

Extensive-Form Games and Strategic Complementarities *

Federico Echenique

Humanities
and Social Sciences, 228-77
Caltech
Pasadena, CA 91125

email: fede@hss.caltech.edu

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Abstract

I prove the subgame-perfect equivalent of the basic result for Nash equilibria in normal-form games of strategic complements: the set of subgame-perfect equilibria is a non-empty, complete lattice—in particular, subgame-perfect Nash equilibria exist. For this purpose I introduce a device that allows the study of the set of subgame-perfect equilibria as the set of fixed points of a correspondence. My results are limited because extensive-form games of strategic complementarities turn out—surprisingly—to be a very restrictive class of games.

Keywords: Strategic complementarities, supermodular games, subgame-perfect equilibrium, dynamic games, Tarski's fixed point theorem.

1 Introduction

In this paper I define extensive-form games of strategic complementarities, and prove the subgame-perfect equivalent of the basic result for Nash equilibria in normal-form games of strategic complements: the set of subgame-perfect Nash equilibria (SPNE) is a non-empty, complete lattice. This has strong implications; not only does it give a general existence proof, it also allows the use of comparative statics techniques. While this seems to be a promising result, I also show that, in extensive-form games, the assumption of strategic complementarities is—surprisingly—very restrictive.

Equilibria are usually analyzed by means of fixed-point methods. This has not been the case for SPNE. A methodological contribution of this paper is the introduction of a device, the “extended best-response correspondence,” with the property that the set of SPNE of a game coincides with the set of fixed points of the extended best-response correspondence. The model of extensive-form games that I use allows time to be continuous, so the extended best-response correspondence can also be used to analyze SPNE of continuous-time games.

SPNE exist in finite games. Harris, Reny, and Robson (1995) present an example of a game without an SPNE that is a two-stage game with a finite number of players, and where only one player has an infinite strategy space. Hence, existence of SPNE is not guaranteed after a minimal departure from finite games. Proofs of existence of SPNE in non-finite games are provided by Harris, Reny, and Robson (1995) for games of almost-perfect information where a public randomization device is present, by Harris (1985a), Harris (1985b) and Hellwig and Leininger (1987) for games of perfect information; and by Fudenberg and Levine (1983) for classes of games with strong “continuity at infinity” properties.

In this paper I show that the existence of SPNE follows from strategic complementarities; concretely, that the set of SPNE of a game whose normal-form is a game of strategic complementarities, is a non-empty, complete lattice. My results apply to continuous-time games and

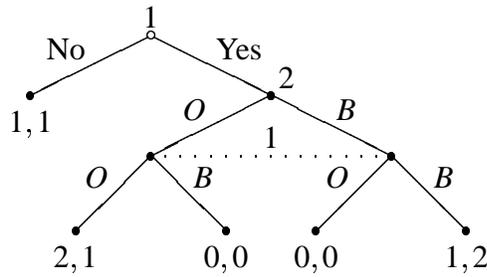


Figure 1: Optional Battle of the Sexes

to games of imperfect information that are not necessarily games of almost-perfect information (I am not aware of other existence results for continuous-time games). The important problem with my results is that the assumption of complementarities in extensive-form games is very strong.

Games of strategic complementarities were first studied by Topkis (1979), and introduced into economics by Vives (1990). Topkis (1998) and Vives (1999) contain a definition of (normal form) complementarities, with up-to-date references. There are many examples of economic models that are games of strategic complements (see Milgrom and Roberts (1990), Topkis (1998), and Vives (1999)). By now it is fair to say that complementarities in normal-form games is a very useful and common structure. Here I show that, while still very useful, complementarities in dynamic contexts are rare.

To illustrate the problem, consider the game in Figure 1. This is “Optional Battle of the Sexes.” Here, player One chooses first to say Yes or No. If One says No then payoffs are 1 each. If One says Yes then they play a Battle of the Sexes game: they simultaneously choose an element of $\{O, B\}$. If the choice is (O, O) then player One gets 2 while Two gets 1. If they choose (O, B) or (B, O) then both get a payoff of 0. If the outcome is (B, B) then payoffs are (1, 2)

It is easy to see that Battle of the Sexes (BoS, the simultaneous-move game that follows after One chooses Yes) is a game of strategic complementarities. Player One’s best response to

Two playing B is to play B and One's response to Two playing O is to play O . So, a change by Two from B to O makes One change in the same direction. This is also true for player Two: a change by One from B to O makes Two change in the same direction. Imposing an order on the players' strategies, we can say that O is "larger" than B . Then the best response of each player is increasing in the other player's choice of strategy, this is the crucial property of a game of strategic complementarities (indeed it is easily seen that BoS satisfies the definition of a game of strategic complementarities in e.g. Milgrom and Roberts (1990)).

Now, consider the extensive-form game Optional BoS and let us impose an order on the set $\{\text{No}, \text{Yes}\}$. Let the action "Yes" at One's initial decision node be larger than "No". Then the strategy No- O (say No at the initial node and plan to play O in Battle of the Sexes) is smaller than Yes- O and No- B is smaller than Yes- B . But, when One is playing No- B it is optimal for Two to play O , while if we increase One's strategy to Yes- B then it is uniquely optimal for Two to play B . This implies that Two's strategy is not increasing in One's strategy choice. We could try to fix this by saying that B is larger than O , but then the problem would arise when One increases the strategy from No- O to Yes- O .

It turns out that it is possible to make Optional BoS a game of strategic complementarities. The solution is to say that the action Yes is smaller than No. This shows that extensive-form games of strategic complementarities are not trivial.¹ But unfortunately the simple solution in Optional BoS is not feasible in general. I shall show how a complication of Optional BoS yields a game that cannot be transformed into a game of strategic complementarities. I shall argue also that most dynamic games of economic interest cannot be transformed into games of strategic complementarities.

The situation contrasts with the study of Markov-Perfect equilibria in stochastic games with complementarities (Curtat (1996), Amir (1989) and Amir (1996a)). Markov strategies limit dynamic strategic interactions, and thus allows complementarities to have leverage, but when more general strategies are allowed, this breaks down. I shall illustrate the situation with

examples in Section 4.

Section 2 presents definitions and notation. Section 3 introduces the extended best-response correspondence and the main results of the paper. Section 4 shows how complementarities are a restrictive assumption by discussing some examples.

2 Generalized Extensive-Form Games

2.1 Basic Definitions and Notation

A detailed discussion of the concepts defined in this subsection is in Topkis (1998). A set X with a transitive, reflexive, antisymmetric binary relation \preceq is a **lattice** if whenever $x, y \in X$, both $x \wedge y = \inf \{x, y\}$ and $x \vee y = \sup \{x, y\}$ exist in X . It is **complete** if, for every nonempty subset A of X , $\inf A, \sup A$ exist in X . For two subsets A, B of X , say that A is smaller than B in the **strong set order** if $a \in A, b \in B$ implies $a \wedge b \in A, a \vee b \in B$. Let $\phi : X \rightarrow X$ be a correspondence. Say that ϕ is **increasing in the strong set order** if, whenever $x \preceq y$, $\phi(x)$ is smaller in the strong set order than $\phi(y)$.

The **order-interval topology** on a lattice is obtained by taking the closed intervals $[x, y] = \{z \in X : x \preceq z \preceq y\}$ as a sub-basis for the closed sets. All lattices in the paper will be endowed with the order-interval topology. All products of partially ordered sets are endowed with the product order. All products of topological spaces are endowed with the product topology.

