Representations of Dirac Structures and Implicit Port-Controlled Lagrangian Systems

Hiroaki Yoshimura[*] *and Jerrold E. Marsden*[†]

1 Introduction

The idea of multiport systems has been known as a useful tool when regarding a system as an *interconnection* of physical elements throughout principle of power invariance, which has been widely used in electrical circuits and networks (see, for instance, [4, 14]), where the principle of power invariance is known as Tellegen's theorem in electrical network theory (see [6]). From the viewpoint of the analogy between mechanical and electrical systems, much effort has been done to develop a network-theoretic approach to nonlinear mechanical systems such as multibody systems in the context of interconnected systems (see, for instance, [15]). Recently, it was shown by [12] and [3] that such interconnections can be represented by Dirac structures, which may be a generalization of both symplectic as well as Poisson structures (see [8, 7]) and also that interconnections of L-C circuits can be modeled by Dirac structures and then incorporated into the context of implicit Hamiltonian systems (see also [1, 2]). On the Lagrangian side, a notion of implicit Lagrangian systems, namely, a Lagrangian analogue of implicit Hamiltonian systems, was developed by [16, 17], where nonholonomic mechanical systems and L-C circuits as degenerate Lagrangian systems were shown to be formulated in the context of implicit Lagrangian systems in which induced Dirac structures were systematically introduced. Furthermore, it was shown by [19] that even for the case in which a given Lagrangian is degenerate, an implicit Hamiltonian system can be constructed

[*]Applied Mechanics and Aerospace Engineering, Waseda University. Research supported in part by JSPS Grant 20560229; email: yoshimura@waseda.jp

[†]Control and Dynamical Systems, California Institute of Technology. Research partially supported by NSF grant DMS-0505711; email: marsden@cds.caltech.edu

from an implicit Lagrangian system by the generalized Legendre transformation.

From the viewpoint of control theory, a notion of *implicit port-controlled Hamiltonian (IPCH) systems*, which is an implicit Hamiltonian system with external control inputs such as control forces and voltages, was developed by [12]. For the case of regular Lagrangian systems, the equivalence between *controlled Lagrangian (CL) systems* and *controlled Hamiltonian (CH)* systems was shown by [5]. For the general case in which a given Lagrangian is degenerate, an implicit Lagrangian analogue of IPCH systems, namely, a notion of *implicit port-controlled Lagrangian (IPCL) systems* for electrical circuits was constructed by [18], where it was shown that L-C transmission lines can be represented in the context of the IPLC system by employing induced Dirac structures with Lagrange multipliers.

In this paper, we will develop two different representations for induced Dirac structures and their associated IPCL systems; namely, (1) a standard representation with Lagrange multipliers; and (2) a representation without Lagrange multipliers. Those representations are consistent with those developed by [8, 7, 3, 9]. Specifically, the second representation without using Lagrange multipliers may be crucial in formulation of constrained mechanical systems since it systematically enables one to eliminate unnecessary constraint forces. In mechanics, it is known that the elimination of constraint forces can be done by the *orthogonal complement method* or the *null space method* (see, for instance, [15, 10]), although the link with Dirac structures has not been clarified. The present paper fills this gap to show that the orthogonal complement method can be incorporated into the context of Dirac structures and the associated IPCL systems and we demonstrate those representations for Dirac structures and implicit Lagrangian systems by an example of L-C circuits.

2 Dirac Structures

Dirac Structures. Recall from [7, 8] that a Dirac structure D on a vector space V is defined as a maximally isotropic subspace of $V \oplus V^*$ such that $D = D^{\perp}$, where V^* is the dual space of V and $D^{\perp}$ is the orthogonal of D relative to the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, which is a symmetric paring on $V \oplus V^*$ given by

$$\langle\!\langle\, (v, \alpha), (\bar{v}, \bar{\alpha})\, \rangle\!\rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle, \tag{1}$$

for $(v, \alpha), (\bar{v}, \bar{\alpha}) \in V \oplus V^*$. In the above, $\langle \cdot\,, \cdot \rangle$ is the natural paring between V^* and V. For the case of a smooth manifold M whose tangent bundle is denoted as TM and whose cotangent bundle is denoted as T^*M, an *(almost) Dirac structure* on M is a subbundle $D \subset TM \oplus T^*M$ that is a Dirac structure in the sense of vector spaces at each point $x \in M$. In geometric mechanics, (almost) Dirac structures provide a simultaneous generalization of both two-forms (not necessarily closed, and possibly degenerate) as well as almost Poisson structures (that is brackets that need not satisfy the Jacobi identity) (see, for instance, [11]). An *integrable Dirac structure*, which corresponds in geometric mechanics to assuming the two-form is closed or to assuming Jacobi's identity for the Poisson tensor, is one that satisfies

$$\langle \pounds_{X_1} \alpha_2, X_3 \rangle + \langle \pounds_{X_2} \alpha_3, X_1 \rangle + \langle \pounds_{X_3} \alpha_1, X_2 \rangle = 0,$$

for all pairs of vector fields and one-forms (X_1, α_1), (X_2, α_2), $(X_3, \alpha_3) \in D$.

