variable, equation 10.11 must be added to the magnetohydrodynamics equations, where $w = w(\rho, \sigma)$ and $p = \rho^2(\partial w/\partial\rho + (\sigma/\rho)\partial w/\partial\sigma)$. The Lie-Poisson bracket for this case lives on the dual of the Lie algebra of $\mathcal{D} \times F(\Omega) \times \Lambda^1(\Omega) \times F(\Omega)$, the action of $\mathcal{D}$ being push forward. The bracket has the expression 9.14 to which 9.12 is added.

To obtain the Lie-Poisson description of the magnetohydrodynamic equations when div $B = 0$ and $B = dA$, we proceed in the following way. There are two obvious group homomorphisms

$$
\begin{array}{ccc}
D \times F(\Omega)\Lambda^1(\Omega) & \xrightarrow[\text{on last factor}]{\text{projection}} & D \times F(\Omega)\Lambda^2(\Omega)/\text{exact 1-forms} \\
 & & \Big\downarrow \begin{array}{l}\text{Exterior derivative}\\ \text{on last factor}\end{array} \\
 & & D \times F(\Omega)\Lambda^2(\Omega)
\end{array}
$$

Dualizing the induced Lie algebra mappings, we obtain Poisson maps

$$
\begin{aligned}
\mathfrak{X}^* \times F(\Omega)^* \times \Lambda^2(\Omega) &\to \mathfrak{X}^* \times F(\Omega)^* \times \{\alpha \in \Lambda^2(\Omega) \mid d\alpha = 0\} \\
&\to \mathfrak{X}^* \times F(\Omega)^* \times \Lambda^1(\Omega).
\end{aligned}
$$

In this way we obtain the Lie-Poisson formulation for magnetohydrodynamics in physical variables (M, ρ, B), with div $B = 0$ (variables: M, ρ, B), or with a magnetic potential (variables: M, ρ, A, for $B = dA$). For the case div $B = 0$ the bracket is still, 9.14, whereas for the case $B = dA$ the same bracket takes the form

$$
\begin{aligned}
\{F, G\}_+(M, \rho, A) \;=\; & \int_\Omega M \cdot \left[\left(\frac{\delta G}{\delta M} \cdot \nabla\right)\frac{\delta F}{\delta M} - \left(\frac{\delta F}{\delta M} \cdot \nabla\right)\frac{\delta G}{\delta M}\right] dx \\
& + \int_\Omega \rho \left[\frac{\delta G}{\delta M} \cdot \left(\nabla \frac{\delta F}{\delta \rho}\right) - \frac{\delta F}{\delta M} \cdot \left(\nabla \frac{\delta G}{\delta \rho}\right)\right] dx \\
& + \int_\Omega (\nabla \times \mathbf{A}) \cdot \left(\frac{\delta F}{\delta A} \times \frac{\delta G}{\delta M} - \frac{\delta G}{\delta A} \times \frac{\delta F}{\delta M}\right) dx \\
& + \int_\Omega \mathbf{A} \cdot \left[\left(\nabla \cdot \frac{\delta G}{\delta A}\right)\frac{\delta F}{\delta M} - \left(\nabla \cdot \frac{\delta F}{\delta A}\right)\frac{\delta G}{\delta M}\right] dx.
\end{aligned}
$$

where we identified the one-form A with $\mathbf{A} = (A_1, A_2, A_3)$. If variable entropy is present, the term 9.12 is added to the bracket.

In the next section we shall describe the Euler-Maxwell equations (fluid electrodynamics), which incorporates the full Maxwell equations rather than just the magnetic field B. The way one makes the transition is as follows.

Vlasov-Maxwell

↓ Poisson map generated by the action of $\mathcal{D} \times F$ (see 9.4 and 10.3)

Euler-Maxwell

↓ limiting operations that eliminate the electric field (and possibly modify multi-species to a single species).