If X is a lattice, a function $f : X \rightarrow \mathbf{R}$ is **quasisupermodular** if for any $x, y \in X$, $f(x) \geq f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y)$ implies $f(x \vee y) > f(y)$. Quasisupermodularity is an ordinal notion of complementarities, it was introduced by Milgrom and Shannon (1994). Let T be a partially ordered set. A function $f : X \times T \rightarrow \mathbf{R}$ satisfies the **single-crossing condition** in (x, t) if whenever $x \prec x'$ and $t \prec t'$, $f(x, t) \leq f(x', t)$ implies that $f(x, t') \leq f(x', t')$ and $f(x, t) < f(x', t)$ implies that $f(x, t') < f(x', t')$. The restriction of a function $f : X \rightarrow Y$ to a subset $X' \subseteq X$ is denoted $f|_{X'}$.

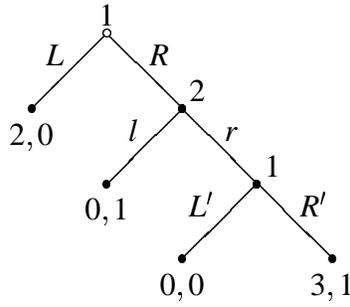


Figure 2: An extensive-form game of strategic complementarities

2.2 Definition of Generalized Extensive-Form Games

I present a definition of extensive-form games that has information sets, as opposed to decision nodes, as primitives. It is really only a slight variation on the usual rules for drawing game trees, but it results in a considerably more general framework because it allows time to be continuous and does not impose perfect recall or partitioned information structures.² I hope that the benefits of having results that apply to continuous-time games are important enough to balance the cost of a slightly unfamiliar framework. Besides its generality, this model of extensive-form games is more parsimonious than the usual one, therefore the proofs are easier and sharper than they would be otherwise.

I shall use the following example to illustrate the concepts as they are introduced.

Example 1 Consider the game in Figure 2. First, player One selects an element in $\{L, R\}$. If she selects L then the game “ends” and the payoffs are 2 to player One and 0 to player Two. If she selects R then Two gets to choose between l and r . If he chooses l then she gets 1 while One gets 0. If he chooses r then player One gets to choose an element in $\{L', R'\}$. Payoffs are $(0, 0)$ and $(3, 1)$ after One chooses L' and R' , respectively. Let a_1 be the first node at which player One moves, a_2 be the second node at which she moves and b be the node at which player Two moves.

Now I shall describe some helpful notions that will build up to the definition of a general-

ized extensive-form game. The exact definition of a generalized extensive-form game is below (Definition 3).

Let $N = \{1, \dots, n\}$ be the set of players. Let H be a set; the elements of H will be referred to as “information sets.” Let $\mathcal{H} = \{H_\alpha : \alpha \in I\}$ be a collection of subsets of H and $\{H^i : i \in N\}$ a partition of H . The interpretation will be that player i is endowed with a collection H^i of information sets and that the elements of $H = \cup_{i=1}^n H^i$ are the information sets of the game. For each $\alpha \in I$, $H_\alpha \subseteq H$ should be interpreted as a “subgame” of H (this interpretation is made precise below). I shall refer to a subgame H_α both as α and H_α .

In Example 1, $N = \{1, 2\}$, $H = \{a_1, a_2, b\}$, $H^1 = \{a_1, a_2\}$ and $H^2 = \{b\}$. There are three subgames in the example, let $I = \{\alpha_0, \alpha_1, \alpha_2\}$, $H_{\alpha_0} = \{a_1, a_2, b\}$, $H_{\alpha_1} = \{a_2, b\}$, and $H_{\alpha_2} = \{a_2\}$; $\mathcal{H} = \{H_{\alpha_0}, H_{\alpha_1}, H_{\alpha_2}\}$.

I make two assumptions about \mathcal{H} , the collection of “subgames.” First, that H itself belongs to this class of subgames: Let $\alpha_0 \in I$ satisfy $H = H_{\alpha_0}$. Second, if $\{H_\alpha : \alpha \in I'\} \subseteq \mathcal{H}$ is a subcollection of \mathcal{H} such that any $\alpha, \alpha' \in I'$ satisfy either $H_\alpha \subseteq H_{\alpha'}$ or $H_{\alpha'} \subseteq H_\alpha$, then $\cup_{\alpha \in I'} H_\alpha \in \mathcal{H}$ —so I assume that \mathcal{H} is closed under increasing unions. For every $\alpha \in I$, $\{H^i : i \in N\}$ induces a partition on H_α : let $H_\alpha^i = H_\alpha \cap H^i$ be the set of player i ’s information sets in subgame H_α . Note that $\mathcal{H}^i = \{H_\alpha^i : \alpha \in I\}$ is also closed under increasing unions. It is easy to verify that the example in Figure 2 satisfies these assumptions, as does any well-defined game tree.

The players choose actions at each of their information sets. For each $h \in H$, let $A(h)$ be the set of actions available to the player that moves at information set h . Each $A(h)$ is endowed with a Hausdorff topology. The set of all possible actions is denoted by $\mathcal{A} = \cup_{h \in H} A(h)$. Player i ’s strategy space in subgame $\alpha \in I$ is $S_\alpha^i = \{s : H_\alpha^i \rightarrow \mathcal{A} : s(h) \in A(h) \text{ for all } h \in H_\alpha^i\} = \prod_{h \in H_\alpha^i} A(h)$. Let $S_\alpha = \prod_{i \in N} S_\alpha^i$ and $S_\alpha^{-i} = \prod_{j \in N \setminus \{i\}} S_\alpha^j$; s_α^{-i} is refers to a generic element of S_α^{-i} . Note that if $A(h)$ is a complete lattice for each h , then S_α^i , S_α^{-i} and S_α are complete lattices in the product order.

Each player is endowed with preferences over strategy profiles in subgame $\alpha \in I$. These

preferences are represented by a collection of payoff functions $u_\alpha^i : S_\alpha \rightarrow \mathbf{R}$.

Consider the game in Example 1: $A(a_1) = \{L, R\}$, $A(b) = \{l, r\}$ and $A(a_2) = \{L', R'\}$, so that $\mathcal{A} = \{L, L', R, R', l, r\}$. The strategy space for player 1 for the whole game is $S_{\alpha_0}^1 = \{LL', LR', RL', RR'\}$ —where LL' means that 1 plans to play L at her first decision node (a_1) and then L' at her second decision node (a_2) and so on. The strategy space for player 2 for the whole game is simply $S_{\alpha_0}^2 = \{l, r\}$. The strategy spaces for the other subgames are $S_{\alpha_1}^1 = \{L', R'\}$, $S_{\alpha_1}^2 = \{l, r\}$, $S_{\alpha_2}^1 = \{L', R'\}$ and $S_{\alpha_2}^2 = \{\emptyset\}$. The choice of \emptyset for player 2 in subgame α_2 formalizes that only 1 makes a choice in this subgame. The players' preferences in each subgame are immediate from Figure 2, $u_{\alpha_0}^1(LL', l) = 2, u_{\alpha_0}^2(LL', l) = 0$, $u_{\alpha_1}^1(L', l) = 0$, $u_{\alpha_1}^2(L', l) = 1$, $u_{\alpha_2}^1(L', \emptyset) = 0$, $u_{\alpha_2}^2(L', \emptyset) = 0$, etc.

