HAMILTONIAN MECHANICS ON LIE GROUPS AND HYDRODYNAMICS

J. MARSDEN AND R. ABRAHAM

Introduction. Some of the most classical and important examples in mechanics are systems whose configuration space is a Lie group. The particular examples we have in mind are the rigid body (on the Lie group SO(3)) and the perfect fluid (on the Lie group of volume preserving diffeomorphisms).

Most of what we have to say is classical and well known. What we do is to put it in the language of global analysis with perhaps some simplification. Our sources are mainly the papers of Arnold and Blancheton [3], [4].

The paper is divided into two parts. In the first we present the general theory. In the second we describe the case of hydrodynamics. Some connections will be made with the calculus of variations in the future. In addition, a more complete exposition of the present work will appear in lecture note forms shortly [11].[1]

1. **Abstract theory.** Let G be a Lie group.[2] By this we mean that G is a smooth manifold modelled on a locally convex topological vector space (locally convex since, amongst other reasons, the Hahn-Banach theorem is needed) and is also a group such that group multiplication and inversion are C^∞ mappings. The tangent bundle of G is denoted TG and the fiber over $x \in G$ is written T_xG.

Let g be a (weak) Riemannian metric[3] on G. This means that the tensor g is an inner product on each T_xG, but inducing a different topology in general. For each $x \in G$, $e_x \in T_xG$ and $f_x \in T_xG$ we write $\langle e_x, f_x \rangle = g(x)\cdot(e_x, f_x)$.

A *weak symplectic form* on a manifold M is a closed two form ω such that the mapping $\omega_b: TM \to T^*M$ defined by $\omega_b(e_x)\cdot f_x = \omega(x)\cdot(e_x, f_x)$ is injective on each fiber. (If ω_b is an isomorphism on each fiber, we call ω a *symplectic form.*)

For infinite dimensional mechanics (continuum mechanics and quantum mechanics for example), if one wishes to work with a symplectic form it is necessary to use domains for vectorfields in the same sense as occurs in semigroup theory; see [8]. If, however, one wishes to exploit differentiability directly and have the vectorfield defined and smooth in the usual sense, then it is necessary to use weak symplectic forms instead. The reason will become evident shortly. We shall use weak symplectic forms here.

Except in unusual and artificial circumstances (these can be obtained for the wave equation using the spaces in [10]), the manifolds one needs to get a smooth

[1] See also [14].

[2] The term "Lie Group" may be misleading since, in the infinite dimensional case, the usual Lie theorems do not hold. The term ILH Lie Group, or Fréchet Lie Group may be better.

[3] g is assumed smooth.

vectorfield are usually Fréchet and not Banach, the C^∞ functions for example. In that case the standard flow theorem for vectorfields is false. Instead one must use techniques of Browder, Kato and others. For the case of hydrodynamics with viscosity a good part of a classical book [6] is needed. Also, the flow is possibly only local, that is, cannot be extended for all time. For the nonviscous case we are concerned with, see Kato [5].[4]

Let M be a manifold and ω a weak symplectic form. A one form α *can be lifted* when there exists a vectorfield (unique) X on M such that $X^b = \alpha$, where $X^b(m) = 2\omega_b(m)\cdot X(m)$, $m \in M$. We write $X = \alpha^\#$.

For a smooth function $f: M \to R$, such that df can be lifted, we write $X_f = (df)^\#$, and we call X_f a *Hamiltonian vectorfield.*

It will be necessary to recall a few theorems about Hamiltonian systems.

THEOREM 1. *Let (M, ω) be a weak symplectic manifold and X_H a Hamiltonian vectorfield with a local smooth flow F_t. Then*

(i) $H \circ F_t = H$ *(conservation of energy), and*

(ii) $F_t^*\omega = \omega$, *or F_t is symplectic (preserves the form ω).*

See [1] or [8] for the proof.

Recall that if M is a manifold then T^*M admits a natural weak symplectic structure given by (locally)

$$\omega(\alpha_x)\cdot((e_1, \alpha_1), (e_2, \alpha_2)) = [\alpha_2(e_1) - \alpha_1(e_2)]/2$$

for $e_i \in T_xM$ and $\alpha_i \in T_x^*M$. [ω is symplectic iff M is modelled on a semireflexive space.] See [8, Theorem 2.4].