MHD

The last step is analogous to other limiting operations in physics such as $c \to \infty$ (Maxwell-Vlasov $\to$ Poisson-Vlasov) and three body problem $\to$ restricted three body problem. We expect that the convergence of Poisson structures for these limits can be understood in the context of Weinstein [1983].

10 Multi-Species Fluid Electrodynamics

The next system we consider is multispecies fluid electrodynamics following Spencer and Kaufman [1982] and Spencer [1982]. Our treatment will hold for an arbitrary number of species although the case of two oppositely charged species is most commonly discussed.

Each fluid species, labelled with subscripts s, is composed of particles of mass m_s and charge q_s. We define $a_s = q_s/m_s$. Then, in terms of the fluid velocities u_s, mass densities p_s, specific entropies σ_s, electric field E, and magnetic field B, the equations of ideal multi-fluid dynamics, in rationalized units, are

$$\dot{E} = \nabla \times B - \sum_s a_s \rho_s u_s \qquad \dot{B} = -\nabla \times E \tag{10.1}$$

$$\nabla \cdot E = \sum_s a_s \rho_s + \rho_{\text{ext}} \nabla \cdot B = 0 \tag{10.2}$$

$$\dot{\rho}_s = -\nabla \cdot (\rho_s u_s) \tag{10.3}$$

$$\dot{\sigma}_s = -u_s \cdot \nabla \sigma_s \tag{10.4}$$

$$\dot{u}_s = -(u_s \cdot \nabla) u_s + a_s (E + u_s \times B) - \rho_s^{-1} \nabla p_s \tag{10.5}$$

where the specific internal energy $U_s(\rho_s, \sigma_s)$, expressed as an equation of state, yields the (partial) pressure p_s according to

$$p_s = \rho_s^2 \partial U_s / \partial \rho_s. \tag{10.6}$$

Equations (10.1)– (10.5) are the Maxwell equations, which we have already encountered in §5, equations 5.2- 5.5. The only difference between the two versions is the set of dynamical variables in which the current and charge densities are expressed. In addition, we have allowed here for the presence of a static external charge density ρ_{ext}. It can be shown that the inclusion of this term does not upset the conservation of energy, while an analogous term J_{ext} to allow for an external current density does. Equation 10.3 is the continuity equation in physical space for the fluid species s, and may be derived in essentially the same fashion as was equation 5.1, which is a continuity equation in phase space. We neglect heat flow, and therefore express entropy convection by the adiabatic equation 10.4, which states that the convective derivative of the entropy is zero. Hence the entropy of each fluid element is constant. Equation 10.5 is the equation of motion. The force is comprised of two terms, due to electromagnetic effects (the Lorentz force term) and stress effects (the pressure gradient term).

As we have seen in the preceding sections, our constructions are made more natural by using momentum variables, rather than velocity variables. We therefore replace the velocity fields u_s by the corresponding momentum densities $M_s \equiv \rho_s u_s$. Then the phase space consists of the set of dynamical variables $(M_s, \rho_s, \sigma_s, E, B)$, and the energy of the system is

$$\begin{aligned} H(M_s, \rho_s, \sigma_s, E, B) &= \sum_s \int \left(\frac{1}{2} \rho_s^{-1} |M_s|^2 + \rho_s U_s \right) d^3x \\ &+ \int \left(\frac{1}{2} |E|^2 + \frac{1}{2} |B|^2 \right) d^3x \end{aligned} \tag{10.7}$$

(We shall sometimes use the notation $M_s \equiv (M_1, \ldots, M_k)$ for k species. Whether this is the case, or whether M_s refers to the single species s, will always be clear from context). The first integral has terms for the kinetic energy of the fluid and for the internal energy of each species. The internal energy term is absent in the Vlasov formulation because of our assumption there of a collisionless plasma. It is present here and in MHD in order to take into account conservative particle interactions which, in contrast to the case of a plasma described by the Vlasov model, are important and often dominate fluid motion. The second integral in 10.7 represents the energy content of the electromagnetic field.

