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ON THE MOTION OF INCOMPRESSIBLE FLUIDS

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We are concerned with the initial value problem for an incompressible inviscid fluid. Specifically, given a bounded domain M in R^3 (or any compact Riemannian manifold which may have boundary) and a smooth vector field V_0 tangent to bdy(M), we seek a time dependent vector field $V(t)$ satisfying :

$$(E)\qquad \begin{array}{ll} \dfrac{\partial V(t)}{\partial t} + \nabla_{V(t)} V(t) = - \operatorname{grad} p & V(t) \text{ tangent to bdy} M \\ \operatorname{div}(V) = 0 & V(0) = V_0 \end{array}$$

Where ∇ is the affine connection in M $\left(\text{so in } R^3 \nabla_V V = \sum_{j=1}^{3} V_j \dfrac{\partial V_i}{\partial X_j}\right)$ div means divergence, and grad p is the gradient of a time dependent function on M, which is determined implicitly.

We have shown [2] that given V_0, there exists a unique V satisfying (E), defined on a time interval depending on V_0. In this report we will discuss the methods of [2].

Our approach to the problem is patterned after the work of Arnold, [1] ; that is, we translate the problem into a Hamiltonian system on a certain non-linear infinite dimensional space. We show that this space has a natural Riemannian structure and we solve the problem by finding geodesics on this space.

We first present a typical situation from mechanics : Let X be differential manifold with Riemannian structure (,) ; let TX be its tangent bundle and T^*X the cotangent bundle. (,) defines an isomorphism between TX and T^*X, and T^*X has a natural sympectic two form Ω (cf. [3], p. 86). By means of the isomorphism, we can consider Ω as a sympectic form on TX. We define Kinetic energy $K : TX \to R$ by $K(V) = \frac{1}{2}(V, V)$. Then there exists a unique vector field Z on TX which satisfies the equation : $\Omega(Z, Y) = - Y(K)$, for Y any vector field on TX. The integral curves of Z project to geodesics on X, and also they are the curves of motion of the Hamiltonian system with energy K.

We proceed to find X for the problem of fluid motion. We assume that the manifold M is filled with fluid and let $\varphi_t : M \to M$ be the map which takes each particle of fluid from its position p at time zero to its position $\varphi_t(p)$ at time t. Since the fluid is incompressible φ_t will preserve the volume element of M. (For a domain in R^3, this means that the Jacobian of φ_t is everywhere 1). From this and the assumption that φ_t is onto, it follows that φ_t must be a volume-preserving diffeomorphism of M. Thus, we let X equal $\mathcal{D}_\mu$ the set of all volume preserving diffeomorphisms of M (μ being the volume element), and our next task is to endow $\mathcal{D}_\mu$ with a differentiable structure. For simplicity, we shall assume that M has no boundary.

First consider $\mathcal{D}$, the set of smooth diffeomorphisms of M, with the C^∞ topology. This space is locally like $C^\infty(T)$, the Frechet space of smooth vector fields on M. Since one cannot in general solve ordinary differential equations on Frechet spaces, we enlarge $\mathcal{D}$ so that it is locally a Hilbert space.

Specifically, we let $\mathcal{D}^s$ be the set of bijective maps $\eta : M \to M$ such that η and η^{-1} are both of class H^s. That is, when written in local coordinates, η(and η^{-1}), together with all partial derivatives up to order s, are square integrable. For $s > n/2 + 1$, ($n = \dim M$) the smoothness of η does not depend on the choice of coordinates ; furthermore, $\mathcal{D}^s$, with the H^s topology, is a topological group which is continuously included in the group of C^1-diffeomorphisms.

Now we construct a differentiable structure for $\mathcal{D}^s$. Let $H^s(T_\eta)$ be the space H^s vector fields over η ; i.e.

$$H^s(T_\eta) = \{V : M \to TM \mid V \in H^s \quad \text{and} \quad \pi \circ V = \eta\}$$

($\pi : TM \to M$ is the bundle projection). It is a Hilbert space with the H^s topology. Let $e : TM \to M$ be the exponential map of M coming from its Riemannian Structure. Then $\Omega_e : H^s(T_\eta) \to \mathcal{D}^s$, defined by $\Omega_e(V) = e \circ V$ has domain a neighborhood of the origin of $H^s(T_\eta)$ and is a homeomorphism from some neighborhood of 0 in $H^s(T_\eta)$ to a neighborhood of η.

For each η, $\Omega_e : H^s(T_\eta) \to \mathcal{D}^s$ provides a chart about η and using the fact that $e : TM \to M$ is C^∞, one can show that the transitions between charts are smooth. Thus $\mathcal{D}^s$ is a C^∞-manifold and for each η, the tangent space to $\mathcal{D}^s$ at η (denoted $T_\eta \mathcal{D}^s$) is identified with $H^s(T_\eta)$.

Using the manifold structure of $\mathcal{D}^s$ we will derive a manifold structure for $\mathcal{D}^s\mu = \{\eta \in \mathcal{D}^s \mid \eta^*(\mu) = \mu\}$, where μ is the volume element of M and $\eta^*(\mu)$ is the usual pull-back of an n-form μ by η.

