AN ALMOST POISSON STRUCTURE FOR THE GENERALIZED RIGID BODY EQUATIONS

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Abstract: In this paper we introduce almost Poisson structures on Lie groups which generalize Poisson structures based on the use of the classical Yang-Baxter identity. Almost Poisson structures fail to be Poisson structures in the sense that they do not satisfy the Jacobi identity. In the case of cross products of Lie groups, we show that an almost Poisson structure can be used to derive a system which is intimately related to a fundamental Hamiltonian integrable system — the generalized rigid body equations. *Copyright© 2000 IFAC*

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1. INTRODUCTION

The theory of Poisson-Lie groups is of great interest in the theory of both classical and quantum systems. The basic theory of Poisson manifolds is discussed, for example, in Weinstein [1983], while the theory of Poisson-Lie groups (groups endowed with a Poisson structure) was developed in the work of Drinfeld [1983], Semenov-Tian-Shansky [1985], and Lu and Weinstein [1990], among other works.

A key aspect of much of this work is the so-called modified Yang Baxter equation. Operators satisfying this equation are classical r-matrices and may be used to define Poisson structures on Lie groups. This yields, for example, the Sklyanin bracket which is used in Deift and Li [1991] to analyze the Toda lattice flow and its relationship to the SVD (singular value decomposition) flow.

To our knowledge no such Poisson-Lie structure has been found for other key classical integrable systems such as the generalized rigid body equations. In this paper we define a structure on a group which is not Poisson-Lie, but is very close to being Poisson in a way which we will make precise and which yields the generalized rigid body equations in a new form. This structure is based on a form of the equations originally derived in the work of Bloch and Crouch [1996] and Bloch, Brockett, and Crouch [1997] and inspired by op-

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timal control problems (see also Brockett [1994]). A connection of this work to discrete rigid body equations may be found in Bloch, Crouch, Marsden, and Ratiu [1998], [2000].

2. PRELIMINARY DEFINITIONS AND BACKGROUND THEORY

In this section we introduce some notation and review some principal results on Poisson structures. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\langle\ ,\ \rangle$ be an Ad-invariant inner product on $\mathfrak{g}$. A key object in defining Poisson structures on groups is the notion of left and right derivatives of C^∞ functions on G (see Semenov-Tian-Shansky [1985] and Deift and Li [1991]).

Definition 1. Let $\phi \in C^\infty(G)$. Then the ***right derivative*** $D_R\phi_g^\#$ of ϕ at g is given by

$$D_R\phi_g(X) = \left.\frac{d}{dt}\right|_{t=0} \phi(ge^{tX}) = \langle D_R\phi_g^\#, X\rangle \quad (2.1)$$

where $X \in \mathfrak{g}$, $D_R\phi_g \in \mathfrak{g}^*$ and $D_R\phi_g^\# \in \mathfrak{g}$. Similarly, the ***left derivative*** of $\phi \in C^\infty(G)$ is given by

$$D_L\phi_g(X) = \left.\frac{d}{dt}\right|_{t=0} \phi(e^{tX}g) = \langle D_L\phi_g^\#, X\rangle \quad (2.2)$$

where $X \in \mathfrak{g}$, $D_L\phi_g \in \mathfrak{g}^*$ and $D_L\phi_g^\# \in \mathfrak{g}$.

Let $Ad_g : \mathfrak{g} \to \mathfrak{g}$, $Ad_g(X) = \frac{d}{dt} ge^{tX}g^{-1}|_{t=0}$ denote the adjoint action of G on its Lie algebra (see, e.g., Marsden and Ratiu [1994]). Thus $Ad_g = L_{g*}R_{g^{-1}*}$ where L_{g*} and R_{g*} denote the left and right action of G on its tangent bundle respectively; L_g and R_g are the left and right actions of the group on itself. Then we have:

Lemma 2.

$$D_R\phi_g \circ Ad_{g^{-1}} = D_L\phi_g \quad (2.3)$$

$$Ad_g(D_R\phi_g^\#) = (D_L\phi_g^\#)\,. \quad (2.4)$$

Proof. We have

$$D_R\phi_g(X) = \left.\frac{d}{dt}\right|_{t=0} \phi(L_g e^{tx}) = \phi_* L_{g*}X$$
$$D_L\phi_g(X) = \left.\frac{d}{dt}\right|_{t=0} \phi(R_g e^{tx}) = \phi_* R_{g*}X\,.$$

