The Dynamics of Two Coupled Rigid Bodies

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Abstract

In this paper we derive a Poisson bracket on the phase space $so(3)^* \times so(3)^* \times SO(3)$ such that the dynamics of two three dimensional rigid bodies coupled by a ball and socket joint can be written as a Hamiltonian system.

§1. Introduction

In this paper we introduce a Poisson bracket on the phase space

$$so(3)^* \times so(3)^* \times SO(3),$$

where $so(3)^*$ is the dual of the Lie algebra of $SO(3)$, so that the dynamics of two rigid bodies coupled by a ball and socket joint can be written as the Hamilitonian system $\dot{H} = \{F, H\}$. This sets the stage so that the stability and asymptotics of the system can be studied using the energy Casimir method as in Holm, Marsden, Ratiu and Weinstein [1985] and Krishnaprasad [1985]; so that chaotic solutions can be found using the Melnikov method such as in Holmes and Marsden [1983]; so that bifurcations of the system can be described using the techniques in Golubitsky and Stewart [1986] and Lewis, Marsden and Ratiu [1986]; and so that control issues can be studied, as in Sanchez de Alvarez [1986].

The dynamics of planar coupled rigid bodies has been studied using similar ideas in Sreenath, Krishnaprasad and Marsden [1986].

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§2. Kinematics

In this section we will derive the Lagrangian describing the free motion of two rigid bodies coupled by a ball and socket joint. At time 0 we assume that the two coupled rigid bodies are in a reference configuration denoted $\mathcal{B}$. Fix an inertial frame and let Q denote a point in the reference configuration $\mathcal{B}$. Let $\mathcal{B}_1$ denote those points $Q \in \mathcal{B}$ which belong to body 1 and let $\mathcal{B}_2$ denote those points which belong to body 2.

The configuration at time t is determined by a smooth map

$$\eta : \mathcal{B} \longrightarrow \mathbf{R}^3, \quad Q \to q(Q,t).$$

We can also specify the configuration at time t as follows. First we specify the position of the joint with respect to the inertial frame. Denote this as $w(t)$. Fix a frame centered at the joint and parallel to the inertial frame. With respect to this frame, the configuration of body 1 is determined as usual by three Euler angles. The Euler angles determine the orientation of a body fixed frame relative to the spatial frame centered at the joint. Alternatively these two frames are related by an element $A_1(t) \in \mathrm{SO}(3)$. Similarly the configuration of body 2 is determined by an element $A_2(t) \in \mathrm{SO}(3)$. We conclude that the configuration space is

$$\mathcal{C} = \mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathbf{R}^3$$

and that

$$\begin{aligned} q(Q,t) &= A_1(t)Q + w(t), \quad \text{for } Q \in \mathcal{B}_1 \\ q(Q,t) &= A_2(t)Q + w(t), \quad \text{for } Q \in \mathcal{B}_2. \end{aligned} \tag{2.1}$$

We now proceed to compute the kinetic energy of the system. This requires that we keep track of the centers of mass of the two bodies and the center of mass of the system relative to the fixed inertial frame as well as the frame centered at the joint. Let m_1 and m_2 denote the masses of the two bodies and let m denote the total mass. Let S_1^0 denote the center of mass of body 1 in the reference configuration relative to the inertial frame and let S_2^0 denote the center of mass for body 2. Let $\mathbf{r}_1(t)$ denote the center of mass of body 1 at time t relative to the inertial frame and let $\mathbf{r}_2(t)$ denote the center of mass for body 2. Let $s_1(t)$ and $s_2(t)$ denote the center of mass of bodies 1 and 2, respectively, measured with respect to the frame centered at the joint. Finally let $\alpha(t)$ denote the center of mass of the ensemble measured with respect to the inertial frame. Figures 1 and 2 show the relationships of these quantites. For example the following equations can be read off from the figures

$$\begin{aligned} s_1(t) &= A_1(t)S_1^0 \qquad & \mathbf{r}_1(t) &= w(t) + s_1(t) \\ s_2(t) &= A_2(t)S_2^0 \qquad & \mathbf{r}_2(t) &= w(t) + s_2(t) \end{aligned} \tag{2.2}$$