THEOREM 2. *Let G be a Lie group and M a manifold with $\Phi: G \times M \to M$ a smooth action of G on M, which extends naturally to an action Φ^* of G on T^*M. Suppose X_H is a Hamiltonian vectorfield on T^*M and H is invariant under the action Φ^*. Then the following functions P_X are invariant under the flow of X_H. Let X be an infinitesimal generator of Φ, so X is a vectorfield on M and define $P_X: T^*M \to R$ by $P_X(\alpha_m) = \alpha_m(X(m))$. (We call P_X the momentum of X.)*

For the proof see [8, Theorem 5.3]. This is the basic conservation law of mechanics.

In case g is a weak Riemannian metric on M inducing a map $g_b: TM \to T^*M$, then we deduce (using [8, Theorem 5.2]) that if X_H is a Hamiltonian vectorfield on TM (with respect to the form $\omega_g(e_x)\cdot((e_1,f_1), (e_2,f_2)) = [\langle f_2, e_1\rangle - \langle f_1, e_2\rangle]/2$) and H is invariant under the induced (adjoint) action on TM then the function $P_X(e_m) = \langle e_m, X(m)\rangle$ is invariant under the flow of X_H.

The latter situation is the one which arises naturally in the case of a Lagrangian system and will be of concern to us below.

Let us now return to the setting of Lie groups. We shall require our group and metric g to have certain regularity properties introduced as follows.

[4] The general nonviscous existence problem is settled in [14].

DEFINITION.[5] Let G be a Lie group and g a weak Riemannian metric on G. We say that g is *compatible* with G iff

(i) g is left invariant; that is, for each $x \in G$, $L_x^* g = g$ where L_x is the diffeomorphism defined by $L_x(y) = xy$;

(ii) the kinetic energy function $T: TG \to \boldsymbol{R}$ defined by $T(e_x) = \langle e_x, e_x \rangle/2$ can be lifted to a (smooth) vectorfield X_T using the weak symplectic form

$$\omega_g((e_1, f_1), (e_2, f_2)) = [\langle f_2, e_1 \rangle - \langle f_1, e_2 \rangle]/2; \text{ and}$$

(iii) X_T possesses a local smooth flow (called the *geodesic flow* of g).

Also, we say that G is a *regular Lie group* iff every (smooth) left invariant vectorfield on G has a flow.

In the finite dimensional case these conditions are of course redundant. In the infinite dimensional case it is (iii) above which is difficult to verify.

The main conservation theorem is the following (the cases of hydrodynamics and a rigid body are due to Euler):

THEOREM 3. *Let G be a regular Lie group and X_H a Hamiltonian vectorfield on TG with H invariant by left translations ($H \circ TL_x = H$ for each $x \in G$, $TL_x: TG \to TG$ being the tangent map). Then for each $v \in T_eG$ (e = identity element of G) the function P_v is invariant under the flow of X_H, where*

$$P_v: TG \to \boldsymbol{R}$$

is defined by

$$P_v(u_x) = \langle T_eR_x \cdot v, u_x \rangle$$

where $u_x \in T_xG$, R_x is right translation by x and T_eR_x is the tangent of R_x evaluated at $e \in G$.

COROLLARY 4. *Let G be a regular Lie group and g a compatible weak Riemannian metric. Then the functions P_v are invariant under the geodesic flow of g. Further, this flow is a symplectic (local) diffeomorphism and conserves (kinetic) energy.*

PROOF. Let G act on itself by left translations. Each $v \in T_eG$ determines an exponential flow on G by assumption, and its derivative is the infinitesimal generator X. Let E_t be the exponential map of v. Then

$$X(y) = d(E_t(y))/dt = d(E_t(e) \cdot y)/dt \text{ at } t = 0.$$

Now $E_t(e) \cdot y = R_y \circ F_t(e)$ and so by the composite mapping theorem, $d(E_t(e) \cdot y)/dt = T_eR_y \cdot d(E_t(e))/dt = T_eR_y \cdot v$. The result is now an immediate consequence of the conservation theorem. ■

In practice it is usually most convenient to work in the Lie algebra T_eG by pulling back the flow to T_eG by left translation (in the so called "body coordinates"). The pull back of the vectorfield X_T to T_eG is the *Euler equations*. They are determined as follows:

[5] In the rest of section one, "left" and "right" can be interchanged, and this introduces a minus sign in Theorem 5 and Lemma 6.