Hence $D_R\phi_g \circ Ad_{g^{-1}} = D_L\phi_g$ proving 2.3. Thus

$$\langle D_R\phi_g^\#, Ad_{g^{-1}}X\rangle = \langle D_L\phi_g^\#, X\rangle$$

or

$$\langle Ad_g D_R\phi_g^\#, X\rangle = \langle D_L\phi_g^\#, X\rangle$$

by Ad-invariance of the inner product. This gives (2.4). ■

Now in order to check the Jacobi identity in our Poisson bracket we also need the second derivatives of functions $\phi \in C^\infty(G)$. We make the following (natural) definitions:

Definition 3. Let $X, Y \in \mathfrak{g}$. We define

$$D_R^2\phi_g(X,Y) = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \phi(ge^{sY}e^{tX}) \quad (2.5)$$

$$D_L^2\phi_g(X,Y) = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \phi(e^{tX}e^{sY}g) \quad (2.6)$$

$$D_R D_L\phi_g(X,Y) = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \phi(e^{tX}ge^{sY}) \quad (2.7)$$

$$D_L D_R\phi_g(X,Y) = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \phi(e^{sY}ge^{tX})\,. \quad (2.8)$$

Then we have

Lemma 4.

(i) $D_R D_L\phi_g(X,Y) = D_L D_R\phi_g(Y,X)$ (2.9)

(ii) $D_R^2\phi_g(X,Y) - D_R^2\phi_g(Y,X) = D_R\phi_g([X,Y])$ (2.10)

(iii) $D_L^2\phi_g(X,Y) - D_L^2\phi_g(Y,X) = D_L\phi_g([Y,X])$. (2.11)

Proof. (i) follows directly from the definitions. To prove (ii) and (iii) note that $(D_R\phi_g)(X) = X_g^L(\phi)$ and $(D_L\phi_g)(X) = X_g^R(\phi)$ where $X_g^L = L_{g*}X$ and $X_g^R = R_{g*}X$ are the left- and right-invariant vector fields determined by $X \in \mathfrak{g}$, respectively. Thus, $D_R^2\phi_g(X,Y) = X_g^L(Y^L(\phi))$ and, by a similar argument, $D_R^2\phi_g(Y,X) = Y_g^L(X^L(\phi))$. This gives (ii) and (iii) follows in the same way. ■

2.1 *The Poisson Bracket Structure*

We now introduce (see Semenov-Tian-Shansky [1985], Deift and Li [1991]) an important class of Poisson structures on Lie groups via classical r-matrices, or, more specifically, via operators satisfying the Yang-Baxter identity (sometimes called the modified Yang-Baxter equation). Operators of this type yield a Poisson bracket usually known as the Sklyanin bracket.

Definition 5. A linear operator A is said to satisfy the ***Yang-Baxter identity*** if $A : \mathfrak{g} \to \mathfrak{g}$ satisfies

(i) $[AX, AY] - A([AX,Y] + [X,AY]) = [X,Y]$

(ii) $\langle AX, Y\rangle + \langle X, AY\rangle = 0\,.$ (2.12)

An operator A is called a ***classical*** r***-matrix*** if

$$[X,Y]_A = \frac{1}{2}([AX,Y]+[X,AY]) \tag{2.13}$$

defines a Lie bracket on $\mathfrak{g}$. $(\mathfrak{g},[\cdot,\cdot]_A)$ is called a ***Baxter Lie algebra***.

Semenov-Tian-Shansky showed the following:

Lemma 6. An operator satisfying the Yang-Baxter identity is a classical r-matrix.

Lemma 7. If A satisfies the Yang-Baxter identity, then

$$\{\phi,\psi\}_S = \frac{1}{2}(AD_R\phi^\#, D_R\phi^\#) - \frac{1}{2}(AD_L\phi^\#, D_L\phi^\#) \tag{2.14}$$

defines a Poisson structure on G called the ***Sklyanin bracket***.

Deift and Li [1991] used an extension of this bracket to analyze the Toda lattice equations. This extension employed operators satisfying the Yang-Baxter identity. Here we describe a generalization of the Sklyanin bracket making explicit the exact role of the Yang-Baxter identity in the proof of the Jacobi identity for the corresponding Poisson structure.

