

Conifold transitions and five-brane condensation in M-theory on $Spin(7)$ manifolds

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Abstract

We conjecture a topology-changing transition in M-theory on a non-compact asymptotically conical $Spin(7)$ manifold, where a 5-sphere collapses and a $\mathbb{C}P^2$ bolt grows. We argue that the transition may be understood as the condensation of M5-branes wrapping S^5 . Upon reduction to ten dimensions, it has a physical interpretation as a transition of D6-branes lying on calibrated submanifolds of flat space. In yet another guise, it may be seen as a geometric transition between two phases of type IIA string theory on a G_2 holonomy manifold with either wrapped D6-branes, or background Ramond–Ramond flux. This is the first non-trivial example of a topology-changing transition with only 1/16 supersymmetry.

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1. Introduction

Topology-changing transitions in string theory are of great interest [1]. These have been well studied in compactifications of type II string theory on Calabi–Yau manifolds, where the residual $\mathcal{N} = 2$ supersymmetry provides much control over the dynamics. There are two prototypical examples. The flop transition, in which a two-cycle shrinks and is replaced by a different two-cycle, proceeds smoothly in string theory [2, 3]. In contrast, the conifold transition, in which a three-cycle shrinks and a two-cycle emerges, is accompanied by a phase transition in the low-energy dynamics which can be understood as the condensation of massless black holes [4, 5].

In the past year, there has been great progress in understanding similar effects in compactifications of M-theory on manifolds of G_2 holonomy, where the resulting four-dimensional theories have $\mathcal{N} = 1$ supersymmetry. There is, once again, an analogue of the flop transition; this time three-cycles shrink and grow and, as with the Calabi–Yau example, the process is smooth [6, 7]. Other G_2 geometrical transitions involving shrinking $\mathbb{C}\mathbb{P}^2$ s have also been discussed [7]. These proceed via a phase transition but, unlike the conifold transition, do not appear to be related to condensation of any particle state⁴ (for related work, see [8]).

The purpose of this paper is to study geometrical transitions in M-theory on eight-dimensional manifolds with $Spin(7)$ holonomy. Since the physics is very similar to the conifold transition in Calabi–Yau manifolds, let us briefly recall what happens in that case. As the name indicates, the conifold is a cone over a five-dimensional space which has topology $\mathbf{S}^2 \times \mathbf{S}^3$ (see figure 1). Two different ways to desingularize this space—called the deformation and the resolution—correspond to replacing the singularity by a finite size \mathbf{S}^3 or \mathbf{S}^2 , respectively. In type IIB string theory, the two phases of the conifold geometry correspond to different branches in the four-dimensional $\mathcal{N} = 2$ low-energy effective field theory. In the deformed conifold phase, D3-branes wrapped around the 3-sphere give rise to a low-energy field q , with mass determined by the size of the \mathbf{S}^3 . In the effective four-dimensional supergravity theory these states appear as heavy, point-like, extremal black holes. On the other hand, in the resolved conifold phase the field q acquires an expectation value reflecting the condensation of these black holes. Of course, in order to make the transition from one phase to the other, the field q must become massless somewhere and this happens at the conifold singularity, as illustrated in figure 1.

In this paper, we will argue that a similar phenomenon occurs in M-theory on a $Spin(7)$ manifold with a certain conical singularity. Apart from related orbifold constructions, there are essentially only two types of conical singularity which are known, at present, to admit a resolution to a smooth complete metric with $Spin(7)$ holonomy. These are listed in table 1. The first corresponds to a cone over $SO(5)/SO(3) = \mathbf{S}^7$ and was constructed a long time ago in [9]. The resolution of this conical singularity leads to a smooth non-compact $Spin(7)$ manifold isomorphic to an \mathbb{R}^4 bundle over \mathbf{S}^4 . Extending the ansatz to asymptotically locally conical

⁴ However, we shall argue below that this interpretation can be given to the same transition in type IIA string theory.

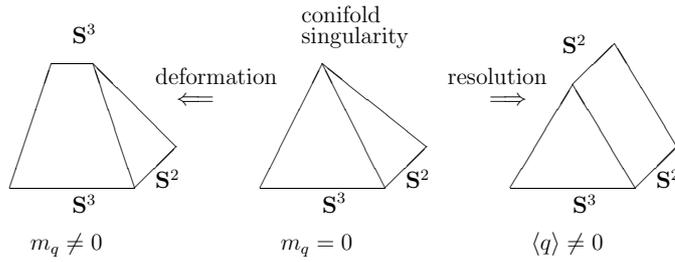


Figure 1. Conifold transition in type IIB string theory.

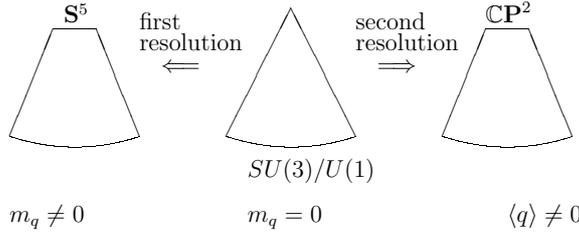


Figure 2. Conifold transition in M-theory on a manifold with $Spin(7)$ holonomy.

Table 1. The two cases of $Spin(7)$ conical singularity studied in this paper.

X	Topology	Base of cone
$\mathbb{R}^4 \times \mathbf{S}^4$	Chiral spin bundle of \mathbf{S}^4	$\mathbf{S}^7 = SO(5)/SO(3)$
$\mathbb{R}^4 \times \mathbf{CP}^2$	Universal quotient bundle	$N_{1,-1} = SU(3)/U(1)$
$\mathbb{R}^3 \times \mathbf{S}^5$	\mathbb{R}^3 bundle over \mathbf{S}^5	

(ALC) metrics, in which a circle stabilizes at finite size asymptotically, it was shown that [10] there are two families of topologically distinct resolutions of this cone, labelled $\mathbb{B}_8 \cong \mathbb{R}^4 \times \mathbf{S}^4$ and $\mathbb{A}_8 \cong \mathbb{R}^8$. One might therefore expect that \mathbb{A}_8 and \mathbb{B}_8 are different phases of M-theory on the same conical singularity. However, we shall argue below that this is *not* the case.

The second conical singularity discussed in the literature corresponds to a cone over $SU(3)/U(1)$ [11–13]. It is in this case that we suggest an interesting phase transition. We conjecture that there exist two possible ways of resolving this singularity, illustrated in figure 2. A well-known resolution consists of gluing in a copy of \mathbf{CP}^2 in place of the singularity. This leads to a one-parameter family of complete metrics with $Spin(7)$ holonomy on the universal quotient bundle \mathcal{Q} of \mathbf{CP}^2 , labelled by the volume of the \mathbf{CP}^2 bolt. They have topology,

$$\mathcal{Q} \cong \mathbb{R}^4 \times \mathbf{CP}^2. \tag{1.1}$$

A less well-known resolution of this $Spin(7)$ conifold may be obtained by blowing up a copy of the 5-sphere. Some numerical evidence for the existence of such a metric was presented in [12]. It remains an open problem to find an explicit $Spin(7)$ metric with these properties; a way to approach this, and a review of the known results, are presented in the appendix. In section 2, using the relationship with singularities of coassociative submanifolds in \mathbb{R}^7 , we provide further strong evidence for the existence of a complete $Spin(7)$ metric with an \mathbf{S}^5 bolt:

$$X \cong \mathbb{R}^3 \times \mathbf{S}^5. \tag{1.2}$$

Furthermore, we hypothesize that the manifolds with topology (1.1) and (1.2) are analogous to the resolution and deformation of the conifold, respectively. In other words, (1.1) and (1.2) are two phases of what one might call a *Spin(7) conifold*.

As with the conifold transition [4, 5], the topology-changing transition in M-theory on the *Spin(7)* cone over $SU(3)/U(1)$ has a nice interpretation in terms of the low-energy effective field theory. We argue that the effective dynamics of M-theory on the cone over $SU(3)/U(1)$ is described by a three-dimensional $\mathcal{N} = 1$ Abelian Chern–Simons–Higgs theory. The Higgs field q arises upon quantization of the M5-brane wrapped over the S^5 . We propose that, at the conifold point where the 5-sphere shrinks, these M5-branes become massless as suggested by the classical geometry. At this point, the theory may pass through a phase transition into the Higgs phase, associated with the condensation of these five-brane states.

To continue the analogy with the Calabi–Yau conifold, recall that the moduli space of type II string theory on the Calabi–Yau conifold has three semiclassical regimes. The deformed conifold provides one of these, while there are two large-volume limits of the resolved conifold, related to each other by a flop transition. In fact, the same picture emerges for the *Spin(7)* conifold. In this case, however, the two backgrounds differ not in geometry, but in the G -flux. It was shown in [11] that, due to the membrane anomaly of [14], M-theory on $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}\mathbb{P}^2$ is consistent only for half-integral units of G_4 through the $\mathbb{C}\mathbb{P}^2$ bolt. We will show that, after the transition from $X \cong \mathbb{R}^3 \times S^5$, the G -flux may take the values $\pm 1/2$, with the two possibilities related by a parity transformation. Thus, the moduli space of M-theory on the *Spin(7)* cone over $SU(3)/U(1)$ also has three semiclassical limits: one with the parity-invariant background geometry $\mathbb{R}^3 \times S^5$, and two with the background geometry \mathcal{Q} which are mapped onto each other under parity.

In view of the interesting phenomena associated with branes in the conifold geometry, and their relationship to the conifold transition [15, 16], it would be interesting to learn more about the *Spin(7)* transition using membrane probes in this background, and also to study the corresponding holographic renormalization group flows (for work in this area, see [17–19]).

Finally, we would like to mention a second interpretation of the *Spin(7)* topology-changing transition, which again has an analogue among lower-dimensional manifolds. To see this, let us first recall the story of the G_2 flop [6, 7] and its relationship to the brane/flux duality of the conifold [20]. In this scenario, one starts in type IIA theory with the familiar geometry of the deformed conifold, and wraps an extra D6-brane around the three-cycle. This yields a system with $\mathcal{N} = 1$ supersymmetry in $3 + 1$ dimensions. A natural question one could ask is: what happens if one tries to go through the conifold transition with the extra D6-brane? One possibility could be that the other branch is no longer connected and the transition is not possible. However, this is not what happens. Instead the physics is somewhat more interesting. According to [6, 7, 20], the transition proceeds, but now the two branches are smoothly connected, with the wrapped D6-brane replaced by RR 2-form flux through the S^2 . Since both D6-branes and RR 2-form tensor fields lift to purely geometric backgrounds in M-theory, the geometric transition can be understood as a flop-like transition in M-theory on a G_2 manifold:

$$X \cong \mathbb{R}^4 \times S^3.$$

For example, to obtain the resolved conifold with RR 2-form flux one can choose the ‘M-theory circle’ to be the fibre of the Hopf bundle (see [21] for a recent discussion)

$$S^1 \hookrightarrow S^3 \rightarrow S^2,$$

while choosing instead an embedding of the M-theory circle in \mathbb{R}^4 gives rise to the deformed conifold, with the D6-brane localized on the S^3 fixed point set [6, 22].

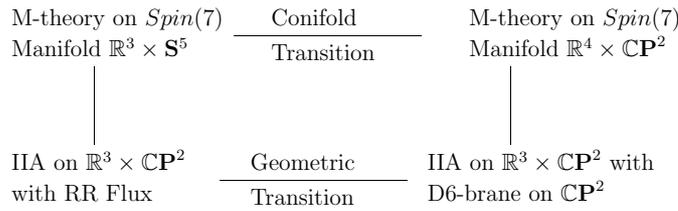


Figure 3. Geometric transition as a conifold transition in M-theory on a $Spin(7)$ manifold.

As we now explain, this procedure works in a slightly different and interesting way for our $Spin(7)$ manifolds with less supersymmetry. Our starting point is type IIA string theory on the G_2 holonomy manifold

$$M^7 \cong \mathbb{R}^3 \times \mathbf{CP}^2,$$

which is obtained by resolving the cone over $SU(3)/U(1)^2$. As we shall see in section 3, the effective low-energy theory is an $\mathcal{N} = 2$ supersymmetric Abelian Higgs model in $2 + 1$ dimensions, and its dynamics is very similar to compactification of M-theory on the same manifold M^7 [7]. In particular, the quantum moduli space consists of three branches, each of which arises from compactification on a manifold of topology M^7 , connected by a singular phase transition. Following the ideas of [6, 20], one could wrap an extra D6-brane over the \mathbf{CP}^2 and ask a similar question: what happens if one tries to go through a phase transition? Using arguments similar to [6, 23], we conjecture that the transition is again possible, via M-theory on a $Spin(7)$ manifold. More precisely, we claim that after the geometric transition one finds type IIA string theory on M^7 , where the D6-brane is replaced by RR flux through $\mathbf{CP}^1 \subset \mathbf{CP}^2$. This leads to a fibration:

$$\mathbf{S}^1 \hookrightarrow \mathbf{S}^5 \rightarrow \mathbf{CP}^2.$$

Hence the M-theory lift of this configuration gives a $Spin(7)$ manifold with the topology $\mathbb{R}^3 \times \mathbf{S}^5$. Similarly, one can identify the lift of M^7 with a D6-brane wrapped around \mathbf{CP}^2 as the $Spin(7)$ manifold $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbf{CP}^2$. Summarizing, we find that the conifold transition in M-theory on a $Spin(7)$ manifold is nothing but a geometric transition in IIA string theory on the G_2 manifold M^7 with branes/fluxes, as shown in figure 3. However, unlike the Calabi–Yau $\rightarrow G_2$ example, in our case of $G_2 \rightarrow Spin(7)$, the transition does not proceed smoothly in M-theory.

The paper is organized as follows. In the following section, we study the relationship between D6-branes on coassociative submanifolds of \mathbb{R}^7 , and their lift to M-theory on manifolds of $Spin(7)$ holonomy. We demonstrate the conifold transition explicitly from the D6-brane perspective. We further discuss several aspects of M-theory on $Spin(7)$ manifolds, including fluxes, anomalies, parity and supersymmetry. In section 3, we turn to the interpretation of the conifold transition from the low-energy effective action. We build a consistent picture in which the geometric transition is understood as a Coulomb to Higgs phase transition in a Chern–Simons–Higgs model. Finally, in section 4, we discuss further aspects of the geometry, and the different reductions to type IIA string theory by quotienting the $Spin(7)$ manifolds. We include explicit constructions of the relevant quotient for the brane/flux transition, as well as the D6-brane loci of section 2. In the appendix, we review the current state of knowledge for the geometry of $X \cong \mathbb{R}^3 \times \mathbf{S}^5$, and use the methods of Hitchin [24] to determine the properties of the metric.

2. D6-branes, M-theory and $Spin(7)$ conifolds

Our main interest in this section is to understand the geometric transition between the asymptotically conical (AC) $Spin(7)$ manifolds \mathcal{Q} and $\mathbb{R}^3 \times \mathbf{S}^5$, both of which are resolutions of the cone on the weak G_2 holonomy Aloff–Wallach space, $N_{1,-1} = SU(3)/U(1)$. Details of the metrics on these spaces are reviewed in the appendix. Here we start by understanding the transition through the relationship to D6-branes spanning coassociative submanifolds of \mathbb{R}^7 .

The key observation is that D6-branes lift to pure geometry in M-theory [25]. We start with a configuration of D6-branes in flat Minkowski space $\mathbb{R}^{1,9}$, with worldvolume $\mathbb{R}^{1,2} \times L$. The branes preserve at least two supercharges ($\mathcal{N} = 1$ supersymmetry in 2 + 1 dimensions) if we choose the four-dimensional locus $L \subset \mathbb{R}^7$ to be a coassociative submanifold [26], calibrated by

$$\Psi^{(4)} = \star\Psi^{(3)} = e^{2457} + e^{2367} + e^{3456} + e^{1256} + e^{1476} + e^{1357} + e^{1234}. \quad (2.3)$$

Upon lifting to M-theory, the D6-brane configuration becomes the background geometry $\mathbb{R}^{1,2} \times X$ where X is an eight-dimensional manifold equipped with a metric of $Spin(7)$ holonomy. When L is smooth, matching of states in the IIA and M-theory descriptions leads to the homology relations between L and X [11],

$$h_0(L) = h_2(X) + 1, \quad H_i(L, \mathbb{Z}) \cong H_{i+2}(X, \mathbb{Z}) \quad i > 0. \quad (2.4)$$

These relations imply, among other things, that the Euler numbers of X and L should be the same. In all our examples one can easily check that this is indeed true, say, via cutting out L inside X and showing that the Euler number of the remaining manifold with boundary is zero.

In this section, we examine the coassociative 4-fold geometry discussed by Harvey and Lawson [27]. Using equations (2.4), we show that certain transitions between D6-branes can be reinterpreted as geometric transitions between $Spin(7)$ manifolds of the type described in the introduction. Of course, one can also work in reverse and, given M-theory on a non-compact $Spin(7)$ manifold X , we may attempt to find a IIA description in terms of D6-branes on coassociative submanifolds of \mathbb{R}^7 . This is possible if X admits a $U(1)$ isometry, which we identify as the M-theory circle, such that the quotient becomes,

$$X/U(1) \cong \mathbb{R}^7. \quad (2.5)$$

In this case, all information about the topology of X is stored in the fixed point set L . This set has an interpretation as the locus of D6-branes in type IIA. The task of identifying L given X is somewhat involved and we postpone the calculations to section 4, where we explicitly construct the quotient to find the locus L .