THEOREM 5. *Let G be a (regular) Lie group, g a compatible metric and X_T the corresponding Hamiltonian vectorfield with flow $F_t: TG \to TG$ (which may be just local). Define H_t on T_eG by*

$$H_t(v) = T_{x(t)}L_{x(t)^{-1}} \cdot F_t v$$

(where defined), where $F_t v \in T_{x(t)}G$. Then H_t is a smooth flow on T_eG and has vectorfield Y uniquely determined by: $Y: T_eG \to T_eG$, $\langle Y(u), v\rangle = \langle [u, v], u\rangle$ where $[u, v]$ is the Lie bracket in T_eG.

PROOF. It is easily checked that H_t is a flow. To compute Y we must compute $dH_t(v)/dt$ at $t = 0$. For this we use the following:

LEMMA 6. *Let $x(t)$ be a smooth curve in G, $v \in T_eG$ and $v(t) = \mathrm{Ad}_{x(t)^{-1}} \cdot v$ where*

$$\mathrm{Ad}_x = T_xR_{x^{-1}} \circ T_eL_x = T_e(R_{x^{-1}} \circ L_x): T_eG \to T_eG.$$

Then we have

$$dv/dt = [v(t), T_{x(t)}L_{x(t)^{-1}} \cdot (dx/dt)].$$

We omit the proof, as it is more or less standard; see [11] or [13].

To prove the theorem, we start with

$$d\langle F_t u, T_eR_{x(t)}v\rangle/dt = 0$$

by the conservation law, for each $u, v \in T_eG$. By left invariance of g,

$$\begin{aligned}\langle F_t u, T_eR_{x(t)} \cdot v\rangle &= \langle T_{x(t)}L_{x(t)^{-1}} \cdot F_t u, \mathrm{Ad}_{x(t)^{-1}} v\rangle \\ &= \langle H_t u, \mathrm{Ad}_{x(t)^{-1}} v\rangle.\end{aligned}$$

Differentiating, using Leibnitz's rule and the lemma, gives at $t = 0$,

$$\langle Y(u), v\rangle + \langle u, [v, u]\rangle = 0$$

where we use the fact that $dx/dt = F_t u$ (which is because we have a second order equation). ■

If one finds the flow of Y on T_eG, the problem of finding F_t is solved using the equation

$$dx/dt = T_eL_{x(t)} \cdot H_t u = F_t u$$

and the fact that F_t is left invariant. For groups of diffeomorphisms and in particular for hydrodynamics, this is just a problem in ordinary differential equations.

COROLLARY 7. *If v_0 is a critical point of Y on T_eG then $x(t) = \exp(tv_0) \in G$ is the geodesic with initial value v_0.*

PROOF. We have $H_t v_0 = v_0$ and so

$$F_t v_0 = T_eL_{x(t)} \cdot v_0 = dx/dt.$$

This is the equation for $x(t) = \exp(tv_0)$. ■

By left invariance, the conservation law and conservation of energy pull back to the Lie algebra T_eG.

For many circumstances it is important to work in a Hilbert space. Therefore we could complete T_eG with respect to $\langle , \rangle$, to obtain a Hilbert space H. It is absolutely crucial to recognize that on H, the vectorfield Y is not smooth and is not everywhere defined. Also, the flow H_t is *not* defined on all of H. (This can be seen for the diffeomorphism group where T_eG corresponds to C^∞ vectorfields, so H would be L_2 vectorfields. For flows of nonsmooth vectorfields see [9].)

In the finite dimensional case, or if the energy function T had D^2T continuous on H (so Y would be smooth on H say), then definiteness of the quadratic form D^2T[6] at a stationary point v_0 would imply its stability (the quadratic form is computed in Arnold [3]). Unfortunately for hydrodynamics this is not the case, so it is unknown whether or not this criterion for stability is valid, as far as we know.

2. **Hydrodynamics of perfect fluids.** Let D be a compact orientable n-manifold with boundary (or without boundary). Let g_0 be a Riemannian metric on D and Ω a volume (orientation) on D (that is, a nonvanishing n-form) which is generally one derived from g_0. For tangent vectors on D we write $v \cdot u$ or $\langle v, u \rangle$ for the inner product with respect to g_0.

Let G be the group of volume preserving diffeomorphisms on D. Leslie [6], and Omori [11] show that the group of all diffeomorphisms has a structure modelled on a Fréchet space for which the group is a Lie group. The procedure has become a more or less standard one in manifolds of maps. See also [2]. A recent theorem of D. Ebin[7] tells us that G is also a Lie group (in fact a Lie subgroup). Then T_eG may be identified with C^∞ vectorfields X on D such that

(i) $\operatorname{div}_\Omega X = 0$, and

(ii) $i_* i_X \Omega = 0$ where $i: \operatorname{bd}(D) \to D$ is the inclusion map.