Let $H^{s-1}(\Lambda^n)$ be the space of H^{s-1} n-forms of M and let

$$A = \left\{ \omega \in H^{s-1}(\Lambda^n) \mid \int_M \omega = \int_M \mu \right\}.$$

A is clearly a closed linear subspace of $H^{s-1}(\Lambda^n)$ of co-dimension 1.

Let $\Psi : \mathcal{D}^s \to A$ by $\Psi(\eta) = \eta^*(\mu)$. Ψ is a smooth map and is a surjection ; i.e., the tangent map $T_\eta \Psi : T_\eta \mathcal{D}^s \to T_{\Psi(\eta)} A$ is onto. From the implicit function theorem it follows that $\Psi^{-1}(\mu) = \mathcal{D}^s_\mu$ a submanifold of $\mathcal{D}^s$. It is also a subgroup. Furthermore, at the identity $id \in \mathcal{D}^s$, one computes that $T_{id}\Psi : T_{id}\mathcal{D}^s \to T_\mu A$ satisfies $T_{id}\Psi(V) = L_V(\mu)$ where "L" means Lie derivative. Therefore,

$$T_{id}\mathcal{D}^s_\mu = \{V \in H^s(T) \mid L_V(\mu) = 0\},$$

the set of divergence free vector fields. Also,

$$T_\eta \mathcal{D}^s_\mu = \{V \in H^s(T_\eta) \mid \operatorname{div}(V \circ \eta^{-1}) = 0\}.$$

Our next step is to define a Riemannian structure on $\mathcal{D}^s\mu$. On $T_\eta\mathcal{D}^s = H^s(T_\eta)$, we define $(V, W) = \int_M \langle V, W\rangle \mu$ where $\langle\ ,\ \rangle$ is the Riemannian metric of M. This gives a weak Riemannian metric on $\mathcal{D}^s$. That is $(\ ,\ )$ has all the usual properties except that on each tangent space $T_\eta\mathcal{D}^s$, $(\ ,\ )$ does not induce the H^s-topology. However, $(\ ,\ )$ is invariant under right multiplication by elements of $\mathcal{D}^s_\mu$. $\mathcal{D}^s_\mu$ inherits a weak Riemannian structure because it is a submanifold of $\mathcal{D}^s$. Also the right invariance of $(\ ,\ )$ means that $\mathcal{D}^s_\mu$ is actually a weak Riemannian homogeneous space.

Our final task is to find the geodesics on $\mathcal{D}^s_\mu$. To do this we look for geodesics of $\mathcal{D}^s$ and these can be found virtually by inspection. Indeed to find a geodesic $\eta(t)$ from η_0 to η_1 in $\mathcal{D}^s$ we want to minimize the energy :

$$\int_M \left\{ \int_0^1 \langle \eta'(t), \eta'(t)\rangle\, dt \right\} \mu,$$

and here the integral in braces is, for each $p \in M$, the energy of the path $t \to \eta(t)(p)$. If each such path is a geodesic, the integral will be minimal at each point of M and hence the total energy will be minimal. Thus, the geodesics of $\mathcal{D}^s$ are those curves $\eta(t)$ in $\mathcal{D}^s$ which have the property that for each $p \in M$, $t \to \eta(t)(p)$ is a geodesic in M.

Of course, the geodesics on $\mathcal{D}^s$ are related to an affine connection on $\mathcal{D}^s$ which we will call $\overline{\nabla}$, and also to a Hamiltonian vector field $\overline{Z}$ on $T\mathcal{D}^s$. Furthermore, since $\mathcal{D}^s_\mu$ is a Riemannian submanifold of $\mathcal{D}^s$, it inherits a connection $\widetilde{\nabla} = P \circ \overline{\nabla}$ where at each tangent space $T_\eta(\mathcal{D}^s)$, P is the orthogonal projection

$$P : T_\eta\mathcal{D}^s \to T_\eta\mathcal{D}^s_\mu\ .$$

Similarly $T\mathcal{D}^s\mu$ gets a Hamiltonian vector field $\widetilde{Z} = TP(\overline{Z})$.

Since our Riemannian structure is weak, it is not clear that P is a smooth map. However, at $id \in \mathcal{D}^s$, $P : T_{id}\mathcal{D}^s \to T_{id}\mathcal{D}^s_\mu$ is simply the projection onto the first summand of the well-known decomposition :

$$H^s(T) = \operatorname{div}^{-1}(0) \oplus \operatorname{grad} \mathcal{F}^{s+1}$$

where the first summand is the set of divergence free H^s vector fields and the second is the set of gradients of H^{s+1} functions on M. This direct sum is topological and P is in fact smooth.

Thus $\mathcal{D}^s_\mu$ has a smooth Hamiltonian vector field which can of course, be integrated to give the required geodesics.

If $\eta(t)$ is such a geodesic and $W(t) = \dfrac{d}{dt}(\eta(t)) \in T_{\eta(t)} s_\mu$, then $V(t) = W(t) \circ \eta(t)^{-1}$ is a time dependent vector field which satisfies our original system (E).

Given V_0 a vector field on M, $V_0 \in T_{id}\mathcal{D}^s_\mu$ and there exists a unique geodesic $\eta(t)$ starting at id in direction V_0. By the above each $\eta(t)$ corresponds to a solution of (E).

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