Definition 2 A collection of payoff functions $\{u_i^\alpha : i \in N, \alpha \in I\}$ is **consistent** if, for every $i \in N$ and $\alpha, \alpha' \in I$, whenever $H_{\alpha'} \subseteq H_\alpha$, $s_\alpha^{-i} \in S_\alpha^{-i}$, $s_\alpha^i, \tilde{s}_\alpha^i \in S_\alpha^i$ and $z_{\alpha'}^i \in S_{\alpha'}^i$, the inequalities $u_\alpha^i(\tilde{s}_\alpha^i, s_\alpha^{-i}) \leq u_\alpha^i(s_\alpha^i, s_\alpha^{-i})$ and $u_{\alpha'}^i(s_\alpha^i|_{H_{\alpha'}^i}, s_\alpha^{-i}|_{H_{\alpha'}^{-i}}) \leq u_{\alpha'}^i(z_{\alpha'}^i, s_\alpha^{-i}|_{H_{\alpha'}^{-i}})$ imply that $u_\alpha^i(\tilde{s}_\alpha^i, s_\alpha^{-i}) \leq u_\alpha^i(s_\alpha^i|_{H_\alpha^i \setminus H_{\alpha'}^i}, z_{\alpha'}^i, s_\alpha^{-i})$. The collection of payoff functions satisfies **continuity** if, for all $\alpha \in I$, $i \in N$, and $s_\alpha^{-i} \in S_\alpha^{-i}$, $s_\alpha^i \mapsto u_\alpha^i(s_\alpha^i, s_\alpha^{-i})$ is an upper semi-continuous function.

Payoffs are consistent if, given opponents' strategy s^{-i} , whenever s_α^i performs better than \tilde{s}_α^i in subgame $H_\alpha \supset H_{\alpha'}$ and $z_{\alpha'}^i$ performs better than s_α^i in subgame $H_{\alpha'}$, it must be that the combined strategy that follows s_α^i in $H_\alpha^i \setminus H_{\alpha'}^i$ and follows $z_{\alpha'}^i$ in $H_{\alpha'}^i$, cannot perform worse than \tilde{s}_α^i .

The payoffs in Example 1 are consistent: Fix the strategy $s_{\alpha_0}^2 = l$ by player 2 in subgame α_0 , the “whole” game. Given any strategy $s_{\alpha_0}^1$, player 1's payoff is independent of choices in node a_2 . In particular, choosing R' , the dominant strategy in subgame α_2 , does not decrease the payoff to following $s_{\alpha_0}^1$. Now, consider $s_{\alpha_0}^2 = r$. The only case where the requirement consistency has bite is for the strategy LL' . In subgame α_0 , LL' is preferred by 1 to RL' . But, in subgame α_2 , R' is better than L' . Consistency then requires that LR' be preferred to RL' in

subgame α_0 —which is satisfied by the specified payoffs.

Example 1 illustrates why payoffs in any well-defined game tree are consistent. Given i 's strategy s_α^i and opponents' strategies s_α^{-i} , if a subgame α' is not reached then i is indifferent among her choices in this subgame and she cannot do worse by picking something that is better in the subgame. On the other hand, if subgame α' is reached then payoffs will be given by choices in α' . Choosing a better strategy in subgame α' can only improve the payoff to s_α^i .

The definition of a generalized extensive-form game is complicated enough to warrant an enumeration of its components:

Definition 3 *The sextuple $\Gamma = \{N, H, \{H_\alpha : \alpha \in I\}, \{H^i : i \in N\}, \{A(h) : h \in H\}, \{u_\alpha^i : i \in N, \alpha \in I\}\}$ is a **generalized extensive-form game** if:*

- $N = \{1, 2, \dots, n\}$ is the set of players;
- H is a set of information sets;
- $\{H^i : i \in N\}$ is a partition of H ;
- $\{H_\alpha : \alpha \in I\}$ is a collection of subsets of H that is closed under increasing unions and such that $H_{\alpha_0} = H$ for some $\alpha_0 \in I$;
- $\{A(h) : h \in H\}$ is a collection of action sets, each endowed with a Hausdorff topology and compact;
- $\{u_i^\alpha : i \in N, \alpha \in I\}$ is a collection of consistent payoff functions that satisfies continuity.

For any subgame $\alpha \in I$, Γ induces naturally an extensive-form game Γ_α : let H_α be the set of information sets of Γ_α , let the subgames $\gamma \in I$ with $H_\gamma \subseteq H_\alpha$ be the subgames of Γ_α ; and let action sets and payoffs be defined as in Γ . I will use “subgame” to denote both the set H_α and its corresponding extensive-form game Γ_α .

A strategy profile s_α is a *subgame-perfect Nash equilibrium* (SPNE) in subgame $\alpha \in I$ if, for every $\gamma \in I$ such that $H_\gamma \subseteq H_\alpha$ and every $i \in N$,

$$s_\alpha^i|_{H_\gamma} \in \operatorname{argmax}_{s_\gamma^i \in S_\gamma^i} u^i(\tilde{s}_\gamma^i, s_\alpha^{-i}|_{H_\gamma}).$$

I shall refer to the SPNE in subgame α_0 , the whole game, as simply SPNE. Note that a strategy profile is a SPNE if and only if its restriction to any subgame is a SPNE in that subgame.

Definition 4 A generalized extensive-form game Γ is an *extensive-form game of strategic complementarities* if $A(h)$ is a complete lattice for all $h \in H$ and if, for all $\alpha \in I$ and $i \in N$, $s_\alpha^i \mapsto u_\alpha^i(s_\alpha^i, s_\alpha^{-i})$ is *quasisupermodular* on S_α^i , and u_α^i satisfies the *single-crossing condition* in $(s_\alpha^i, s_\alpha^{-i})$.

It is slightly cumbersome to show that the game in Figure 1, has strategic complements. I leave this for section 4.

2.3 Examples of Generalized Extensive-Form Games

2.3.1 Optional BoS

I shall present the current notation for the ‘‘Optional Battle of the Sexes’’ game from the introduction. Let a_1 be the initial node, b be the node at which player Two moves and a_2 be One’s information set after that Two has moved. Then, $H = \{a_1, a_2, b\}$, $H^1 = \{a_1, a_2\}$, $H^2 = \{b\}$. There are two subgames, so $I = \{\alpha_0, \alpha_1\}$, $H_{\alpha_0} = \{a_1, a_2, b\}$ and $H_{\alpha_1} = \{b, a_2\}$. Action spaces are $A(a_1) = \{\text{Yes}, \text{No}\}$, $A(b) = A(a_2) = \{O, B\}$. Strategy spaces are $S_{\alpha_0}^1 = \{\text{Yes}O, \text{Yes}B, \text{No}O, \text{No}B\}$, $S_{\alpha_0}^2 = S_{\alpha_1}^1 = S_{\alpha_1}^2 = \{O, B\}$.

2.3.2 Battle of the Sexes in Continuous Time

The game is a Battle of the Sexes played in continuous time. As Anderson (1984) and Simon and Stinchcombe (1989) point out, the map from strategies to outcomes might not be well

defined in continuous-time games, which implies that we cannot define the payoffs resulting from different strategy profiles. I will not spell out the details in Simon and Stinchcombe's (1989) model of continuous-time extensive-form games, I only present a simple example (that in fact falls within Simon and Stinchcombe's framework). My construction of strategies and payoffs is similar to that in Radzik and Goldman (2001) in a class of continuous-time zero-sum games.