For the full diffeomorphism group, (i) is omitted.

Condition (ii) means that X is parallel to the boundary. Also, the Lie bracket in T_eG is the negative of the usual Lie bracket of vectorfields. This requires the observation that, under an action, the map taking the Lie algebra to the infinitesimal generator is an antihomomorphism, a standard result [13]. Letting G act naturally on D gives the stated result.

Define a weak Riemannian structure for G or all diffeomorphisms by setting

$$\langle X, Y \rangle = \int_D X \cdot Y \Omega$$

for X, Y vectorfields. Extend by right translation to all of G.[8]

CONJECTURE.[9] *On the regular Lie group G, $\langle , \rangle$ is compatible with G.*

---

[6] On the foliation described in [3].

[7] See [14] for the proof.

[8] For hydrodynamics one must use right invariant metrics, while it is customary to use left invariant ones for the rigid body.

[9] This conjecture is proven in [14].

That G is regular is simple. Namely if $X \in T_eG$, and X has flow F_t, then the map $f \mapsto f \circ F_t$ is the exponential map of X. This is easily seen. One cannot do this for general X which are just in L^2.

For compatibility of $g = \langle\ ,\ \rangle$ one must show that we get a vectorfield X and that it has a local flow. From §1, it is enough to work with the Euler equations. We shall just show how to get the Euler equations Y (Theorem 5). Conjecture 8 is true for the full diffeomorphism group and suitable metrics. Details may be found in [11].

THEOREM 9. *For G described above, the vectorfield Y of Theorem 5 is given by: for each vectorfield X, there are C^∞ functions f and g (unique up to constants) such that*

$$\widetilde{Y(X)} = -i_X d(\tilde{X}) + df = -L_X\tilde{X} + dg$$

where $\tilde{X}$ denotes the one form obtained from X via g_0, i_X is the inner product, and L_X is the Lie derivative. Traditionally p, defined by $p = f - \langle X, X\rangle/2 = g + \langle X, X\rangle/2$ is called the pressure.[10]

PROOF. First, we note that for vectorfields X, X_0, Y that

$$\int_D \langle X, [X_0, Y]\rangle\Omega = \int_D i_{[X_0, Y]}\tilde{X}\Omega = \int_D (i_{[X_0, Y]} + L_Y i_{X_0})\tilde{X}\Omega$$

since $\int_D L_Y(i_{X_0}\tilde{X})\Omega = 0$ by Stoke's theorem and the boundary condition $i_Y\Omega = 0$ on bd(D). But $i_{[X_0, Y]} + L_Y i_{X_0} = i_{X_0}L_Y$ (see [1]), and so we finally obtain

$$\int_D \langle X, [X_0, X]\rangle\Omega = \int_D i_{X_0} i_Y d\tilde{X}\Omega = \int_D \langle Y, L_{X_0}\tilde{X}\rangle\Omega.$$

Of course in this the fact that the vectorfields are divergence free is essential. (On the full diffeomorphism group the equations are a little different.)[11] If this condition were omitted, the pressure term would be absent.

It is a classical theorem (Hodge theory)[12] that a vectorfield Z can be written uniquely

$$Z = Z_0 + \operatorname{grad}(f)$$

where $\operatorname{grad}(f) = df$, and Z_0 and $\operatorname{grad}(f)$ are orthogonal and $\operatorname{div}_\Omega Z_0 = 0$, and Z_0 is parallel to the boundary. (The function f is obtained by solving Laplace's equation $\nabla^2 f = \operatorname{div} Z$.) One easily sees the orthogonality directly, as follows:

For any X in T_eG, observe that $\langle X, \operatorname{grad}(f)\rangle = \int X \cdot \operatorname{grad}(f)\Omega = 0$ since

$$\int X \cdot \operatorname{grad}(f)\Omega = \int L_X f\Omega = \int L_X(f\Omega) = \int di_X(f\Omega) = 0,$$

[10] The equations for Y are usually written in the equivalent form $dv/dt + \nabla_v v = dp$ where ∇ is the covariant derivative and v is an integral curve of Y.

[11] For the right invariant metric on the full diffeomorphism group, the equations are $dv/dt + v(\operatorname{div} v) + L_v v = 0$. The existence problem is unknown for these equations.