Definition 8. A linear mapping $J:\mathfrak{g}\to\mathfrak{g}$ is called an ***equivariant Yang-Baxter operator*** if, for some smooth function F_J,

$$[JX,JY] - J[JX,Y] - J[X,JY] = F_J(X,Y),$$

where

$$Ad_g F_J(X,Y) = F_J(Ad_g X, Ad_g Y)$$

and if

$$\langle JX,Y\rangle + \langle X,JY\rangle \equiv 0\,. \tag{2.15}$$

We have the following result.

Theorem 9. Let J_1 and J_2 be equivariant Yang-Baxter operators with $F_{J_1}=F_{J_2}\equiv F_J$. Then

$$\{\phi,\psi\}^\pm = \frac{1}{2}(\langle D_L\phi^\#, J_1 D_L\psi^\#\rangle \pm \langle D_R\phi^\#, J_2, D_R\psi^\#\rangle) \tag{2.16}$$

is a Poisson bracket on G.

Proof. The bracket is clearly skew symmetric and bilinear. To prove the Jacobi identity, proceed as follows:

$$\begin{aligned}
&4\{\phi_1,\{\phi_2,\phi_3\}^\pm\}^\pm \\
&= 2\langle D_L\phi_1^\#, J_1(D_L\{\phi_2,\phi_3\}^\pm)\rangle \\
&\quad \pm 2\langle D_R\phi_1^\#, J_2(D_R\{\phi_2,\phi_3\}^\pm)^\#\rangle \\
&= \langle D_L\phi_1^\#, J_1(D_L\langle D_L\phi_2^\#, J_1 D_L\phi_3^\#\rangle)^\#\rangle \\
&\quad \pm \langle D_L\phi_1^\#, J_1(D_L\langle D_R\phi_2^\#, J_2 D_R\phi_3^\#\rangle)^\#\rangle \\
&\quad \pm \langle D_R\phi_1^\#, J_2(D_R\langle D_L\phi_2^\#, J_1, D_L\phi_3^\#\rangle)^\#\rangle \\
&\quad + \langle D_R\phi_1^\#, J_2(D_R\langle D_R\phi_2^\#, J_2 D_R\phi_3^\#\rangle)^\#\rangle\,.
\end{aligned}$$

Since J_1 and J_2 are skew and $\langle D_{L(R)}\phi^\#, X\rangle = D_{L(R)}\phi(X)$ by definition, we may rewrite this expression as

$$\begin{aligned}
4\{\phi_1,\{\phi_2,\phi_3\}^\pm\}^\pm = &-D_L^2\phi_2(J_1D_L\phi_3^\#, J_1D_L\phi_1^\#) \\
&+ D_L^2\phi_3(J_1D_2\phi_2^\#, J_1D_L\phi_1^\#) \\
&- D_R^2\phi_2(J_2D_R\phi_3^\#, J_2D_R\phi_1^\#) \\
&+ D_R^2\phi_3(J_2D_R\phi_2^\#, J_2D_R\phi_1^\#) \\
&\mp D_LD_R\phi_2(J_2D_R\phi_3^\#, J_1D_L\phi_1^\#) \\
&\pm D_LD_R\phi_3(J_2D_R\phi_2^\#, A_1D_2\phi_1^\#) \\
&\mp D_RD_L\phi_2(J_1D_L\phi_3^\#, J_2D_R\phi_1^\#) \\
&\pm D_RD_L\phi_3(J_1D_L\phi_2^\#, J_2D_R\phi_1^\#)\,.
\end{aligned} \tag{2.17}$$

We need to show that this expression plus its cyclic permutations is identically zero. To do this, represent the first two terms in (2.17) in obvious fashion as $-(2,3,1)+(3,2,1)$. This plus its permutations is the expression

$$\begin{aligned}
&-(2,3,1)+(3,2,1)\\
&-(3,1,2)+(1,3,2)\\
&-(1,2,3)+(2,1,3)\,.
\end{aligned}$$

Using Lemma 4 (iii) this equals

$$\begin{aligned}
&\langle D_L\phi_2^\#, [J_1D_L\phi_3^\#, J_1D_L\phi_1^\#]\rangle \\
&\quad + \langle D_L\phi_1^\#, [J_1D_L\phi_2^\#, J_1D_L\phi_3^\#]\rangle \\
&\quad + \langle D_L\phi_3^\#, [J_1D_L\phi_1^\#, J_1D_L\phi_2^\#]\rangle\,.
\end{aligned}$$