Distributions. Recall from [2] that any Dirac structure D on M naturally defines a distribution $\Delta_M \subset TM$, which is denoted by, for each $x \in M$,

$$\Delta_M(x) = \{v_x \in T_xM \mid (v_x, 0) \in D(x)\}$$

and a codistribution $\Gamma_M \subset T^*M$ whose fibers are given by

$$\Gamma_M(x) = \{\alpha_x \in T_x^*M \mid \exists v_x \text{ such that } (v_x, \alpha_x) \in D(x)\}\,.$$

In this paper, we assume that all distributions are regular, i.e., smooth and constant in dimension at each point $x \in M$. Since D is isotropic, one has $\Delta_M(x) \subset \Gamma_M^\circ(x)$, where $\Gamma_M^\circ(x) \subset T_xM$ is the annihilator of $\Gamma_M(x) \subset T_x^*M$, which is defined by

$$\Gamma_M^\circ(x) = \{v_x \in T_xM \mid \langle \alpha_x, v_x \rangle = 0 \text{ for all } \alpha_x \in \Gamma_M(x)\}\,.$$

Similarly, one can read $\Gamma_M(x) \subset \Delta_M^\circ(x)$, where $\Delta_M^\circ(x) \subset T_x^*M$ is the annihilator of $\Delta_M(x)$, which is given by

$$\Delta_M^\circ(x) = \{\alpha_x \in T_x^*M \mid \langle \alpha_x, v_x \rangle = 0 \text{ for all } v_x \in \Delta_M(x)\}\,.$$

Since D is maximally isotropic, it reads $\Delta_M = \Gamma_M^\circ$ and equivalently, $\Gamma_M = \Delta_M^\circ$.

Induced Dirac Structures. Let us review an induced Dirac structure for the setting of *implicit port-controlled Lagrangian systems* by following [16].

Let Q be an n-dimensional configuration manifold, whose kinematic constraints are given by a constraint distribution $\Delta_Q \subset TQ$, given by, at each $q \in Q$,

$$\Delta_Q(q) = \{v \in T_qQ \;\mid\; \langle \omega^a(q), v \rangle = 0,\; a = 1, ..., m\},$$

where ω^a are m one-forms on Q. Define the distribution Δ_{T^*Q} on T^*Q by

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,$$

where $T\pi_Q : TT^*Q \to TQ$ is the tangent map of the cotangent bundle projection $\pi_Q : T^*Q \to Q$, while the annihilator of Δ_{T^*Q} can be defined by, for each $z \in T_q^*Q$,

$$\Delta_{T^*Q}^\circ(z) = \{\alpha_z \in T_z^*T^*Q \mid \langle \alpha_z,\, w_z \rangle = 0 \;\text{ for all } w_z \in \Delta_{T^*Q}(z)\}.$$

Let Ω be the canonical symplectic structure on T^*Q and $\Omega^\flat : TT^*Q \to T^*T^*Q$ be the associated bundle map. Then, a Dirac structure D_{Δ_Q} on T^*Q induced from Δ_Q can be defined by, for each $z \in T_q^*Q$,

$$D_{\Delta_Q}(z) = \{\, (w_z, \alpha_z) \in T_zT^*Q \times T_z^*T^*Q \;\mid\; w_z \in \Delta_{T^*Q}(z), \\ \text{and } \; \alpha_z - \Omega^\flat(z) \cdot w_z \in \Delta_{T^*Q}^\circ(z)\,\}.$$

Local Expression. Let us choose local coordinates q^i on Q so that Q is locally represented by an open set $U \subset \mathbf{R}^n$. The constraint set Δ_Q defines a subspace of TQ, which we denote by $\Delta(q) \subset \mathbf{R}^n$ at each point $q \in U$. If the dimension of $\Delta(q)$ is $n - m$, then we can choose a basis $e_{m+1}(q), e_{m+2}(q), \ldots, e_n(q)$ of $\Delta(q)$.