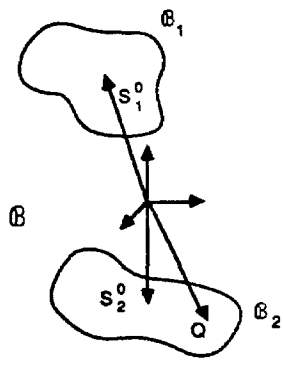

Figure 1

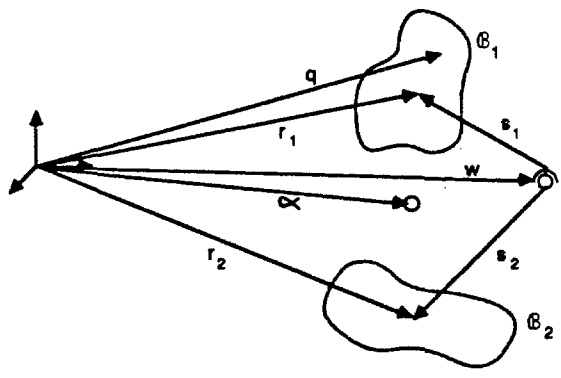

Figure 2

Given two vectors $v = (v^1, v^2, v^3)$ and $w = (w^1, w^2, w^3)$, denote their inner product relative to the standard Euclidean structure as

$$(v, w) = \sum_{i=1}^{3} v^i w^i,$$

and the corresponding norm by $|v|$. Let $\mu(Q)$ denote the mass measure of the ensemble in the reference configuration. The kinetic energy KE of the configuration is

$$\begin{aligned}
\mathrm{KE} &= \frac{1}{2}\int_{\mathcal{B}} |\dot{q}(t)|\, \mathrm{d}\mu(Q) \\
&= \frac{1}{2}\int_{\mathcal{B}_1} |\dot{A}_1(t)Q + w(t)|^2\, \mathrm{d}\mu(Q) + (1 \leftrightarrow 2) \\
&= \frac{1}{2}\int_{\mathcal{B}_1} (\dot{A}^i_{1k}Q^k + \dot{w}^i)\cdot(\dot{A}^j_{2l}Q^l + \dot{w}^j)\, \mathrm{d}\mu(Q) + (1 \leftrightarrow 2) \\
&= \left\{\frac{1}{2}\int_{\mathcal{B}_1} Q^k Q^l\, \mathrm{d}\mu(Q)\right\} \dot{A}^i_{1k}\dot{A}^j_{1l} + \left\{\frac{1}{2}\int_{\mathcal{B}_1} \mathrm{d}\mu(Q)\right\} \dot{w}^i\dot{w}^j \\
&\quad + \left\{\frac{1}{2}\int_{\mathcal{B}_1} Q^k\, \mathrm{d}\mu(Q)\right\} \dot{A}^i_{1k}\dot{w}^j + \left\{\frac{1}{2}\int_{\mathcal{B}_1} Q^l\, \mathrm{d}\mu(Q)\right\} \dot{A}^j_{1l}\dot{w}^i + (1 \leftrightarrow 2).
\end{aligned} \tag{2.3}$$

Let I_1 denote the coefficient of inertia matrix of body 1, defined by

$$(I_1)^{ij} = \int_{\mathcal{B}_1} Q^i Q^j\, \mathrm{d}\mu(Q), \qquad \text{for } i, j = 1, 2, 3. \tag{2.4}$$

The coefficient of inertia $\tilde{I}_2$ of body 2 is defined similarly. Using these definitions and the defintion of the center of mass, we can rewrite the expressions above as

$$\begin{aligned}
\mathrm{KE} &= \frac{1}{2}\mathrm{tr}\,(\dot{A}_1 I_1 \dot{A}_1^t) + \frac{m_1}{2}(\dot{w}, \dot{w}) + m_1(\dot{A}_1 S_1^0, \dot{w}) + (1 \leftrightarrow 2) \\
&= \frac{1}{2}\mathrm{tr}\,(\dot{A}_1 I_1 \dot{A}_1^t) + m_1(\dot{A}_1 S_1^0, \dot{w}) + (1 \leftrightarrow 2) + \frac{m}{2}(\dot{w}, \dot{w})
\end{aligned} \tag{2.5}$$

The Lagrangian is simply the total kinetic energy. To summarize, the velocity phase space for our system is $T\mathrm{SO}(3) \times T\mathrm{SO}(3) \times T\mathbf{R}^3$ and the Lagrangian is given by (2.5).