Now use the facts that J_1 is skew and $\langle [X,Y],Z\rangle + \langle Y,[X,Z]\rangle = 0$ to rewrite this as

$$\begin{aligned}
&\langle D_L\phi_1^\#, [J_1D_L\phi_2^\#, J_1D_L\phi_3^\#]\rangle \\
&-\langle D_L\phi_1^\#, J_1[D_L\phi_2^\#, J_1D_L\phi_3^\#]\rangle \\
&-\langle D_L\phi^\#, J_1[J_1D_L\phi_2^\#, D_L\phi_3^\#]\rangle\,.
\end{aligned} \tag{2.18}$$

Since J_1 is an equivariant Yang-Baxter operator this equals

$$-\langle D_L\phi_1^\#, F_J(D_L\phi_2^\#, D_L\phi_3^\#)\rangle\,. \tag{2.19}$$

Similarly, now using Lemma 4 (ii), we obtain for the second two terms of (2.17),

$$+\langle D_R\phi_1^\#, F_J(D_R\phi_2^\#, D_R\phi_3^\#)\rangle\,. \tag{2.20}$$

By Lemma 2, we have $(D_L\phi_g)^\# = Ad_g(D_R\phi_g^\#)$. Thus, since $\langle\ ,\ \rangle$ is Ad-invariant and F_J is Ad-equivariant, the cyclic sum of terms 1 and 2 in (2.17) is identically zero.

Now consider the last four terms of (2.17). By (i) of Lemma 4 we may rewrite these terms as

$$\begin{array}{l} \mp D_L D_R \phi_2 (J_2 D_R \phi_3^{\#}, J_1 D_L \phi_1^{\#}) \\ \pm D_L D_R \phi_3 (J_2 D_R \phi_2^{\#}, J_1 D_L \phi_1^{\#}) \\ \mp D_L D_R \phi_2 (J_2 D_R \phi_1^{\#}, J_1 D_L \phi_3^{\#}) \\ \pm D_L D_R \phi_3 (J_2 D_R \phi_1^{\#}, J_1 D_L \phi_2^{\#}) \,. \end{array}$$

Again, represent this as

$$\mp(2,3,1) \pm (3,2,1) \mp (2,1,3) \pm (3,1,2).$$

Then the cyclic sum is

$$\begin{array}{l} \mp(2,3,1) \pm (3,2,1) \mp (2,1,3) \pm (3,1,2) \\ \mp(3,1,2) \pm (1,3,2) \mp (3,2,1) \pm (1,2,3) \\ \mp(1,2,3) \pm (2,1,3) \mp (1,3,2) \pm (2,3,1) \end{array}$$

which is identically zero. Hence the Jacobi identity is satisfied. ■

The proof of lemma 7 follows directly from this result by noting that $F_J(X,Y) = [X,Y]$, is clearly Ad-invariant.

3. DYNAMICS WITH RESPECT TO AN ALMOST POISSON STRUCTURE

In this section we consider dynamics on groups under the bracket (2.16), but in the case where F_J is generally *not* Ad_g equivariant – such structures will be called ***almost Poisson structures*** on Lie groups. We begin by considering the general case and then specialize to the case of particular interest to us – cross products of Lie groups.

Such a bracket is an instance of an *almost* Poisson structure (see da Silva and Weinstein [1999]):

Definition 10. An ***almost Poisson manifold*** is a pair $(M, \{,\})$ where M is a smooth manifold and (i) $\{,\}$ defines an ***almost Lie algebra structure*** on the C^∞ functions on M, i.e. the bracket satisfies all conditions for a Lie algebra except that the Jacobi identity is not satisfied and (ii) $\{,\}$ is a derivation in each factor.

Given an almost Poisson structure with local coordinate expression $\pi^{ij}(z)$ (the generalization of the Poisson tensor) and a function H on M, one defines an ***almost Poisson*** vector field on M by

$$\dot{z}^i = \pi^{ij}(z) \frac{\partial H}{\partial z^j} \,. \tag{3.1}$$

Now in our setting we have:

Theorem 11. The equation

$$\dot{\phi} = \{\phi, H\}^{\pm} \tag{3.2}$$

where $\{\ ,\ \}^{\pm}$ is the bracket (2.16) on a Lie group G yields the equation

$$\dot{g} = \frac{1}{2} R_{g*} J_1 D_L H^{\#} \pm \frac{1}{2} Lg_* J_2 D_R H^{\#} \,. \tag{3.3}$$