2.1. D6-branes and the cone over $SO(5)/SO(3)$

Before we examine the example relevant for the conifold transition, let us first start with the $Spin(7)$ holonomy metric on the cone over $\mathbf{S}^7 \cong SO(5)/SO(3)$. As we shall see, the coassociative locus L for this example is intimately related to the $N_{1,-1}$ case of primary interest. The $Spin(7)$ holonomy metric on the cone over \mathbf{S}^7 has a resolution to the chiral spin bundle of \mathbf{S}^4 ,

$$X = \Sigma^- \mathbf{S}^4 \cong \mathbb{R}^4 \times \mathbf{S}^4. \quad (2.6)$$

This manifold has isometry group $Sp(2) \times Sp(1)$. Since $H^2(X; U(1))$ is trivial, M-theory compactified on X has no further symmetries arising from the C -field [11]. As we shall describe in detail in section 4, the D6-brane locus L arises if we choose the M-theory circle

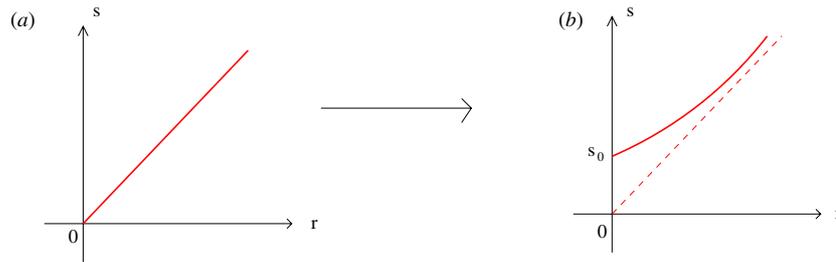


Figure 4. A curve in the r - s plane, whose $Sp(1)$ orbits sweep out coassociative submanifolds: (a) the singular cone over a squashed 3-sphere and (b) $L = H^1$.

to be embedded diagonally within the full isometry group. Upon reduction to type IIA string theory, this symmetry group is broken⁵ to $Sp(1) \times U(1)$, which acts on the locus L . The $Sp(1)$ -invariant coassociative cones in \mathbb{R}^7 can be completely classified [28–30]. Without going into details of these methods, we simply mention that the result derives from classification of three-dimensional simple subalgebras in the Cayley algebra. This leads, essentially, to two distinct families of coassociative cones: one discussed by Harvey and Lawson [27], and one constructed by Mashimo [30]. As we will argue below, it is the first case which is relevant to our problem. We leave the analysis of the second case to the interested reader.

In order to describe the locus L , it will prove useful to decompose \mathbb{R}^7 in terms of the quaternions \mathbb{H} ,

$$\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4 = \text{Im } \mathbb{H} \oplus \mathbb{H}. \tag{2.7}$$

The advantage of this notation is that it makes manifest a natural $Sp(1)$ action. To see this, let $x \in \text{Im } \mathbb{H}$ and $y \in \mathbb{H}$. Then for $q \in Sp(1)$, we have the action

$$q : (x, y) \rightarrow (qx\bar{q}, y\bar{q}). \tag{2.8}$$

This acts on the \mathbb{R}^3 factor as the usual $Sp(1) \sim SO(3)$ action. The action on the second factor \mathbb{H} may be understood in terms of the usual action of $Spin(4) \sim SO(4)$ on \mathbb{R}^4 , where we write $Spin(4) = Sp(1)_L \times Sp(1)_R$ and take $Sp(1) = Sp(1)_R$.

The utility of the action (2.8) on $\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}$ lies in the fact that it preserves the G_2 structure. This fact was employed by Harvey and Lawson to construct $Sp(1)$ -invariant calibrated submanifolds. Define a radial coordinate s on $\text{Im } \mathbb{H}$ and a radial coordinate r on \mathbb{H} , and consider the curve in the r - s plane given by

$$s(4s^2 - 5r^2)^2 = \rho \geq 0. \tag{2.9}$$

Then it can be shown that, under the action of $Sp(1)$, we sweep out a coassociative submanifold L of \mathbb{R}^7 [27]. To describe L explicitly, let us introduce a fixed unit vector $\epsilon \in \text{Im } \mathbb{H}$. Then

$$L = \{(sq\epsilon\bar{q}, r\bar{q}) : q \in Sp(1), (s, r) \in \mathbb{R}^+ \times \mathbb{R}^+, s(4s^2 - 5r^2)^2 = \rho\}. \tag{2.10}$$

When the deformation parameter vanishes, $\rho = 0$, the curve (2.9) has two solutions. For now we will not consider the simplest branch, $s = 0$, but instead restrict attention to

$$s = \frac{\sqrt{5}}{2}r, \tag{2.11}$$

for which the coassociative 4-fold L described in (2.10) is a cone over the squashed 3-sphere. It is on this submanifold that we place a single D6-brane, as depicted in figure 4(a). Now consider

⁵ The global structure is $U(2)$.

resolving the conical singularity of L by turning on $\rho > 0$. Again, there are two branches, and we restrict attention to $s > \sqrt{5}r/2$ as shown in figure 4(b). At large distances, $s, r \gg \rho$, L is asymptotic to a cone over a squashed 3-sphere. However, at small distances $r \rightarrow 0$, the coordinate s stabilizes at the finite value s_0 ,

$$s_0 = \left(\frac{\rho}{16}\right)^{1/5}. \quad (2.12)$$

At this point the principal \mathbf{S}^3 orbit therefore collapses to an \mathbf{S}^2 bolt at $r = 0$, and the global topology of the surface L can be identified with the spin bundle of \mathbf{S}^2 which we denote as H^1 ,

$$L = H^1 \cong \mathbb{R}^2 \times \mathbf{S}^2. \quad (2.13)$$

For this smooth L , we may use formulae (2.4) to determine the homology of X , the M-theory lift. We see that X is indeed described by a manifold of topology (2.6) as advertised. In section 4, we show explicitly that this L coincides with the fixed point set of a suitable circle action on $X = \Sigma^- \mathbf{S}^4$. In this construction, the $Sp(1)$ action of Harvey and Lawson sweeps out the $Spin(7)$ manifold X with a family of submanifolds. We will further show that the deformation parameter ρ , which measures the size of the \mathbf{S}^2 bolt of L , is related to the radius of the \mathbf{S}^4 bolt of X . Thus, the coassociative cone over the squashed 3-sphere (2.11) describes the reduction of the $Spin(7)$ cone over the squashed 7-sphere.

For this case, the D6-brane picture shows no sign of a geometrical transition to a manifold with different topology. Let us now turn to an example where such a transition does occur.

2.2. D6-branes and the cone over $SU(3)/U(1)$

We now turn to the main theme of the paper; the geometrical transition that occurs in the $Spin(7)$ cone over the weak G_2 holonomy Aloff–Wallach space, $N_{1,-1} = SU(3)/U(1)$. Let us start with the familiar resolution of this space to

$$\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}\mathbf{P}^2. \quad (2.14)$$

The isometry group of this space is $U(3)$. A further symmetry arises from the C -field. The relevant cohomology groups are [7, 11]

$$H^2(\mathcal{Q}; U(1)) = U(1)_J = H^2(N_{1,-1}; U(1)), \quad (2.15)$$

which ensure that there is a single unbroken global symmetry, denoted by $U(1)_J$, in the low-energy dynamics of M-theory compactified on \mathcal{Q} .

Remarkably, the calibrated D6-brane locus which lifts to the manifold \mathcal{Q} is described by the same Harvey–Lawson curve (2.9) that we met in the previous section. To see this, consider the conical D6-brane described by the branch (2.11). To this we simply add a further flat D6-brane, lying on the locus

$$s = 0. \quad (2.16)$$

The final configuration is depicted in figure 5(b). These two coassociative submanifolds coincide at the conical singularity $s = r = 0$. To resolve this singularity, we may deform the upper branch by $\rho \neq 0$ in the manner described in the previous section, while leaving the flat D6-brane described by (2.16) unaffected. The resulting coassociative 4-manifold L , shown in figure 5(c), is simply the disjoint union

$$L = H^1 \cup \mathbb{R}^4. \quad (2.17)$$

From equation (2.4), we see that this indeed has the requisite topology in order to lift to \mathcal{Q} . From the D6-brane perspective, the global $U(1)_J$ symmetry corresponds to the unbroken gauge symmetry on the flat D6-brane. It is worth noting that this result is reminiscent of the

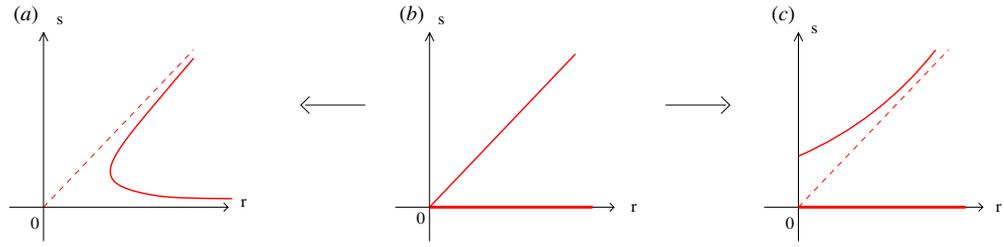


Figure 5. A curve in the r - s plane whose $Sp(1)$ orbits sweep out coassociative submanifolds: (a) $L = \mathbf{S}^3 \times \mathbb{R}$, (b) the singular cone and (c) $L = H^1 \cup \mathbb{R}^4$.

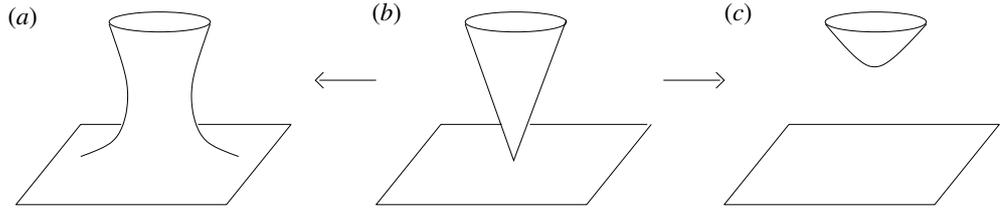


Figure 6. The D6-brane in spacetime, lying on the coassociative submanifolds: (a) $L = \mathbf{S}^3 \times \mathbb{R}$, (b) the singular cone and (c) $L = H^1 \cup \mathbb{R}^4$.

D6-brane description of the G_2 holonomy manifolds with topology $\mathbb{R}^3 \times \mathbf{S}^4$ and $\mathbb{R}^3 \times \mathbb{C}\mathbf{P}^2$ [7, 17]. In this case, the addition of an extra flat D6-brane is also responsible for the difference between the \mathbf{S}^4 and $\mathbb{C}\mathbf{P}^2$ non-contractible cycle.

We are now in a position to see the geometrical transition from the D6-brane perspective. We simply note that there is a second resolution of the singular locus L given by the union of (2.11) and (2.16). This arises as another branch of the smooth curve (2.9), where

$$0 < s < \frac{\sqrt{5}}{2}r. \tag{2.18}$$

In this situation, the two disjoint D6-branes smoothly join to lie on this branch as shown in figure 5(a). The turning point of the curve occurs at $s = \frac{1}{2}r = (\rho/2^8)^{1/5}$. Note that every point on the curve $s(r)$ is mapped onto a 3-sphere under the $Sp(1)$ action (2.8). Since this branch has both $s > 0$ and $r > 0$, there are no degenerate orbits and we conclude that the topology of the coassociative submanifold is given by

$$L \cong \mathbf{S}^3 \times \mathbb{R}. \tag{2.19}$$

This branch of $s(r)$ has two asymptotic components, $s \sim \frac{\sqrt{5}}{2}r$ and $s \sim 0$, whose $Sp(1)$ orbits coincide with the boundary of $H^1 \cup \mathbb{R}^4$. Taking the $\rho \rightarrow 0$ limit returns us again to the singular description of two D6-branes. Figure 6 depicts a diagram of the D6-brane transition in spacetime.

From the homology relations (2.4), we see that a D6-brane placed on $L \cong \mathbf{S}^3 \times \mathbb{R}$ lifts to a $Spin(7)$ manifold of topology

$$X \cong \mathbb{R}^3 \times \mathbf{S}^5. \tag{2.20}$$

For this manifold $H^2(X; U(1))$ is trivial, ensuring that there are no symmetries associated with the C -field in M-theory. This reflects the fact that there is no longer a flat D6-brane in the IIA

picture. We conclude that the geometrical transition between coassociative submanifolds of topology $H^1 \cup \mathbb{R}^4 \leftrightarrow \mathbf{S}^3 \times \mathbb{R}$ lifts in M-theory to a geometrical transition of $Spin(7)$ holonomy manifolds of topology $\mathbb{R}^4 \times \mathbf{CP}^2 \leftrightarrow \mathbb{R}^3 \times \mathbf{S}^5$.

We should point out that there are probably many other non-compact $Spin(7)$ manifolds with conical singularities that arise as M-theory lifts of D6-branes on coassociative submanifolds in \mathbb{R}^7 . Thus, we already mentioned coassociative cones constructed in [30]. It would be interesting to study the geometry, and especially the physics, of these examples in more detail.

2.3. Deformations of coassociative cones

As we have argued, D6-branes on coassociative cones L in \mathbb{R}^7 lift to conical $Spin(7)$ manifolds in M-theory. In the following section, we will be interested in the dynamics of M-theory on such geometries. Therefore, it will be important to study their deformations and determine whether or not they are L^2 normalizable. The latter aspect determines the interpretation in the low-energy theory: L^2 -normalizable deformations correspond to dynamical fields, whereas non-normalizable deformations have infinite kinetic energy and should rather be interpreted as true moduli, or coupling constants.

In M-theory on a non-compact (asymptotically conical) manifold X deformations of the metric, δg , may be L^2 normalizable if the asymptotic conical metric is approached suitably quickly. A specific criterion can be obtained by looking at the L^2 norm of δg [7]:

$$|\delta g|^2 = \int_X d^d x \sqrt{g} g^{ii'} g^{jj'} \delta g_{ij} \delta g_{i'j'}. \quad (2.21)$$

It follows that the deformation is L^2 normalizable if and only if $\delta g/g$ goes to zero faster than $r^{-d/2}$, where d is the dimension of X . In other words, the critical exponent that triggers L^2 normalizability is given by half the dimension of X . In our first example, $X \cong \mathbb{R}^4 \times \mathbf{S}^4$, the explicit form of the metric is available, and one can directly check that the deformation corresponding to the change of the size of the \mathbf{S}^4 is not normalizable [11]

$$|\delta g|^2 \rightarrow \infty. \quad (2.22)$$

For the $Spin(7)$ manifolds $X \cong \mathbb{R}^3 \times \mathbf{S}^5$ and $Q \cong \mathbb{R}^4 \times \mathbf{CP}^2$, no explicit asymptotically conical metric is known. For this reason, we turn to the D6-brane locus to extract the relevant information. At finite string coupling, the D6-brane configuration lifts to an asymptotically locally conical $Spin(7)$ metric, which retains a finite asymptotic circle. The conditions for normalizability in such a metric are weaker than those of the corresponding asymptotically conical space. Hence, if the D6-brane deformation is non-normalizable, we can draw the same conclusion about the deformation of the $Spin(7)$ geometry. If the D6-brane locus is normalizable, no such conclusion may be reached.

The question of deformations of branes on non-compact, special Lagrangian submanifolds has been addressed recently by Lambert [31]. For a submanifold L of flat space, it was shown that the critical exponent that determines the L^2 normalizability of the deformation is given by half the dimension of L . To see this, one must study the Lagrangian for the deformation modes [31]

$$L_{\text{eff}} = \int_L d^p \sigma \sqrt{-\det(g + \delta g)} - \int_L d^p \sigma \sqrt{-\det(g)}, \quad (2.23)$$

where $g_{ij} = \eta_{ij} + \partial_i x^I \partial_j x^J \delta_{IJ}$ is the induced metric on the D6-brane, and x^I are the D-brane embedding coordinates.

Applied to the two examples of coassociative cones discussed above, this criterion says that the deformation parameter ρ in equation (2.9) is L^2 normalizable only if

$$\frac{\delta x}{\delta \rho} \sim r^{-\alpha}, \quad \alpha > 2. \tag{2.24}$$

It is easy to show that this condition does not hold, and therefore ρ corresponds to a non-normalizable deformation in our models. In order to evaluate the derivative of x^I with respect to the modulus ρ it is convenient to gauge fix the asymptotic D6-brane worldvolume to be along \mathbb{H} (this is precisely what we have for $X \cong \mathbb{R}^3 \times \mathbb{S}^5$). Since the distance to $\mathbb{H} \subset \mathbb{R}^7$ is measured by the radial variable s , we essentially need to evaluate the asymptotic behaviour of $ds/d\rho$. From the defining polynomial (2.9) we compute

$$\frac{ds}{d\rho} \times [(4s^2 - 5r^2)^2 + 16s^2(4s^2 - 5r^2)] = 1, \tag{2.25}$$

and therefore,

$$\frac{ds}{d\rho} = [(\rho/s) + 16s^{3/2}\rho^{1/2}]^{-1}, \tag{2.26}$$

which at large r (large s) goes to zero as $r^{-3/2}$. This proves that ρ is indeed a non-normalizable deformation, in agreement with (2.22).

In a similar situation, it was conjectured by Joyce [32] that deformations of (strongly asymptotically conical) special Lagrangian cones $L \subset \mathbb{C}^3$ are, in fact, topological. Namely, it was argued that the total number of deformations of L is given by $b_1(L) + b_0(\partial L) - 1$. This is to be compared with deformations of compact special Lagrangian submanifolds, parametrized by $b_1(L)$ [33]. One might think that a similar topological formula holds for deformations of coassociative submanifolds $L \subset \mathbb{R}^7$, in which case the number of deformations is likely to be given by $b_2^+(L)$ (as in the compact case) plus some correction due to the non-compactness of L .

2.4. Anomalies and supersymmetry

The D6-brane configurations of the form $\mathbb{R}^{1,2} \times L$, where L is given by (2.13) or (2.17), are, as they stand, anomalous. In order to cancel the anomaly one must turn on a half-integral flux of the gauge field strength \mathcal{F} through the \mathbb{S}^2 bolt. Using the results of [34], we now argue that this flux does not break supersymmetry.