Recall that the constraint sets can be also represented by the annihilator of $\Delta(q)$, which is denoted by $\Delta^\circ(q)$ spanned by such one-forms that we write as $\omega^1, \omega^2, \ldots, \omega^m$. Using $\pi_Q : T^*Q \to Q$ locally denoted by $z = (q,p) \mapsto q$ and $T\pi_Q : TT^*Q \to TQ; (q,p,\dot{q},\dot{p}) \mapsto (q,\dot{q})$, it follows that

$$\Delta_{T^*Q} \cong \{(q,p,\dot{q},\dot{p}) \mid q \in U, \dot{q} \in \Delta(q)\}.$$

Let points in T^*T^*Q be locally denoted by (q,p,β,u), where β is a covector and u is a vector. Then, the annihilator of Δ_{T^*Q} is locally represented as

$$\Delta^\circ_{T^*Q} \cong \{(q,p,\beta,u) \mid q \in U,\ \beta \in \Delta^\circ(q) \text{ and } u = 0\}.$$

Since we have the local formula $\Omega^\flat(q,p) \cdot w_{(q,p)} = (q,p,-\dot{p},\dot{q})$, the condition $\alpha_{(q,p)} - \Omega^\flat(q,p) \cdot w_{(q,p)} \in \Delta^\circ_{T^*Q}$ reads $\alpha + \dot{p} \in \Delta^\circ(q)$, and $w - \dot{q} = 0$, where $\alpha_{(q,p)} = (q,p,\alpha,w)$ and $w_{(q,p)} = (q,p,\dot{q},\dot{p})$. So, the induced Dirac structure is locally represented by

$$D_{\Delta_Q}(q,p) = \{((\dot{q},\dot{p}),(\alpha,w)) \mid \dot{q} \in \Delta(q),\ w = \dot{q},\ \alpha + \dot{p} \in \Delta^\circ(q)\}. \tag{2}$$

Representation (I). Let us introduce a matrix representation of D_{Δ_Q} given in equation (2). First, let $\mathbf{N}^T(q)$ be an $n \times m$ matrix whose m-column vectors $\omega^1(q), ..., \omega^m(q)$ span the basis of $\Delta^\circ(q)$, namely, $\mathbf{N}^T(q) = [\omega^1(q), ..., \omega^m(q)]$ and the distribution $\Delta(q) \subset \mathbf{R}^n \cong T_qQ$ may be represented by

$$\Delta(q) = \{\dot{q} \in \mathbf{R}^n \mid \mathbf{N}(q)\,\dot{q} = 0\}.$$

So, using Lagrange multipliers $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbf{R}^m$, one has

$$\Delta^\circ(q) = \left\{\beta \in (\mathbf{R}^\mathbf{n})^* \mid \beta = \mathbf{N}^T(q)\lambda\right\}.$$

Thus, the induced Dirac structure can be represented by

$$D_{\Delta_Q}(q,p) = \left\{((\dot{q},\dot{p}),(\alpha,w)) \mid \mathbf{N}(q)\dot{q} = 0,\ w = \dot{q},\ \alpha + \dot{p} = \mathbf{N}^T(q)\lambda\right\}. \tag{3}$$

Example: L-C Circuits. Consider an illustrative example of L–C circuits shown in Fig.1, which was also investigated in [3, 13]. Refer also to [16, 17, 19]. In the L–C circuit, the configuration space W is a 4-dimensional vector space, that is, $W = \mathbf{R}^4$. Then, we have $TW\,(\cong W \times W)$ and $T^*W\,(\cong W \times W^*)$. Let $q = (q_L, q_{C_1}, q_{C_2}, q_{C_3}) \in W$ denote charges and $\dot{q} = (\dot{q}_L, \dot{q}_{C_1}, \dot{q}_{C_2}, \dot{q}_{C_3}) \in T_qW$ currents associated with the L–C circuit. The set of currents satisfying the KCL (Kirchhoff current law) constraints forms a constraint KCL space $\Delta \subset TW$, which is given for each $q = (q^1, q^2, q^3, q^4) = (q_L, q_{C_1}, q_{C_2}, q_{C_3}) \in W$, by the 2-dimensional distribution $\Delta(q) \subset \mathbf{R}^4 \cong T_qW$ such that

$$\Delta(q) = \left\{\dot{q} \in \mathbf{R}^4 \mid \mathbf{N}(q)\,\dot{q} = 0\right\},$$