If $H \in I(G)$, the functions on G invariant under conjugation, then $D_L H = D_R H = DH$ and (3.2) becomes

$$\dot{g} = \frac{1}{2} R_{g^*} J_1 DH^{\#} \pm \frac{1}{2} L_{g^*} J_2 DH^{\#} \tag{3.4}$$

Proof. Equation (3.2) may be written as

$$\frac{d}{dt}\phi(g) = \frac{1}{2} D_L \phi (J_1 D_L H^{\#}) \pm \frac{1}{2} D_R \phi (J_2 D_R H^{\#}).$$

But

$$D_L \phi = \phi_* R_{g^*}, D_R \phi = \phi_* L_{g^*} \text{ and } \frac{d}{dt}\phi(g) = \phi_* \dot{g}.$$

Hence we obtain (3.3).

Now, if $\phi \in I(G)$, i.e., $\phi(g) = \phi(hgh^{-1})$, for all $h \in G$, then $\phi_* X = \phi_* Ad_h X$, for all $h \in G$, $X \in \mathfrak{g}$. Hence $\phi_* = \phi_* Ad_h$ or $\phi_* R_{h_*} = \phi_* L_{h_*}$ and thus

$$D_L \phi_g = \phi_* R_{g_*} = \phi_* L_{g_*} = D_R \phi_g \,. \quad ■$$

We now consider the case of interest to us, where the Lie group is

$$D = G \times G \tag{3.5}$$

with Lie algebra $\mathfrak{g} \times \mathfrak{g} \equiv \mathfrak{g}^D$. We define the map $J : \mathfrak{g}^D \to \mathfrak{g}^D$ given by

$$J(X_1, X_2) = (X_2, -X_1) \tag{3.6}$$

for $(X_1, X_2) \in \mathfrak{g}^D$.

We shall shortly define a bracket using J but note that J does *not* define a Poisson structure, for while J is clearly skew, if $X, Y \in \mathfrak{g}^D$

$$\begin{aligned} F_J(X,Y) &= [JX, JY] - J[JX, Y] - J[X, JY] \\ &= \begin{pmatrix} [X_1 + X_2, Y_1 + Y_2] \\ [X_1 + X_2, Y_1 + Y_2] \end{pmatrix} \\ &\quad - \begin{pmatrix} [X_1, Y_1] \\ [X_2, Y_2] \end{pmatrix} \end{aligned}$$

which is clearly not Ad_g invariant.

We now make the following definitions in this setting in order to set up the equations of motion:

Definition 12. Let e be the identity in G, $g \in D$, $(X_1, X_2) \in \mathfrak{g}^D$, then we define

$$\begin{aligned} (D_R^1 \phi)_g (X_1) &= \frac{d}{dt}\Big|_{t=0} \phi(g(e^{tX_1}, e)) \\ (D_R^2 \phi)_g (X_2) &= \frac{d}{dt}\Big|_{t=0} \phi(g(e, e^{tX_2})) \\ (D_L^1 \phi)_g (X_1) &= \frac{d}{dt}\Big|_{t=0} \phi((e^{tX_1}, e) g) \\ (D_L^2 \phi)_g (X_2) &= \frac{d}{dt}\Big|_{t=0} \phi((e, e^{tX_2}) g) \\ (D_R \phi)_g (X_1, X_2) &= ((D_R^1 \phi)_g, (D_R^2 \phi)_g)(X_1, X_2) \\ (D_L \phi)_g (X_1, X_2) &= ((D_L^1 \phi)_g, (D_L^2 \phi)_g)(X_1, X_2) \,. \end{aligned}$$

Using the inner product on $\mathfrak{g}$ and $\mathfrak{g}^D$ we get obvious definitions of $(D_L^1 \phi_g)^{\#} \in \mathfrak{g}$, etc. Then the + bracket (2.16) becomes

$$\{\phi, \psi\} = \frac{1}{2}\langle D_L^1\phi^\#, D_L^2\psi^\#\rangle - \frac{1}{2}\langle D_L^2\phi^\#, D_L^1\psi^\#\rangle$$
$$+\frac{1}{2}\langle D_R^1\phi^\#, D_R^2\psi^\#\rangle$$
$$-\frac{1}{2}\langle D_R^2\phi^\#, D_R^1\psi^\#\rangle \,. \qquad (3.7)$$

Using (3.3), we have

Corollary 13. The equations $\dot{\phi} = \{\phi, H\}$ induced by the bracket (3.7) on D are given by

$$\dot{g}_1 = \frac{1}{2}R_{g_1^*}(D_L^2 H_{(g_1,g_2)})^\# + \frac{1}{2}L_{g_1^*}(D_R^2 H_{(g_1,g_2)})^\#$$
$$\dot{g}_2 = -\frac{1}{2}R_{g_2^*}(D_L^1 H_{(g_1,g_2)})^\# - \frac{1}{2}L_{g_2^*}(D_R^1 H_{(g_1,g_2)})^\# \,. \qquad (3.8)$$

We now prove the main result of the paper, which provides a special case of these equations and turns out to be intimately related to the generalized rigid body equations on $so(n)$.