§3. Reduction by the Euclidean Group

Consider the following action of an element g of the Euclidean group $E(3)$

$$g = \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix},$$

where $A \in \mathrm{SO}(3)$ and $b \in \mathbf{R}^3$, on a point (A_1, A_2, w) in the configuration space $\mathcal{C}$:

$$g \cdot (A_1, A_2, w) = (BA_1, BA_2, Bw + b). \tag{3.1}$$

It is easy to check that the Lagrangian (2.5) is invariant under this action. Since the Lagrangian is invariant under E(3) , so is the Hamiltonian. The purpose of this section is to perform the reduction by this group. This will be done in two steps. First we will reduce by $\mathbf{R}^3$; this accounts for conservation of total linear momentum. Then we will reduce by SO(3); this accounts for conservation of total angular momentum.

We begin by rewriting the Lagragian in terms of the linear momentum p. Using (2.2), we can write the total linear momentum as

$$\begin{aligned}
p &= m\dot{\alpha} = m_1\dot{r}_1 + m_2\dot{r}_2 \\
&= m\dot{w} + m_1\dot{A}_1 S_1^0 + m_2\dot{A}_2 S_2^0.
\end{aligned} \tag{3.2}$$

It is convenient to introduce the expression

$$\rho = \frac{m_1}{m}\dot{A}_1 S_1^0 + \frac{m_2}{m}\dot{A}_2 S_2^0. \tag{3.3a}$$

and write

$$\begin{aligned} w &= \alpha - \frac{m_1}{m}A_1 S_1^0 - \frac{m_2}{m}A_2 S_2^0 \\ \dot{w} &= \dot{\alpha} - \rho \end{aligned} \tag{3.3b}$$

Substituting (3.3a and b) into (2.5) and simplifying gives the following form for the Lagrangian

$$L = \frac{1}{2}\mathrm{tr}\,(\dot{A}_1 I_1 \dot{A}_1^t) + \mathrm{tr}\,(\dot{A}_2 I_2 \dot{A}_2^t) + \frac{1}{2m}(p,p) - \frac{m}{2}(\rho,\rho). \tag{3.4}$$

The Legendre transformation FL induces a symplectic structure on

$$TC = T\mathrm{SO}(3) \times T\mathrm{SO}(3) \times T\mathbf{R}^3$$

and the tangent of the action of $\mathbf{R}^3$ on C (see (3.1)) is symplectic. To verify this statement and the ones that follow, see Abraham and Marsden [1978], chapters 3 and 4. The momentum map for this action is given by

$$\begin{aligned} J : T\mathrm{SO}(3) \times T\mathrm{SO}(3) \times T\mathbf{R}^3 &\longrightarrow \mathbf{R}^{3*} \\ (A_1, \dot{A}_1, A_2, \dot{A}_2, w, \dot{w}) &\mapsto p. \end{aligned}$$

The corresponding reduced space at p is:

$$\left(J^{-1}(p)\right)/\mathbf{R}^3 = T\mathrm{SO}(3) \times T\mathrm{SO}(3).$$

From (3.4) we see that the Lagrangian on the reduced space is simply

$$L = \frac{1}{2}\mathrm{tr}\,(\dot{A}_1 I_1 \dot{A}_1^t) + \frac{1}{2}\mathrm{tr}\,(\dot{A}_2 I_2 \dot{A}_2^t) - \frac{1}{2m}\left|m_1 \dot{A}_1 S_1^0 + m_2 \dot{A}_2 S_2^0\right|^2 + \frac{|p|^2}{2m} \tag{3.5}$$

Since p is constant, we can drop the last term. This completes the first stage of the reduction.

We now perform the reduction corresponding to conservation of total angular momentum. This time we use Poisson reduction; see Krishnaprasad and Marsden [1986] for a summary of Poisson reduction. Consider the map

$$\begin{aligned} \lambda : T^*\left(\mathrm{SO}(3) \times \mathrm{SO}(3)\right) &\longrightarrow \mathrm{so}(3)^* \times \mathrm{so}(3)^* \times \mathrm{SO}(3) \\ (\pi_{A_1}, \pi_{A_2}) &\mapsto (\Pi_1, \Pi_2, A) = \left(T^* L_{A_1}\pi_{A_1}, T^* L_{A_2}\pi_{A_2}, A_1^{-1}A_2\right). \end{aligned} \tag{3.6}$$

We will define a Poisson bracket on the target space so that the map λ becomes a Poisson map with respect to this bracket and the canonical bracket on the cotangent bundle $T^*\left(\mathrm{SO}(3) \times \mathrm{SO}(3)\right)$.