As shown by Freed and Witten [35], due to a global anomaly for fundamental strings ending on type II D-branes, the ‘ $U(1)$ gauge field’ \mathcal{A} on a D-brane worldvolume W should be interpreted globally as a spin^c connection. This means that, when W is not a spin manifold, the quantization law for the field strength $\mathcal{F} = d\mathcal{A}$ is shifted from standard Dirac quantization. Specifically, for all two-cycles $U_2 \subset W$ we have

$$\int_{U_2} \frac{\mathcal{F}}{2\pi} = \frac{1}{2} \int_{U_2} w_2(W) \pmod{\mathbb{Z}}, \tag{2.27}$$

where $w_2(W)$ is the second Stiefel–Whitney class of W . Consider a D6-brane whose worldvolume has a component $\mathbb{R}^{1,2} \times L$, where $L = H^1 \cong \mathbb{R}^2 \times \mathbb{S}^2$. It is easy to see that H^1 does not admit a spin structure, so that $w_2(L) \neq 0$. Consequently, in order to cancel the anomaly found by Freed and Witten, one must turn on a half-integral flux of \mathcal{F} through $\mathbb{S}^2 \subset H^1$. Thus

$$\int_{\mathbb{S}^2} \frac{\mathcal{F}}{2\pi} \in \mathbb{Z} + \frac{1}{2}. \tag{2.28}$$

It is crucial that this flux does not break supersymmetry. Fortunately, supersymmetric type II D-brane configurations with non-zero gauge field strengths were studied in [34]. The result relevant for us is that, for a D-brane wrapped over a coassociative submanifold, one can turn on a flux of the gauge field without breaking supersymmetry, provided

$$\mathcal{F} \cdot \Psi^{(3)} = 0. \quad (2.29)$$

For the explicit choice of $\Psi^{(3)}$ in equation (2.3), we find that the flux is supersymmetric if and only if it is anti-self-dual. The 2-form dual to the \mathbf{S}^2 bolt of H^1 is indeed anti-self-dual (the self-intersection number of \mathbf{S}^2 is -1), and we therefore do not break any supersymmetry by turning on the flux (2.28).

Let us now turn to the interpretation of this \mathcal{F} -flux in the M-theory lift of these configurations. It was shown in [11] that the \mathcal{F} -flux on the D6-brane may be identified with the G -flux in M-theory, using the relation

$$H^4(X; \mathbb{Z}) \cong H^2(L; \mathbb{Z}). \quad (2.30)$$

Since the Freed–Witten anomaly requires the existence of \mathcal{F} -flux, one may suspect that a similar consistency requirement leads to the presence of G -flux in the M-theory lift. Indeed, it was argued in [11] that the Freed–Witten anomaly for D6-branes wrapping a locus L is equivalent to Witten’s membrane anomaly for M-theory on the manifold X . Recall that the membrane path-integral is well defined only if the G -field satisfies the shifted quantization condition [14]

$$a \equiv \left[\frac{G}{2\pi} \right] - \frac{\lambda}{2} \in H^4(X; \mathbb{Z}), \quad (2.31)$$

where $\lambda(X) = p_1(X)/2 \in H^4(X; \mathbb{Z})$ is an integral class for a spin manifold X . If λ is even, one may consistently set $G = 0$. However, if λ is not divisible by 2 as an element of $H^4(X; \mathbb{Z})$, one must turn on a half-integral G -flux in order to have a consistent vacuum. The relationship between type IIA string theory and M-theory then leads to an identification of $w_4(X) \cong \lambda(X) \bmod 2$ and $w_2(L)$, under the mod 2 reduction of the isomorphism (2.30).

As explained in [11], for both $X = \Sigma^- \mathbf{S}^4 \cong \mathbb{R}^4 \times \mathbf{S}^4$ and $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}\mathbf{P}^2$ one can show that $\lambda(X)$ generates $H^4(X; \mathbb{Z}) \cong \mathbb{Z}$, and therefore one must turn on a half-integral G -flux through the bolt \mathbf{S}^4 or $\mathbb{C}\mathbf{P}^2$, respectively. Finiteness of the kinetic energy $\int G \wedge *G$, together with the equations of motion, requires G to be an L^2 -normalizable harmonic 4-form on X . In the case of $X = \Sigma^- \mathbf{S}^4$, such a 4-form G was constructed explicitly in [36]. We do not have the explicit asymptotically conical metric on $X = \mathcal{Q}$. However, the following result of Segal and Selby [37] ensures the existence of an L^2 -normalizable harmonic 4-form G on X which represents the generator of $H_{\text{cpt}}^4(X; \mathbb{R}) \cong \mathbb{R}$. On a complete manifold X , a harmonic form is necessarily closed and co-closed, so that G defines a cohomology class on X . In [37] it was argued that if the natural map $f : H_{\text{cpt}}^p(X) \mapsto H^p(X)$ takes a non-trivial compactly supported cohomology class b to a non-trivial ordinary cohomology class $f(b)$, then there is a non-zero L^2 -normalizable harmonic p -form on X representing b . For both *Spin*(7) manifolds $X = \Sigma^- \mathbf{S}^4$ and $X = \mathcal{Q}$, the natural map $f : H_{\text{cpt}}^4(X; \mathbb{Z}) \rightarrow H^4(X; \mathbb{Z})$ maps the generator of $H_{\text{cpt}}^4(X; \mathbb{Z}) \cong \mathbb{Z}$ to the generator of $H^4(X; \mathbb{Z}) \cong \mathbb{Z}$. In particular, the Thom class which generates $H_{\text{cpt}}^4(X)$ is represented by an L^2 -normalizable harmonic 4-form. Thus the existence of the 4-form found explicitly in [36] is guaranteed by the general result of [37]. We therefore set G equal to the L^2 -normalizable harmonic 4-form predicted by [37], appropriately normalized so that G satisfies the quantization condition (2.31).

We have seen previously that, from the D6-brane perspective, turning on a half-integral anti-self-dual \mathcal{F} -flux through the \mathbf{S}^2 bolt in H^1 does not break supersymmetry. Moreover, in

M-theory, this flux is dual to turning on a half-integral G -flux through the S^4 or $\mathbb{C}P^2$ bolt of Σ^-S^4 or \mathcal{Q} , respectively. We choose conventions such that the parallel spinor of the $Spin(7)$ manifold has positive chirality (in the $\mathfrak{8}_s$), ensuring that the Cayley form is self-dual⁶. In these conventions, the L^2 -normalizable 4-form constructed in [36] is self-dual. We thus learn that the half-integral self-dual G -flux does not break supersymmetry. It would be interesting to derive this directly from supersymmetry conditions in M-theory, extending a similar analysis of fluxes on compact $Spin(7)$ manifolds [38, 40].

In this subsection, we have seen that, in order to satisfy certain anomaly constraints, we must turn on the background G -flux for M-theory on $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}P^2$. This is related to the background \mathcal{F} -flux for the D6-brane configurations of sections 2.1 and 2.2. However, we have not yet determined which value of the G -flux arises after the conifold transition from $X = \mathbb{R}^3 \times S^5$ to \mathcal{Q} . For this, we must examine the boundary data more carefully.

2.5. Flux at infinity

In order to define the problem of M-theory on an asymptotically conical $Spin(7)$ manifold, we must specify both the base of the cone Y and the details of the 3-form. So far we have concentrated on the former. Here we turn our attention to the latter. We shall find that for vanishing asymptotic flux, M-theory on a $Spin(7)$ manifold that is asymptotic to the cone on the Aloff–Wallach space $Y = N_{1,-1}$ has three branches⁷. Two of these branches correspond to the two choices of sign for the half-integral G -flux through $\mathbb{C}P^2 \subset \mathcal{Q}$, whereas the other branch corresponds to a change of topology to $\mathbb{R}^3 \times S^5$.

As shown in [39], M-theory vacua which cannot be connected by domain walls are classified, in part, by $H^4(Y; \mathbb{Z})$. However, $H^4(Y; \mathbb{Z})$ is trivial for both $X = \Sigma^-S^4$ and \mathcal{Q} [11]. Therefore, the only asymptotic, non-geometric, datum needed to specify the model is the value of the total flux at infinity [39]

$$\Phi_\infty = N_{M2} + \frac{1}{192} \int_X (P_1^2 - 4P_2) + \frac{1}{2} \int_X \frac{G}{2\pi} \wedge \frac{G}{2\pi}. \tag{2.32}$$

Here N_{M2} is the number of membranes filling three-dimensional spacetime and, to preserve supersymmetry, we require $N_{M2} \geq 0$. The first and second Pontryagin forms of X are,

$$P_1 = -\frac{1}{8\pi^2} \text{Tr } R^2, \quad P_2 = -\frac{1}{64\pi^4} \text{Tr } R^4 + \frac{1}{128\pi^4} (\text{Tr } R^2)^2. \tag{2.33}$$

Note that anomaly cancellation requires $\Phi_\infty = 0$ for a compact space X [14, 41]. Indeed, when X is compact, the R^4 terms give

$$\frac{1}{192} \int_X (P_1^2 - 4P_2) = \frac{1}{192} (p_1(X)^2 - 4p_2(X)), \tag{2.34}$$

where $p_1^2(X)$ and $p_2(X)$ are Pontryagin numbers of X . When the structure group of the tangent bundle of X admits a reduction from $Spin(8)$ to $Spin(7)$, one can show that $p_1^2 - 4p_2 = -8\chi$, where⁸ $\chi(X)$ is the Euler number of X (see, for example, [42, 43]). This is equivalent to the existence of a nowhere vanishing spinor field on X . We, therefore, get the usual anomaly term $\chi(X)/24$ familiar in Calabi–Yau 4-fold compactifications.

⁶ This agrees with the conventions of [10] and [38], and allows comparison with the formulae for Calabi–Yau 4-folds in [39]. However, it is the opposite convention to [36]. Note also that the duality of the Cayley form is correlated with the use of D6-branes versus anti-D6-branes in sections 2.1 and 2.2.

⁷ We thank E Witten for explanations and very helpful discussions on these points.

⁸ The minus sign is correlated with our choice of orientation of X by choosing the non-vanishing spinor field to be in the $\mathfrak{8}_s$ representation.

When X is non-compact, things are a little more complicated. It will be crucial for us to compute the value of Φ_∞ for our backgrounds, but we are presented with an immediate problem since, on a non-compact manifold, the integral of the Pontryagin forms over X is not a topological invariant and we do not know explicitly the metric on X .

Fortunately, there is an effective way to compute the total flux at infinity, provided that a dual D6-brane model is available. We have already provided one such dual picture in section 2.2, consisting of D6-branes wrapping coassociative cycles in flat \mathbb{R}^7 . However, in this case the non-trivial part of the D6-brane worldvolume is also non-compact which does nothing to ameliorate our task. Thankfully, a second dual D6-brane model exists for the resolution of the cone to $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}\mathbf{P}^2$. This was described in the introduction, and will be dealt with in greater detail in section 4. In this case, the D6-brane wraps the coassociative four-cycle $B = \mathbb{C}\mathbf{P}^2$ of the G_2 holonomy manifold $M^7 = \Lambda_+^2(\overline{\mathbb{C}\mathbf{P}^2})$, the bundle of self-dual⁹ 2-forms over $\overline{\mathbb{C}\mathbf{P}^2}$. This space has topology $M^7 \cong \mathbb{R}^3 \times \mathbb{C}\mathbf{P}^2$.

The anomaly condition (2.32) relates the flux at infinity to the number of space-filling membranes, the integral of the Pontryagin forms and the G -flux. After reduction to type IIA theory the effective membrane charges become the effective charge of space-filling D2-branes. What is the type IIA interpretation of the anomaly formula (2.32)?

Since from the type IIA perspective the three-dimensional effective theory is obtained by compactification on a seven-dimensional G_2 manifold M^7 , there is no contribution to the D2-brane charge from the bulk. However, in type IIA theory we also have a space-filling D6-brane wrapped on the coassociative four-cycle $B = \mathbb{C}\mathbf{P}^2$ inside M^7 . Due to the non-trivial embedding of the D6-brane worldvolume in spacetime, the Ramond–Ramond fields in the bulk couple to the gauge field strength \mathcal{F} on the D6-brane. Specifically, we have

$$I_{WZ} = - \int_{\mathbb{R}^3 \times B} C_* \wedge \text{ch}(\mathcal{F}) \wedge \sqrt{\frac{\hat{A}(TB)}{\hat{A}(NB)}}, \quad (2.35)$$

where TB (respectively NB) denotes the tangent (respectively normal) bundle of $B = \mathbb{C}\mathbf{P}^2$ inside M^7 , and the Dirac genus \hat{A} can be expressed in terms of the Pontryagin forms as follows [44]:

$$\hat{A} = 1 - \frac{P_1}{24} + \frac{7P_1^2 - 4P_2}{5760} + \dots \quad (2.36)$$

Comparing the C_3 coupling on the right-hand side of (2.35) with formula (2.32), we see that the type IIA analogue of the latter is

$$N_{D2} - \int_B \sqrt{\hat{A}(TB)/\hat{A}(NB)} - \frac{1}{2} \int_B \frac{\mathcal{F}}{2\pi} \wedge \frac{\mathcal{F}}{2\pi}, \quad (2.37)$$

where N_{D2} is the number of space-filling D2-branes. This is naturally identified with N_{M2} in M-theory. Recall that in the previous section we identified the (shifted) gauge field strength on the D6-brane with the (shifted) G -flux in M-theory, via the isomorphism (2.30). This suggests that the last terms in (2.32) and (2.37) are also naturally identified. Since a D6-brane wrapped

⁹ Note that we choose the orientation of the bolt to have $b_+^2 = 0$ and $b_-^2 = 1$. With this convention, deformations of the bolt are parametrized by $b_+^2 = 0$ [33]. We thank B Acharya for explanations of these points. Throughout the rest of sections 2 and 3, we simply refer to the bolt as $\mathbb{C}\mathbf{P}^2$, with the understanding that the opposite orientation to the canonical one is to be used. However, one must be careful to note that the bundle $\Lambda_+^2(\mathbb{C}\mathbf{P}^2)$ is an *entirely different* bundle from $\Lambda_+^2(\overline{\mathbb{C}\mathbf{P}^2})$, as one can see by comparing Pontryagin classes. It is the total space of the latter bundle on which the G_2 metric is defined.

on the bolt $\mathbb{C}\mathbf{P}^2 \subset M^7 = \mathbb{R}^3 \times \mathbb{C}\mathbf{P}^2$ is anomalous unless one turns on a half-integral flux of the gauge field strength [35], we have

$$\int_{\mathbb{C}\mathbf{P}^1} \frac{\mathcal{F}}{2\pi} = k \in \mathbb{Z} + \frac{1}{2}, \tag{2.38}$$

which means the last term in (2.37) takes the value $\frac{1}{2} \int (\mathcal{F}/2\pi)^2 = -k^2/2$. The minus sign occurs since the orientation of $\overline{\mathbb{C}\mathbf{P}^2}$ in M^7 is such that it has $b_-^2 = 1$. The contribution from the G -flux through $\mathbb{C}\mathbf{P}^2 \subset \mathcal{Q}$ is

$$\int_{\mathbb{C}\mathbf{P}^2} \frac{G}{2\pi} = k \in \mathbb{Z} + \frac{1}{2}. \tag{2.39}$$

Using the fact that the self-intersection number of $\mathbb{C}\mathbf{P}^2$ inside \mathcal{Q} is equal to +1 in our conventions, we find that the last term of (2.32) is given by $+k^2/2$. Finally, it is now natural to equate the remaining purely geometric terms, so that

$$\frac{1}{8} \int_X (P_1^2 - 4P_2) = \frac{1}{2} \int_B (P_1(TB) - P_1(NB)). \tag{2.40}$$

We should stress here that the right-hand side of this formula is computed on a G_2 manifold M^7 and is manifestly a topological invariant when B is compact, whereas the left-hand side is computed on the corresponding non-compact 8-manifold X of $Spin(7)$ holonomy. Thus, we are able to compute the integral of the Pontryagin forms by computing locally the two-brane charge which is induced on the D6-branes. Formula (2.40) may be proved in the compact case using a combination of G -index theorems. Details will be presented elsewhere.

In [11] the Pontryagin classes in (2.40) relevant for the B -picture for $X = \Sigma^- \mathbf{S}^4$ and $X = \mathcal{Q}$ were computed. The right-hand side of (2.40) is then given by $((-4) - 0)/2 = -2$ and $((-3) - 3)/2 = -3$, respectively. Remarkably, these are (minus) the topological Euler characters of X .

Putting these considerations together, we may now compute Φ_∞ for the case of $X = \mathcal{Q}$:

$$\Phi_\infty = N_{M2} - \frac{3}{24} + \frac{1}{2}k^2, \tag{2.41}$$

where the half-integer k determines the G -flux (2.39). Thus, for $N_{M2} = 0$ and the minimal value of $k = 1/2$ or $k = -1/2$, we find $\Phi_\infty = 0$.

Using this result for \mathcal{Q} , together with some index theorems, we will now be able to compute the integral of the R^4 terms for the $Spin(7)$ manifold $\mathbb{R}^3 \times \mathbf{S}^5$. For zero G -flux, $G = 0$, we shall again find that $\Phi_\infty = 0$ with $N_{M2} = 0$.

The particular combination of Pontryagin forms of interest can be written in terms of index densities for various elliptic operators on X . Specifically, we have (see, for example, [42])

$$\frac{1}{16} (P_1^2 - 4P_2) = \text{ind } \mathcal{D}_1 - \text{ind } \mathcal{D}_- - \frac{1}{2} \text{ind } d. \tag{2.42}$$

Here the index density for the exterior derivative d is nothing but the usual Euler density. The twisted Dirac operators \mathcal{D}_1 and \mathcal{D}_- are the usual Dirac operators on X coupled to the bundles Λ^1 of 1-forms and Δ^- of negative chirality spinors (in the $\mathfrak{8}_c$). The key observation is to note that the existence of a non-vanishing spinor ξ in the $\mathfrak{8}_s$ representation provides us with a natural isomorphism between the bundles Λ^1 and Δ^- . Explicitly, we can convert between the two using the formulae

$$\psi = V \cdot \xi \qquad V^\mu = \bar{\xi} \Gamma^\mu \psi, \tag{2.43}$$

where V and ψ are an arbitrary (co)-vector and negative chirality spinor, respectively.

For our supersymmetric compactification with G -flux, we have a covariantly constant spinor ξ in the $\mathfrak{8}_s$, where the derivative depends on the G -flux

$$D_\mu(t)\xi \equiv \left(\nabla_\mu - \frac{t}{288} G_{\rho\lambda\sigma\tau} (\Gamma_\mu^{\rho\lambda\sigma\tau} - 8\delta_\mu^\rho \Gamma^{\lambda\sigma\tau}) \right) \xi = 0, \quad (2.44)$$

where we need to set $t = 1$ in this equation. The usual covariant derivative on X is then given by setting $t = 0$: $D(0) = \nabla$. We may now define twisted Dirac operators $\mathcal{D}_1(t)$ and $\mathcal{D}_-(t)$. The index densities in (2.42) are then for $\mathcal{D}_1(0) = \mathcal{D}_1$ and $\mathcal{D}_-(0) = \mathcal{D}_-$, respectively.