Theorem 14. Consider the flow (3.8) in the case

$$H = 4\,\mathrm{Trace}(I_1 g_2^{-1} g_1 + I_2 g_1 g_2^{-1}) \qquad (3.9)$$

where I_1 and I_2 are symmetric, positive definite matrices. Let J_1 and J_2 be defined by

$$J_1^{-1}(X) = I_1 X + X I_1$$
$$J_2^{-1}(X) = I_2 X + X I_2.$$

Then the flow (3.8) is given by

$$\dot{g}_1 = J_2^{-1}(g_2 g_1^{-1} - g_1 g_2^{-1})g_1 - g_1 J_1^{-1}(g_2^{-1}g_1 - g_1^{-1}g_2)$$
$$\dot{g}_2 = J_2^{-1}(g_2 g_1^{-1} - g_1 g_2^{-1})g_2$$
$$- g_2 J_1^{-1}(g_2^{-1}g_1 - g_1^{-1}g_2)\,. \qquad (3.10)$$

Proof. Let $g = (g_1, g_2)$ and compute as follows:

$$D_R^2 H_g(X) = 4\,\mathrm{Trace}(-I_1 X g_2^{-1} g_1 - I_2 g_1 X g_2^{-1})$$
$$= 4\,\mathrm{Trace}(-J_1^{-1}(X) g_2^{-1} g_1 + X I_1, g_2^{-1} g_1 - g_2^{-1} I_2 g_1 X)$$
$$= 4\,\mathrm{Trace}(-X J_1^{-1}(g_2^{-1} g_1) + X I_1, g_2^{-1} g_1 - g_2^{-1} I_2 g_1 X)$$
$$= 2\,\mathrm{Trace}(-X J_1^{-1}(g_2^{-1} g_1 - g_1^{-1} g_2)$$
$$+ X(I_1 g_2^{-1} g_1 - g_1^{-1} g_2 I_1) + (g_1^{-1} I_2 g_2 - g_2^{-1} I_2 g_1) X)$$

since $X \in so(n)$ and $g_1 g_2 \in SO(n)$. Therefore,

$$(D_R^2 H_g)^\# = -2 J_1^{-1}(g_2^{-1} g_1 - g_1^{-1} g_2)$$
$$+ 2(I_1 g_2^{-1} g_1 - g_1^{-1} g_2 I_1) + 2(g_1^{-1} I_2 g_2 - g_2^{-1} I_2 g_1).$$

Computing similarly we find:

$$(D_L^2 H_g)^\# = -2 J_2^{-1}(g_1 g_2^{-1} - g_2 g_1^{-1})$$
$$+ 2(g_1 g_2^{-1} I_2 - I_2 g_2 g_1^{-1}) + 2(g_2 I_1 g_1^{-1} - g_1 I_1 g_2^{-1})$$
$$(D_L^1 H_g)^\# = 2 J_2^{-1}(g_1 g_2^{-1} - g_2 g_1^{-1})$$
$$+ 2(g_2 g_1^{-1} I_2 - I_2 g_1 g_2^{-1}) + 2(g_1 I g_2^{-1} - g_2 I_1 g_1^{-1})$$
$$(D_R^1 H_g)^\# = 2 J_1^{-1}(g_2^{-1} g_1 - g_1^{-1} g_2)$$
$$+ 2(I_1 g_1^{-1} g_2 - g_2^{-1} g_1 I_1) + 2(g_2^{-1} I_2 g_1 - g_1^{-1} I_2 g_2)\,.$$

Substitution into (3.8) gives the result. ■

4. THE RIGID BODY EQUATIONS

In this last section we relate the system (3.10) to the rigid body system on $SO(n)$. We show

Theorem 15. The equations (3.10) in the case $J_1^{-1} = 0$, $J_2 = J$ are locally equivalent to the generalized rigid body equations.