Introduce the body angular velocities of each of the bodies

$$\begin{aligned} \hat{\Omega}_1 &= A_1^{-1}\dot{A}_1 \\ \hat{\Omega}_2 &= A_2^{-1}\dot{A}_2, \end{aligned} \tag{3.7}$$

where $\hat{\Omega}$ is the linear map $v \mapsto \Omega \times v$ on $\mathbf{R}^3$. We will also need the moment of inertia J_1 of body 1 given by

$$\begin{aligned} (J_1)^{ij} &= -\int_{\mathcal{B}_1} Q^i Q^j \, d\mu(Q), \qquad \text{if } i \neq j \\ &= \int_{\mathcal{B}_1} \left(|Q|^2 - (Q^i)^2\right) d\mu(Q), \qquad \text{if } i = j. \end{aligned} \tag{3.8}$$

The moment of inertia J_2 of body 2 is defined in a similar way.

In order to derive an explicit exprssion for the angular momenta of the system, it is helpful to write the total kinetic energy as a quadratic form. After a certain amount of algebra, the total kinetic energy (3.5), can be written

$$(\Omega_1^t, \Omega_2^t)\begin{pmatrix} \tilde{J}_1 & \epsilon\Lambda \\ \epsilon\Lambda^t & \tilde{J}_2 \end{pmatrix}(\Omega_1, \Omega_2), \tag{3.9a}$$

where

$$\begin{aligned} \tilde{J}_1 &= J_1 + \frac{m_1^2}{m}\left(|S_1^0|^2 \cdot 1 - S_1^0 \otimes S_1^0\right) \\ \tilde{J}_2 &= J_2 + \frac{m_2^2}{m}\left(|S_2^0|^2 \cdot 1 - S_2^0 \otimes S_2^0\right) \\ \Lambda &= \left(\hat{S}_1^0\right)^t A \left(\hat{S}_2^0\right) \\ \epsilon &= \frac{m_1 m_2}{m} \end{aligned} \tag{3.9b}$$

Let $\mathbf{J}$ denote the matrix

$$\mathbf{J} = \begin{pmatrix} \tilde{J}_1 & \epsilon\Lambda \\ \epsilon\Lambda^t & \tilde{J}_2 \end{pmatrix}. \tag{3.9c}$$

From the Legendre transformation applied to Lagrangians quadratic in the velocities, the angular momenta are found to be

$$\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} = \mathbf{J}\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}. \tag{3.9d}$$

The inertial orientation matrices A_1, A_2 determine the relative orientation matrix

$$A = A_1^{-1} A_2. \tag{3.10}$$

Using the definitions (3.7) - (3.10), we can rewrite the Lagragian (3.5) as

$$L = \frac{1}{2}\Pi \cdot \mathbf{J}^{-1}\Pi \tag{3.11}$$

and conclude, using the fact that the Lagragian is quadratic in the momenta, that the Hamiltonian $H(\Pi_1, \Pi_2, A)$ is also given by (3.11).

We next derive the Poisson bracket on $\mathrm{so}(3)^* \times \mathrm{so}(3)^* \times \mathrm{SO}(3)$. Given a function F on $\mathrm{so}(3)^* \times \mathrm{so}(3)^* \times \mathrm{SO}(3)$, define a function F_λ on $T^*(\mathrm{SO}(3) \times \mathrm{SO}(3))$ by

$$F_\lambda = F \circ \lambda. \tag{3.12}$$

The canonical bracket on $T^*(\mathrm{SO}(3) \times \mathrm{SO}(3))$ is

$$\begin{aligned} \{F_\lambda, G_\lambda\} = {} & D_{A_1} F_\lambda \cdot \tfrac{\delta H_\lambda}{\delta \pi_1} - D_{A_1} H_\lambda \cdot \tfrac{\delta F_\lambda}{\delta \pi_1} \\ & + D_{A_2} F_\lambda \cdot \tfrac{\delta H_\lambda}{\delta \pi_2} - D_{A_2} H_\lambda \cdot \tfrac{\delta F_\lambda}{\delta \pi_2}. \end{aligned} \tag{3.13}$$

Using the chain rule, we can introduce a bracket on $\mathrm{so}(3)^* \times \mathrm{so}(3)^* \times \mathrm{SO}(3)$, so that λ becomes a Poisson map. This is straghtforward but tedious. To organize the computation, it is helpful to note the following facts.