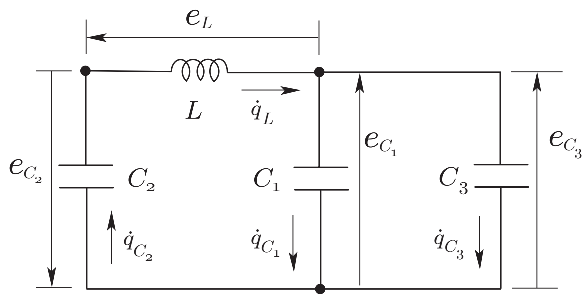

Figure 1. *L-C Circuit*

where the matrix $\mathbf{N}(q)$ is given by

$$\mathbf{N}(q) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}. \tag{4}$$

On the other hand, the transpose of $\mathbf{N}(q)$ span the 2-dimensional annihilator $\Delta^\circ(q) \subset (\mathbf{R}^4)^* \cong T_q^*W$ as

$$\Delta^\circ(q) = \left\{\beta \in (\mathbf{R}^4)^* \mid \beta = \mathbf{N}^T(q)\lambda\right\}$$

where $\lambda = (\lambda_1, \lambda_2)$ denotes Lagrange multipliers. Note that the *constraint voltage space* is spanned by the transpose of $N(q)$, namely, $N^T(q) = [\omega^1(q), \omega^2(q)]$, where ω^1 and ω^2 are one-forms on W associated with the KCL constraints.

Consistent with the general theory, the induced distribution Δ_{T^*W} on T^*W is defined by the KCL constraint distribution $\Delta \subset TW$ as

$$\Delta_{T^*W} = (T\pi_W)^{-1}(\Delta) \subset TT^*W,$$

where $\pi_W : T^*W \to W$ is the canonical projection and $T\pi_W : TT^*W \to TW$. Now writing the projection map $\pi_W : T^*W \to W$ locally as $(q,p) \mapsto q$, its tangent map is locally given by $T\pi_W : (q,p,\dot{q},\dot{p}) \mapsto (q,\dot{q})$. Then, we can represent Δ_{T^*W} as

$$\Delta_{T^*W} \cong \{(q,p,\dot{q},\dot{p}) \mid q \in U,\ \mathbf{N}(q)\dot{q} = 0\}.$$

The annihilator of Δ_{T^*W} is denoted by

$$\Delta^\circ_{T^*W} \cong \left\{(q,p,\beta,u) \mid q \in U,\ \beta = \mathbf{N}^T(q)\lambda \text{ and } u = 0\right\}.$$

Recall also from equation (2) that the Dirac structure D_Δ on T^*W induced from the KCL constraint distribution Δ is locally given, for each $(q,p) \in T^*W$, by

$$D_\Delta(q,p) = \left\{((q,p,\dot{q},\dot{p}),(q,p,\alpha,w)) \mid \mathbf{N}(q)\dot{q} = 0,\ w = \dot{q},\ \alpha + \dot{p} = \mathbf{N}^T(q)\lambda\right\}.$$

Representation (II). In Representation (I) for the induced Dirac structure in equation (3), we utilized the Lagrange multipliers, which represent *constraint forces* in constrained mechanical systems. Here, let us develop another representation of D_{Δ_Q} on T^*Q without using the Lagrange multipliers.

Let us choose an $n \times (n-m)$ matrix $\mathbf{B}(q) = [e_{m+1}(q), e_{m+2}(q), \ldots, e_n(q)]$, whose column vectors span the basis of $\Delta(q)$. Then, it follows that the distribution $\Delta(q) \subset \mathbf{R}^n \cong T_qQ$ can be also represented by

$$\Delta(q) = \{\dot{q} \in \mathbf{R}^n \mid \dot{q} = \mathbf{B}(q)\, u\},$$

where $u = (u^{m+1}, u^{m+2}, ..., u^n) \in \mathbf{R}^{n-m}$. It is needless to say that the orthogonality condition between $\mathbf{N}(q)$ and $\mathbf{B}(q)$ holds as $\mathbf{B}^T(q)\mathbf{N}^T(q) = 0$. The above condition naturally comes from the fact that Δ° is the annihilator of the distribution Δ; namely, in other words, the basis $e_{m+1}(q), ..., e_n(q)$ is orthogonal to the dual basis $\omega^1(q), ..., \omega^m(q)$ at each $q \in Q$. Therefore, one can read that

$$\Delta^\circ(q) = \left\{\beta \in (\mathbf{R}^n)^* \mid \mathbf{B}^T(q)\, \beta = 0\right\}.$$

Thus, the induced Dirac structure $D_{\Delta_Q} \subset TT^*Q \oplus T^*T^*Q$ can be represented without using the Lagrange multipliers as

$$D_{\Delta_Q}(q,p) = \left\{((\dot{q},\dot{p}),(\alpha,w)) \mid \mathbf{N}(q)\dot{q} = 0,\ w = \dot{q},\ \mathbf{B}^T(q)(\alpha + \dot{p}) = 0\right\}. \qquad (5)$$