[12] Cf. Morrey, *Multiple integrals in the calculus of variations*, Chapter 7 and also [11], [14].

using Stoke's theorem and $i_X\Omega = 0$ or $\mathrm{bd}(D)$. Now for the theorem we let f be such that $i_X d(\tilde{X}) - df$ is divergence free and then observe, by our remarks, that

$$\begin{aligned} -\langle Y(X), X_0\rangle = \langle [X, X_0], X\rangle &= \int_D i_{X_0} i_X d(\tilde{X})\Omega \\ &= \langle i_X d\tilde{X}, X_0\rangle \\ &= \langle i_X d\tilde{X} - \mathrm{grad}(f), X_0\rangle. \end{aligned}$$

Note that $i_X d(\tilde{X})$ might not be parallel to the boundary but $L_X(\tilde{X})$ is, and these differ by a gradient. By nondegeneracy we conclude the result.

The pressure thus constrains the motion to being divergence free.

Observe that the motion of a stationary point X_0 is just its own flow. See Corollary 7. This holds for harmonic vectorfields for example.

Finally we give a theorem classically known as *Kelvin's circulation theorem.* One should note that the theorem as proven in standard hydrodynamics books lacks rigor.

THEOREM 10. *Let $X(t)$ be the velocity vectorfield in T_eG above. Let l be a smooth closed loop in D (a compact 1-manifold) and $l_t = F_t(l)$, where F_t is the motion of the fluid (geodesic). Then $\int_{l_t} \tilde{X}(t)$ is independent of t.*

PROOF. Since F_t is a diffeomorphism,

$$\int_{l_t} \tilde{X}(t) = \int_l F_t^* \tilde{X}(t).$$

Now $d(F_t^*\tilde{X}(t))/dt = F_t^* L_X \tilde{X} + F_t^* d\tilde{X}/dt$, which by Theorem 9 equals $F_t^* dg = d(F_t^* g)$. Thus by Stoke's theorem,

$$\frac{d}{dt}\int_{l_t} \tilde{X}(t) = \int_l d(F_t^* g) = 0. \blacksquare$$

Curiously this is not true for the geodesic flow on the full diffeomorphism group.

This theorem is quite analogous to the circulation theorem for mechanics. It also holds for TG, so we give it

THEOREM 11. *Let M be a (weak) symplectic manifold with form $\omega = d\theta$. Let F_t be a smooth Hamiltonian flow (or local flow) on M. Let A be a compact two manifold in M with boundary $l = \mathrm{bd}(A)$, and $l_t = F_t(l)$. Then $\int_{l_t} \theta$ is independent of t.*

PROOF. By Stoke's theorem,

$$\int_{l_t} \theta = \int_{A_t} \omega = \int_A F_t^* \omega = \int_A \omega,$$

since $F_t^*\omega = \omega$.

REFERENCES

1. R. Abraham and J. Marsden, *Foundations of mechanics*, Benjamin, New York, 1967.

2. R. Abraham, *Lectures on global analysis*, mimeographed, Princeton Univ. Press, Princeton, N.J., 1968.

3. V. Arnold, *Sur la geometric differentialle des groupes de Lie de dimension infinite et ses applications à l'hydrodynamique de fluides parfaits*, Ann. Inst. Fourier, (Grenoble), **16** (1966).

4. E. Blancheton, *Mechanique analytique des milieus continus*, Ann. Inst. H. Poincaré, Sect. A, **7** (1967), 189–213.

5. T. Kato, *On classical solutions of the two dimensional non-stationary Euler equation*, Arch. Rational Mech. Anal. **25** (1967), 188–200.

6. O. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York, 1963.

7. J. Leslie, *On a differential structure for the group of diffeomorphisms*, Topology, **6** (1967), 263–271.

8. J. Marsden, *Hamiltonian one parameter groups*, Arch. Rational Mech. Anal. **28** (1968), 362–396.

9. ———, *Generalized Hamiltonian mechanics*, Arch. Rational Mech. Anal. **28** (5), (1968), 323–361.

10. ———, *A Banach space of analytic functions for constant coefficient equations of evolution*, Canad. Math. Bull. **11** (1968).

11. ———, *Hamiltonian mechanics, infinite dimensional Lie groups, geodesic flows and hydrodynamics*, mimeographed lecture notes, University of California, Berkeley.

12. H. Omori, *On the group of diffeomorphisms on a compact manifold* (preprint).

13. P. Tondeur, *Introduction to Lie groups and transformation groups*, Springer Lecture Note Series, (1965).

14. D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid* (to appear).

UNIVERSITY OF CALIFORNIA, BERKELEY AND SANTA CRUZ