To avoid problems with the map from strategies to outcomes, I impose that players are only allowed to switch infrequently from one action to the other. Time is indexed by $t \in [0, 1]$. Assume that players choose either O or B at time $t = 0$. Their decisions remain fixed for a period $\delta \in (0, 1)$; at time $t = \delta$ they can choose to switch, represented by action 1, or not to change their time 0 choice, represented by action 0. At any posterior time t , players are allowed to choose 1 only if they have chosen 0 in $[t - \delta, t)$. That is, switches are irreversible for a length of time δ . The players' "flow" payoffs are as in BoS in the introduction. If, at time t , they both choose O then One gets a payoff of 2 while Two gets a payoff of 1; when they both choose B , then One gets 1 while Two gets 2. If they choose different actions at time t then they both get a payoff of 0.

For any $t \in [0, 1]$, the events until time t are described by a pair of vectors $h_t = (h_t^1, h_t^2)$, with $h_t^i = (t_1^i, \dots, t_k^i)$ and where t_l^i is the time of the l th switch by player i . By the description above, we must have $t_l^i - t_{l-1}^i \geq \delta$ for $l = 2, \dots, k$ and $t_k^i \leq t$. Player i starts with action O if $t_1^i = 0$, with action B if $t_1^i > 0$. Any feasible h_t is called a time- t history. Let H_t be the set of all time- t histories.

The set of all information sets is $H = \cup_{t \in [0, 1]} H_t$. Any history h_t starts a subgame $H_{h_t} = \{h_\tau \in H : t \leq \tau, h_\tau|t = h_t\}$, where $h_\tau|t = h_t$ means that h_τ and h_t coincide on switches before time t . For any history h_t , the actions available to player i are $\{0, 1\}$ or, if she has switched recently (so $t - t_k < \delta$), $\{0\}$. I will show that the collection of information sets is closed under increasing unions. Let $\{H_{h_t} : h_t \in \tilde{H}\}$ be an increasing collection of subgames and let

$\underline{t} = \inf \{t : h_t \in \tilde{H}\}$. Then, all histories h_t coincide on switches up to \underline{t} , let $h_{\underline{t}}$ be this common history. It is immediate that $H_{h_{\underline{t}}} = \cup_{h_t \in \tilde{H}} H_{h_t}$. Similarly, the collection of information sets is closed under intersections.

Strategies are maps $h_t \mapsto s^i(h_t) \in A^i(h_t)$, where $A^i(h_t)$ is $\{0\}$ if the last switch in h_t was later than $t - \delta$ and $A^i(h_t)$ is $\{0, 1\}$ else. A pair of strategies define, recursively, a finite collection of switches. A finite collection of switches gives, through the definition of flow payoffs above, the payoff associated to the strategy profile. Additivity of payoffs (from flow payoffs) implies immediately that payoffs are consistent.

3 The Extended Best-Response Correspondence and Strategic Complementarities

3.1 Main Results

In this paper I shall focus on subgame-perfect equilibria. In order to keep track of the best responses to opponents' strategies in each subgame, I need to introduce the set

$$S^i = \left\{ s^i \in \prod_{\alpha \in I} S_{\alpha}^i : s_{\alpha_0}^i|_{H_{\alpha}^i} = s_{\alpha}^i \right\}.$$

This is the set of lists $s^i \in \prod_{\alpha \in I} S_{\alpha}^i$ so that the component $s_{\alpha}^i \in S_{\alpha}^i$ that corresponds to subgame H_{α} coincides with the restriction of $s_{\alpha_0}^i \in S_{\alpha_0}^i$ —the strategy for the whole game—to subgame H_{α} . Let $S = \prod_{i \in N} S^i$. For the example in Figure 2, recall that $S_{\alpha_0}^1 = \{LL', LR', RL', RR'\}$, $S_{\alpha_1}^1 = \{L', R'\}$, and $S_{\alpha_2}^1 = \{L', R'\}$. So,

$$S^1 = \{(LL', L', L'), (LR', R', R'), (RL', L', L'), (RR', R', R')\},$$

which is really the same set as $S_{\alpha_0}^1$, S^1 is an accounting device. In general S^i and S^i are isomorphic: identify $s^i \in S^i$ with $s_{\alpha_0}^i \in S^i$. In the rest of the paper I will frequently identify S and \mathcal{S} .

Definition 5 *Player i 's extended best-response correspondence $\beta^i : S \rightarrow S^i$ is defined by:*

$$\beta^i(s) = \{s^i \in S^i : u_\alpha^i(s_\alpha^i, s_\alpha^{-i}) \geq u_\alpha^i(\hat{s}_\alpha^i, s_\alpha^{-i}) \text{ for all } \hat{s}_\alpha^i \in S_\alpha^i, \text{ for all } \alpha \in I\}.$$

The game's extended best-response correspondence is $\beta : S \rightarrow S$, defined as $\beta(s) = \prod_{i \in N} \beta^i(s)$.

Player i 's extended best-response correspondence assigns a strategy that is a best response in each subgame to her opponents' strategy. A game Γ 's SPNE can be analyzed by means of its extended best-response correspondence β . Lemma 6 shows the usefulness of the extended best-response construction. By the construction of β , and the definition of SPNE, one can easily see the validity of the following first lemma—so I omit its proof.

Lemma 6 *The set of SPNE of a generalized extensive-form game equals the fixed points of its extended best-response correspondence.*

The next lemma, Lemma 7, shows that β is not a vacuous construction. The idea behind its proof is simple. Given opponents' strategies $s_{\alpha_0}^{-i}$, if $s_{\alpha_0}^i$ is a best response for player i in the whole game H_{α_0} , then $s_{\alpha_0}^i$ should prescribe a best response for subgames that are reached under $s_{\alpha_0}^{-i}$. Also, i is indifferent between strategies on subgames that are not reached. Modifying $s_{\alpha_0}^i$ to play a best response to $s_{\alpha_0}^{-i}$ also on non-reached subgames yields, by consistency of payoffs, a strategy that is still a best response to $s_{\alpha_0}^{-i}$ in the original game H_{α_0} . Repeating this operation “subgame by subgame” we can obtain an element in $\beta(s)$.

The reasoning “subgame by subgame” suggests a proof by induction. Even in simple games (like infinitely repeated bimatrix games) the set of subgames is uncountable, so a proof by classical induction is not possible. The proper tool turns out—expectedly—to be Zorn's Lemma.

Lemma 7 *For all $s \in S$, $\beta(s)$ is not empty.*

Proof: Let $s \in S$ and fix $i \in N$. For any $\alpha \in I$, let $\beta_\alpha^i(s^{-i}) = \operatorname{argmax}_{s^i \in S_\alpha^i} u_\alpha^i(s^i, s_\alpha^{-i})$. Tychonoff's Theorem implies that S_α^i is compact, so $\beta_\alpha^i(s^{-i})$ is nonempty because S_α^i is compact and payoffs

are upper semi-continuous in the player's own strategy. Let $\Omega = \{(s_\alpha^i, H_\alpha^i) : \alpha \in I, s_\alpha^i \in \beta_\alpha^i(s^{-i})\}$ be the set of pairs of best responses and subgames. Order Ω by \preceq , where $(s_{\alpha'}^i, H_{\alpha'}^i) \preceq (s_\alpha^i, H_\alpha^i)$ if and only if $H_{\alpha'}^i \subseteq H_\alpha^i$ and $s_\alpha^i|_{H_{\alpha'}^i} = s_{\alpha'}^i$. It is immediate to verify that \preceq is a partial order on Ω .