In accordance with our general scheme, the vacuum Maxwell equations, discussed fully in §7, and the equations for ideal fluid flow, discussed in §9, must now be coupled, in order to yield equations 10.1 and 10.3 in the form

$$\dot{F} = \{F, H\}, \tag{10.8}$$

where F represents a function of the dynamical variables.

The Hamiltonian, written in terms of the canonical momenta $N_s = M_s + a_s \rho_s A$ and the other pre-reduction variables, is

$$\begin{aligned} H(N_s, \rho_s, \sigma_s, A, Y) &= \sum_s \int \left[\frac{1}{2} \rho_s^{-1} |N_s - a_s \rho_s A|^2 + \rho_s U_s \right] d^3x \\ &+ \frac{1}{2} \int [|\nabla \times A|^2 + |Y|^2] \, d^3x. \end{aligned} \tag{10.9}$$

To apply reduction to the full phase space $\mathcal{P} = \{(N_s, \rho_s, \sigma_s, A, Y)\} \equiv \mathfrak{g}_s^* \times T^*\mathcal{A}$, the action of the gauge group G on $T^*\mathcal{A}$, $A \mapsto A+\nabla\varphi, \varphi : \mathbb{R}^3 \to \mathbb{R}$, must be extended to an action Φ_φ of G on all of $\mathcal{P}$. We require that $H \circ \Phi_\varphi = H$, and that Φ_φ preserves Poisson brackets of functionals F, G on $\mathcal{P}$, *i.e.*, $\{F \circ \Phi_\varphi, G \circ \Phi_\varphi\} = \{F, G\} \circ \Phi_\varphi$. It is obvious from Equation 10.9 that the action

$$\Phi_\varphi(N_s, \rho_s, \sigma_s, A, Y) = (N_s + a_s\rho_s\nabla\varphi, \rho_s, \sigma_s, A + \nabla\varphi, Y) \tag{10.10}$$

satisfies the first requirement, and one can show that it also satisfies the second. Indeed, the action of φ on the fluid variables is just a piece of the left action on $\mathcal{D} \times F$ on $(\mathfrak{X} \times F)^*$ described in 9.4, where G is identified with F.

To obtain the momentum map $J : \mathcal{P} \to \mathfrak{g}^*$ for the action Φ, it suffices to calculate $J_s : g_s^* \to \mathfrak{g}^*$, the momentum map on $g_s^* \equiv \{(N_s, \rho_s, \sigma_s)\}$. This is a straightforward calculation which is similar to 8.1; we get

$$J_s(N_s, \rho_s, \sigma_s) = -a_s\rho_s. \tag{10.11}$$

Therefore, the momentum on all of $\mathcal{P}$, obtained by summing

$$J(N_s, \rho_s, \sigma_s, A, Y) = J_{\mathcal{A}}(A, Y) + \sum_s J_s(N_s, \rho_s, \sigma_s)$$

is

$$J(N_s, \rho_s, \sigma_s, A, Y) = -\nabla \cdot Y - \sum_s a_s\rho_s. \tag{10.12}$$

With $E = -Y$ as in §7, reduction at the external charge density ρ_{ext} then specifies that the dynamics takes place on the level set of constant external charge:

$$\left.\begin{array}{rcl} J^{-1}(\rho_{\text{ext}}) & = & \{(N_s, \rho_s, \sigma_s, A, Y) \in \mathcal{P}| \\ \nabla \cdot E & = & \rho_{\text{ext}} + \sum_s a_s, \rho_s, E = -Y \end{array}\right\}.$$

Coordinates on the reduced phase space $J^{-1}(\rho_{\text{ext}})/G$ are now given by :

Proposition 10.1.

$$\left.\begin{array}{rcl} J^{-1}(\rho_{\text{ext}})/G & = & \{(M_s, \rho_s, \sigma_s, B, E)| \\ \nabla \cdot E & = & \rho_{\text{ext}} + \sum_s a_s, \rho_s \nabla \cdot B = 0 \end{array}\right\}.$$

Proof To elements $(N_s, \rho_s, \sigma_s, A, Y)$ of $\mathcal{P}$, associate quintuples $(M_s, \rho_s, \sigma_s, B, E)$, where $M_s = N_s - a_s\rho_s A, B = \nabla \times A$, and $E = -Y$. Then the proposition follows from the momentum map constructed above, and from a simple verification that two elements of $J^{-1}(\rho_{\text{ext}})$ are associated to the same quintuple if and only if they are related by the gauge transformation, 10.10. QED

It remains now to compute the Poisson structure on $J^{-1}(\rho_{\text{ext}})/G$.