To do this, we recall some background information from Bloch, Brockett, and Crouch [1997] (see also Bloch, Crouch, Marsden, and Ratiu [1998, 2000]). We recall that the rigid body equations on $SO(n)$ (or generally on any compact Lie group – see e.g. Marsden and Ratiu [1994], Ratiu [1980]) may be written as

$$\dot{Q} = \Omega Q$$
$$\dot{M} = [\Omega, M] \qquad (4.1)$$

where $Q \in SO(n)$ denotes the configuration space variables, $\Omega \in so(n)$ is the angular velocity, and $M = J\Omega = \Lambda\Omega + \Omega\Lambda$ is the angular momentum. Here J is a symmetric positive definite operator defined by the diagonal positive definite matrix Λ. We remark that the rigid body equations here are written in right-invariant as opposed to the commonly used left-invariant form in order to be consistent with the conventions used in the remainder of the paper. This results in a sign change in the second of equations (4.1). The classical rigid body equations (4.1) are of course Hamiltonian on $T^*SO(n)$ with respect to the canonical symplectic structure. We now consider the following equations:

$$\dot{Q} = \Omega Q$$
$$\dot{P} = \Omega P, \qquad (4.2)$$

where $\Omega = J^{-1}M$ and $M = PQ^T - QP^T$ for $Q, P \in SO(n)$. We then can easily check that:

Proposition 16. The mapping $(Q, P) \mapsto (Q, M)$, from $SO(n) \times SO(n)$ to $T^*SO(n)$ takes all solutions of equation (4.2) onto solutions of the generalized rigid body equations (4.1).

Proof. Differentiating $M = PQ^T - QP^T$ and using the equations (4.2) gives the second of equations (4.1). ■.

Conversely, given the rigid body equations (4.1) we may solve for the variable P in the expression

$$M = PQ^T - QP^T$$

in a neighborhood of $M = 0$. Locally, in a neighborhood of $M = 0$, where $\sinh^{-1}$ is well defined

$$P = \left(e^{\sinh^{-1} M/2}\right) Q\,. \qquad (4.3)$$

This follows from the observation that

$$M = e^{\sinh^{-1} M/2} - e^{-\sinh^{-1} M/2}.$$

For $so(n)$, however, sinh is many to one, so the two representations are not entirely equivalent. (For more details giving the precise region for M for which one can solve for P, see Bloch, Crouch, Marsden, and Ratiu [2000]. This reference also contains a discussion of how this system is related to certain optimal control problems.)

Observe, however, that this proves Theorem 15, as this new form of the rigid body equation is exactly of the type given to us by our almost Poisson structure, and the system of equations (3.10), under the conditions of theorem 15

We would like to say some more, however, about the Hamiltonian structure of these equations, again following Bloch, Brockett, and Crouch [1997].

Note firstly that the generalized rigid body flow naturally reduces to a flow in the variable M on an adjoint orbit of $so(n)$ and we can view the map which takes $PQ^T - QP^T$ to M as reduction. In fact, the map $(Q, P) \mapsto (Q, M)$ given above is a canonical transformation from the symplectic structure on $T^*gl(n)$ to that on $T^*SO(n)$ which intertwines the Hamiltonian equations (4.2) on $T^*gl(n)$ with the Hamiltonian equations (4.1) on $T^*SO(n)$.

While the classical rigid body equations (4.1) are Hamiltonian on $T^*SO(n)$ with respect to the canonical symplectic structure, on the group we have

Proposition 17. The generalized rigid body equations in the form (4.2) are Hamiltonian on $T^*gl(n)$ with respect to the canonical symplectic structure and the Hamiltonian

$$H = \frac{1}{4}\left\langle J^{-1}(PQ^T - QP^T), PQ^T - QP^T\right\rangle, \tag{4.4}$$

where $\langle \xi, \eta \rangle = \mathrm{Trace}(\xi^T \eta)$.

This is a straightforward computation.

We remark that here P and Q are natural coordinates for $T^*gl(n)$ and, for $P(0), Q(0) \in SO(n)$, $P(t)$ and $Q(t)$ evolve in $SO(n)$ under the flow of H. Hence $SO(n) \times SO(n)$ is an invariant manifold for the flow of H. Note also that this Hamiltonian is equivalent to $H = (1/4)\left\langle J^{-1}M, M\right\rangle$, as in Ratiu [1980].

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