Let $\tilde{\Omega} \subseteq \Omega$ be any linearly ordered subset of Ω . Say $\tilde{I} \subseteq I$ is such that $\tilde{\Omega} = \{(s_\alpha^i, H_\alpha^i) \in \Omega : \alpha \in \tilde{I}\}$. Let $\hat{H} = \cup_{\alpha \in \tilde{I}} H_\alpha^i$. Since \mathcal{H}^i is closed under increasing unions, $\hat{H} \in \mathcal{H}^i$. Let $\gamma \in I$ satisfy, $H_\gamma^i = \hat{H}$. For any $h \in H_\gamma$, there is $\alpha \in \tilde{I}$ such that $h \in H_\alpha$; construct $s_\gamma^i \in S_\gamma^i$ by setting $s_\gamma^i(h) = s_\alpha^i(h)$. Since $\tilde{\Omega}$ is linearly ordered, s_γ^i is well defined.

I shall show that (s_γ^i, H_γ^i) is an upper bound on $\tilde{\Omega}$ in Ω . Clearly, $H_\alpha^i \subseteq H_\gamma^i$ and $s_\gamma^i|_{H_\alpha^i} = s_\alpha^i$ for all $\alpha \in \tilde{I}$. Let $\bar{s}_\gamma^i \in \beta_\gamma^i(s^{-i})$ and let $\{s_\gamma^i(\alpha)\}_{\alpha \in \tilde{I}}$ be the net in S_γ^i obtained by $s_\gamma^i(\alpha)|_{H_\alpha^i} = s_\alpha^i|_{H_\alpha^i}$, $s_\gamma^i(\alpha)|_{H_\gamma^i \setminus H_\alpha^i} = \bar{s}_\gamma^i|_{H_\gamma^i \setminus H_\alpha^i}$, and directing \tilde{I} by set inclusion (of the H_α with $\alpha \in \tilde{I}$). Note that $s_\gamma^i(\alpha) \rightarrow s_\gamma^i$ in the product topology. Fix any $\tilde{s}_\gamma^i \in S_\gamma^i$. Then $\bar{s}_\gamma^i \in \beta_\gamma^i(s^{-i})$ implies that $u_\gamma^i(\tilde{s}_\gamma^i, s_\gamma^{-i}) \leq u_\gamma^i(\bar{s}_\gamma^i, s_\gamma^{-i})$. Now, $s_\gamma^i|_{H_\alpha^i} = s_\alpha^i \in \beta_\alpha^i(s^{-i})$ implies that $u_\alpha^i(\bar{s}_\gamma^i|_{H_\alpha^i}, s_\alpha^{-i}) \leq u_\alpha^i(s_\gamma^i|_{H_\alpha^i}, s_\alpha^{-i})$. By consistency of payoffs, then, for any $\alpha \in \tilde{I}$, $u_\gamma^i(\tilde{s}_\gamma^i, s_\gamma^{-i}) \leq u_\gamma^i(s_\gamma^i(\alpha), s_\gamma^{-i})$. But then $u_\gamma^i(\tilde{s}_\gamma^i, s_\gamma^{-i}) \leq u_\gamma^i(s_\gamma^i, s_\gamma^{-i})$, as $s_\gamma^i(\alpha) \rightarrow s_\gamma^i$ and payoffs are upper semi-continuous. This shows that $(s_\gamma^i, H_\gamma^i) \in \Omega$, so (s_γ^i, H_γ^i) is an upper bound on $\tilde{\Omega}$.

The linearly ordered set $\tilde{\Omega}$ was arbitrary. By Zorn's lemma there is a maximal element, say $(s_\alpha^{*i}, H_\alpha^{*i})$, of Ω . Suppose $H_\alpha^{*i} \neq H_{\alpha_0}^i$. Let $\bar{s}_{\alpha_0}^i \in \beta_{\alpha_0}^i(s^{-i})$. Define $s_{\alpha_0}^i \in S_{\alpha_0}^i$ by $s_{\alpha_0}^i|_{H_{\alpha_0}^i \setminus H_\alpha^{*i}} = \bar{s}_{\alpha_0}^i|_{H_{\alpha_0}^i \setminus H_\alpha^{*i}}$ and $s_{\alpha_0}^i|_{H_\alpha^{*i}} = s_\alpha^{*i}$. Now, $(s_\alpha^{*i}, H_\alpha^{*i}) \in \Omega$ implies that $s_\alpha^{*i} \in \beta_\alpha^i(s^{-i})$. Let $\tilde{s}_{\alpha_0}^i \in S_{\alpha_0}^i$, then $u_{\alpha_0}^i(\tilde{s}_{\alpha_0}^i, s_{\alpha_0}^{-i}) \leq u_{\alpha_0}^i(\bar{s}_{\alpha_0}^i, s_{\alpha_0}^{-i})$ and $u_\alpha^i(\bar{s}_{\alpha_0}^i|_{H_\alpha^{*i}}, s_\alpha^{-i}) \leq u_\alpha^i(s_{\alpha_0}^{*i}|_{H_\alpha^{*i}}, s_\alpha^{-i})$. By consistency of payoffs, $u_{\alpha_0}^i(\tilde{s}_{\alpha_0}^i, s_{\alpha_0}^{-i}) \leq u_{\alpha_0}^i(s_{\alpha_0}^i, s_{\alpha_0}^{-i})$. Hence, $s_{\alpha_0}^i \in \beta_{\alpha_0}^i(s^{-i})$, so that $(s_{\alpha_0}^i, H_{\alpha_0}^i) \in \Omega$ and $(s_\alpha^{*i}, H_\alpha^{*i}) \preceq (s_{\alpha_0}^i, H_{\alpha_0}^i)$. But $(s_\alpha^{*i}, H_\alpha^{*i}) \neq (s_{\alpha_0}^i, H_{\alpha_0}^i)$. Contradiction, since $(s_\alpha^{*i}, H_\alpha^{*i})$ is maximal.

Construct $s^{*i} \in S^i$ by setting $s_\alpha^{*i} = s_{\alpha_0}^{*i}|_{H_\alpha^i}$ for all $\alpha \in I$. It is then immediate that $s^{*i} \in \beta^i(s)$ since, by construction, for all $\alpha \in I$, $s_\alpha^{*i} = s_\alpha^{*i} \in \beta_\alpha^i(s^{-i})$. ■

Extended best-response correspondences translate the problem of finding SPNE to a fixed-point problem. By adding the assumption of strategic complementarities, fixed points are ob-

tained by a version of Tarski’s Fixed Point Theorem, and the set of SPNE can be analyzed by “lattice programming” techniques. I include this version of Tarski’s Fixed Point Theorem—due to Zhou (1994)—as Theorem 8. Potentially, though, extended best-response correspondences are useful in other classes of extensive-form games as well.

Theorem 8 (Zhou (1994)) *Let X be a complete lattice, and $\phi : X \rightrightarrows X$ a correspondence such that $\phi(x)$ is a non-empty, subcomplete sublattice for all $x \in X$. If ϕ is increasing in the strong set order, then the set of fixed points of ϕ form a non-empty complete lattice.*

Theorem 9 *If Γ is an extensive-form game of strategic complementarities, then its SPNE form a non-empty, complete lattice.*

Proof: I need to show that β is monotone increasing in the strong set order and takes non-empty, closed values in order to apply Zhou’s (1994) version of Tarski’s fixed point theorem. First I show that β is monotone increasing in the strong set order. Let $s, z \in S$ with $s \prec z$. Let $s' \in \beta(s)$ and $z' \in \beta(z)$. By Theorem 4 in Milgrom and Shannon (1994), for every $\alpha \in I$, $s'^i \vee z'^i \in \operatorname{argmax}_{s \in S_\alpha^i} u_\alpha^i(s, z_\alpha^{-i})$ and $s'^i \wedge z'^i \in \operatorname{argmax}_{s \in S_\alpha^i} u_\alpha^i(s, s_\alpha^{-i})$. Hence, $s' \vee z' \in \beta(z)$ and $s' \wedge z' \in \beta(s)$, proving that β is increasing in the strong set order.