The explicit form of the isomorphism (2.43) together with the fact that $D(1)\xi = 0$ implies that the spectra of the twisted Dirac operators $\mathcal{D}_1(1)$ and $\mathcal{D}_-(1)$ are identical. In particular, the index of these Dirac operators on X with APS boundary conditions are equal. Recall that the APS index theorem for a manifold with boundary takes the following form:

$$\text{Index } D = \int_X \text{ind } D + \int_{\partial X} K - \frac{h + \eta(0)}{2}. \quad (2.45)$$

Here $\text{ind } D$ is the relevant index density for D , K is a boundary term depending on the second fundamental form, h denotes the multiplicity of the zero-eigenvalue of D restricted to ∂X and η is the usual APS function for the elliptic operator D restricted to ∂X . The boundary conditions are global; the projection onto the non-negative part of the spectrum on the boundary is set to zero. We will write the APS theorem even more schematically as

$$\text{Index } D = \int_X \text{ind } D + \text{boundary terms}, \quad (2.46)$$

where the boundary terms depend only on the boundary data. We will not need to worry about the explicit form of these terms. It follows that we may write the integral of the Pontryagin forms as

$$\frac{1}{16} \int_X (P_1^2 - 4P_2) = \text{Index } \mathcal{D}_1(0) - \text{Index } \mathcal{D}_-(0) - \frac{1}{2} \chi(X) + \text{boundary terms}, \quad (2.47)$$

where we have absorbed all of the boundary terms into the last term. As we have already argued,

$$\text{Index } \mathcal{D}_1(1) = \text{Index } \mathcal{D}_-(1). \quad (2.48)$$

Now, importantly, the G -field vanishes at infinity. This was required earlier for finiteness of the energy. Thus the restriction of $\mathcal{D}_1(t)$ or $\mathcal{D}_-(t)$ to the boundary of X is independent of t . In other words, the spinor ξ is a genuine Killing spinor on Y , even in the presence of G -flux. We may now smoothly deform the operators $\mathcal{D}_1(t)$, $\mathcal{D}_-(t)$ from $t = 1$ back to $t = 0$ without changing the boundary data. Because of this last fact, relation (2.48) must continue to hold for all t : by (2.45) the index varies smoothly under a smooth change of D in the interior, and since the index is an integer, it is therefore constant under such deformations. Note that such arguments no longer hold when the deformation is not smooth. For example, we can deform the metric on $X = \mathcal{Q}$ whilst leaving the boundary data fixed by varying the size of the bolt $\mathbb{C}\mathbb{P}^2$. This deformation is smooth, except when we pass the conifold point. In fact, the index jumps as we move from \mathcal{Q} to $\mathbb{R}^3 \times \mathbb{S}^5$, as we shall see presently.

It follows that for a $Spin(7)$ manifold X we have

$$\frac{1}{16} \int_X (P_1^2 - 4P_2) = -\frac{1}{2} \chi(X) + \text{boundary terms}. \quad (2.49)$$

We have just argued, using duality, that for $X = \mathcal{Q}$ the sum of all the boundary terms must vanish, since the left-hand side of relation (2.49) gave precisely the topological result $-\chi(X)/2$. For $X = \mathbb{R}^3 \times \mathbb{S}^5$, the Euler class vanishes since the manifold is contractible to \mathbb{S}^5 .

Moreover, the boundary terms are equal to the boundary terms for \mathcal{Q} since both manifolds have the same asymptotics. But we have just argued that the sum of the boundary terms is zero. Thus, for $X = \mathbb{R}^3 \times \mathbf{S}^5$ with $N_{M2} = 0$ and $G = 0$, we have once again $\Phi_\infty = 0$.

There is a simple physical explanation of this result. For $G = 0$ and no space-filling M2-branes, the only contribution to the total flux (2.32) is from the R^4 terms. There exists a type IIA dual for M-theory on $X = \mathbb{R}^3 \times \mathbf{S}^5$ which involves RR 2-form flux on a G_2 manifold with no D6-branes. This was described briefly in the introduction and will be dealt with in detail in section 4. In the absence of space-filling D2-branes, there is no other contribution to the effective D2-brane charge in this type IIA string theory configuration. Thus the R^4 terms in M-theory must vanish.

It is satisfying that this combination of physical and mathematical arguments is self-consistent.

2.6. Parity transformations

Let us quickly recap: we have examined M-theory on the background which asymptotes to the $Spin(7)$ cone over the Aloff–Wallach space $N_{1,-1}$, with vanishing flux at infinity. We have shown that there are three choices of supersymmetric vacua satisfying these boundary data, and therefore three possible branches for the moduli space of M-theory on this conical singularity. One branch consists of $X = \mathbb{R}^3 \times \mathbf{S}^5$ with $G = 0$. The other two branches correspond to $X = \mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}\mathbf{P}^2$ with either plus or minus a half unit of G -flux through the $\mathbb{C}\mathbf{P}^2$ bolt.

It is natural to wonder how, if at all, these three branches join together. In section 2.2, we presented evidence that one may move from a branch with topology $X = \mathbb{R}^3 \times \mathbf{S}^5$ to a branch of topology \mathcal{Q} . But which sign of G -flux does this transition choose? Our conjecture here is that all three branches meet at a singular point of moduli space. To see that this must be the case, it is instructive to study the action of parity on each of these backgrounds.

In odd spacetime dimensions, parity acts by inverting an odd number of spatial coordinates. In the present case of M-theory compactified on $\mathbb{R}^{1,2} \times X$, it is convenient to take the action of the parity operator to be

$$P : x^1 \rightarrow -x^1, \tag{2.50}$$

for $x^1 \in \mathbb{R}^{1,2}$, with all other coordinates left invariant. The advantage of such a choice is that parity in M-theory coincides with parity in the low-energy three-dimensional theory. With the natural decomposition of gamma matrices into a 3–8 split, it is simple to show that this remains true for the fermions, with a 11-dimensional Majorana fermion ϵ transforming as

$$P : \epsilon \rightarrow \Gamma_1 \epsilon. \tag{2.51}$$

In order for the Chern–Simons interaction $C \wedge G \wedge G$ of 11-dimensional supergravity to preserve parity, we must also choose the action on the 3-form field of M-theory,

$$P : \begin{cases} C_{ijk}(x) \rightarrow +C_{ijk}(Px) & \text{if } i, j \text{ or } k = 1 \\ C_{ijk}(x) \rightarrow -C_{ijk}(Px) & \text{if } i, j \text{ and } k \neq 1. \end{cases} \tag{2.52}$$

While this ensures that M-theory respects parity, certain backgrounds may spontaneously break this symmetry. This is not the case for the geometry $X \cong \mathbb{R}^3 \times \mathbf{S}^5$. However, the requirement of non-vanishing G -flux in the geometry \mathcal{Q} ensures that parity is indeed broken in this background, since

$$P : \int_{\mathbb{C}\mathbf{P}^2} \frac{G}{2\pi} \rightarrow - \int_{\mathbb{C}\mathbf{P}^2} \frac{G}{2\pi}. \tag{2.53}$$

Thus we see that the two branches with $G = \pm\frac{1}{2}$ transform into each other under a parity transformation. It follows that, if either of these branches can be reached from the parity even branch with $X = \mathbb{R}^3 \times \mathbf{S}^5$, then both may be reached. In the following section, we shall argue that this may be understood via the condensation of a parity odd state.

Note that similar comments apply to the $Spin(7)$ manifold $\mathbb{B}_8 \cong \mathbb{R}^4 \times \mathbf{S}^4$. Once again, the presence of G -flux ensures that parity is spontaneously broken. In this case however, there is no third parity-invariant branch, allowing for the possibility that the two branches with flux $\pm\frac{1}{2}$ are disjoint. However, given the similarity between the D6-brane pictures, described in sections 2.1 and 2.2, it seems likely that the branches are once again connected.

3. Low-energy dynamics and condensation of five-branes

In this section, we would like to consider the low-energy dynamics of M-theory compactified on a $Spin(7)$ manifold to three dimensions with $\mathcal{N} = 1$ (two supercharges) supersymmetry. To illustrate our methods we will first consider the similar, but simpler, example of IIA string theory compactified to three dimensions on a manifold of G_2 holonomy. The dynamics of the massless modes was discussed in detail by Atiyah and Witten [7]. Here we include the effects from massive wrapped branes which become light at certain points of the moduli space, and rederive some of the results of Atiyah and Witten [7] in a new fashion. We will then apply the lessons learnt to the $Spin(7)$ case.

3.1. Type II strings on manifolds of G_2 holonomy

3.1.1. The cone over $\mathbb{C}\mathbf{P}^3$. The cone over $Y = \mathbb{C}\mathbf{P}^3$ is the first, and simplest, example considered by Atiyah and Witten [7]. The singularity may be resolved to a manifold of topology

$$X \cong \mathbb{R}^3 \times \mathbf{S}^4. \quad (3.54)$$

Let us consider IIA string theory compactified on X to $d = 2 + 1$ dimensions. This preserves $\mathcal{N} = 2$ supersymmetry. Following [7], we firstly examine the massless Kaluza–Klein modes. The volume of the \mathbf{S}^4 cycle provides a real scalar field,

$$\phi = \text{Vol}(\mathbf{S}^4). \quad (3.55)$$

Importantly, this deformation is normalizable and ϕ is dynamical [7]. The $\mathcal{N} = 2$ supersymmetry requires that ϕ be accompanied by a further scalar field arising from the harmonic L^2 -normalizable 3-form ω such that

$$\int_{\mathbb{R}^3} \omega \neq 0, \quad (3.56)$$

where \mathbb{R}^3 is the fibre over \mathbf{S}^4 . The existence of ω ensures that the RR 3-form C_3 has a zero mode

$$C_3 = \sigma \omega, \quad (3.57)$$

and σ provides the second massless scalar in $\mathbb{R}^{1,2}$. Large gauge transformations of C_3 mean that σ is a periodic scalar. Moreover, the parity transformation (2.52) ensures that it is actually a pseudoscalar. Atiyah and Witten [7] combine ϕ and σ into the complex field $\phi \exp(i\sigma)$, which is the lowest component of a chiral multiplet. Here we choose instead to dualize the scalar σ for a three-dimensional $U(1)$ gauge field,

$$d\sigma = *dA, \quad (3.58)$$

where the Hodge dual \star -operator is defined on the $\mathbb{R}^{1,2}$ Minkowski space transverse to X . In this language, the low-energy dynamics is defined in terms of a $U(1)$ vector multiplet. To see the utility of this duality, let us consider the effect of D4-branes wrapping the coassociative S^4 . The lowest mass state occurs if the field strength on the D4-brane is set to zero. For $\phi \gg 0$, the real mass of the D4-brane is given by

$$M_{D4} \sim \phi + \frac{1}{48} \int_{S^4} (p_1(TS^4) - p_1(NS^4)) \sim \phi - \phi_0, \tag{3.59}$$

where the fractional D0-brane charge induced by the curvature couplings (2.35) was calculated in section 2.5 and contributes a negative, bare, real mass $\phi_0 = 1/12$. (see [45] for a nice discussion). Since the mass is proportional to ϕ , supersymmetry requires that this state is charged under the $U(1)$ gauge field. To see this explicitly, note that the charge of these states is measured by the asymptotic RR flux,

$$\int_{\mathbb{R}^3 \times S^1} dC_3 = \int_{S^1} d\sigma, \tag{3.60}$$

where \mathbb{R}^3 is the fibre of X and S^1 is a large spacelike circle in $\mathbb{R}^{1,2}$ surrounding the point-like D4-brane. The D4-brane is therefore a global vortex in σ or, alternatively, is charged electrically under A . Note that the state corresponding to a single D4-brane is not in the spectrum since it has logarithmically divergent mass. (The same is true for the M5-brane wrapping S^4 in M-theory which leads to a BPS string in the four-dimensional effective theory.) Nevertheless, its effects are still important.

How does this D4-brane appear in the low-energy effective action? The simplest hypothesis is that a correct quantization of the D4-brane wrapped around the calibrated S^4 yields a single, short (BPS) chiral multiplet. To see that this is consistent, let us examine the global symmetries of the model. Recall that these arise from both geometrical isometries of X and gauge symmetries of the C_3 field [7]. The latter are more important for us. The symmetries are determined by large gauge transformations at infinity, while those which can be continued into the interior of X are unbroken symmetries. We have [7],

$$H^2(Y; U(1)) = U(1)_J, \quad H^2(X; U(1)) = 0. \tag{3.61}$$

The model therefore has a single, broken, $U(1)$ global symmetry which acts on the dual photon as $\sigma \rightarrow \sigma + c$. Note, in particular, that there are no further flavour symmetries. While this does not rule out the existence of further matter multiplets, constrained by a suitable superpotential, it does suggest that the simplest possibility is to have a single chiral multiplet, which we denote as q .

However, this is not the full story since, in three dimensions, massive charged particles do not necessarily decouple from the low-energy dynamics. Rather, they lead to the generation of Chern–Simons couplings. Since we have ‘integrated in’ the D4-brane state q to describe our effective theory, we must compensate by the introduction of a bare Chern–Simons coupling in our theory. This is such that, upon integrating out the massive D4-brane, the effective Chern–Simons coupling vanishes. To determine this Chern–Simons coupling, we need both the charge of the chiral multiplet q , and the sign of the mass of the fermions. The former is determined by (3.60) to be $+1$, while the latter may be fixed, by convention, to be positive. Thus, integrating in the fermions associated with q gives rise to a bare Chern–Simons parameter $\kappa = -\frac{1}{2}$. We are thus led to the simplest hypothesis for the matter content; a Maxwell–Chern–Simons theory, with single charged chiral multiplet.

The supersymmetric completion of the Chern–Simons term includes a D-term coupling to ϕ . Physically, this can be understood as arising from integrating in the complex scalar field q . Thus, the potential energy of the low-energy dynamics is given by,

$$V = e^2(|q|^2 - \kappa\phi)^2 + (\phi - \phi_0)^2|q|^2, \tag{3.62}$$

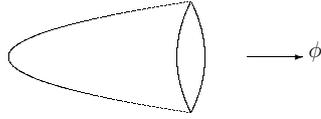


Figure 7. The quantum moduli space of $\mathcal{N} = 2$ three-dimensional Chern–Simons–Maxwell theory with a single chiral multiplet.

where e^2 is the gauge coupling constant. Naively, this theory has no moduli space of vacua. However, if we set $\phi > \phi_0$ then, upon integrating out the chiral multiplet, the renormalized κ vanishes and we see that this is indeed a supersymmetric vacuum state. The moduli space of this theory is thus given by $\phi > \phi_0$, over which the dual photon is fibred, as shown in figure 7. This fibre degenerates at $\phi = 1/12$ where the chiral multiplet is massless, and $U(1)_J$ is restored at this point. Atiyah and Witten argue that this point is smooth [7].

Let us comment briefly on parity. Type IIA string theory on the G_2 manifold $X \cong \mathbb{R}^3 \times \mathbf{S}^4$ is parity invariant. The same is true of our low-energy description, arising after integrating out the D4-brane. Our effective theory also includes higher-dimensional parity-breaking operators, for example $(A \wedge F)^3$. However, as usual, our theory is simply not to be trusted at such scales since we have ignored many other contributions.

It is curious to note that the smooth moduli space depicted in figure 7 may be embedded as the \mathbf{S}^1 fibre of Taub–NUT space. To see this, consider $\mathcal{N} = 4$ supersymmetric SQED with a single hypermultiplet, in three dimensions. It is well known that the four-dimensional Coulomb branch of this model is endowed with the smooth hyper-Kähler Taub–NUT metric, arising at one-loop [46]

$$ds^2 = \left(\frac{1}{e^2} + \frac{1}{\phi} \right) (d\phi^2 + \phi^2 d\theta^2 + \phi^2 \sin^2 \theta d\lambda^2) + \frac{1}{4} \left(\frac{1}{e^2} + \frac{1}{\phi} \right)^{-1} (d\sigma + \cos \theta d\lambda)^2. \quad (3.63)$$

The $\mathcal{N} = 4$ gauge theory has an $SU(2)_N \times SU(2)_R$ R -symmetry group. Of these, only the former acts as an isometry on the Coulomb branch, rotating the three complex structures. The Coulomb branch has a further tri-holomorphic $U(1)_J$ isometry, which rotates the dual photon σ .

One may flow from this $\mathcal{N} = 4$ gauge theory to the $\mathcal{N} = 2$ gauge theory of interest by turning on relevant, supersymmetry-breaking, operators. These operators may be conveniently introduced by weakly gauging the diagonal global symmetry $U(1)_D \subset U(1)_N \times U(1)_R \times U(1)_J \subset SU(2)_N \times SU(2)_R \times U(1)_J$ [47]. This gives masses to precisely half of the fields as required. Moreover, integrating out half of the hypermultiplet gives rise to the bare Chern–Simons coupling $\kappa = -\frac{1}{2}$. However, because the relevant deformation is associated with a symmetry, we may follow it to the infrared, where it may be understood as the generation of a potential on the moduli space of vacua, proportional to the length of the Killing vector corresponding to simultaneous rotations of σ and λ ,

$$V = \phi^2 \sin^2 \theta \left(\frac{1}{e^2} + \frac{1}{\phi} \right) + \frac{1}{4} \left(\frac{1}{e^2} + \frac{1}{\phi} \right)^{-1} (1 + \cos \theta)^2 \quad (3.64)$$

This potential vanishes on the two-dimensional submanifold $\theta = 0$. This submanifold is precisely the moduli space of figure 7.