Example: L-C Circuits. Associated with the matrix in equation (4), namely,

$$\mathbf{N}(q) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix},$$

we can construct its orthogonal complementary matrix $\mathbf{B}(q)$ such that $\mathbf{B}^T(q)\mathbf{N}^T(q) = 0$, which is, by inspection, given by

$$\mathbf{B}^T(q) = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \qquad (6)$$

By using this matrix $\mathbf{B}$, we can eliminate unnecessary Lagrange multipliers, which may correspond to the elimination of constraint forces in mechanics. This lead to the representation

$$\Delta^\circ(q) = \left\{\beta \in (\mathbf{R}^n)^* \mid \mathbf{B}^T(q)\, \beta = 0\right\}.$$

It immediately follows that we can obtain Representation (II) in equation (5).

Remarks. It was shown by [15] that the **dual connection matrices** $\mathbf{N}(q)$ and $\mathbf{B}(q)$ are the orthogonal complement with each other, each of which plays an essential role in eliminating unnecessary constraint forces in constrained mechanical systems such as multibody systems and also that from the viewpoint of analogy between mechanics and circuits, the matrices $\mathbf{N}(q)$ and $\mathbf{B}(q)$ may correspond to the so-called *fundamental cut–set* and *loop matrices* in electrical network theory. In conjunction with optimal control theory, the matrix $\mathbf{B}^T(q)$ is sometimes called the *null space matrix* (see, for instance, [10]).

3 Implicit Port-Controlled Lagrangian Systems

Let us define an implicit port-controlled Lagrangian system, which is an implicit Lagrangian system that has an external control force.

Implicit Port-Controlled Lagrangian Systems. Given an external control force field $F : TQ \to T^*Q$, it induces a horizontal one-form $\tilde{F}$ on T^*Q such that, for $v_q \in TQ$ and $p_q \in T^*Q$,

$$\tilde{F}(p_q) \cdot w = \langle F(v_q), T\pi_Q(w) \rangle ,$$

where $\tilde{F} = (q, p, F(v), 0) \in T^*_{p_q} T^*Q$ and $w = (q, p, \delta q, \delta p) \in T_{p_q} T^*Q$.

Let L be a given Lagrangian on TQ and let $\Delta_Q \subset TQ$ be a constraint distribution. Then, an **implicit port-controlled Lagrangian (IPCL) system** or an **implicit controlled Lagrangian (ICL) system** is defined by a triple (L, F, Δ_Q, X), which satisfies for each $(q, v) \in \Delta_Q \subset TQ$ and with $(q, p) = \mathbf{F}L(q, v)$,

$$(X(q, v, p), \mathcal{D}L(q, v) - \pi_Q^* F(q, v)) \in D_{\Delta_Q}(q, v). \tag{7}$$

In the above, $\mathcal{D}L : TQ \to T^*T^*Q$ denotes the Dirac differential of a given Lagrangian L, given by $\mathcal{D}L(q, v) = (q, \partial L/\partial v, -\partial L/\partial q, v)$ and $X : TQ \oplus T^*Q \to TT^*Q$ is a partial vector field defined at points $(v, p) \in \Delta_Q \times P$ that assigns a vector in T_pT^*Q to each point $(q, v, p) \in \Delta_Q \oplus P$, where $P = \mathbf{F}L(\Delta_Q) \subset TQ$. Let us write $X(q, v, p) = (q, p, \dot{q}, \dot{p})$, so that $\dot{q}$ and $\dot{p}$ are functions of (q, v, p) and it follows that the local expression of IPCL systems in equation (7) may be locally given by

$$\dot{q} = v, \quad \dot{p} - \frac{\partial L}{\partial q} - F(q, v) \in \Delta^\circ(q), \quad \dot{q} \in \Delta(q), \quad p = \frac{\partial L}{\partial v}. \tag{8}$$

The curve $(q(t), v(t), p(t))$, $t_1 \le t \le t_2$ in $TQ \oplus T^*Q$ that satisfies the condition (7) is a solution curve of the IPCL system (L, F, Δ_Q, X).

Power Balance. Let $(q(t), v(t), p(t))$, $t_1 \le t \le t_2$ in $TQ \oplus T^*Q$ be a solution curve of the IPCL system. Let E be the generalized energy, which is given by

$$E(q, v, p) = \langle p, v \rangle - L(q, v)$$

and the time derivative of E reads

$$\frac{d}{dt} E(q(t), v(t), p(t)) = \langle F(q(t), \dot{q}(t)), \dot{q}(t) \rangle ,$$

where we employed $p = \partial L/\partial v$ and $\dot{q} = v$.