That β takes closed values is an immediate consequence of upper semi-continuity of payoffs in each subgame. By Lemma 7, β takes non-empty values. Hence, by Zhou’s version of Tarski’s fixed point theorem, the set of fixed points of β is a complete lattice. Lemma 6 implies that the set of SPNE is a complete lattice. ■

Theorem 9 implies that there is a smallest and a largest SPNE of any extensive-form game of strategic complementarities. Note that the subgames of any extensive-form game of strategic complements are also extensive-form games of strategic complements. Therefore, by Theorem 9, each subgame has a smallest and a largest SPNE strategy profile. It turns out that the extremal SPNE of any subgame are obtained from the extremal SPNE of the whole game. This has important consequences: It can be seen that, in multi-stage games, the extremal equilibria

are Markov-Perfect. At the same time, for the reasons discussed in Section 4, the scope of Theorem 9 on multi-stage games is very limited—hence I choose not to dwell on the implications of Theorem 9 on multi-stage games here, see Echenique (2000) for the details.

The collection of subgames $\{H_\alpha : \alpha \in I\}$ is *closed under intersections* if for any $\alpha, \alpha' \in I$, there is $\xi \in I$ such that $H_\xi = H_\alpha \cap H_{\alpha'}$. Any well-defined game tree has subgames that are closed under intersections (in fact subgames are either nested or disjoint, so they are trivially closed under intersections).

Theorem 10 *Let Γ be an extensive-form game of strategic complementarities with subgames that are closed under intersections; let \underline{s} be its smallest SPNE and \bar{s} its largest SPNE. If H_α , with $\alpha \in I$, is any subgame, then $\underline{s}|_{H_\alpha}$ and $\bar{s}|_{H_\alpha}$ are, respectively, the smallest and largest SPNE of the extensive-form game corresponding to H_α .*

Proof: Suppose, by way of contradiction, that there is a subgame H_α with a smallest SPNE strategy profile s_α that is not equal to $\underline{s}|_{H_\alpha}$. Let $\tilde{\mathcal{S}} = \{\hat{s} \in \mathcal{S} : \hat{s}_\alpha = s_\alpha\}$, by repeating the arguments above we obtain that $\tilde{\mathcal{S}}$ is a complete lattice. Let $\tilde{\beta} : \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{S}}$ be defined by $\tilde{\beta}(s) =$

$$\left\{ \hat{s} \in \tilde{\mathcal{S}} : u_\gamma^i(s_\gamma^i, s_\gamma^{-i}) \leq u_\gamma^i(\hat{s}_\gamma^i, s_\gamma^{-i}) \text{ for all } s_\gamma^i \in S_\gamma^i, s_\gamma^i|_{H_\alpha} = s_\alpha^i, i \in N, \text{ and } \gamma \in I, H_\gamma \not\subseteq H_\alpha \right\}.$$

That $\tilde{\beta}$ is monotone increasing in the strong set order and closed-valued follows from arguments similar to those proving that β is monotone increasing and closed-valued. That $\tilde{\beta}$ has non-empty values can be proved by following the steps in the proof of Lemma 7, and restricting the optimizing strategies to equal s_α^i on information sets that also belong to H_α .

By Zhou's version of Tarski's Theorem, there is a fixed point $s \in \tilde{\mathcal{S}}$ of $\tilde{\beta}$. I claim that this is a SPNE of the whole game H_{α_0} . Fix $i \in N$. For any $\gamma \in I$ with $H_\gamma \subseteq H_\alpha$, $s_{\alpha_0}|_{H_\gamma} = s_\alpha|_{H_\gamma}$. But $s_{\alpha_0}^i|_{H_\gamma} \in \beta_\gamma^i(s)$ since H_γ is a subgame of H_α and s_α is a SPNE in H_α . Let $\gamma \in I$ with $H_\gamma \not\subseteq H_\alpha$. Γ has subgames that are closed under intersections, hence there is $\xi \in I$ with $H_\xi = H_\gamma \cap H_\alpha$. Since H_ξ is a subgame of H_α , $s_\xi = s_\alpha|_{H_\xi}$ so $s_\xi \in \beta_\xi^i(s)$ because s_α is a SPNE in subgame H_α .

Then, $s_\gamma^i \in \beta_\gamma^i(s)$ because if there is $\hat{s}_\gamma^i \in S_\gamma^i$ with $u_\gamma^i(s_\gamma^i, s_\gamma^{-i}) < u_\gamma^i(\hat{s}_\gamma^i, s_\gamma^{-i})$ it must be that \hat{s}_γ^i and s_γ^i differ on the subgame $H_\xi = H_\gamma \cap H_\alpha$. But, using consistency of payoffs,

$$u_\gamma^i(s_\gamma^i, s_\gamma^{-i}) < u_\gamma^i(\hat{s}_\gamma^i, s_\gamma^{-i}) \leq u_\gamma^i(\hat{s}_\gamma^i|_{H_\gamma \setminus H_\alpha}, s_\xi^i, s_\gamma^{-i}),$$

impossible since $s \in \tilde{\beta}(s)$. Hence, $s_\gamma^i \in \beta_\gamma^i(s)$ for any $\gamma \in I$ so s is a SPNE.

By Theorem 9 there is a SPNE $\tilde{s} = \underline{s} \wedge s_{\alpha_0}$ because the set of SPNE is a lattice. Since \underline{s} and s_{α_0} differ on H_α , \tilde{s} is smaller than \underline{s} , a contradiction. ■

Comparative statics results for the extremal SPNE follow from well-known results in Monotone Comparative Statics. For comparative statics of non-extremal equilibria, see Echenique (2002).

Definition 11 *Let T be a partially ordered set. The collection $\{\Gamma(t) : t \in T\}$ is an **increasing family of extensive-form games** if, for any $t \in T$, $\Gamma(t) = \{N, H, \{H_\alpha : \alpha \in I\}, \{H_i : i \in N\}, \{A_t(h) : h \in H\}, \{u_{\alpha t}^i : i \in N, \alpha \in I\}\}$ is an extensive-form game of strategic complementarities; if for all $h \in H$, $A_t(h)$ is increasing in the strong set order in t and if, for all $i \in N$ $\alpha \in I$, $u_{\alpha t}^i$ satisfies the single-crossing condition in (s_α^i, t) .*

Theorem 12 *Let $\{\Gamma(t) : t \in T\}$ be an increasing family of extensive-form games. Let $t, t' \in T$ with $t \preceq t'$. The smallest (largest) SPNE of $\Gamma(t)$ is smaller, as an element of S , than the smallest (largest) SPNE of $\Gamma(t')$.*

Proof: Let β_t and $\beta_{t'}$ be the extended best-response correspondences of $\Gamma(t)$ and $\Gamma(t')$, respectively. An argument similar to the proof that the extended best-response function is monotone increasing in the proof of Theorem 9 establishes that, for any $s \in S$, $\beta_t(s)$ is smaller than $\beta_{t'}(s)$ in the strong set order. The result then follows from Theorem 2.5.2 in Topkis (1998). ■

4 Complementarities are restrictive

How common is the existence of complementarities in extensive-form games? There are two answers to this question. First, one can argue that the order on strategies is not part of the description of a game, so one has a “degree of freedom” in checking for complementarities, one can try to find an order on strategies such that a game has complementarities: So, how often can one find an order on strategies such that a game has complementarities? Very often—I shall not expand on why here, but it is a direct application of the characterization in Echenique (2001); Echenique’s (2001) results require, though, that one knows first the number of equilibria of the game, so it does not provide sufficient conditions for existence, or for comparative statics.