3.1.2. The cone over $SU(3)/U(1)^2$. The next example we consider is the cone over $Y = SU(3)/U(1)^2$ which was also discussed by Atiyah and Witten [7]. The conical singularity may be resolved to a manifold of topology

$$X \cong \mathbb{R}^3 \times \mathbf{CP}^2. \quad (3.65)$$

The story is similar to that above. Once again, the volume of the four-cycle yields a dynamical real scalar, $\phi = \text{Vol}(\mathbb{CP}^2) > 0$, and a normalizable harmonic 3-form provides the supersymmetric partner σ [7]. This latter, periodic, pseudoscalar is dualized in favour of a $U(1)$ gauge field. The only question is what charged matter arises from a D4-brane wrapped on \mathbb{CP}^2 . This time the analysis is somewhat different. Although \mathbb{CP}^2 is a supersymmetric cycle, the lack of spin structure implies that we cannot wrap a D4-brane on it without including suitable worldvolume field strengths. These require fluxes,

$$\int_{\mathbb{CP}^1} \frac{\mathcal{F}}{2\pi} = k \in \mathbb{Z} + \frac{1}{2}. \tag{3.66}$$

For supersymmetry [34], the flux must satisfy (2.29). In our conventions, this requires that \mathcal{F} is anti-self-dual which, since the bolt $B = \overline{\mathbb{CP}^2}$ has $b_2^+(B) = 0$ and $b_2^-(B) = 1$, is indeed the case. In our analysis, we wish to include only the states which become light in some regime of moduli space. From the semiclassical mass formula, for $\phi \gg 0$ we have

$$M_{D4} \sim \phi - \frac{1}{2} \int_{\mathbb{CP}^2} \frac{\mathcal{F}}{2\pi} \wedge \frac{\mathcal{F}}{2\pi} + \frac{1}{48} \int_{\mathbb{CP}^2} (p_1(T\mathbb{CP}^2) - p_1(N\mathbb{CP}^2)) \geq \phi. \tag{3.67}$$

The last term was calculated in section 2.5 (see the paragraph above (2.41)), and yields a negative contribution to the mass; $-\phi_0 = -1/8$. Nevertheless, the fact that the anomaly cancellation requires a non-zero, anti-self-dual, flux ensures that the inequality (3.67) holds, and is saturated only by the two states with minimal flux, $k = \pm 1/2$. All states with larger values of flux remain massive throughout moduli space. We therefore include in the low-energy description only the two states with minimal flux.

What quantum numbers do these two states carry in the low-energy effective theory? From (3.60), a D4-brane with either sign of flux has charge +1 under the $U(1)$ gauge group. However, the low-energy fermion fields have equal, but opposite, mass. To see this, note that under parity we have $k \rightarrow -k$. This follows from the fact that \mathcal{F} transforms in the same way as the NS–NS 2-form field, whose own transformation properties may be deduced from (2.52). Since the action of parity in IIA string theory coincides with the action in the low-energy three-dimensional theory¹⁰, the signs of the fermion masses are reversed. It is conventional in $\mathcal{N} = 2$, $d = 2 + 1$ theories to adjust the complex structure of chiral multiplets so that the sign of the gauge field charge coincides with the sign of the fermion mass. We therefore find the low-energy effective dynamics to be governed by a three-dimensional, $\mathcal{N} = 2U(1)$ gauge theory with two chiral multiplets q and \tilde{q} of charges +1 and -1 , respectively, and equal, but opposite, fermion masses. Unlike the previous case, the ‘integrating in’ of two chiral multiplets with opposite fermion masses means that no bare Chern–Simons term is generated.

The presence of two, minimally coupled, chiral multiplets endows the low-energy dynamics with a flavour symmetry, under which both q and \tilde{q} transform with the same charge. This symmetry may be seen from M-theory, where the C-field yields the following global symmetries [7]:

$$H^2(Y; U(1)) = U(1)_J \times U(1)_F, \quad H^2(X; U(1)) = U(1)_F. \tag{3.68}$$

The interpretation of this is that there exist two $U(1)$ global symmetries, one of which, $U(1)_J$, is spontaneously broken. We have suggestively labelled the unbroken symmetry $U(1)_F$. To see that it is indeed the above flavour symmetry, it suffices to note that the corresponding,

¹⁰ It also acts as worldsheet parity for the IIA string.

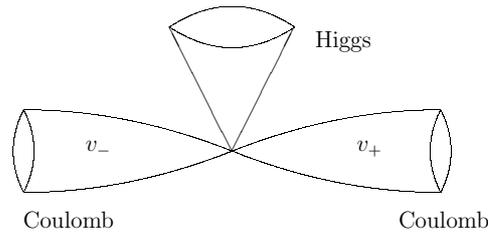


Figure 8. The quantum moduli space of $\mathcal{N} = 2$ three-dimensional SQED with two chiral multiplets.

non-dynamical, gauge potential is given by

$$A_F = \int_{\mathbb{CP}^1} C_3, \quad (3.69)$$

where $\mathbb{CP}^1 \subset \mathbb{CP}^2$. This ensures that a D4-brane with flux indeed carries the requisite charge.

IIA string theory on this background also contains an object arising from the D2-brane wrapping the $\mathbb{CP}^1 \subset \mathbb{CP}^2$. From the above discussion, we learn that this state carries flavour, but no gauge, charge. In fact, it is simple to construct this state in the low-energy theory. Its identity follows from the observation that it may be constructed from a D4-brane with $k = +1/2$ bound to an anti-D4-brane also with $k = +1/2$. This relates the D2-brane to the $q\tilde{q}$ bound state, which indeed carries the correct quantum numbers. At first sight, there appears to be a contradiction. The D2-brane is not wrapped on a calibrated cycle, and does not therefore give rise to a BPS state. In contrast, the operator $q\tilde{q}$ is holomorphic. However, it is a dynamical question whether the bound state of q and \tilde{q} saturates the BPS bound and, using mirror symmetry, one may argue that it does not. To see this, recall that this state is dual to a vortex state on the Higgs branch of a theory with two oppositely charged chiral multiplets [48]. But no classical BPS vortex solution exists in this theory¹¹.

Finally, let us turn to the moduli space of this theory, and ask what happens as $\phi \rightarrow 0$. The classical scalar potential is given by

$$V = e^2(|q|^2 - |\tilde{q}|^2)^2 + \phi^2(|q|^2 + |\tilde{q}|^2). \quad (3.70)$$

So far we have restricted attention to the Coulomb branch, with $\phi \neq 0$. However, for $\phi = 0$, there exists a one complex-dimensional Higgs branch in which $U(1)_F$ is spontaneously broken, and $U(1)_J$ is unbroken. The quantum dynamics of this three-dimensional gauge theory was examined in [48, 49]. Here it is shown that the Coulomb branch bifurcates into two cigar-shaped branches, joined together at $\phi = 0$ where they meet the Higgs branch. It is thought that at the junction of the three branches there lives an interacting superconformal field theory. The two Coulomb branches are parametrized asymptotically by $v_{\pm} = \exp(\pm\phi \pm i\sigma)$. The final quantum moduli space is sketched in figure 8. The authors of [48] further conjecture that, at strong coupling, the theory enjoys a triality symmetry which interchanges the two Coulomb branches and the Higgs branch. They argue that the physics is thus dual to the Landau–Ginzburg model with three chiral multiplets, Φ_i , $i = 1, 2, 3$, and the superpotential

$$\mathcal{W} = \Phi_1\Phi_2\Phi_3. \quad (3.71)$$

This is in perfect agreement with the results of [7], where the existence of three branches was deduced using a discrete S_3 symmetry group of the G_2 holonomy manifold X . Each of the

¹¹ This follows from the fact that a line bundle of negative degree cannot have a non-zero holomorphic section.

three branches corresponds to a manifold of topology $X \cong \mathbb{R}^3 \times \mathbb{C}\mathbf{P}^2$, with only the unbroken $U(1)$ symmetry group distinguishing them.

In [17] it was shown that X has a non-normalizable deformation which preserves both homology and holonomy. There are three such ways to perform this deformation, each of which preserves only one of the three branches of moduli space. Let us see how to capture this behaviour from the perspective of the low-energy dynamics. Since both supersymmetries and global symmetries of the low-energy dynamics are left intact, this deformation can correspond to only two possible parameters; a real mass parameter m or a FI parameter ζ . Including both, the scalar potential reads

$$V = e^2(|q|^2 - |\tilde{q}|^2 - \zeta)^2 + (\phi + m)^2|q|^2 + (-\phi + m)^2|\tilde{q}|^2. \tag{3.72}$$

There are indeed precisely three such combinations of these parameters which preserve a given branch of the vacuum moduli space

$$\begin{aligned} \zeta \neq 0, \quad m = 0 & \quad \text{Higgs survives} \\ \zeta = -m \neq 0 & \quad v_+ \text{ Coulomb survives} \\ \zeta = +m \neq 0 & \quad v_- \text{ Coulomb survives.} \end{aligned}$$

3.2. M-theory on the $Spin(7)$ cone over $SU(3)/U(1)$

We now turn to the main topic of this paper: the dynamics of M-theory compactified on the cone over $Y = SU(3)/U(1)$. We will use the intuition gleaned from the previous sections to provide a consistent picture of the topology-changing transition from $X \cong \mathbb{R}^3 \times \mathbf{S}^5$ to $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}\mathbf{P}^2$.

Let us start with M-theory compactified on $X \cong \mathbb{R}^3 \times \mathbf{S}^5$. As for the G_2 examples discussed in the previous subsection, the volume of the \mathbf{S}^5 yields a real parameter,

$$\phi = \text{Vol}(\mathbf{S}^5). \tag{3.73}$$

However, there is an important difference with the G_2 holonomy examples discussed in the previous section. As shown in section 2.3, using the relationship with coassociative cones, fluctuations of ϕ are non-normalizable. Therefore ϕ plays the role of a modulus in the low-energy dynamics. Of course, one could imagine compactifying the $Spin(7)$ manifold, with the geometry $\mathbb{R}^3 \times \mathbf{S}^5$ providing a good description in the neighbourhood of a conical singularity, in which case ϕ is once again promoted to a dynamical field.

We now turn to the massless modes arising from the C -field. Although no explicit harmonic 3-form (or, indeed, a metric!) is known on $\mathbb{R}^3 \times \mathbf{S}^5$, there is a simple argument to ensure the existence of such an object. To see this, consider the symmetries of M-theory on $\mathbb{R}^3 \times \mathbf{S}^5$. The global symmetry at infinity arising from the C -field is given by

$$H^2(Y; U(1)) \cong U(1)_J, \tag{3.74}$$

where the generator is dual to the \mathbf{S}^2 fibre of $Y \rightarrow \mathbf{S}^5$. This $U(1)$ symmetry is spontaneously broken in the interior

$$H^2(X; U(1)) \cong 0. \tag{3.75}$$

Thus, in the conical limit $\phi = 0$, where the \mathbf{S}^5 bolt collapses to zero size, the low-energy theory has a global $U(1)_J$ symmetry. This is spontaneously broken for $\phi > 0$, with

$$\delta C = d\Lambda, \tag{3.76}$$

where Λ is a 2-form dual to the \mathbf{S}^2 fibre of $Y \rightarrow \mathbf{S}^5$. Since $U(1)_J$ acts non-trivially on the low-energy theory, it must give rise to a Goldstone mode,

$$C = \sigma \omega_3, \tag{3.77}$$

predicting the existence of a harmonic 3-form ω_3 which represents the generator of the compactly supported cohomology $H_{\text{cpt}}^3(X) \cong \mathbb{R}$.

There remains the question of whether this 3-form is L^2 normalizable, or, equivalently, whether the periodic pseudoscalar σ is dynamical. This remains an open problem. We will denote normalization of the kinetic term for σ as e^2 ; the non-normalizable limit corresponds to $e^2 \rightarrow \infty$. As in previous examples, we dualize σ in favour of a $U(1)$ gauge potential A . In terms of these new variables, the field strength kinetic term is normalized as the usual $1/e^2$. Note that if ω_3 is non-normalizable, we are dealing with the strong coupling limit of the gauge theory.

As in the G_2 -holonomy examples of the previous section, extra massive states arise from wrapped branes. In the present case, these come from M5-branes wrapping $\mathbb{R}^3 \times \mathbf{S}^5$. In the semiclassical limit $\phi \gg 0$, these give rise to states of real mass ϕ , charged under the $U(1)$ gauge field. From the perspective of the coassociative D6-brane locus L of figure 5(a), this M5-brane corresponds to a D4-brane with topology of a four-disc D^4 , whose boundary \mathbf{S}^3 wraps the minimal volume three-cycle of L . What type of matter does quantization of the wrapped M5-brane yield? As the \mathbf{S}^5 is not a calibrated submanifold, we again do not expect a supersymmetric multiplet. But, since $\mathcal{N} = 1$ supersymmetry in three dimensions does not admit BPS particle states, this is no great limitation. The simplest possible matter content, consistent with the symmetries of the theory, occurs if the M5-brane gives rise to a single complex scalar multiplet q . We assume that this is the case and that, as in [4, 5], this is the only single particle state to become light as $\phi \rightarrow 0$.

As in the G_2 example, ‘integrating in’ the M5-brane state requires the introduction of a bare Chern–Simons coupling $k = -\frac{1}{2}$, in order to cancel the induced Chern–Simons coupling when it is subsequently integrated out. The bosonic part of the low-energy effective action is therefore given by,

$$\mathcal{L}_{\mathbb{R}^3 \times \mathbf{S}^5} = \int_{\mathbb{R}^{1,2}} \frac{1}{e^2} F \wedge \star F + \frac{k}{4\pi} A \wedge F + |Dq|^2 + \phi^2 |q|^2. \quad (3.78)$$

Note that the D-term contribution to the potential energy, given in equations (3.62) and (3.72) for previous examples, is absent in this case. This is because, in three-dimensional theories with $\mathcal{N} = 1$ supersymmetry, D-terms arise from scalar, rather than vector, multiplets and, in the present case, the ϕ field is non-normalizable. In contrast, note that, even in the $e^2 \rightarrow \infty$ limit, the gauge field retains a single derivative kinetic term.

Although the Lagrangian (3.78) is not parity invariant due to the presence of the Chern–Simons term, after integrating out the fermionic superpartner of q , this term is cancelled and parity is restored at low energies as required by the discussion in section 2.6.

What happens as the \mathbf{S}^5 shrinks to zero size? With only $\mathcal{N} = 1$ supersymmetry for protection, it is difficult to make any concrete statements about the strong coupling physics. Nevertheless, we shall present a consistent picture which passes several tests. We conjecture that, as suggested classically, the state arising from the M5-brane wrapping \mathbf{S}^5 becomes light in this limit. From section 2, we have learnt that the manifold $X = \mathbb{R}^3 \times \mathbf{S}^5$ can undergo a geometrical transition to the topologically distinct manifold $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}\mathbf{P}^2$. From the perspective of the low-energy dynamics, the natural interpretation of this is as a transition onto the Higgs branch [4, 5] through condensation of M5-branes,

$$|q|^4 \sim \text{Vol}(\mathbb{C}\mathbf{P}^2). \quad (3.79)$$

Since the state which condenses is non-BPS, it is hard to prove explicitly that this occurs. Still, there are several checks we can perform to see if such an interpretation holds water.

Firstly, consider the symmetries of M-theory compactified on $X \cong \mathbb{R}^4 \times \mathbb{C}\mathbb{P}^2$. The relevant cohomology groups are

$$H^2(Y; U(1)) = U(1)_J, \quad H^2(X; U(1)) = U(1)_J, \tag{3.80}$$

implying that a global $U(1)_J$ symmetry is left unbroken on this branch. This is indeed the case on the Higgs branch of our low-energy effective theory, if we identify $U(1)_J$ with the action on the dual photon.

Further agreement arises from examining the various extended objects that exist in M-theory on \mathcal{Q} , arising from wrapped M5- or M2-branes. At first sight, it appears that we may wrap an M5-brane over the four-cycle $\mathbb{C}\mathbb{P}^2 \subset \mathcal{Q}$ to get a domain wall in $d = 3$. However, there is a subtle obstruction to doing this. Specifically, on an M5-brane worldvolume W propagates a chiral 2-form with self-dual field strength T . This satisfies the relation [50]

$$dT = G|_W. \tag{3.81}$$

In particular, if there is a cohomologically non-trivial G -flux over W , one cannot wrap a five-brane. But this is precisely the case for $X = \mathcal{Q}$ since the membrane anomaly requires a half-integral flux of G over $\mathbb{C}\mathbb{P}^2$. Thus there is no wrapped five-brane.

However, we are free to wrap an M2-brane over $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^2$. This non-BPS state has a semiclassical mass proportional to the volume of $\mathbb{C}\mathbb{P}^1$, and is electrically charged under the global $U(1)_J$ symmetry with the identification (3.80). From the D6-brane perspective of section 2, this state is a fundamental string stretched between the two disjoint D6-branes depicted in figure 5(c). What does this state correspond to from the perspective of our low-energy gauge theory? The fact that it is charged under $U(1)_J$ implies that it must be a vortex. Vortices in non-supersymmetric Maxwell–Chern–Simons theories have been studied in the literature [52]. The mass of such an object is expected to be proportional to $|q|^2$. Comparing to the mass of the M2-brane state, we are led to the relationship (3.79).

We turn now to the question of parity breaking. In the previous section, we have argued that the transition onto the manifold $\mathcal{Q} \cong \mathbb{R}^3 \times \mathbb{C}\mathbb{P}^2$ spontaneously breaks parity due to the presence of non-zero G -flux. Since we have chosen the parity transformation (2.50) to act on $\mathbb{R}^{1,2}$, this should be reflected in parity breaking of the low-energy theory. This occurs naturally in our picture due to the Chern–Simons term coupling in (3.78). Recall that on the Coulomb branch, this was cancelled upon integrating out the Dirac fermionic superpartner of q , denoted by ψ , to result in a parity-invariant theory. However, if the mass of ψ vanishes, or indeed becomes negative, this cancellation no longer occurs and parity is broken.

While we have presented a plausible scenario for the low-energy description of the topology-changing transition, we should point out that, with such little supersymmetry, other possibilities exist. For example, the $U(1)$ gauge symmetry may, instead, be broken by a Cooper pair of ψ . Parity is then, once again, broken in the Higgs branch as required. This is somewhat similar to the phenomenon of p-wave superconductivity.

Finally, we may sketch the moduli space of M-theory on the $Spin(7)$ cone over the Aloff–Wallach space $N_{1,-1}$. It consists of three branches: a two-dimensional Coulomb branch corresponding to the geometry $X \cong \mathbb{R}^3 \times \mathbb{S}^5$, and two, one-dimensional Higgs branches, related by parity, each corresponding to the geometry $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{C}\mathbb{P}^2$. The resulting picture is drawn in figure 9.

4. Geometric quotients

In this section, we describe the proposed $Spin(7)$ conifold transition in M-theory in terms of two equivalent, but rather different, type IIA duals. These dual pictures correspond to choosing different M-theory circles on which to reduce.