Representation (I). Using Representation (I) in equation (3) for the induced Dirac structure, an IPCL system in equation (8) can be represented in matrix by

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial L}{\partial q} - F(q, v) \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{N}^T(q)\, \lambda \end{pmatrix}, \\ 0 &= \mathbf{N}(q)\, \dot{q}, \\ p &= \frac{\partial L}{\partial v}. \end{aligned}$$

In the above, the IPCL system may be represented by differential–algebraic equations for the $3n$-dimensional local coordinates (q, v, p) for $TQ \oplus T^*Q$ and with the m Lagrange multipliers $\lambda = (\lambda_1, ..., \lambda_m)$. The term on the right-hand side

$$Y(q,p) = \begin{pmatrix} 0 \\ \mathbf{N}^T(q)\,\lambda \end{pmatrix}$$

denotes the *vertical vector field* $Y : T^*Q \to TT^*Q$ associated to the constraint force field $F^c : TQ \to T^*Q$, which is referred to as a *special horizontal one-form* $\tilde{F}^c = -\mathbf{i}_Y\Omega$ on T^*Q that satisfies

$$\tilde{F}^c(p_q) \cdot w = \left\langle F^c(v_q), T_{v_q}\pi_Q(w) \right\rangle = 0$$

for $v_q \in TQ$, $p_q \in T^*Q$ and $w = (q, p, \delta q, \delta p) \in \Delta_{T^*Q}(p_q)$.

Example: L-C Circuits. Let $T : TW \to \mathbf{R}$ be the magnetic energy of the L-C circuit, which is defined by the inductance L as

$$T_q(f) = \frac{1}{2} L\,(f_L)^2,$$

and let $V : W \to \mathbf{R}$ be the electric potential energy of the L-C circuit, which is defined by capacitors C_1, C_2, and C_3 such that

$$V(q) = \frac{1}{2}\frac{(q_{C_1})^2}{C_1} + \frac{1}{2}\frac{(q_{C_2})^2}{C_2} + \frac{1}{2}\frac{(q_{C_3})^2}{C_3}.$$

Then, we can define the Lagrangian of the L–C circuit $\mathcal{L} : TW \to \mathbf{R}$ by

$$\mathcal{L}(q,f) = T_q(f) - V(q) = \frac{1}{2} L\,(f_L)^2 - \frac{1}{2}\frac{(q_{C_1})^2}{C_1} - \frac{1}{2}\frac{(q_{C_2})^2}{C_2} - \frac{1}{2}\frac{(q_{C_3})^2}{C_3}.$$

It is obvious that $\mathcal{L} : TW \to \mathbf{R}$ of the L-C circuit is degenerate, since

$$\det \left[\frac{\partial^2 \mathcal{L}}{\partial f^i \partial f^j} \right] = 0; \quad i, j = 1, ..., 4.$$

The constraint distribution $\Delta \subset TW$ is given by using the matrix $\mathbf{N}(q)$ as in equation (4), while the *constraint flux linkage subspace* is given by using the Legendre transform as

$$P = \mathbf{F}\mathcal{L}(\Delta) \subset T^*W.$$

In coordinates, $(q, p) = \mathbf{F}\mathcal{L}(q, f) \in T^*W$, and it follows

$$(p_L, p_{C_1}, p_{C_2}, p_{C_3}) = \left(\frac{\partial \mathcal{L}}{\partial f_L}, \frac{\partial \mathcal{L}}{\partial f_{C_1}}, \frac{\partial \mathcal{L}}{\partial f_{C_2}}, \frac{\partial \mathcal{L}}{\partial f_{C_3}} \right),$$

from which we obtain $p_L = L\,f_L$ and with the constraints $p_{C_1} = 0$, $p_{C_2} = 0$, $p_{C_3} = 0$, which correspond to **primary constraints in the sense of Dirac**. Needless to say, the primary constraints form the constraint flux linkage subspace $P \subset T^*W$.

Let $X : TW \oplus T^*W \to TT^*W$ be a partial vector field on T^*W, defined at each point in P, with components denoted by

$$X(q,f,p) = (\dot{q}_L, \dot{q}_{C_1}, \dot{q}_{C_2}, \dot{q}_{C_3}, \dot{p}_L, 0, 0, 0)\,.$$

The Dirac differential of L, namely, $\mathcal{DL}(q,f) = (q, \partial\mathcal{L}/\partial v, -\partial\mathcal{L}/\partial q, f)$ is given by

$$\mathcal{DL}\,(q,f) = \left(q_L, q_{C_1}, q_{C_2}, q_{C_3}, p_L, 0, 0, 0, 0, \frac{q_{C_1}}{C_1}, \frac{q_{C_2}}{C_2}, \frac{q_{C_3}}{C_3}, f_L, f_{C_1}, f_{C_2}, f_{C_3}\right)$$

together with $p = \partial\mathcal{L}/\partial f$.