Second, given games with some kind of “natural” order for which heuristically one *should* get complementarities—for example a dynamic variant of a static game with complementarities—the answer is negative. In the discussion below I shall give examples of games with and without extensive-form complementarities, I believe these examples explain where the problems arise.

The situation contrasts with the study of Markov-Perfect equilibria in stochastic games with complementarities. For example, Curtat (1996) (implicitly also Amir (1989) and Amir (1996a)) imposes supermodularity conditions on payoffs, and is able to obtain results on the set of Markov-Perfect equilibria using lattice-theoretic tools. The reason is that Markov-Perfect equilibria limit the strategic interaction over time enough so that the effect of complementarities is preserved in dynamic contexts—this will, I hope, be clear from the examples below.

1. Optional BoS in the introduction is a game of strategic complementarities. Optional BoS II in Figure 3 cannot be made into a game of strategic complementarities. Say that Yes is larger than No and repeat the argument from the introduction: An increase in One’s strategy from No-*OO* to Yes-*OO* makes Two shift from *B* to *O* in the game following Yes. Strategic complementarities requires then that the action *O* is larger than *B* at this information set. But this gives a decreasing response when we shift from No-*OB* to Yes-*OB*. The solution in the

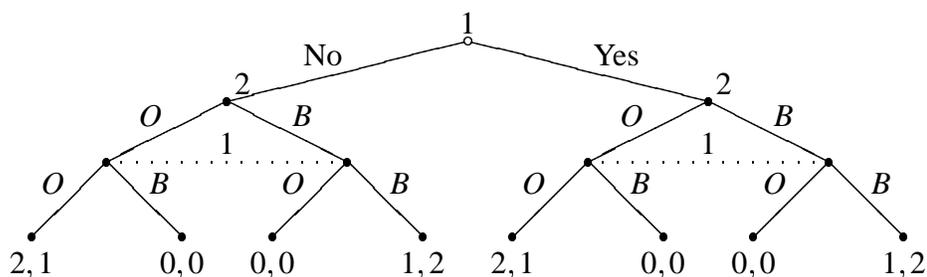


Figure 3: Optional BoS II

introduction was to change the order on $\{\text{Yes}, \text{No}\}$. But here this clearly gives rise to the same situation in the subgame following No. ³

The problem is that, off the path specified by a strategy profile, players are indifferent between actions. This indifference makes the single-crossing property kick in. If the information set h is off the path of a strategy profile (s^i, s^{-i}) then player i is indifferent between strategies that differ on $A(h)$. Now, if s'^{-i} is a larger strategy, and h is on the path of (s^i, s'^{-i}) , then it must be that payoffs are such that, if $a, a' \in A(h)$ and a is smaller than a' , then a is preferred to a' . That is, preferences have to coincide with the order on actions for every strategy profile that has h “on its path.”

2. I shall show that, even in simple dynamic games, while “per-period payoffs” are supermodular, (extended) best responses are not monotone increasing, there is no largest SPNE, and the set of SPNE is not a lattice. The root of the problem is that the strategic interactions in dynamic games destroy the effect of complementarities in per-period payoffs.

First I present a finite-horizon example. Then I present an infinite-horizon example to show that there are additional problems with dynamic complementarities in infinite-horizon games (I am indebted to an anonymous referee for pointing out that there are additional problems in infinite-horizon games). The point of these examples is that there is no hope of generalizing—with a different notion of extensive-form complementarities—the results in Section 3 to many games of economic interest.

a) The example in item 1 is a dynamic game that intuitively should satisfy definition 4, but fails to do so. Theorems 9, 10 and 12 depend on the property that best responses are monotone increasing, and not directly on the supermodularity/single-crossing assumptions on payoffs in Definition 4—these assumptions are merely sufficient for monotone (extended) best responses, not necessary. I shall show that, even in simple dynamic games, while “per-period payoffs” are supermodular, (extended) best responses are not monotone increasing, there is no largest SPNE, and the set of SPNE is not a lattice.

Consider the matrix game in Figure 4. It is easy to check that this game has strategic complementarities when each player’s strategy set is ordered by $\alpha < \beta < \gamma$. Indeed, as each player’s strategy set is totally ordered, payoffs are quasisupermodular in the player’s own strategies. The single-crossing property holds too, note that for player “Row” the single-crossing property only kicks in when “Column” plays β and Row compares the payoff of β with that of the smaller strategy α —the payoff to Row of β is higher, and it remains higher when Column increases her strategy from β to γ , so Row’s payoffs satisfy the single-crossing property. Similarly with Column’s payoffs.

Now suppose that Row and Column play the twice-repeated version of the matrix game. Suppose that they discount per-period payoffs with a discount factor $\delta \geq 1/2$. I shall show that best responses in the twice-repeated version of the game are not monotone increasing, that there is no largest SPNE, and that there are non-monotone SPNE outcomes.

Consider the following strategy for Column: “play γ in period 1, play α in period 2 if the

	α	β	γ
α	3,3	0,1	0,0
β	1,0	1,1	3,0
γ	0,0	0,3	2,2

Figure 4: A game with complementarities.

outcome in period 1 was (γ, γ) , play β in period 2 if not.” A best-response to this strategy by Row is to do the same, i.e. “play γ in period 1, play α in period 2 if the outcome in period 1 was (γ, γ) , play β in period 2 if not.” Suppose now that Column increases her strategy to “play γ in period 1, play β in period 2 regardless of the period-1 outcome.” Then, any best-response by Row involves playing β in period 1—so no best response to the increase in Column’s strategy involves an increase in Row’s strategy.

One SPNE *outcome* is $((\gamma, \gamma), (\alpha, \alpha))$ —meaning that (γ, γ) is the outcome the first time the game is played, and (α, α) is the outcome the second time. Also, $((\beta, \beta), (\beta, \beta))$ is a SPNE outcome. Now, $((\gamma, \gamma), (\alpha, \alpha))$ and $((\beta, \beta), (\beta, \beta))$ are unordered as vectors in $\{\alpha, \beta, \gamma\}^4$, and there are no larger SPNE outcomes—the vectors $((\gamma, \gamma), (\beta, \beta))$ and $((\gamma, \gamma), (\gamma, \gamma))$ are larger, but they cannot be the result of a SPNE. So there is no largest SPNE outcome, which implies that there is no largest SPNE, and that the set of SPNEs is not a lattice.

This example is important because it gets quickly hard to check that a game satisfies Definition 4, so it would help if it were enough to check complementarities on per-period payoffs. Then, even if Definition 4 is hard to satisfy, there is little hope that it can be weakened in a fruitful way, as any such weakening should reasonably imply that the example above is a game with strategic complementarities.

The negative conclusions about complementarities and SPNE contrast with existing results on Markov-Perfect equilibria (Amir 1996a, Curtat 1996). Curtat’s (1996) results delivers one monotone Markov-Perfect equilibrium for games with “per-period” complementarities. For example, by using as state-space the strategy profile chosen last period, one gets two constant (and thus monotone) Markov-Perfect equilibria: i) play (α, α) each period no matter what the state and ii) play (β, β) each period no matter what the state.

b) I shall discuss finitely- and infinitely-repeated prisoner’s dilemma. I assume that the reader is familiar with a textbook analysis of the game at the level of, for example, Fudenberg and Tirole (1991) or Osborne and Rubinstein (1994).