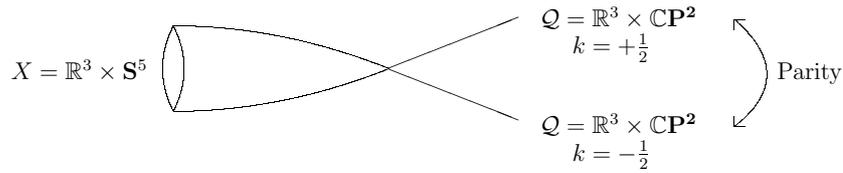


Figure 9. The moduli space of M-theory on the $Spin(7)$ cone over the Aloff–Wallach space $N_{1,-1} = SU(3)/U(1)$. The G -flux is measured by $\int_{\mathbb{CP}^2} G/2\pi = k$. Parity acts as a reflection in the horizontal axis.

In general, given a $U(1)$ isometry of X , one can choose to embed the M-theory circle along the $U(1)$ orbits. If $U(1)$ acts freely—that is, there are no fixed points of the circle action—then the quotient space $X/U(1)$ is a manifold, and M-theory on X is dual to type IIA string theory on the quotient $X/U(1)$ with, in general, a non-trivial RR 1-form potential A^{RR} and dilaton field φ . The field strength $F^{RR} = dA^{RR}$ is then interpreted geometrically as the curvature of the M-theory circle bundle

$$U(1) \hookrightarrow X \rightarrow X/U(1). \tag{4.82}$$

A reduction to type IIA also exists when the $U(1)$ action has a fixed point set W of codimension 4 in X . Then the quotient $X/U(1)$ may be given a manifold structure, with W embedded as a codimension 3 submanifold. The circle fibration (4.82) now degenerates over W , which is interpreted as the locus of a D6-brane in type IIA string theory. The field strength $F^{RR}/2\pi$ is no longer closed. Rather, its integral over a small 2-sphere S^2 linking W in $X/U(1)$ is equal to 1.

We shall find two interesting type IIA duals of our conifold transition in M-theory, corresponding to different choices of M-theory circle, which we refer to as the B -picture and L -picture [7, 11, 17]. Let us briefly summarize these dual pictures. We begin with the B -picture.

The starting point is to consider type IIA string theory on the asymptotically conical G_2 manifold which is an \mathbb{R}^3 bundle over \mathbb{CP}^2

$$\mathbb{R}^3 \times \mathbb{CP}^2. \tag{4.83}$$

The low-energy effective theory has $\mathcal{N} = 2$ supersymmetry in $d = 3$, and was discussed in section 3.1.2. One may now wrap¹² a space-filling D6-brane around the calibrated bolt \mathbb{CP}^2 , thus breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$, and lift the whole configuration to M-theory. In M-theory, this configuration is described by the $Spin(7)$ manifold $\mathcal{Q} \cong \mathbb{R}^4 \times \mathbb{CP}^2$.

The conifold transition in M-theory corresponds, in the B -picture, to shrinking the \mathbb{CP}^2 bolt to zero size, and blowing up a different copy of \mathbb{CP}^2 . Thus, in the B -picture, the conifold transition looks like a flop transition in which one copy of \mathbb{CP}^2 collapses and another blows up. The space-filling D6-brane turns into a single unit of RR flux through $\mathbb{CP}^1 \subset \mathbb{CP}^2$. This brane/flux transition is illustrated in figure 3.

The second type IIA dual, which we refer to as the L -picture, was discussed in section 2. Here the M-theory conifold transition corresponds, in type IIA string theory, to a transition of coassociative submanifolds in *flat space*, with D6-branes wrapped on the calibrated submanifolds. The coassociative submanifolds in question were first constructed in the seminal paper of Harvey and Lawson [27]. In order to produce this dual picture, we simply choose a different M-theory circle on which to reduce.

¹² One must also include a suitable half-integral flux of the gauge field strength on the D6-brane, as discussed earlier.

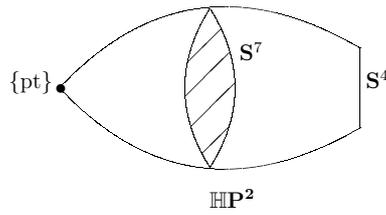


Figure 10. A foliation of $\mathbb{H}\mathbb{P}^2$ space by S^7 principal orbits. The two special orbits consist of a point and a 4-sphere.

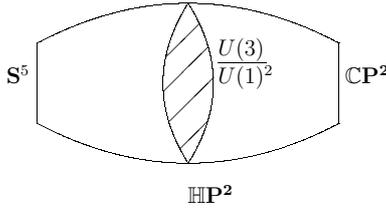


Figure 11. A foliation of $\mathbb{H}\mathbb{P}^2$ space by $U(3)/U(1)^2$ principal orbits. The two special orbits are S^5 and $\mathbb{C}\mathbb{P}^2$.

In the remainder of the section we describe in detail how these two pictures emerge. In section 4.1, we describe the B -picture, and its relation with the quaternionic projective plane. In section 4.2, we construct the quotients $X/U(1) \cong \mathbb{R}^7$ explicitly. The topology of X is completely encoded in the fixed point set L , as exemplified by equations (2.4). We also discuss how the symmetries of M-theory on X reduce to symmetries of the L -picture.

4.1. Brane/flux transitions

In this subsection, we will describe more precisely the duality between the proposed $Spin(7)$ conifold transition and the B -picture dual outlined above. Roughly speaking, in the B -picture, the conifold transition corresponds to a transition in which D6-branes are replaced with RR 2-form flux. This type of transition is by now familiar. However, there are some additional subtleties in our case. As we shall see presently, there is a curious relation between all of the $Spin(7)$ manifolds discussed in this paper and the quaternionic projective plane. We therefore begin with a discussion of $\mathbb{H}\mathbb{P}^2$.

Our starting point is to consider the orbit structure of the quaternionic projective plane $\mathbb{H}\mathbb{P}^2 = Sp(3)/Sp(2) \times Sp(1)$ under the two subgroups $Sp(2) \times Sp(1)$ and $U(3)$ of $Sp(3)$. Note that these are also the isometry groups of the $Spin(7)$ manifolds in table 1. In both cases the generic orbit of the action on $\mathbb{H}\mathbb{P}^2$ is of codimension 1. On rather general grounds, one therefore knows that there will be two special orbits of higher, and generally unequal, codimensions. The generic orbit is then necessarily a sphere bundle over each of the special orbits. Filling in each sphere bundle and gluing back-to-back give a construction of the manifold compatible with the group action. The orbit structures in each case are illustrated in figures 10 and 11.

Consider the orbit structure of $\mathbb{H}\mathbb{P}^2$ under the subgroup $Sp(2) \times Sp(1) \subset Sp(3)$. This is illustrated in figure 10. If we denote homogeneous coordinates on $\mathbb{H}\mathbb{P}^2$ as (u_1, u_2, u_3) with $u_i \in \mathbb{H}$, then the two special orbits are the point $(0, 0, 1)$ and the copy of $\mathbb{H}\mathbb{P}^1 = S^4$

consisting of the points $(u_1, u_2, 0)$. The generic orbit \mathbf{S}^7 is the distance sphere¹³ from the point $(0, 0, 1)$.

Using this information, we may now construct the B -picture for the $Spin(7)$ manifold $\Sigma^- \mathbf{S}^4$. The latter may be obtained from $\mathbb{H}\mathbf{P}^2$ by simply deleting the special orbit $(0, 0, 1)$. The isometry group of the $Spin(7)$ manifold $\Sigma^- \mathbf{S}^4$ is precisely $Sp(2) \times Sp(1)$. If we now take the $U(1)$ subgroup given by $U(1)_c \equiv U(1) \subset Sp(1)$ in the last factor of $Sp(2) \times Sp(1)$, then the fixed point set is the special orbit \mathbf{S}^4 . The quotient space is therefore given by¹⁴

$$\Sigma^- \mathbf{S}^4 / U(1)_c \cong \Lambda^- \mathbf{S}^4. \quad (4.84)$$

Since the \mathbf{S}^4 descended from a fixed point set, in type IIA we have a D6-brane wrapped on the bolt of the G_2 manifold $\Lambda^- \mathbf{S}^4$, the bundle of anti-self-dual 2-forms over \mathbf{S}^4 . The low-energy physics of type IIA string theory on this G_2 manifold (without the D6-brane) was discussed in section 3.1.2.

Consider now the orbit structure of $\mathbb{H}\mathbf{P}^2$ under $U(3) \subset Sp(3)$. This is illustrated in figure 11. In this case, discussed in [7], the generic orbit is the Aloff–Wallach space $N_{1,-1} = U(3)/U(1)^2$, where $U(1)^2$ is generated by elements of the form $\text{diag}(\mu, \lambda, \lambda^{-1}) \in U(3)$. The two special orbits are $\mathbb{C}\mathbf{P}^2 = U(3)/U(2) \times U(1)$ and $\mathbf{S}^5 = U(3)/U(1) \times SU(2)$. The copy of $\mathbb{C}\mathbf{P}^2$ is merely the subset of $\mathbb{H}\mathbf{P}^2$ in which all the homogeneous coordinates are purely complex. If we delete this special orbit from $\mathbb{H}\mathbf{P}^2$, we obtain a manifold which is an \mathbb{R}^3 bundle over the \mathbf{S}^5 special orbit. This is in fact the underlying manifold for the $Spin(7)$ geometry of interest. On the other hand, deleting the \mathbf{S}^5 special orbit from $\mathbb{H}\mathbf{P}^2$ gives the manifold \mathcal{Q} , which is an \mathbb{R}^4 bundle over the $\mathbb{C}\mathbf{P}^2$ special orbit.

To get the B -picture, we consider dividing out by the circle action on $\mathbb{H}\mathbf{P}^2$ generated by the diagonal subgroup $U(1)_D \subset U(3)$. The fixed point set of this circle action is precisely the special orbit $\mathbb{C}\mathbf{P}^2$. The generic orbit descends to a copy of the so-called twistor space of $\mathbb{C}\mathbf{P}^2$, which is the coset space $U(3)/U(1)^3$. This is also the sphere bundle of $\Lambda^- \mathbb{C}\mathbf{P}^2$.

If we delete the fixed copy of $\mathbb{C}\mathbf{P}^2$, we obtain a *free* action of $U(1) = U(1)_D$ on $\mathbb{R}^3 \times \mathbf{S}^5$. The $U(1)$ acts on the \mathbf{S}^5 special orbit by a Hopf map over a *dual* copy of $\mathbb{C}\mathbf{P}^2$, given by $\widetilde{\mathbb{C}\mathbf{P}^2} = U(3)/U(1) \times U(2)$. The quotient is therefore

$$(\mathbb{R}^3 \times \mathbf{S}^5) / U(1)_D \cong \Lambda^- \mathbb{C}\mathbf{P}^2, \quad (4.85)$$

with a single unit of RR 2-form flux through $\mathbb{C}\mathbf{P}^1 \subset \mathbb{C}\mathbf{P}^2$. Deleting the special orbit \mathbf{S}^5 from $\mathbb{H}\mathbf{P}^2$ and taking the quotient also give

$$\mathcal{Q} / U(1)_D \cong \Lambda^- \mathbb{C}\mathbf{P}^2, \quad (4.86)$$

but now the fixed point set $\mathbb{C}\mathbf{P}^2$ becomes a D6-brane wrapped on the zero section of $\Lambda^- \mathbb{C}\mathbf{P}^2$. We thus have a picture of the transition in which D6-branes are replaced with RR flux. Note that in this asymptotically conical case, the dilaton blows up at infinity, so that the type IIA solution is not really valid at large distances. However, there exist asymptotically locally conical versions of the two metrics for which the dilaton stabilizes to a finite value at infinity. An ALC $Spin(7)$ metric on \mathcal{Q} was constructed in [11].

Note that although the right-hand sides of (4.85) and (4.86) are diffeomorphic, they are not the ‘same’ manifold. The bolt of (4.86) came from the fixed $\mathbb{C}\mathbf{P}^2$, whereas the bolt of (4.85) came from a dual copy $\widetilde{\mathbb{C}\mathbf{P}^2}$. These are not the same copy of $\mathbb{C}\mathbf{P}^2$. Put another way, given a G_2 cone on the twistor space $SU(3)/U(1)^2$, there are three choices of $\mathbb{C}\mathbf{P}^2$ that we may make

¹³ In fact, the weak G_2 metric on the squashed 7-sphere, up to homothety, is embedded in the quaternionic projective plane in this way, for an appropriate geodesic distance [53].

¹⁴ In the remainder of the paper we choose orientation conventions such that the G_2 manifold is the bundle of anti-self-dual 2-forms over B .

to form the resolution $\Lambda^- \mathbb{C}\mathbb{P}^2$, permuted by the Weyl group Σ_3 of $SU(3)$. If we start with a D6-brane wrapped on the $\mathbb{C}\mathbb{P}^2$ bolt of the G_2 manifold $\Lambda^- \mathbb{C}\mathbb{P}^2$, then as we shrink the bolt to zero size, another copy of $\mathbb{C}\mathbb{P}^2$ blows up (namely $\widetilde{\mathbb{C}\mathbb{P}^2}$), with the D6-brane replaced with RR flux through $\mathbb{C}\mathbb{P}^1 \subset \widetilde{\mathbb{C}\mathbb{P}^2}$.

4.2. D6-branes on coassociative submanifolds of \mathbb{R}^7

In this subsection, we describe the construction of the L -picture discussed in section 2. The aim is to identify the appropriate $U(1)$ subgroup such that $X/U(1) \cong \mathbb{R}^7$, along with the corresponding fixed point sets, and also to determine how the symmetries in M-theory are realized in the L -picture.

Since the construction of these quotients is a little involved, it is useful at this stage to give a brief summary of the approach that we will take. The initial problem is to find the appropriate $U(1)$ isometry along which to embed the M-theory circle. In fact, this is related in a curious way to the B -picture, as described at the end of this introduction. The quotient itself is constructed in much the same way as the quotients in [7]. Roughly speaking, one foliates the manifold X by a family of $U(1)$ -invariant submanifolds. This family is acted on by a certain $Sp(1)$ subgroup of the symmetry group of X . After we take the quotient by $U(1)$ to reduce to type IIA on \mathbb{R}^7 , this $Sp(1)$ action describes the sweeping out of \mathbb{R}^7 in a form of generalized polar coordinates. This is precisely the Harvey and Lawson action (2.8). Recall that this $Sp(1)$ is a subgroup of G_2 which preserves the decomposition $\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}$. If one takes an appropriate curve in the r - s plane, where s and r are radial coordinates on each factor in the decomposition of $\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}$, then under the action of $Sp(1)$ we sweep out a coassociative 4-fold L , as described in section 2. At the same time, this $Sp(1)$ action sweeps out the fixed point set L of the circle action. In mathematical terms, we have therefore constructed an $Sp(1)$ -equivariant map from X to \mathbb{R}^7 . On X this $Sp(1)$ is simply a part of the symmetry group. On \mathbb{R}^7 it is the Harvey and Lawson action (2.8).

A mysterious duality. Before we proceed, we pause to point out a curious relation between the L - and B -pictures which emerges via the embedding in $\mathbb{H}\mathbb{P}^2$ described in the last subsection. Firstly, it is a bizarre enough fact that the explicitly known AC $Spin(7)$ and G_2 manifolds, together with their isometry groups, are related to $\mathbb{H}\mathbb{P}^2$ at all. We have already discussed the $Spin(7)$ case in this paper, and two of the three AC G_2 manifolds and their relation with $\mathbb{H}\mathbb{P}^2$ were discussed in [7]. However, in the $Spin(7)$ case, we also have the following curious fact. The $U(1)$ subgroup of $Sp(3)$ that produces the B -picture for the $Spin(7)$ manifold $\Sigma^- \mathbb{S}^4$ is the *same* $U(1)$ subgroup that produces the L -picture for the $Spin(7)$ manifold \mathcal{Q} , namely $U(1) = U(1)_c$. Moreover, the converse is also true! That is, the $U(1)$ subgroup $U(1)_D$ that produces the B -picture for the $Spin(7)$ manifold \mathcal{Q} also produces the L -picture for the $Spin(7)$ manifold $\Sigma^- \mathbb{S}^4$. This is most peculiar, and it is not clear to us whether or not there is any deep underlying reason for this fact.

4.2.1. *The cone over $SO(5)/SO(3)$.* The asymptotically conical $Spin(7)$ manifold $X = \Sigma^- \mathbb{S}^4$ was first constructed in [9], and is a resolution of the cone on the weak G_2 holonomy squashed 7-sphere Y . There is a single modulus $a > 0$ corresponding to the radius of the \mathbb{S}^4 bolt. The isometry group is $Sp(2) \times Sp(1)$. A generic principal orbit $Y = \mathbb{S}^7$ may therefore be viewed as the coset space

$$Y = Sp(2) \times Sp(1)_c / Sp(1)_{a+c} \times Sp(1)_b, \tag{4.87}$$

where we have labelled $Sp(1)_a \times Sp(1)_b \subset Sp(2)$ and $Sp(1)_{a+c}$ denotes the diagonal subgroup of $Sp(1)_a \times Sp(1)_c$. The 7-sphere Y then fibres over $\mathbf{S}^4 = Sp(2) \times Sp(1)_c / Sp(1)_a \times Sp(1)_b \times Sp(1)_c$ with fibres being copies of $\mathbf{S}^3 = Sp(1)_a \times Sp(1)_b \times Sp(1)_c / Sp(1)_{a+c} \times Sp(1)_b$. This is the quaternionic Hopf fibration.

In this notation, the $U(1)$ subgroup we require to produce the quotient (2.5) is given by the diagonal $U(1)_D = U(1)_{a+b+c} \subset Sp(1)_{a+b+c}$.

The circle action on $\mathbb{H}^3 \setminus \{0\}$ is free, but when we descend to \mathbb{HP}^2 the special orbit \mathbb{CP}^2 under the action of $U(3)$ is fixed. The circle action on the \mathbf{S}^5 special orbit merely rotates around the Hopf fibres over the dual copy of \mathbb{CP}^2 . This was described in the last subsection in relation to the B -picture for the other $Spin(7)$ geometry.

The quotient of \mathbb{HP}^2 by this action was shown to be $\mathbb{HP}^2 / U(1)_D = \mathbf{S}^7$ in [7]. The manifold X is obtained by deleting the point $(0, 0, 1)$ from \mathbb{HP}^2 . This descends to a point in \mathbf{S}^7 , and we have therefore shown that

$$\Sigma^- \mathbf{S}^4 / U(1)_D = \mathbf{S}^7 \setminus \{\text{pt}\} = \mathbb{R}^7. \tag{4.88}$$

The fixed point set is given by deleting the point $(0, 0, 1)$ from the fixed special orbit \mathbb{CP}^2 to give

$$L = \mathbb{CP}^2 \setminus \{\text{pt}\} = H^1, \tag{4.89}$$

where H^1 denotes the total space of the spin bundle of \mathbf{S}^2 . By supersymmetry, this embedding will be coassociative with respect to some G_2 structure on \mathbb{R}^7 .