Fact 1. *If $\pi_A \in T_A^* SO(3)$, then*

$$\Pi := T_e^* L_A \cdot \pi_A = A^t \pi_A \in so(3)^* \tag{3.14}$$

Fact 2. *Let $H_\lambda(A_1, \pi_1, A_2, \pi_2)$ be a function on $T^*(SO(3) \times SO(3))$ and let $\frac{\delta H_\lambda}{\delta \pi_1}$ denote the functional derivative of H_λ with respect to π_1. Then*

$$\tfrac{\delta H_\lambda}{\delta \pi_1} = A \tfrac{\delta H}{\delta \Pi_1}.$$

Fact 3. *Let $F_\lambda(A_1, \pi_1, A_2, \pi_2)$ be a function on $T^*(SO(3) \times SO(3))$, where we have, by abuse of notation, written an element in the cotangent space at A_1 as (A_1, π_1). Then*

$$D_{A_1} F_\lambda(A_1, \pi_1, A_2, \pi_2)(\delta A_1) = \left\langle \tfrac{\delta F}{\delta \Pi_2}, (\delta A_2)^t \pi_2 \right\rangle + \left\langle \tfrac{\delta F}{\delta A}, A_1^{-1}(\delta A_2) \right\rangle.$$

It is now straightforward to combine these facts to see that the canonical bracket (3.13) may be written in terms of the fuctions F and G using the chain rule as

$$\begin{aligned} \{F, H\}(\Pi_1, \Pi_2, A) = {} & -\left\langle \Pi_1, \left[\tfrac{\delta F}{\delta \Pi_1}, \tfrac{\delta H}{\delta \Pi_1}\right] \right\rangle - \left\langle \Pi_2, \left[\tfrac{\delta F}{\delta \Pi_2}, \tfrac{\delta H}{\delta \Pi_2}\right] \right\rangle \\ & - \left\langle \tfrac{\delta F}{\delta A}, \tfrac{\delta H}{\delta \Pi_1} A - A \tfrac{\delta H}{\delta \Pi_2} \right\rangle + \left\langle \tfrac{\delta H}{\delta A}, \tfrac{\delta F}{\delta \Pi_1} A - A \tfrac{\delta F}{\delta \Pi_2} \right\rangle. \end{aligned} \tag{3.15}$$

To summarize, we have

Theorem 1. *(i) With the canonical bracket on $T^*(SO(3) \times SO(3))$ and the bracket (3.15) on $so(3)^* \times so(3)^* \times SO(3)$, the map*

$$\lambda : T^*(SO(3) \times SO(3)) \longrightarrow so(3)^* \times so(3)^* \times SO(3)$$

given by (3.6) is a Poisson map. (ii) The dynamics on the reduced space is given by

$$\dot{F} = \{F, H\}.$$

§4. Further Remarks

1. We begin with a brief discussion of the Casimirs. Consider the momentum map

$$\begin{aligned} J : T^*(\mathrm{SO}(3) \times \mathrm{SO}(3)) &\longrightarrow \mathrm{so}(3)^* \\ (\pi_{A_1}, \pi_{A_2}) &\mapsto (T^*R_{A_1}\pi_{A_1} + T^*R_{A_2}\pi_{A_2}). \end{aligned} \tag{4.1}$$

The composition of this with the Casimir

$$\begin{aligned} C : \mathrm{so}(3)^* &\longrightarrow \mathbf{R} \\ \Pi &\mapsto |\Pi|^2 \end{aligned} \tag{4.2}$$

produces a collective Hamiltonian on $T^*(\mathrm{SO}(3) \times \mathrm{SO}(3))$ whose Hamiltonian vector field is tangent to the G–orbits (see Guillemin and Sternberg [1980] or Holmes and Marsden [1983]) and therefore induces a Casimir $\tilde{C}$ on the space $\mathrm{so}(3)^* \times \mathrm{so}(3)^* \times \mathrm{SO}(3)$ via the Poisson map (3.6).

Tracing through the diagram shows that $\tilde{C} = |\Pi_1 + A\Pi_2|^2$; and, hence, any function of the form $\Phi\left(|\Pi_1 + A\Pi_2|^2\right)$ is a Casimir for the bracket (3.15).

2. The symplectic leaves in the nine dimensional space $\mathrm{so}(3)^* \times \mathrm{so}(3)^* \times \mathrm{SO}(3)$ appear to be eight dimensional (level sets of the function $|\Pi_1 + A\Pi_2|^2$) and in the case of $J = 0$, (given by (4.1),) the six dimensional space $T^*\mathrm{SO}(3)$; and, finally, if $\Pi_1 = 0, \Pi_2 = 0$, a two dimensional space S^2 of trival equilbria. We expect to explore the geometry of these leaves and the other topics listed in the introduction in a future publication.

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