Consider finitely-repeated prisoner's dilemma. Let us order the players' actions such that "cooperate" is larger than "defect" for each player—nothing depends on that order, we can change it and get the same result. One-shot prisoner's dilemma has strategic complementarities because best-responses are constant (defect no matter what the opponent does), and thus monotone increasing. One-shot prisoner's dilemma also has a unique equilibrium, which gives us a game with strategic complementarities in each step of the backward-induction process, as the future consequences of any pair of chosen strategies are the same. Strategic complementarities are thus inherited from one step to the next in the backward-induction process (see Amir (1996b) for a discussion in the context of recursive decision problems).

Now consider infinitely repeated prisoner's dilemma with a large discount factor. For any history h of play, there is a SPNE where both players cooperate after history h . Hence there cannot be a largest SPNE, as the strategies "cooperate after any history" are not a SPNE. One can thus not hope to obtain a result like Theorem 9 in this game. The root of the problem is again that extended best-responses are not monotone increasing: if player 2 plays the strategy "cooperate at any history where both have always cooperated, defect at any other history" it is a best response for player 1 to play the strategy "cooperate at any history where both have always cooperated, defect at any other history." Now if player 2 increases her strategy to "cooperate at all histories," no best response by 1 will involve cooperating at the null history. Thus player 1's best response does not increase.

Again, in this example per-period complementarities are sufficient to get the existence of a monotone Markov-Perfect equilibrium—but then some monotone Markov-Perfect equilibria are trivial: use a singleton state-space and play "defect" for ever. The point is that per-period complementarities are not enough to yield the results in Section 3.

3. The problem in the examples in item 2 is that complementarities are endogenous. I now explain in more detail how this problem arises.

Suppose there are two players in a two-stage game. Each player i chooses, in time t , a real

number a_t^i , the players' choices in each time period are simultaneous. Suppose that player i 's payoff from outcome $(a_1^1, a_1^2, a_2^1, a_2^2)$ is $u^i(a_1^1, a_1^2, a_2^1, a_2^2)$, suppose that u^i has the single-crossing property for all possible pairs of its arguments. Note that, if choices were simultaneous, this would be a game with strategic complementarities. Now let's look at the backward induction solution to this game. From e.g. Milgrom and Shannon's (1994) results, there is a Nash equilibrium of each second-stage subgame that is monotone increasing in the first-period choices. That is, let $(a_2^1(a_1^1, a_1^2), a_2^2(a_1^1, a_1^2))$ be a Nash equilibrium in the (a_1^1, a_1^2) -subgame, and $(a_2^1(a_1^1, a_1^2), a_2^2(a_1^1, a_1^2))$ increase as (a_1^1, a_1^2) increases.

It is now easy to see that the induced first-period game may not have complementarities. Let $a_1^1 < \bar{a}_1^1$, $a_1^2 < \bar{a}_1^2$, and suppose that

$$u^i(a_1^1, a_1^2, a_2^1(a_1^1, a_1^2), a_2^2(a_1^1, a_1^2)) \leq u^i(\bar{a}_1^1, \bar{a}_1^2, a_2^1(\bar{a}_1^1, \bar{a}_1^2), a_2^2(\bar{a}_1^1, \bar{a}_1^2)).$$

To obtain single-crossing in the induced first-period payoffs, we need

$$u^i(a_1^1, \bar{a}_1^2, a_2^1(a_1^1, \bar{a}_1^2), a_2^2(a_1^1, \bar{a}_1^2)) \leq u^i(\bar{a}_1^1, \bar{a}_1^2, a_2^1(\bar{a}_1^1, \bar{a}_1^2), a_2^2(\bar{a}_1^1, \bar{a}_1^2)).$$

But the single-crossing assumptions on u^i only guarantee that

$$u^i(a_1^1, \bar{a}_1^2, a_2^1(a_1^1, \bar{a}_1^2), a_2^2(a_1^1, \bar{a}_1^2)) \leq u^i(\bar{a}_1^1, \bar{a}_1^2, a_2^1(\bar{a}_1^1, \bar{a}_1^2), a_2^2(\bar{a}_1^1, \bar{a}_1^2)),$$

and it is easy to generate examples where that is not enough for single crossing the first-period payoffs. ⁴

The problem is that existence of complementarities in period 1 depends on the period-two *equilibrium*, so static complementarities assumptions are insufficient to guarantee extensive-form complementarities. Definition 4 puts enough structure on across-subgames complementarities so that the induced first-period payoffs do have complementarities. It would be nice to know if there are weaker conditions that guarantee this, but I think it is very unlikely that there are any.

	R	L
L'	x	$x \vee y$
R'	$x \wedge y$	y
	s^2	

	R	L
L'	0	2
R'	3	2
	r	

	R	L
L'	0	2
R'	0	2
	l	

Figure 5: Payoffs to Player One in the game in Figure 2

4. Finally, I will show that the game in Example 1 is a game of strategic complementarities. Order the actions so that L is larger than R , l is larger than r and L' is larger than R' . Consider Figure 5. The two matrices to the right show the payoffs to player One for each of the eight possible strategy profiles in this game.

Fix Two's strategy $s^2 = r$. We need to check that $s^1 \mapsto u^1(s^1, r)$ is quasisupermodular. Here, $u^1(RL', r) = 0 < u^1(RL' \wedge LR', r) = u^1(RR', r) = 3$, so the requirement in the definition of quasisupermodularity is vacuous. Fix $s^2 = l$, then $u^1(RL' \wedge LR', l) = u^1(RR', l) = 0$ and $u^1(RL', l) = 0$ so we need $u^1(LR', l) \leq u^1(RL' \vee LR', l)$. Which is satisfied since $u^1(RL' \vee LR', l) = u^1(LL', l) = 2$ and $u^1(LR', l) = 2$. Now, we need to check that the players' payoffs satisfy the single-crossing property. Note from Figure 5 that, when $s^2 = r$, no increase in a strategy by player One is profitable. Since $r < l$ the single-crossing property in player One's payoffs is satisfied vacuously.

	R	L
L'	0	0
R'	-1	0
	s^2	

Figure 6: Gain from increasing Two's strategy: $u^2(s^1, l) - u^2(s^1, r)$.

To see that Two's payoffs satisfy the single-crossing property, consider Figure 6. The figure shows the gain to Two $u^2(s^1, l) - u^2(s^1, r)$ from increasing his strategy from r to l . It is seen directly from the figure that whenever $u^2(s^1, l) - u^2(s^1, r) \geq 0$ and $s^1 < s^{l'}$ then

$u^2(s^l, l) - u^2(s^l, r) \geq 0$. This establishes that the game in Figure 2 is a game of strategic complementarities.

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Notes

¹They are not dominance-solvable, as might be suggested by the discussion above.

²See Fudenberg and Tirole (1991) and Osborne and Rubinstein (1994) for the two usual definitions.

³ Optional BoS II shows that the property of having complementarities is not robust to the addition of an irrelevant move. This is also true about complementarities in normal-form games, so this non-robustness does not cause complementarities to be especially restrictive in extensive-form games.

⁴For example, $u^i(a_1^1, a_1^2, a_2^1, a_2^2) = a_1^1 a_1^2 a_2^1 a_2^2 + a_1^1(1 - a_1^1) + a_2^1(1 - a_2^1)$ works if we restrict $a_i^i \geq 0$.