Thus, the L-C circuit can be represented in the context of implicit Lagrangian systems $(\mathcal{L}, \Delta, X)$ by requiring that, for each $(q,f) \in \Delta \subset TW$,

$$(X(q,f,p), \mathcal{DL}(q,f)) \in D_\Delta(q,p)$$

holds and with the Legendre transform $(q,p) = \mathbf{F}\mathcal{L}(q,f)$. Therefore, the implicit Lagrangian system for this L-C circuit may be locally described by

$$\begin{pmatrix} \dot{q}_L \\ \dot{q}_{C_1} \\ \dot{q}_{C_2} \\ \dot{q}_{C_3} \\ \hline \dot{p}_L \\ 0 \\ 0 \\ 0 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} 0 \\ \frac{q_{C_1}}{C_1} \\ \frac{q_{C_2}}{C_2} \\ \frac{q_{C_3}}{C_3} \\ \hline f_L \\ f_{C_1} \\ f_{C_2} \\ f_{C_3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

together with the Legendre transformation $p_L = L\,f_L$. The above equations of motion are supplemented by the KCL constraints

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \dot{q}_L \\ \dot{q}_{C_1} \\ \dot{q}_{C_2} \\ \dot{q}_{C_3} \end{pmatrix}.$$

Thus, we obtain the differential-algebraic equations as

$$\begin{aligned} &\dot{q}_L = f_L,\ \dot{q}_{C_1} = f_{C_1},\ \dot{q}_{C_2} = f_{C_2},\ \dot{q}_{C_3} = f_{C_3}, \\ &\dot{p}_L = -\lambda_1,\ \lambda_2 = -\frac{q_{C_1}}{C_1},\ \lambda_1 = -\lambda_2 + \frac{q_{C_2}}{C_2},\ \lambda_2 = -\frac{q_{C_3}}{C_3}, \\ &p_L = L\,f_L,\ \dot{q}_L = \dot{q}_{C_2},\ \dot{q}_{C_1} = \dot{q}_{C_2} - \dot{q}_{C_3}. \end{aligned}$$

Representation (II). Using Representation (II) in (5) for the induced Dirac structure, it follows that an IPCL system in equation (8) can be represented by

$$0 = \mathbf{B}^T(q)\left\{\dot{p} - \frac{\partial L}{\partial q} - F(q,v)\right\},$$

$$
\begin{aligned}
0 &= \mathbf{N}(q)\,\dot{q}, \\
\dot{q} &= v, \\
p &= \frac{\partial L}{\partial v}.
\end{aligned}
$$

In the above, the constraint forces represented by Lagrange multipliers $\lambda = (\lambda_1, ..., \lambda_m)$ are eliminated by premultiplying the matrix $\mathbf{B}^T(q)$ and the equations of motion for the system are given by the set of differential–algebraic equations for the $3n$-dimensional local coordinates (q, v, p) for $TQ \oplus T^*Q$.

Example: L-C Circuits. By analogy with mechanics, dynamics of electric circuits can be described in Representation (II) without Lagrange multipliers. As to the representation (II) of the L-C circuit in Fig.1, the equations of motion

$$
0 = \mathbf{B}^T(q)\left\{\dot{p} - \frac{\partial L}{\partial q} - F(q, v)\right\}
$$

are given, in coordinates, by

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \left[\begin{pmatrix} \dot{p}_L \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{q_{C_1}}{C_1} \\ \frac{q_{C_2}}{C_2} \\ \frac{q_{C_3}}{C_3} \end{pmatrix} \right],
$$

while the KCL constraints $0 = \mathbf{N}(q)\,\dot{q}$ are given, in coordinates, by

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \dot{q}_L \\ \dot{q}_{C_1} \\ \dot{q}_{C_2} \\ \dot{q}_{C_3} \end{pmatrix},
$$

together with $p_L = Lf_L$ and $\dot{q}_L = f_L,\ \dot{q}_{C_1} = f_{C_1},\ \dot{q}_{C_2} = f_{C_2},\ \dot{q}_{C_3} = f_{C_3}$.