Theorem 10.2. *For two functionals F, G of the field variables $(M_s, \rho_s, \sigma_s, E, B)$, the Poisson bracket is given by*

$$\begin{aligned}
&\{F, G\}(M_s, \rho_s, \sigma_s, E, B) \\
&\quad = \sum_s \{F, G\}(M_s, \rho_s, \sigma_s) + \{F, G\}(E, B) \\
&\qquad + \sum_s \int \left(\frac{\delta F}{\delta M_s} \cdot \frac{\delta G}{\delta E} - \frac{\delta G}{\delta M_s} \cdot \frac{\delta F}{\delta E} + B \cdot \left[\frac{\delta F}{\delta M_s} \times \frac{\delta G}{\delta M_s} \right] \right) a_s \rho_s d^3x
\end{aligned} \tag{10.13}$$

where the first term is the sum of equations 10.9 and 9.12 and the second is defined by Proposition 7.1.

Proof Given F and G, define $\bar{F}$ on $\mathcal{P}$ according to $\bar{F}(N_s, \rho_s, \sigma_s, A, Y) = F(M_s, \rho_s, \sigma_s, B, E)$. Define $\bar{G}$ similarly. Then $\{F, G\}$ is found by computing $\{\bar{F}, \bar{G}\}$ as the sum of equations, 9.10 and , 9.12, written for unreduced variables, and the canonical bracket on $T^*\mathcal{A}$, and by expressing the result in terms of the variables on $J^{-1}(\rho_{\text{ext}})/G$. QED

We observe that the first term of equation, 10.12 involves only the fluid variables and that the second is purely electromagnetic, while the third provides the coupling of the fluids to the electric and magnetic fields. Bilinearity, skew symmetry, and the Jacobi identity all follow for equation 10.12 by the method used in its derivation. In addition it is readily verified that the correct evolution equations for the phase space variables follow from equations 10.12 and 10.7. Note that the set of evolution equations so obtained will not be precisely equations 10.1 and 10.3, but rather equivalent equations involving the M_s as dynamical variables. Additional body forces, such as gravity, can easily be incorporated by the inclusion of an appropriate term in the Hamiltonian. Finally, equations 10.2, rather than being postulated separately as initial conditions, follow from the gauge invariance of electromagetism.

Finally we note that the gauge group for the variables (A, Y) in both the Maxwell-Vlasov case and here commutes with the $\mathcal{D} \times F$ action described in 9.4. Thus, as $\mathcal{D} \times F$ clearly induces a Poisson map $(f, A, Y) \mapsto (M, \rho, A, Y)$ before reduction, it must also induce one after reduction. Thus we arrive at :

Theorem 10.3. *The map $(f_s, E, B) \mapsto (M_s, \rho_s, E, B)$ from the multispecies Maxwell-Vlasov phase space to the (isentropic) Euler-Maxwell phase space given*

$$M_s = \int v f_s(x, v) dv, \rho_s = \int f_s(x, v)\, dv$$

is a Poisson map with respect to the symplectic structures 9.1 and 10.12 (with entropy deleted).

Thus one can arrive at the same fluid electrodynamic bracket either by the direct reduction construction given by 10.1 and 10.2 or by collapsing the Maxwell-Vlasov bracket by a naturally occuring momentum map. For the bracket in (N, E, B) variables and another connection with semi-direct products, see Marsden, Ratiu and Weinstein [1983].

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