The symmetry group of M-theory on X consists entirely of geometric symmetries $Sp(2) \times Sp(1)$. There are no symmetries associated with the C -field in this case since $H^2(X; U(1))$ is trivial. When we pass to type IIA, we shall find that the symmetry group gets broken to

$$Sp(2) \times Sp(1) \mapsto U(1)_J \times Sp(1) \times U(1). \tag{4.90}$$

The $U(1)_J$ is associated with the M-theory circle, and is of course just $U(1)_D$ in this case. We would like to understand the action of the remaining factor of $Sp(1) \times U(1) \sim U(2)$ on the type IIA geometry. Specifically, this will be a symmetry of the embedding of $L = H^1$ in \mathbb{R}^7 . In order to understand this, and its relation with the Harvey and Lawson geometries, it will be convenient to understand the quotient just constructed in a rather different way.

The idea is to foliate X by a two-parameter family of invariant 6-manifolds whose quotients by $U(1)_D$ may be identified with a parametrization of \mathbb{R}^7 in terms of generalized polar coordinates. Specifically, if one denotes

$$\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4 = \text{Im } \mathbb{H} \oplus \mathbb{H}, \tag{4.91}$$

then for a unit imaginary quaternion $x \in \text{Im } \mathbb{H}$, we define

$$\mathbb{R}^5(x) = (tx, y) \quad \text{where } t \in \mathbb{R}, \quad y \in \mathbb{H}. \tag{4.92}$$

As the unit vector x varies over the unit 2-sphere, the spaces $\mathbb{R}^5(x)$ sweep out \mathbb{R}^7 . Note that $\pm x$ give the same five-space.

More precisely, this process of sweeping out $\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}$ is achieved by fixing some value of x and then acting with $Sp(1)$ via the Harvey and Lawson action (2.8). This action on $\mathbb{R}^7 = \text{Im } \mathbb{O}$ preserves the G_2 structure and corresponding splitting into $\text{Im } \mathbb{H} \oplus \mathbb{H}$. Note that our fixed point set (4.89) is topologically the same as that of the Harvey and Lawson submanifold (2.13). The deformation parameter ρ , which measures the size of the \mathbf{S}^2 bolt, corresponds to the deformation parameter a of the $Spin(7)$ geometry which is essentially the radius of the \mathbf{S}^4 bolt. As we take both parameters to zero, we obtain a conical geometry in each case: a cone on the weak G_2 squashed 7-sphere in M-theory and a coassociative cone on a squashed 3-sphere for the D6-brane worldvolume in type IIA.

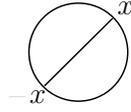


Figure 12. The quotient of \mathbf{S}^4 by $U(1)$ is a 3-disc D^3 . The boundary is identified with the fixed \mathbf{S}^2 . The image of the 2-sphere \mathbf{S}_x^2 is the line segment joining x to $-x$.

It remains to identify the $Sp(1) \subset G_2$ action of Harvey and Lawson with the part of the unbroken symmetry group in (4.90), and also to understand the action of the additional $U(1)$ factor. To do this we will follow a similar route to [7], although we will not spell out all of the details.

Let us view the \mathbf{S}^4 bolt of X as the unit sphere in \mathbb{R}^5 . Then $Sp(2) \subset Sp(2) \times Sp(1)$ acts on this 4-sphere via the usual action of $Spin(5) \cong Sp(2)$. Consider the subgroup $U(2) \subset Sp(2)$. Then $U(1)_D$ restricts to the diagonal $U(1) = U(1)_{a+b}$ and we get a decomposition

$$\mathbb{R}^5 = \mathbb{R}^2 \oplus \mathbb{R}^3, \tag{4.93}$$

where the two-plane \mathbb{R}^2 is rotated and the copy of \mathbb{R}^3 is fixed. The set of fixed points on \mathbf{S}^4 is therefore a copy of $\mathbf{S}^2 \subset \mathbb{R}^3$, which is also the subset of $\mathbb{H}\mathbf{P}^1 = \mathbf{S}^4$ in which the homogeneous coordinates are complex. When we take the quotient, since \mathbf{S}^2 has codimension 2 in \mathbf{S}^4 , this fixed point set becomes a boundary in the reduced space (see figure 12). Specifically, $\mathbf{S}^4/U(1)_D \cong D^3$, the closed 3-disc. The boundary ∂D^3 of this 3-disc is thus identified with the fixed \mathbf{S}^2 . Conversely, the $Sp(1) \sim SO(3)$ subgroup of $U(2) \sim U(1) \times SO(3)$ acts canonically on the factor of \mathbb{R}^3 in (4.93), and thus acts on the fixed 2-sphere $\mathbf{S}^2 \subset \mathbb{R}^3$. If we now view this 2-sphere as the unit sphere in $\text{Im } \mathbb{H}$, then $Sp(1)$ acts by conjugation of quaternions. This is precisely the action we were looking for. Thus the $Sp(1)$ factor in the unbroken symmetry group (4.90) is identified with $Sp(1) \cong SU(2) \subset U(2)$. The remaining factor of $U(1)$ is identified with $U(1)_c \subset Sp(1)_c$ which acts trivially on \mathbf{S}^4 . We shall discuss this factor further below.

Now fix a point $x \in \mathbf{S}^2 \subset \mathbb{R}^3 = \text{Im } \mathbb{H}$ and consider the 2-sphere \mathbf{S}_x^2 in \mathbf{S}^4 which lives in the three-space

$$\mathbb{R}^2 \oplus \mathbb{R}_x \subset \mathbb{R}^2 \oplus \mathbb{R}^3 = \mathbb{R}^5. \tag{4.94}$$

Here \mathbb{R}_x denotes the line in \mathbb{R}^3 which contains the point $x \in \mathbf{S}^2$ and the origin. Then, as we vary x by acting with $Sp(1)$, we sweep out \mathbf{S}^4 with a two-parameter family of 2-spheres \mathbf{S}_x^2 . Note that for fixed x , the image of \mathbf{S}_x^2 in the 3-disc D^3 is the line segment joining $[-x, x]$.

Denote the restriction of the \mathbb{R}^4 bundle $X = \Sigma^- \mathbf{S}^4$ to \mathbf{S}_x^2 as X_x . As we vary x , these invariant 6-manifolds sweep out X . We find that $X_x/U(1) = \mathbb{R}^5(x)$. Thus, as one varies x over the set of fixed points one sweeps out \mathbb{R}^7 with a two-parameter family of five-spaces $\mathbb{R}^5(x)$. In fact, one can show using similar techniques to [7] that, as a bundle with $U(1)$ action, X_x is given by

$$X_x = H^1(1) \oplus H^{-1}(1), \tag{4.95}$$

and that the quotient of this 6-manifold by $U(1)$ is indeed $\mathbb{R}^5(x)$. The details are left as an exercise for the interested reader, or, alternatively, may be found in [54].

We are now ready to identify the remaining group actions. The zero section of $X = \Sigma^- \mathbf{S}^4$ descends to the closed 3-disc, D^3 . Deleting the fixed point set gives the open 3-disc $\overset{\circ}{D}^3$ which is of course diffeomorphic to $\mathbb{R}^3 = \text{Im } \mathbb{H}$. Thus the quotient of X , minus its fixed points, is

$$\text{Im } \mathbb{H} \oplus \mathbb{H}, \tag{4.96}$$

with the second factor coming from the fibres of X . The splitting (4.96) is naturally the spin bundle of $\mathbb{R}^3 = \text{Im } \mathbb{H}$. Indeed, if one deletes the fixed \mathbf{S}^2 from \mathbf{S}^4 , then over the open disc $\overset{\circ}{D}^3$ the chiral spin bundle is simply the spin bundle of the disc. The $Sp(1)$ action by conjugation on $\text{Im } \mathbb{H}$ that we have already found therefore induces an $Sp(1)$ action on the \mathbb{H} factor. This is the right action [33]

$$y \mapsto y\bar{q}. \tag{4.97}$$

Thus the total action of $Sp(1) \subset U(2)$ on \mathbb{R}^7 is precisely the $Sp(1)$ action of Harvey and Lawson (2.8). Including the fixed points also gives \mathbb{R}^7 with (4.96) embedded as a dense open subset.

Finally, there is the factor $U(1) = U(1)_c$ which acts purely on the fibres of X . The Harvey and Lawson geometry is $Sp(1)$ invariant by construction, but there is indeed another $U(1)$ symmetry which corresponds to $U(1)_c$ under our isomorphism. Namely, the action

$$(x, y) \mapsto (x, \lambda y). \tag{4.98}$$

This $U(1)$ is a subgroup of G_2 which also preserves the corresponding decomposition of \mathbb{R}^7 . It acts by rotating the fibres of $H^1 = L$. This is indeed a symmetry of this geometry. A principal $Sp(1) = \mathbf{S}^3$ orbit is a squashed 3-sphere. The isometry group is $U(2) \sim Sp(1) \times U(1)$, with the $Sp(1)$ acting on the base \mathbf{S}^2 of the Hopf fibration, and $U(1)$ acting on the fibres. In sum, we have an action of $Sp(1) \times U(1)$ given by

$$(x, y) \mapsto (qx\bar{q}, \lambda y\bar{q}). \tag{4.99}$$

Note that $(-1, -1) \in Sp(1) \times U(1)$ acts trivially, and so the group that acts effectively here is $(Sp(1) \times U(1))/\mathbb{Z}_2 \cong U(2)$. The symmetry group (4.90) of X is actually more precisely $(Sp(2) \times Sp(1))/\mathbb{Z}_2$ with the \mathbb{Z}_2 generated by $(-1, -1)$. Thus the symmetry breaking may be written more precisely as

$$(Sp(2) \times Sp(1))/\mathbb{Z}_2 \mapsto U(1)_J \times U(2). \tag{4.100}$$

This completes our analysis.

4.2.2. The cone over $SU(3)/U(1)$. The asymptotically conical $Spin(7)$ manifolds \mathcal{Q} and $\mathbb{R}^3 \times \mathbf{S}^5$ are both resolutions of the cone on the weak G_2 holonomy Aloff–Wallach space, $N_{1,-1} = SU(3)/U(1)$. Numerical evidence for the existence of these solutions was given in [12]. The isometry group is $U(3)$. In each case, there is a single modulus $a > 0$ corresponding to the size of the $\mathbb{C}\mathbf{P}^2$ or \mathbf{S}^5 bolt, respectively.

Our aim in this section is to find a $U(1)$ subgroup of $U(3)$ such that the quotient spaces may be identified with \mathbb{R}^7 (2.5). We shall find that the fixed point sets in each case are given, respectively, by

$$L = H^1 \cup \mathbb{R}^4 \quad \text{and} \quad L = \mathbf{S}^3 \times \mathbb{R}. \tag{4.101}$$

In the conical limit $a \rightarrow 0$, we therefore have

$$L \rightarrow C(\mathbf{S}^3 \cup \mathbf{S}^3) = \mathbb{R}^4 \cup \mathbb{R}^4. \tag{4.102}$$

Remarkably, these are precisely the coassociative submanifolds (2.17) and (2.19) considered in section (2.2). If we place a D6-brane on the coassociative submanifold (2.17) and lift to M-theory, we obtain the $Spin(7)$ manifold \mathcal{Q} . The symmetry group associated with the C -field in M-theory is given by $H^2(\mathcal{Q}; U(1)) \cong U(1)$ which becomes the axial $U(1)$ on the D6-branes (the diagonal $U(1)$ decouples as usual). The circle reduction breaks the geometric symmetry group to

$$U(3) \mapsto U(2) \times U(1)_J. \tag{4.103}$$

If we write $U(2) \times U(1) \subset U(3)$ in the obvious way, we shall find that $U(1)_J$ is given by the $U(1)$ factor, and the $U(2)$ symmetry of the Harvey and Lawson geometry gets identified with the $U(2)$ factor. Note this is the same $U(2)$ subgroup of $Sp(3)$ found in the last section, but now we have $U(1)_J = U(1)_c$. The $Sp(1)$ that sweeps out L is given by $Sp(1) \cong SU(2) \subset U(2)$. Note also that the coassociative plane $\mathbb{R}^4 = \mathbb{H}$ is just the copy of \mathbb{H} in $\text{Im } \mathbb{H} \oplus \mathbb{H}$ at $s = 0$, and this is also swept out under

$$(x, y) \mapsto (qx\bar{q}, y\bar{q}) \tag{4.104}$$

on taking $x = 0$ (rather than $x = \epsilon$ a unit vector, which describes the H^1 component of L).

Conversely, if we place a D6-brane on (2.19) and lift to M-theory, we obtain the $Spin(7)$ manifold $\mathbb{R}^3 \times \mathbb{S}^5$. The symmetry group $H^2(X; U(1))$ associated with the C -field in M-theory is now trivial, which corresponds to the fact that there is only one connected component of L in this resolution. The circle reduction again breaks the geometric symmetry group as in (4.103).

In the remainder of the paper, we give an explicit construction of the L -picture quotients.

The first resolution. We would like to construct the isomorphism $\mathcal{Q}/U(1)_c \cong \mathbb{R}^7$ where the codimension 4 fixed point set is $L = H^1 \cup \mathbb{R}^4$, thus making contact with the coassociative geometry described in section 2. The bundle \mathcal{Q} is a chiral spin bundle (or, more precisely, a spin^c bundle), but in this subsection we shall find the following description of $X = \mathcal{Q}$ more useful.

The universal quotient bundle \mathcal{Q} fits into the following short exact sequence of vector bundles

$$0 \rightarrow L_{\mathbb{C}}^{(2)} \rightarrow \mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3 \rightarrow \mathcal{Q} \rightarrow 0. \tag{4.105}$$

Here $L_{\mathbb{C}}^{(2)}$ denotes the canonical complex line bundle over $\mathbb{C}\mathbb{P}^2$. It is a sub-bundle of the trivial bundle $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3$ defined by

$$L_{\mathbb{C}}^{(2)} = \{(l, z) \in \mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3 \mid z \in l\}. \tag{4.106}$$

That is, the fibre of $L_{\mathbb{C}}^{(2)}$ above the point $l \in \mathbb{C}\mathbb{P}^2$ is the complex line through l . In fact, one may make a similar definition of $L_{\mathbb{K}}^{(n)}$ as the canonical line bundle over $\mathbb{K}\mathbb{P}^n$, for any of the associative normed division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. For example, the topology of X in the last subsection is $L_{\mathbb{H}}^{(1)}$.

The group $U(3)$ acts on $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3$ by the diagonal action on each factor, and, in this way, $U(3)$ acts on the line bundle $L_{\mathbb{C}}^{(2)}$ and therefore on the quotient \mathcal{Q} , which we may take to be $(L_{\mathbb{C}}^{(2)})^{\perp}$ with respect to a flat metric on \mathbb{C}^3 .

The $U(1)$ subgroup we require is $U(1) = U(1)_c$ generated by elements $(1, 1, \lambda^2)$ inside the maximal torus of $U(3)$. The fixed point set on the zero section $\mathbb{C}\mathbb{P}^2$ consists of two components. In terms of homogeneous coordinates on $\mathbb{C}\mathbb{P}^2$, the circle action fixes the point $A = (0, 0, 1)$, together with a copy of $B = \mathbb{C}\mathbb{P}^1 = \mathbb{S}^2$ given by the points $(z_1, z_2, 0)$.

The fibre of $L_{\mathbb{C}}^{(2)}$ above the fixed point A is $(0, 0, z) \in \mathbb{C}^3$ and is acted on by our $U(1)$ subgroup with weight 2. Conversely, the $\mathbb{R}^4 = \mathbb{C}^2$ fibre of \mathcal{Q} above A consists of the points $(z_1, z_2, 0) \in \mathbb{C}^3$, which is clearly fixed under the $U(1)$ action. Thus the total fixed point set above A is a copy of \mathbb{R}^4 .

The total fixed point set above $B = \mathbb{C}\mathbb{P}^1$ in $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3$ is $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}^2$. Over B , the line bundle $L_{\mathbb{C}}^{(2)}$ restricts to $L_{\mathbb{C}}^{(1)}$, and we have the following short exact sequence:

$$0 \rightarrow L_{\mathbb{C}}^{(1)} \rightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}^2 \rightarrow E \rightarrow 0. \tag{4.107}$$

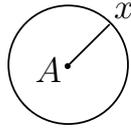


Figure 13. The quotient of $\mathbb{C}\mathbb{P}^2$ by $U(1)$ is also a 3-disc D^3 . The boundary is identified with the fixed $\mathbb{C}\mathbb{P}^1 = \mathbb{S}^2$, and the centre is the fixed point A . The image of the 2-sphere \mathbb{S}_x^2 is the line segment joining A to x .

The quotient bundle E is a line bundle and is just $(L_{\mathbb{C}}^{(1)})^{-1}$, which is thus the total fixed point set in \mathcal{Q} above B . This line bundle has first Chern class one, and so is also H^1 . In sum, the total fixed point set is

$$L = \mathbb{R}^4 \cup H^1. \tag{4.108}$$

Since L has codimension 4 in $X = \mathcal{Q}$, the quotient will be a manifold, and is in fact diffeomorphic to \mathbb{R}^7 . Rather as before, we shall find that \mathcal{Q} may be foliated by a two-parameter family of invariant 6-manifolds whose quotients by $U(1)$ may be identified with a parametrization of \mathbb{R}^7 in terms of generalized polar coordinates. Specifically, if one denotes

$$\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}, \tag{4.109}$$

then for a unit vector $x \in \text{Im } \mathbb{H}$, we define

$$\mathbb{R}_+^5(x) = (tx, y) \quad \text{where } t \in \mathbb{R}_+, \quad y \in \mathbb{H}. \tag{4.110}$$

As the unit vector x varies under the natural action of $Sp(1)$, the half-spaces $\mathbb{R}_+^5(x)$ sweep out \mathbb{R}^7 , with the boundary of the half-space as axis. In fact, this axis will turn out to be the fixed \mathbb{R}^4 above the point A .

