MATCHING AND STABILIZATION OF THE UNICYCLE WITH RIDER

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Abstract: In this paper we apply matching techniques for controlled Lagrangians to the stabilization problem of a nonholonomic system consisting of a unicycle with rider. We show how generalized matching results may be applied to the Routhian associated with this nonholonomic system.

Keywords: Feedback stabilization, Lyapunov methods, Nonlinear control

1. INTRODUCTION

In this paper we apply the method of controlled Lagrangians to the stabilization of slow vertical steady state motions of the unicycle with rider. The controlled Lagrangian approach for stabilization was introduced in Bloch, *et al.* (1997) for underactuated holonomic systems with the control force acting along the symmetry directions. Later on this approach was extended to handle certain systems with broken symmetry, see (Bloch, *et al.*, 1999; Auckly, *et al.*, 1998; Hamberg, 1999).

This method requires that specific *matching conditions* are satisfied. These conditions allow one to introduce a *controlled Lagrangian* and to rewrite the equations for the controlled (forced) system as the Euler-Lagrange equations for this controlled Lagrangian.

The method proposed here extends the technique of controlled Lagrangians to a class of nonholonomic systems and gives a systematic procedure for control design in both the linear and the nonlinear settings.

2. NONHOLONOMIC MATCHING

The system considered here, the unicycle with rider, is modeled by a homogeneous disk that moves on a horizontal plane without slipping and has a mass and a pendulum attached. The pendulum is free to move in the plane orthogonal to the disk, while the attached mass stays in the disk's plane. In this system only the sideways motion of the rider (such as the rider's limbs) is modeled and not any pedaling control. The configuration space is $Q = S^1 \times S^1 \times S^1 \times SE(2)$, which we parametrize with coordinates $(r^1, r^2, \psi, \phi, x, y)$, as in Figure 1. This mechanical system is $SO(2) \times SE(2)$ invariant; the group

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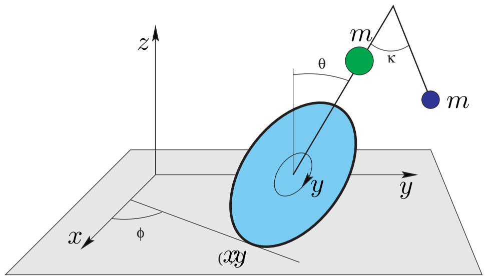

Fig. 1. The configuration variables for the unicycle with rider.

that is in the ψ variable, while the group $SE(2)$ represents the Euclidean symmetry of the overall system.

The equations of motion with a control torque u on the pendulum are those derived in the standard way from the Lagrange-d'Alembert principle:

$$\frac{d}{dt}\frac{\partial \mathcal{R}}{\partial \dot r^1} = \nabla_1 \mathcal{R}, \qquad \frac{d}{dt}\frac{\partial \mathcal{R}}{\partial \dot r^2} = \nabla_2 \mathcal{R} + u, \tag{1}$$

$$\frac{dp_1}{dt} = \mathcal{D}^b_{1\alpha} p_b \dot r^\alpha, \qquad \frac{dp_2}{dt} = \mathcal{D}^b_{2\alpha} p_b \dot r^\alpha, \tag{2}$$

where $\mathcal{R} = \frac{1}{2} g_{\alpha\beta} \dot r^\alpha \dot r^\beta - U(r,p)$ is the *Routhian*, U is the *amended potential*, (p_1, p_2) is the nonholonomic momentum where p_1 is conjugate to ϕ, p_2 is conjugate to ψ, and the covariant derivatives in the shape equations (1) are defined by

$$\nabla_\alpha = \partial_{r^\alpha} + \mathcal{D}^b_{a\alpha} p_b \partial_{p_a}.$$

See (Zenkov, *et al.*, 1998, 1999) and references therein. The full dynamics is governed by equations (1) and (2) coupled with the *reconstruction equation* for the group variables ψ, ϕ, x, y. This reconstruction equation is not needed here as it does not affect the evolution of the shape and the momentum variables, and thus is not used in our stabilization analysis.

Our key observation is that the vertical steady state motions of the unicycle

$$r^1 = 0, \quad r^2 = 0, \quad p_1 = 0, \quad p_2 = c \tag{3}$$

for each value of c are dynamically equivalent to the equilibria of an auxiliary holonomic system, the inverted double pendulum. The Lagrangian for this auxiliary system is the Routhian of the original system restricted to the level set $p = (0, c)$ of the nonholonomic momentum. Applying holonomic matching techniques, we obtain the *controlled metric* $\widetilde g_{\alpha\beta}$ and the *controlled amended potential* $\widetilde U$ and form the *controlled Routhian* $\widetilde{\mathcal{R}} = \frac{1}{2}\widetilde g_{\alpha\beta}\dot r^\alpha \dot r^\beta - \widetilde U$. The *controlled energy* corresponding to this controlled Routhian can be chosen, with appropriate choices of gains, to be positive definite at equilibrium (3). We then show that there exist *controlled covariant derivatives* $\widetilde\nabla_\alpha$, such that the equations

coupled with (2) are equivalent to the original equations (1) and (2). Equations (2) and (4) linearized at (3) have two zero and four pure imaginary eigenvalues. By adding appropriate dissipative terms to the control input, we force the four nonzero eigenvalues to the left half plane. By the Lyapunov-Malkin theorem, the slow vertical motion of the unicycle (3) (with small c) becomes orbitally stable. See (Zenkov, *et al.*, 1998, 1999) for the details on the Lyapunov-Malkin theorem and on the dissipative terms.

The explicit formula for the control u is

$$u = \nabla_2 U - g_{2\beta}\widetilde g^{\alpha\beta}\widetilde\nabla_\alpha \widetilde U - g_{2\gamma}(\widetilde\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta})\dot r^\alpha \dot r^\beta + \{\text{dissipative terms}\},$$

where $\Gamma^\gamma_{\alpha\beta}$ and $\widetilde\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of the metrics $g_{\alpha\beta}$ and $\widetilde g_{\alpha\beta}$.

3. CONCLUSION

The method developed here extends the matching technique to the class of nonholonomic systems with no curvature terms in the shape equation and the momentum equation in the form of a parallel transport equation. We intend in a future publication to consider more general nonholonomic systems, in particular with control inputs acting along some of the symmetry directions as well.

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