Finally, the resultant equations of motion without Lagrange multipliers are given by the following implicit differential-algebraic equations:

$$
\begin{aligned}
&\dot{q}_L = f_L,\ \dot{q}_{C_1} = f_{C_1},\ \dot{q}_{C_2} = f_{C_2},\ \dot{q}_{C_3} = f_{C_3}, \\
&\dot{p}_L = -\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2},\ 0 = -\frac{q_{C_1}}{C_1} - \frac{q_{C_3}}{C_3}, \\
&p_L = L\,f_L,\ \dot{q}_L = \dot{q}_{C_2},\ \dot{q}_{C_1} = \dot{q}_{C_2} - \dot{q}_{C_3}.
\end{aligned}
$$

4 Conclusions

In this paper, we showed how implicit port-controlled Lagrangian (IPCL) systems can be developed in the context of Dirac structures with two representations: (1) a standard representation with Lagrange multipliers and (2) a representation using orthogonal complementary matrices that eliminates Lagrange multipliers. Especially, the latter representation enables one to obtain lower order system equations. Finally, we demonstrated our theory of IPCL systems by an example of L-C circuits.

Bibliography

[1] Blankenstein, G. [2000], *Implicit Hamiltonian Systems: Symmetry and Interconnection*, Ph.D. Dissertation. University of Twente.

[2] Blankenstein, G. and T. S. Ratiu [2004], Singular reduction of implicit Hamiltonian systems, *Rep. Math. Phys.* **53**, 211–260.

[3] Bloch, A. M. and P. E. Crouch [1997], Representations of Dirac structures on vector spaces and nonlinear L-C circuits. Differential Geometry and Control (Boulder, CO, 1997), 103–117, *Proc. Sympos. Pure Math.* **64**. Amer. Math. Soc., Providence, RI.

[4] Brayton, R. K. and J. K. Moser [1964], A theory of nonlinear networks I, II. In *Quart. Appl. Math.*, **22**, pages 1–33, 81–104.

[5] Chang, D., A. M. Bloch, N. Leonard, J. E. Marsden, and C. Woolsey [2002], The equivalence of controlled Lagrangian and controlled Hamiltonian systems, *Control and the Calculus of Variations (special issue dedicated to J.L. Lions)* **8**, 393–422.

[6] Chua, L. O., C. A. Desoer, and D. A. Kuh [1987], *Linear and Nonlinear Circuits*. McGraw-Hill Inc.

[7] Courant, T. J. [1990a], Dirac manifolds, *Trans. Amer. Math. Soc.* **319**, 631–661.

[8] Courant, T. and A. Weinstein [1988], Beyond Poisson structures. In *Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986)*, volume 27 of *Travaux en Cours*, pages 39–49. Hermann, Paris.

[9] Dalsmo, M. and A. J. van der Schaft [1998], On representations and integrability of mathematical structures in energy-conserving physical systems, *SIAM J. Control Optim.* **37** (1), 54–91.

[10] Leyendecker, S., S. Ober-Blöbaum, J. E. Marsden and M. Ortiz [2007], Discrete mechanics and optical control for constrained multibody dynamics, *Proceedings of International Design Engineering Technical Conferences & Computers and Information in Engineering Conference*. DETC2007-34574, 1–9, September 4-7, 2007, Las Vegas.

[11] Marsden, J. E. and T. S. Ratiu [1999], *Introduction to Mechanics and Symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, second edition.

[12] van der Schaft, A. J. and B. M. Maschke [1995], The Hamiltonian formulation of energy conserving physical systems with external ports, *Archiv für Elektronik und "Ubertragungstechnik* **49**, 362–371.

[13] van der Schaft, A. J. [1998], Implicit Hamiltonian systems with symmetry, *Rep. Math. Phys.* **41**, 203–221.

[14] Wyatt, J. L. and L. O. Chua [1977], A theory of nonenergic n-ports, *Circuit Theory and Applications* **5**, 181-208.

[15] Yoshimura, H. [1995], *Dynamics of Flexible Multibody Systems*, Doctoral Dissertation. Waseda University.

[16] Yoshimura, H. and J. E. Marsden [2006a], Dirac structures in Lagrangian mechanics Part I: Implicit Lagrangian systems, *J. Geom. and Phys.*, **57**, 133–156.

[17] Yoshimura, H. and J. E. Marsden [2006b], Dirac structures in Lagrangian mechanics Part II: Variational structures, *J. Geom. and Phys.*, **57**, 209–250.

[18] Yoshimura, H. and J. E. Marsden [2006c], Dirac structures and implicit Lagrangian systems in electric networks, Proc. of the 17th International Symposium on Mathematical Theory of Networks and Systems, Paper WeA08.5, pages 1–6, July 24-28, 2006, Kyoto.

[19] Yoshimura, H. and J. E. Marsden [2007], Dirac structures and the Legendre transformation for implicit Lagrangian and Hamiltonian systems, *Lagrangian and Hamiltonian Methods in Nonlinear Control 2006*, volume 366 of *Lecture Notes in Control and Information Science Series*, pages 233–247, Springer-Verlag.