The quotient of $\mathbb{C}\mathbb{P}^2$ by $U(1)$ is again the closed 3-disc D^3 (see figure 13). The image of the point A lies at the centre of D^3 , and the boundary ∂D^3 of D^3 is identified with the fixed $\mathbb{C}\mathbb{P}^1: \partial D^3 \cong B$. For each fixed point $x \in B$, we now define the 2-sphere \mathbb{S}_x^2 to be the copy of $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^2$ containing the points x and A . The 2-sphere B is acted on by the subgroup $Sp(1) \cong SU(2) \subset U(2)$ in (4.103). In this way we sweep out $\mathbb{C}\mathbb{P}^2$ with a two-parameter family of 2-spheres, with axis A . If we view B as the unit sphere in $\text{Im } \mathbb{H}$, then this $Sp(1)$ action is by conjugation of quaternions. Note that the diagonal $U(1) = U(1)_{a+b}$ acts trivially on B . This $U(1)$ will become the extra $U(1)$ symmetry in the Harvey and Lawson geometry.

We again define X_x to be the restriction of $X = \mathcal{Q}$ to \mathbb{S}_x^2 , so that the corresponding 6-manifolds sweep out \mathcal{Q} , with axis $\mathbb{R}_A^4 = X_A$, as x varies over the fixed $\mathbb{S}^2 = B$. The quotient spaces $X_x/U(1)$ will turn out to be half-spaces $\mathbb{R}_+^5(x)$ which sweep out \mathbb{R}^7 as x varies.

Let us fix $x \in B$. Without loss of generality, we may take $x = (1, 0, 0) \in B \subset \mathbb{C}\mathbb{P}^2$. Each 6-manifold X_x is a \mathbb{C}^2 bundle over \mathbb{S}_x^2 . This will decompose into the sum of two line bundles. Thus, as a bundle with $U(1)$ action, we must have

$$X_x = H_x^k(m) \oplus H_x^l(n), \tag{4.111}$$

for some integers k, l, m, n . In order to work out this splitting, we may look at the total weights of the $U(1)$ action over the two fixed points x and A .

The fibre of X_x above the north pole x is given by points in $\mathbb{C} \oplus \mathbb{C} = \{(0, z_2, z_3)\}$. The first copy is fixed, and the second is acted on with weight 2. Thus the weights on the tangent space and the two copies of \mathbb{C} above the north pole x are, respectively

$$(2, 0, 2). \tag{4.112}$$

The fibre above the south pole A is fixed, so the weights are

$$(-2, 0, 0). \tag{4.113}$$

This completely determines the integers k, l, m, n to be $0, 1, 0, 1$, respectively. We therefore find that a trivial direction splits off to give

$$X_x = H_x^1(1) \times \mathbb{C}_x, \tag{4.114}$$

where the trivial factor \mathbb{C}_x is fixed under $U(1)$ and the H_x^1 factor is rotated. This splitting is also consistent with the fact that the first Chern class of X_x should be the generator of $H^2(\mathbb{S}_x^2; \mathbb{Z}) \cong \mathbb{Z}$. Note that the fixed \mathbb{C}_x above the point $x \in B$ is the fibre of the fixed H_B^1 above the point x . However, above the point A (which is contained in every \mathbb{S}_x^2) this \mathbb{C}_x lies in the fixed $\mathbb{R}^4 = X_A$. Conversely, the rotated fibre of H_x^1 above x coincides with the rotated fibre \mathbb{C}_B above x .

The quotient space is given by

$$X_x/U(1) = (H_x^1(1)/U(1)) \times \mathbb{C}_x = \mathbb{R}_+^3(x) \times \mathbb{C}_x = \mathbb{R}_+^5(x), \tag{4.115}$$

where we have used the fact that $H^1(1)/U(1) = \mathbb{R}_+^3$ [7]. The boundary of the half-space $\mathbb{R}_+^5(x)$ is the fixed copy of \mathbb{H} , which is $\{0\} \times \mathbb{H}$ in $\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}$. The diagonal $U(1)$ subgroup of $U(2)$ acts on the fixed H^1 by rotating the fibre with weight 1 (it acts trivially on the zero section), and acts on the fixed $\mathbb{R}^4 = \mathbb{C} \oplus \mathbb{C}$ with weights $(1, 1)$. This is the same action that we found in the last subsection, and thus we identify this $U(1)$ with the extra $U(1)$ symmetry of the Harvey and Lawson geometry.

The second resolution. To complete the picture, we would like to construct the isomorphism $\mathbb{R}^3 \times \mathbb{S}^5/U(1) \cong \mathbb{R}^7$ with codimension 4 fixed point set $L = \mathbb{S}^3 \times \mathbb{R}$.

The isometry group is again $U(3)$ with the M-theory circle being $U(1) = U(1)_c$. The bolt \mathbb{S}^5 is acted on by the $U(3)$ symmetry group in the obvious way, viewing \mathbb{S}^5 as the unit sphere in \mathbb{C}^3 . The $U(1)$ action then decomposes

$$\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}, \tag{4.116}$$

with the copy of \mathbb{C} rotated and the \mathbb{C}^2 fixed. Hence the fixed point set on \mathbb{S}^5 is a copy of \mathbb{S}^3 , the unit sphere in \mathbb{C}^2 . The restriction of the \mathbb{R}^3 bundle $\mathbb{R}^3 \times \mathbb{S}^5$ to \mathbb{S}^3 is isomorphic to the product space $\mathbb{S}^3 \times \mathbb{R}^3$. Our $U(1)$ acts on this space. Specifically, above each fixed point on the 3-sphere, we get a copy of \mathbb{R}^3 on which $U(1)$ acts. There are essentially only two choices. Either the whole fibre is fixed, or else we get a splitting

$$\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}, \tag{4.117}$$

with the \mathbb{C} rotated (with some weight) and the factor of \mathbb{R} fixed. Since we know from the previous subsection that the fixed point set on the sphere bundle of $\mathbb{R}^3 \times \mathbb{S}^5$ is $\mathbb{S}^3 \cup \mathbb{S}^3$, this rules out the first possibility, and we conclude that the total fixed point set is

$$L = \mathbb{S}^3 \times \mathbb{R}, \tag{4.118}$$

in agreement with the Harvey and Lawson geometry. Now fix a point $w \in \mathbb{S}^3$ and define the 2-sphere \mathbb{S}_w^2 as the unit 2-sphere in

$$\mathbb{R}_w \oplus \mathbb{C} \subset \mathbb{C}^2 \oplus \mathbb{C} = \mathbb{C}^3, \tag{4.119}$$

where \mathbb{R}_w is the line in $\mathbb{C}^2 = \mathbb{R}^4$ through the points $\pm w$. The restriction of the \mathbb{R}^3 bundle $X = \mathbb{R}^3 \times \mathbb{S}^5$ to \mathbb{S}_w^2 must split as

$$X_w = H_w^k(n) \times \mathbb{R}(w), \tag{4.120}$$

for some bundle with $U(1)$ action over \mathbf{S}_w^2 given by $H_w^k(n)$. We will not actually need to determine this bundle precisely. We simply observe that, since L has codimension 4, the quotient is a manifold, and the only¹⁵ values of (k, n) for which this is possible are given by $(k, n) = (2, 0)$ or $(0, 2)$. In either case the quotient is $H^k(n)/U(1) = \mathbb{R}^3$, and we conclude that

$$X_w/U(1) = \mathbb{R}^3(w) \times \mathbb{R}(w) = \mathbb{R}^4(w). \quad (4.121)$$

As w varies over the fixed \mathbf{S}^3 , these four-spaces sweep out \mathbb{R}^7 in generalized polar coordinates

$$\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3, \quad (4.122)$$

where for a unit vector $w \in \mathbb{R}^4$ we define

$$\mathbb{R}^4(w) = (tw, v), \quad \text{where } t \in \mathbb{R}, \quad v \in \mathbb{R}^3. \quad (4.123)$$

The symmetry group again breaks according to

$$U(3) \mapsto U(2) \times U(1)_J. \quad (4.124)$$

We may again identify the $U(2)$ action with the Harvey and Lawson symmetry group. Specifically, we see from the above construction that $U(2)$ acts on the \mathbf{S}^3 factor via its embedding in \mathbb{C}^2 . The diagonal $U(1)_D \subset U(2)$ Hopf fibres the 3-sphere over a copy of \mathbf{S}^2 , and the $Sp(1) \cong SU(2) \subset U(2)$ part acts transitively on $\mathbf{S}^3 \cong Sp(1)$. This is precisely the action of the $U(2)$ symmetry group on the Harvey and Lawson coassociative geometry.

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Appendix. Geometry of the $Spin(7)$ conifold

In this section, we describe the geometry of the $Spin(7)$ cone on $SU(3)/U(1)$ and its complete resolutions

$$\mathbb{R}^4 \times \mathbb{C}\mathbf{P}^2, \quad \mathbb{R}^3 \times \mathbf{S}^5. \quad (\text{A.125})$$

Even though the existence of a $Spin(7)$ metric in both cases is suggested by the dual configuration of D6-branes (see section 2), the explicit AC metric is not known. The most systematic and convenient way to find this metric appears to be via the technique developed by Hitchin [24], which has already been applied with great success in the case of G_2 manifolds [55, 56]. Therefore, one particular goal of the discussion below will be to explain this technique and compare the results with what is known in the literature.

¹⁵ The case $(k, n) = (1, \pm 1)$ is also ruled out since this would require that the entire \mathbb{R}^3 fibre above either the north or south pole of \mathbf{S}_w^2 be fixed, which we know is not the case.

In general, given a compact 7-manifold Y , the problem is to find a complete $Spin(7)$ metric on a manifold X with principal orbits being copies of Y . To do this, one picks a family of 4-forms with a fixed homology class on Y :

$$\rho(x_1, \dots, x_r) \in \Omega_{\text{exact}}^4(Y), \tag{A.126}$$

where x_1, \dots, x_r are parameters (below functions of t). One can choose x_i such that

$$\rho = \sum_i x_i u_i, \tag{A.127}$$

where each u_i is exact, i.e. can be written in the form:

$$u_i = d(v_i), \quad v_i \in \Omega^3(Y)/\Omega_{\text{closed}}^3(Y). \tag{A.128}$$

Then, the following (indefinite) bilinear form is non-degenerate [24]:

$$Q(u_i, u_j) = \int_Y u_i \wedge v_j. \tag{A.129}$$

Given $\rho \in \Lambda^4 V^*$, one can define its Hodge dual $\sigma \in \Lambda^3 V \otimes \Lambda^7 V^*$. If ρ has the explicit form $\rho_{ijkl} dx^i dx^j dx^k dx^l$, then we can write σ as σ^{ijk} . Taking any $v, w \in V^*$, one can construct a top degree 7-tensor:

$$v_a \sigma^{aij} w_b \sigma^{bkl} \sigma^{mnp} \in (\Lambda^7 V^*)^2. \tag{A.130}$$

Or, one can think of it as a map:

$$H: V^* \rightarrow V \otimes (\Lambda^7 V^*)^2. \tag{A.131}$$

Evaluating $\det H \in (\Lambda^7 V^*)^{12}$, one can define

$$\phi(\rho) = |\det H|^{1/12}. \tag{A.132}$$

By taking the total volume, one can define the following functional of $x_1(t), \dots, x_r(t)$:

$$V(\rho) = \int_Y \phi(\rho). \tag{A.133}$$

Once we have $Q(u_i, u_j)$ and $V(\rho)$, we can write down the gradient flow equations:

$$\frac{dx_i}{dt} = -Q^{(-1)ij} \frac{dV}{dx^j}, \tag{A.134}$$

which give the desired metric with $Spin(7)$ holonomy. Indeed, according to [24] solutions to these equations define the $Spin(7)$ structure on the 8-manifold X :

$$\Psi = dt \wedge * \rho + \rho. \tag{A.135}$$

Now, let us see how this works in the case we are interested in, namely, when Y is the Aloff–Wallach space

$$N_{1,-1} \cong SU(3)/U(1).$$

To describe the geometry of this space more explicitly, we define left-invariant 1-forms L_A^B on $SU(3)$ (with $A = 1, \dots, 3$) satisfying $L_A^A = 0$, $(L_A^B)^\dagger = L_B^A$, together with the exterior algebra

$$dL_A^B = iL_A^C \wedge L_C^B. \tag{A.136}$$

One must now split the generators into those that lie in the coset $SU(3)/U(1)$ and those that lie in the denominator $U(1)$. In particular, one must specify the $U(1)$ generator Q . In the case of $Y = N_{1,-1}$ we have

$$Q \equiv -L_1^1 + L_2^2. \tag{A.137}$$

One may now introduce the 1-forms

$$\sigma \equiv L_1^3, \quad \Sigma \equiv L_2^3, \quad \nu \equiv L_1^2, \quad (\text{A.138})$$

together with the $U(1)$ generator

$$\lambda \equiv L_1^1 + L_2^2. \quad (\text{A.139})$$

Note that λ and Q generate the maximal torus of $SU(3)$. Finally, since the forms (A.138) are complex, one may split them into real and imaginary parts

$$\sigma \equiv \sigma_1 + i\sigma_2, \quad \Sigma \equiv \Sigma_1 + i\Sigma_2, \quad \nu \equiv \nu_1 + i\nu_2 \quad (\text{A.140})$$

In order to find possible resolutions of the $Spin(7)$ cone on $Y = N_{1,-1}$, one needs to find a basis of left-invariant 4-forms u_i . Since the 4-forms u_i are also required to be exact, one starts with left-invariant 3-forms v_i , which in the present case are defined by the condition

$$v([Q, g_1], g_2, g_3) + v(g_1, [Q, g_2], g_3) + v(g_1, g_2, [Q, g_3]) = 0, \quad (\text{A.141})$$

for all $g_i \in su(3)/u(1)$. In our case, $Y = N_{1,-1}$, we find six left-invariant 3-forms v_i , two of which turn out to be closed. Therefore, we end up with only four independent left-invariant exact 4-forms $u_i = dv_i$:

$$\begin{aligned} u_1 &= d(\lambda\sigma_1\sigma_2) & u_2 &= d(\lambda\Sigma_1\Sigma_2) & u_3 &= d(\lambda\nu_1\nu_2) \\ u_4 &= 4(-\nu_1\nu_2\sigma_1\sigma_2 + \nu_1\nu_2\Sigma_1\Sigma_2 + \sigma_1\sigma_2\Sigma_1\Sigma_2). \end{aligned} \quad (\text{A.142})$$

Using the definition of the bilinear form Q one can easily compute

$$Q = \begin{pmatrix} 0 & -2 & -2 & 4 \\ -2 & 0 & 0 & 4 \\ -2 & 0 & 0 & -4 \\ 4 & 4 & -4 & 0 \end{pmatrix}. \quad (\text{A.143})$$

Finally, we define the $U(1)$ -invariant 4-form ρ in terms of the basis u_i :

$$\rho = x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4. \quad (\text{A.144})$$

Following the general prescription, we now have to compute the invariant functional $V(\rho)$ and derive a system of first-order gradient flow equations that follow from this $V(\rho)$. However, since in total we have only four independent functions x_i , under a suitable change of variables the resulting system is guaranteed to be equivalent to the system of differential equations obtained for the following metric ansatz:

$$ds_8^2 = dt^2 + a(t)^2 (\sigma_1^2 + \sigma_2^2) + b(t)^2 (\Sigma_1^2 + \Sigma_2^2) + c(t)^2 (\nu_1^2 + \nu_2^2) + f(t)^2 \lambda^2. \quad (\text{A.145})$$

The requirement of $Spin(7)$ holonomy requires the functions to satisfy the following system of first-order differential equations,

$$\begin{aligned} \frac{\dot{a}}{a} &= \frac{b^2 + c^2 - a^2}{abc} & \frac{\dot{b}}{b} &= \frac{a^2 + c^2 - b^2}{abc} - \frac{f}{b^2} \\ \frac{\dot{c}}{c} &= \frac{a^2 + b^2 - c^2}{abc} + \frac{f}{c^2} & \frac{\dot{f}}{f} &= -\frac{f}{c^2} + \frac{f}{b^2}, \end{aligned} \quad (\text{A.146})$$

where $\dot{a} = \frac{da}{dt}$, etc. These were studied by Cvetic *et al* [12], where a detailed numerical analysis was performed. They find a family of metrics for $\mathbb{R}^4 \times \mathbb{C}\mathbb{P}^2$ which may be expanded near the bolt at $t = 0$,

$$\begin{aligned} a &= t - \frac{1}{2}(1+q)t^3 + \dots & b &= 1 + \frac{5}{6}t^2 + \dots \\ c &= 1 + \frac{2}{3}t^2 + \dots & f &= t + qt^3 + \dots \end{aligned}$$

From this we see that b and c are non-vanishing which, comparing to (A.145), ensures that a $\mathbb{C}\mathbb{P}^2$ stabilizes at the centre of the space. Note that we have fixed the overall scale of the manifold. The metric functions may be extended to regular solutions over the whole space only for the parameter q in the range $q \geq q_0$ where q_0 is a numerical constant. For all values of $q > q_0$, the metric is asymptotically locally conical, with the circle dual to λ having finite size at infinity. For the specific value of $q = 13/9$, a complete analytic solution was found in [11]. For $q = q_0$ the metric is asymptotically conical.

The second family of asymptotic expansions around the bolt is given by [12]

$$\begin{aligned} a &= 1 - \frac{1}{3}qt + \left(1 - \frac{5}{18}q^2\right)t^2 + \left(\frac{7}{45} - \frac{167}{810}q^2\right)qt^3 + \dots \\ b &= 1 + \frac{1}{3}qt + \left(1 - \frac{5}{18}q^2\right)t^2 - \left(\frac{7}{45} - \frac{167}{810}q^2\right)qt^3 + \dots \\ c &= 2t + \frac{4}{27}(q^2 - 9)t^3 + \dots \\ f &= q + \frac{2}{3}q^3t^2. \end{aligned} \tag{A.147}$$

In contrast to the previous solution, a , b and f are all non-vanishing at $t = 0$, ensuring that the bolt has topology S^5 . The parameter q now takes values in $0 < q \leq q_0 \sim 0.87$. Even though complete $Spin(7)$ metrics of this type are not known explicitly, our analysis in sections 2 and 4, and the numerical analysis in [12], strongly suggest that there is a manifold with short-distance asymptotics (A.147) and asymptotically locally conical behaviour at large distances, for all values of q apart from q_0 . In the latter case, the metric describes an asymptotically conical manifold with topology $\mathbb{R}^3 \times S^5$.

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